On Symbolic 2-Plithogenic Real Matrices and Their Algebraic Properties

Article · January 2023
DOI: 10.54216/IJNS.210109

Arif Mehmood
Dera Ismail Khan, 29050, Pakistan

On Symbolic 2-Plithogenic Real Matrices and Their Algebraic Properties

Abuobida Mohammed A. Alfahal[1,*], Yaser Ahmad Alhasan[1], Raja Abdullah Abdulfatah[1]

Arif Mehmood[2], Mustafa Talal Kadhim[3]

[1]Deanship of the Preparatory Year, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia
[2]Department of mathematics, institute of numerical sciences, Gomal University, Dera Ismail Khan 29050, Pakistan
[3]Computer Techniques Engineering Department, Al-Mustaqbal University, Babil, Iraq

Emails: a.alfahal@psau.edu.sa; y.alhasan@psau.edu.sa; r.abdulfatah@psau.edu.sa ; mehdaniyal@gmail.com mostafat2000k@gmail.com
Corresponding author: a.alfahal@psau.edu.sa

Abstract

Symbolic n-plithogenic sets came with many generalizations to classical algebraic structures, with many interesting properties and theorems, where the symbolic 2-plithogenic structures are very similar in their algebraic properties to refined neutrosophic algebraic structures. The main goal of this article is to study the algebraic properties of symbolic 2-plithogenic matrices such as the computing on symbolic 2-plithogenic determinants, symbolic 2-plithogenic special values, and symbolic 2-plithogenic representations by linear functions. In addition, many examples will be presented and discussed in terms of theorems to clarify the validity of the content of this paper.

Keywords: symbolic 2-plithogenic matrix; symbolic 2-plithogenic ring; symbolic 2-plithogenic invertibility.

1. Introduction and Preliminaries

Generalizing classical algebraic structures is a new research direction based on using new general systems to build generalized structures over them [15-17]. The symbolic 2-plithogenic algebraic structures are considered as novel generalizations of classical well-known structures such as symbolic plithogenic vector spaces, modules, and rings [1-8].

Symbolic plithogenic algebraic structures are very similar to refined neutrosophic structures with some differences in the definition of the multiplication operation, see [9-14,16-18].

This work discusses the concept of symbolic 2-plithogenic matrices with symbolic 2-plithogenic entries, were determinants, eigen values and vectors, exponents, and diagonalization will be handled in terms of theorems and examples. First, we recall some related concepts:

Definition.

The symbolic 2-plithogenic ring of real numbers is defined as follows:

$$2-SP_R = \{t_0 + t_1P_1 + t_2P_2; t_i \in R, P_1 \times P_2 = P_2 \times P_1 = P_2, {P_1}^2 = {P_2}^2 = P_2\}$$

The addition operation on $2-SP_R$ is defined as follows:

$$(t_0 + t_1P_1 + t_2P_2) + (t_0' + t_1'P_1 + t_2'P_2) = (t_0 + t_0') + (t_1 + t_1')P_1 + (t_2 + t_2')P_2$$

The multiplication on $2-SP_R$ is defined as follows:

$$(t_0 + t_1P_1 + t_2P_2)(t_0' + t_1'P_1 + t_2'P_2)$$
$$= t_0t_0' + (t_0t_1' + t_1t_0' + t_1t_1')P_1 + (t_0t_2' + t_1t_2' + t_2t_2' + t_2t_0' + t_2t_1')P_2$$

DOI: https://doi.org/10.54216/IJNS.210109
Received: February 08, 2023 Revised: April 24, 2023 Accepted: May 06, 2023

Example.

Take $T = 1 + 2P_1 - 5P_2, L = 3 + 4P_1 + 11P_2$, we have:

$T + L = 4 + 6P_1 + 6P_2, T \times L = 3 + (4 + 6 + 8)P_1 + (11 + 22 - 55 - 15 - 20)P_2 = 3 + 418P_1 - 57P_2$

Remark.

If $T = t_0 + t_1P_1 + t_2P_2 \in 2 - SP_R$, then:

$T^{-1} = \frac{1}{T} = \frac{1}{t_0} + \left[\frac{1}{t_0+t_1} - \frac{1}{t_0}\right]P_1 + \left[\frac{1}{t_0+t_1+t_2} - \frac{1}{t_0+t_1}\right]P_2$, with $t_0 \neq 0, t_0 + t_1 \neq 0, t_0 + t_1 + t_2 \neq 0$.

Main Discussion

Definition.

A symbolic 2-plithogenic square real matrix is a matrix with symbolic 2-plithogenic real entries.

Example:

Consider the following 3×3 2-plithogenic matrix:

$$\begin{pmatrix} 3 + P_1 - P_2 & 1 + P_1 & 5 \\ -P_1 + P_2 & 3P_1 & 4P_2 \\ -1 + 2P_1 - P_2 & 5 + 2P_2 & 7 + P_1 + 10P_2 \end{pmatrix}$$

Remark.

If L is a symbolic 2-plithogenic square real matrix, then L can be written as follows:

$L = L_0 + L_1P_1 + L_2P_2$, where L_i are three classical square real matrices.

Example.

$$\begin{pmatrix} 3 + P_1 - P_2 & 1 + P_1 & 5 \\ -P_1 + P_2 & 3P_1 & 4P_2 \\ -1 + 2P_1 - P_2 & 5 + 2P_2 & 7 + P_1 + 10P_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 5 \\ 0 & 0 & 0 \\ -1 & 5 & 7 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}P_1 + \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 4 \\ -1 & 2 & 10 \end{pmatrix}P_2$$

Remark.

Let $L = L_0 + L_1P_1 + L_2P_2, T = T_0 + T_1P_1 + T_2P_2$, be two square 2-plithogenic matrices, then:

$L + T = (L_0 + T_0) + (L_1 + T_1)P_1 + (L_2 + T_2)P_2$.

$L \times T = L_0T_0 + (L_0T_1 + L_1T_0 + L_1T_1)P_1 + (L_0T_2 + L_1T_2 + L_2T_2 + L_2T_0 + L_2T_1)P_2$

We denote the ring of all symbolic 2-plithogenic matrices by $2 - SP_M$.

Theorem.

Let $S = S_0 + S_1P_1 + S_2P_2$ be a symbolic 2-plithogenic square real matrix, then:

1). S is invertible if and only if $S_0, S_0 + S_1, S_0 + S_1 + S_2$ are invertible.

2). If S is invertible then $S^{-1} = S_0^{-1} + [(S_0 + S_1)^{-1} - S_0^{-1}]P_1 + [(S_0 + S_1 + S_2)^{-1} - (S_0 + S_1)^{-1}]P_2$

3). $S^m = S_0^m + [(S_0 + S_1)^m - S_0^m]P_1 + [(S_0 + S_1 + S_2)^m - (S_0 + S_1)^m]P_2$ for $m \in N$.

Proof.

1),2). Assume that $S_0, S_0 + S_1, S_0 + S_1 + S_2$ are invertible, then we put $K = K_0 + K_1P_1 + K_2P_2$, where $K_0 = S_0^{-1}, K_1 = (S_0 + S_1)^{-1} - S_0^{-1}, K_2 = (S_0 + S_1 + S_2)^{-1} - (S_0 + S_1)^{-1}$, then:

$S \times K = S_0K_0 + (S_0K_1 + S_1K_0 + S_1K_1)P_1 + (S_0K_2 + S_1K_2 + S_2K_2 + S_2K_0 + S_2K_1)P_2$

We have:

$S_0K_0 = U_{n\times n}$.

$$\begin{aligned} S_0K_1 + S_1K_0 + S_1K_1 &= S_0(S_0 + S_1)^{-1} - S_0S_0^{-1} + S_0(S_0 + S_1)^{-1} - S_0S_0^{-1} + S_0S_0^{-1} \\ &= (S_0 + S_1)(S_0 + S_1)^{-1} - S_0S_0^{-1} = O_{n\times n} \end{aligned}$$

Also,

$$\begin{aligned} &S_0K_2 + S_1K_2 + S_2K_2 + S_2K_0 + S_2K_1 \\ &= S_0(S_0 + S_1 + S_2)^{-1} - S_0(S_0 + S_1)^{-1} + S_1(S_0 + S_1 + S_2)^{-1} - S_1(S_0 + S_1)^{-1} \\ &+ S_2(S_0 + S_1 + S_2)^{-1} - S_2(S_0 + S_1)^{-1} + S_2S_0^{-1} + S_2(S_0 + S_1)^{-1} - S_2S_0^{-1} \\ &= (S_0 + S_1 + S_2)(S_0 + S_1 + S_2)^{-1} - (S_0 + S_1)(S_0 + S_1)^{-1} = O_{n\times n} \end{aligned}$$

This implies that $K = S^{-1}$.

For the converse, we assume that S is invertible, then there exists $K = K_0 + K_1P_1 + K_2P_2 \in 2 - SP_M$ such that $S \times K = U_{n\times n}$.

$S \times K = U_{n\times n}$ is equivalent to:

$$\begin{cases} S_0K_0 = U_{n\times n} \ldots (1) \\ S_0K_1 + S_1K_0 + S_1K_1 = O_{n\times n} \ldots (2) \\ S_0K_2 + S_1K_2 + S_2K_2 + S_2K_0 + S_2K_1 = O_{n\times n} \ldots (3) \end{cases}$$

Equation (1) implies that S_0 is invertible and $K_0 = S_0^{-1}$.

DOI: https://doi.org/10.54216/IJNS.210109
Received: February 08, 2023 Revised: April 24, 2023 Accepted: May 06, 2023

By adding (1) to (2), we get $(S_0 + S_1)(K_0 + K_1) = U_{n\times n}$, so that $S_0 + S_1$ is invertible and $K_0 + K_1 = (S_0 + S_1)^{-1}$, thus $K_1 = (S_0 + S_1)^{-1} - K_0 = (S_0 + S_1)^{-1} - {S_0}^{-1}$.
By adding (1) to (2) to (3), we get:
$(S_0 + S_1 + S_2)(K_0 + K_1 + K_2) = U_{n\times n}$, thus $S_0 + S_1 + S_2$ is invertible and $K_0 + K_1 + K_2 = (S_0 + S_1 + S_2)^{-1}$, hence $K_2 = (S_0 + S_1 + S_2)^{-1} - (K_0 + K_1) = (S_0 + S_1 + S_2)^{-1} - (S_0 + S_1)^{-1}$.
This implies the proof.
3). For $m = 1$, we get $S^1 = S_0 + [(S_0 + S_1) - S_0]P_1 + [(S_0 + S_1 + S_2) - (S_0 + S_1)]P_2$.
We assume that it is true for $m = k$, then:

$$\begin{aligned} S^{k+1} = S \times S^k &= [S_0 + S_1P_1 + S_2P_2]\left[{S_0}^k + \left[(S_0 + S_1)^k - {S_0}^k\right]P_1 + [(S_0 + S_1 + S_2)^k - (S_0 + S_1)^k]P_2\right] \\ &= {S_0}^{k+1} + \left[S_0(S_0 + S_1)^k - {S_0}^{k+1} + S_1(S_0 + S_1)^k - S_1{S_0}^k + S_1{S_0}^k\right]P_1 \\ &+ \left[S_0(S_0 + S_1 + S_2)^k - S_0(S_0 + S_1)^k + S_1(S_0 + S_1 + S_2)^k - S_1(S_0 + S_1)^k\right. \\ &\left. + S_2(S_0 + S_1 + S_2)^k - S_2(S_0 + S_1)^k + S_2{S_0}^k + S_2(S_0 + S_1)^k - S_2{S_0}^k\right]P_2 \\ &= {S_0}^{k+1} + \left[(S_0 + S_1)(S_0 + S_1)^k - {S_0}^{k+1}\right]P_1 \\ &+ [(S_0 + S_1 + S_2)(S_0 + S_1 + S_2)^k - (S_0 + S_1)(S_0 + S_1)^k]P_2 \\ &= {S_0}^{k+1} + \left[(S_0 + S_1)^{k+1} - {S_0}^{k+1}\right]P_1 + [(S_0 + S_1 + S_2)^{k+1} - (S_0 + S_1)^{k+1}]P_2 \end{aligned}$$

Definition.
Let $L = L_0 + L_1P_1 + L_2P_2 \in 2 - SP_M$, we define:
$detL = det(L_0) + [det(L_0 + L_1) - det(L_0)]P_1 + [det(L_0 + L_1 + L_2) - det(L_0 + L_1)]P_2$.

Example.
Take $L = \begin{pmatrix} 1 + P_1 + P_2 & 3 - P_1 + 2P_2 \\ P_1 & P_1 + P_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}P_1 + \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}P_2 = L_0 + L_1P_1 + L_2P_2$.
$det(L_0) = 0, det(L_0 + L_1) = det\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} = 0, det(L_0 + L_1 + L_2) = det\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} = 2$
So that $detL = 0 + (0 - 0)P_1 + (2 - 0)P_2 = 2P_2$.
If we can compute $detL$ by the normal way, we get:

$$\begin{aligned} detL &= (1 + P_1 + P_2)(P_1 + P_2) - (3 - P_1 + 2P_2)(P_1) = P_1 + P_2 + P_1 + P_2 + P_2 + P_2 - 3 + P_1 - 2P_2 \\ &= 2P_2 \end{aligned}$$

Theorem.
Let $L = L_0 + L_1P_1 + L_2P_2, S = S_0 + S_1P_1 + S_2P_2$, then:
1). S is invertible if and only if $det(S)$ is invertible in $2 - SP_R$.
2). $det(S \times L) = det(S) \times det(L)$.
Proof.
1). According to the previous theorem, the matrix S is invertible if and only if $S_0, S_0 + S_1, S_0 + S_1 + S_2$ are invertible.
This is equivalent to $det(S_0) \neq 0, det(S_0 + S_1) \neq 0, det(S_0 + S_1 + S_2) \neq 0$, thus
$det(S) = det(S_0) + [det(S_0 + S_1) - det(S_0)]P_1 + [det(S_0 + S_1 + S_2) - det(S_0 + S_1)]P_2$ is invertible.
2). $S \times L = S_0 \times L_0 + (S_0L_1 + S_1L_0 + S_1L_1)P_1 + (S_0L_2 + S_1L_2 + S_2L_2 + S_2L_0 + S_2L_1)P_2$.
$det(S \times L) = det(S_0L_0) + [det(S_0 + S_1)(L_0 + L_1) - det(S_0L_0)]P_1 + [det(S_0 + S_1 + S_2)(L_0 + L_1 + L_2) - det(S_0 + S_1)(L_0 + L_1)]P_2 = det(S) \times det(L)$.

Example.
The matrix $\begin{pmatrix} 1 + P_1 + P_2 & 3 - P_1 + 2P_2 \\ P_1 & P_1 + P_2 \end{pmatrix}$ is not invertible, that is because its determinant $det(S) = 2P_2$ is not invertible.

Example.
Consider $X = \begin{pmatrix} 1 + P_1 + P_2 & 2 - P_1 + P_2 \\ 3P_1 + P_2 & 1 + 4P_1 - P_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix}P_1 + \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}P_2 = X_0 + X_1P_1 + X_2P_2$
$det(X_0) = 1, det(X_0 + X_1) = det\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} = 7, det(X_0 + X_1 + X_2) = det\begin{pmatrix} 3 & 2 \\ 4 & 4 \end{pmatrix} = 4$.
$det(X) = 1 + (7 - 1)P_1 + (4 - 7)P_2 = 1 + 6P_1 - 3P_2$.

Eigen Values/Vectors:
Let $X = X_0 + X_1P_1 + X_2P_2 \in 2 - SP_M$, we say that $A = a_0 + a_1P_1 + a_2P_2 \in 2 - SP_R$ is a symbolic 2-plithogenic eigen value if and only if $X.Y = A.Y; Y = y_0 + y_1P_1 + y_2P_2 \in 2 - SP_V$.
Y is called the corresponding symbolic 2-plithogenic eigen vectors.

DOI: https://doi.org/10.54216/IJNS.210109
Received: February 08, 2023   Revised: April 24, 2023   Accepted: May 06, 2023

Theorem.
Let $A = A_0 + A_1P_1 + A_2P_2 \in 2 - SP_M$, then $t = t_0 + t_1P_1 + t_2P_2 \in 2 - SP_R$ is a symbolic 2-plithogenic eigen value of A if and only if t_0 is eigen value of A_0, $t_0 + t_1$ is eigen value of $A_0 + A_1$, and $t_0 + t_1 + t_2$ is eigen value of $A_0 + A_1 + A_2$.
In addition, $X = X_0 + X_1P_1 + X_2P_2$ is the corresponding eigen value vector of t if and only if X_0 is eigen vector of t_0, $X_0 + X_1$ is eigen vector of $t_0 + t_1$, and $X_0 + X_1 + X_2$ is eigen vector of $t_0 + t_1 + t_2$.
Proof.
By considering the equation $A.X = t.X$, we get:

$$\begin{cases} A_0X_0 = t_0X_0 \\ (A_0 + A_1) = (t_0 + t_1)(X_0 + X_1) \\ (A_0 + A_1 + A_2)(X_0 + X_1 + X_2) = (t_0 + t_1 + t_2)(X_0 + X_1 + X_2) \end{cases}$$

Thus, we get the proof.
Example.
Consider the matrix:

$$A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 2 & 5 \end{pmatrix} P_1 + \begin{pmatrix} -1 & 0 \\ 1 & 3 \end{pmatrix} P_2 = \begin{pmatrix} 1 + 2P_1 - P_2 & 0 \\ 3 + 2P_1 + P_2 & 2 + 5P_1 + 3P_2 \end{pmatrix}$$

The eigen values of A_0 are $\{1,2\}$.
The eigen values of $A_0 + A_1$ are $\{3,7\}$.
The eigen values of $A_0 + A_1 + A_2$ are $\{2,10\}$.
We discuss the following possible cases:
Case 1.
If $t_0 = 1, t_0 + t_1 = 3, t_0 + t_1 + t_2 = 2$, then $t = 1 + 2P_1 - P_2$.
Case 2.
If $t_0 = 1, t_0 + t_1 = 7, t_0 + t_1 + t_2 = 2$, then $t = 1 + 6P_1 - 5P_2$.
Case 3.
If $t_0 = 1, t_0 + t_1 = 3, t_0 + t_1 + t_2 = 10$, then $t = 1 + 2P_1 + 7P_2$.
Case 4.
If $t_0 = 1, t_0 + t_1 = 7, t_0 + t_1 + t_2 = 10$, then $t = 1 + 6P_1 + 3P_2$.
Case 5.
If $t_0 = 2, t_0 + t_1 = 3, t_0 + t_1 + t_2 = 2$, then $t = 2 + P_1 - P_2$.
Case 6.
If $t_0 = 2, t_0 + t_1 = 3, t_0 + t_1 + t_2 = 10$, then $t = 2 + P_1 + 7P_2$.
Case 7.
If $t_0 = 2, t_0 + t_1 = 7, t_0 + t_1 + t_2 = 2$, then $t = 2 + 5P_1 - 5P_2$.
Case 8.
If $t_0 = 2, t_0 + t_1 = 7, t_0 + t_1 + t_2 = 10$, then $t = 2 + 5P_1 + 3P_2$.
The eigen vectors of A_0 are $\{u_1 = (1,-3), u_2 = (0,1)\}$.
The eigen vectors of $A_0 + A_1$ are $\left\{u_3 = \left(1, \frac{-5}{4}\right), u_4 = (0,1)\right\}$.
The eigen vectors of $A_0 + A_1 + A_2$ are $\left\{u_5 = \left(1, \frac{-3}{4}\right), u_6 = (0,1)\right\}$.
The possible cases:
Case 1.
If $u_1 = (1,-3), u_3 = \left(1, \frac{-5}{4}\right), u_5 = \left(1, \frac{-3}{4}\right)$, then $X_1 = (1,-3) + \left(0, \frac{7}{4}\right) P_1 + \left(0, \frac{1}{2}\right) P_2$.
Case 2.
If $u_1 = (1,-3), u_3 = \left(1, \frac{-5}{4}\right), u_6 = (0,1)$, then $X_2 = (1,-3) + \left(0, \frac{7}{4}\right) P_1 + \left(-1, \frac{9}{4}\right) P_2$.
Case 3.
If $u_1 = (1,-3), u_4 = (0,1), u_5 = \left(1, \frac{-3}{4}\right)$, then $X_3 = (1,-3) + (-1,4)P_1 + \left(1, \frac{-7}{4}\right) P_2$.
Case 4.
If $u_1 = (1,-3), u_4 = (0,1), u_6 = (0,1)$, then $X_4 = (1,-3) + (-1,4)P_1 + (0,0)P_2$.
Case 5.
If $u_2 = (0,1), u_3 = \left(1, \frac{-5}{4}\right), u_5 = \left(1, \frac{-3}{4}\right)$, then $X_5 = (0,1) + \left(1, \frac{-9}{4}\right) P_1 + \left(0, \frac{1}{2}\right) P_2$.
Case 6.
If $u_2 = (0,1), u_3 = \left(1, \frac{-5}{4}\right), u_6 = (0,1)$, then $X_6 = (0,1) + \left(1, \frac{-9}{4}\right) P_1 + \left(-1, \frac{9}{4}\right) P_2$.
Case 7.

DOI: https://doi.org/10.54216/IJNS.210109

Received: February 08, 2023 Revised: April 24, 2023 Accepted: May 06, 2023

If $u_2 = (0,1), u_4 = (0,1), u_5 = \left(1, \frac{-3}{4}\right)$, then $X_7 = (0,1) + (0,0)P_1 + \left(0, \frac{-7}{4}\right) P_2$.

Case 8.
If $u_2 = (0,1), u_4 = (0,1), u_6 = (0,1)$, then $X_8 = (0,1) + (0,0)P_1 + (0,0)P_2$.

2. The diagonalization problem.

Definition.
Let $Y = Y_0 + Y_1P_1 + Y_2P_2$ be a symbolic 2-plithogenic square matrix, Y is called diagonalizable if there exists an invertible symbolic 2-plithogenic matrix $B = B_0 + B_1P_1 + B_2P_2$ and a diagonal symbolic 2-plithogenic matrix $T = T_0 + T_1P_1 + T_2P_2$ such that $Y = BTB^{-1}$.
The following theorem explains the conditions of diagonalization.

Theorem.
The symbolic 2-plithogenic square matrix $Y = Y_0 + Y_1P_1 + Y_2P_2$ is diagonalizable if and only if $Y_0, Y_0 + Y_1, Y_0 + Y_1 + Y_2$ are diagonalizable.
Proof.
Y is diagonalizable if and only if there exists T and B according to the definition, such that $Y = BTB^{-1}$.
First, we have:
$BT = B_0T_0 + (B_0T_1 + B_1T_0 + B_1T_1)P_1 + (B_0T_2 + B_1T_2 + B_2T_2 + B_2T_0 + B_2T_1)P_2$.
$B^{-1} = {B_0}^{-1} + \left[(B_0 + B_1)^{-1} - {B_0}^{-1}\right]P_1 + [(B_0 + B_1 + B_2)^{-1} - (B_0 + B_1)^{-1}]P_2$.

$$\begin{aligned}
BTB^{-1} = {} & B_0T_0{B_0}^{-1} \\
& + \big[B_0T_0(B_0 + B_1)^{-1} - B_0T_0{B_0}^{-1} + B_0T_1{B_0}^{-1} + B_1T_0{B_0}^{-1} + B_1T_1{B_0}^{-1} \\
& + B_0T_1(B_0 + B_1)^{-1} - B_0T_1{B_0}^{-1} + B_1T_0(B_0 + B_1)^{-1} - B_1T_1{B_0}^{-1} + B_1T_1(B_0 + B_1)^{-1} \\
& - B_1T_1{B_0}^{-1}\big]P_1 \\
& + \big[B_0T_0(B_0 + B_1 + B_2)^{-1} - B_0T_0(B_0 + B_1)^{-1} + B_0T_1(B_0 + B_1 + B_2)^{-1} \\
& - B_0T_1(B_0 + B_1)^{-1} + B_1T_0(B_0 + B_1 + B_2)^{-1} - B_1T_0(B_0 + B_1)^{-1} \\
& + B_1T_1(B_0 + B_1 + B_2)^{-1} - B_1T_1(B_0 + B_1)^{-1} + B_0T_2{B_0}^{-1} + B_2T_0{B_0}^{-1} + B_1T_2{B_0}^{-1} \\
& + B_2T_1{B_0}^{-1} + B_2T_2{B_0}^{-1} + B_0T_2(B_0 + B_1)^{-1} - B_0T_2{B_0}^{-1} + B_2T_0(B_0 + B_1)^{-1} \\
& - B_2T_0{B_0}^{-1} + B_1T_2(B_0 + B_1)^{-1} - B_1T_2{B_0}^{-1} + B_2T_1(B_0 + B_1)^{-1} - B_2T_1{B_0}^{-1} \\
& + B_2T_2(B_0 + B_1)^{-1} - B_2T_2{B_0}^{-1} + B_0T_2(B_0 + B_1 + B_2)^{-1} - B_0T_2(B_0 + B_1)^{-1} \\
& + B_2T_0(B_0 + B_1 + B_2)^{-1} - B_2T_0(B_0 + B_1)^{-1} + B_1T_2(B_0 + B_1 + B_2)^{-1} \\
& - B_1T_2(B_0 + B_1)^{-1} + B_2T_1(B_0 + B_1 + B_2)^{-1} - B_2T_1(B_0 + B_1)^{-1} \\
& + B_2T_2(B_0 + B_1 + B_2)^{-1} - B_2T_2(B_0 + B_1)^{-1}\big]P_2 \\
= {} & B_0T_0{B_0}^{-1} + \left[(B_0 + B_1)(T_0 + T_1)(B_0 + B_1)^{-1} - B_0T_0{B_0}^{-1}\right]P_1 \\
& + [(B_0 + B_1 + B_2)(T_0 + T_1 + T_2)(B_0 + B_1 + B_2)^{-1} - (B_0 + B_1)(T_0 + T_1)(B_0 + B_1)^{-1}]P_2
\end{aligned}$$

The equation $Y = BTB^{-1}$ implies that:

$$\begin{cases} Y_0 = B_0T_0{B_0}^{-1} \\ Y_0 + Y_1 = (B_0 + B_1)(T_0 + T_1)(B_0 + B_1)^{-1} \\ Y_0 + Y_1 + Y_2 = (B_0 + B_1 + B_2)(T_0 + T_1 + T_2)(B_0 + B_1 + B_2)^{-1} \end{cases}$$

Which means that $Y_0, Y_0 + Y_1, Y_0 + Y_1 + Y_2$ are diagonalizable.

3. Algorithm for the diagonalization.

Let $Y = Y_0 + Y_1P_1 + Y_2P_2$, assume that $Y_0, Y_0 + Y_1, Y_0 + Y_1 + Y_2$ are diagonalizable, then to diagonalize Y, follow these steps:
Step 1.
Diagonalize $Y_0, Y_0 + Y_1, Y_0 + Y_1 + Y_2$, which means find three invertible matrices L_0, L_1, L_2 and three diagonal matrices D_0, D_1, D_2 such that $Y_0 = L_0D_0{L_0}^{-1}, Y_0 + Y_1 = L_1D_1{L_1}^{-1}, Y_0 + Y_1 + Y_2 = L_2D_2{L_2}^{-1}$.
Step 2.
Put $B = L_0 + (L_1 - L_0)P_1 + (L_2 - L_1)P_2, T = D_0 + (D_1 - D_0)P_1 + (D_2 - D_1)P_2$.
Step 3.
We get $Y = BTB^{-1}$.

Example.
Consider the symbolic 2-plithogenic square matrix:

DOI: https://doi.org/10.54216/IJNS.210109
Received: February 08, 2023 Revised: April 24, 2023 Accepted: May 06, 2023

$$B=\begin{pmatrix}1+2P_1-P_2 & 0\\ 3+2P_1+P_2 & 2+5P_1+3P_2\end{pmatrix}=\begin{pmatrix}1&0\\3&2\end{pmatrix}+\begin{pmatrix}2&0\\2&5\end{pmatrix}P_1+\begin{pmatrix}-1&0\\1&3\end{pmatrix}P_2=Y_0+Y_1P_1+Y_2P_2$$

We have:

$$Y_0=\begin{pmatrix}1&0\\3&2\end{pmatrix}, Y_0+Y_1=\begin{pmatrix}3&0\\5&7\end{pmatrix}, Y_0+Y_1+Y_2=\begin{pmatrix}2&0\\6&10\end{pmatrix}$$

$$L_0=\begin{pmatrix}1&0\\-3&1\end{pmatrix}, D_0=\begin{pmatrix}1&0\\0&2\end{pmatrix}, L_1=\begin{pmatrix}1&0\\\frac{-5}{4}&1\end{pmatrix}, D_1=\begin{pmatrix}3&0\\0&7\end{pmatrix}, L_2=\begin{pmatrix}1&0\\\frac{-3}{4}&1\end{pmatrix}, D_2=\begin{pmatrix}1&0\\0&10\end{pmatrix}$$

$$L_0^{-1}=\begin{pmatrix}1&0\\3&1\end{pmatrix}, L_1^{-1}=\begin{pmatrix}1&0\\\frac{5}{4}&1\end{pmatrix}, L_2^{-1}=\begin{pmatrix}1&0\\\frac{3}{4}&1\end{pmatrix}$$

We have: $Y_0=L_0D_0L_0^{-1}, Y_0+Y_1=L_1D_1L_1^{-1}, Y_0+Y_1+Y_2=L_2D_2L_2^{-1}$.

We put:

$$B=L_0+(L_1-L_0)P_1+(L_2-L_1)P_2=\begin{pmatrix}1&0\\-3&1\end{pmatrix}+\begin{pmatrix}0&0\\\frac{7}{4}&0\end{pmatrix}P_1+\begin{pmatrix}0&0\\\frac{1}{2}&0\end{pmatrix}P_2=\begin{pmatrix}1&0\\-3+\frac{1}{2}P_1+P_2&1\end{pmatrix}$$

$$T=D_0+(D_1-D_0)P_1+(D_2-D_1)P_2=\begin{pmatrix}1&0\\0&2\end{pmatrix}+\begin{pmatrix}2&0\\0&5\end{pmatrix}P_1+\begin{pmatrix}-1&0\\0&3\end{pmatrix}P_2$$
$$=\begin{pmatrix}1+2P_1-P_2&0\\0&2+5P_1+3P_2\end{pmatrix}$$

$$B^{-1}=L_0^{-1}+\left(L_1^{-1}-L_0^{-1}\right)P_1+\left(L_2^{-1}-L_1^{-1}\right)P_2=\begin{pmatrix}1&0\\3&1\end{pmatrix}+\begin{pmatrix}0&0\\\frac{-7}{4}&0\end{pmatrix}P_1+\begin{pmatrix}0&0\\\frac{-1}{2}&0\end{pmatrix}P_2$$
$$=\begin{pmatrix}1&0\\3-\frac{7}{4}5P_1-\frac{1}{2}P_2&1\end{pmatrix}$$

Now let's compute:

$$BT=\begin{pmatrix}1&0\\-3+\frac{1}{2}P_1+P_2&1\end{pmatrix}\begin{pmatrix}1+2P_1-P_2&0\\0&2+5P_1+3P_2\end{pmatrix}=\begin{pmatrix}1+2P_1-P_2&0\\-3-\frac{3}{4}5P_1+\frac{9}{4}P_2&2+5P_1+3P_2\end{pmatrix}$$

$$BTB^{-1}=\begin{pmatrix}1+2P_1-P_2&0\\-3-\frac{3}{4}5P_1+\frac{9}{4}P_2&2+5P_1+3P_2\end{pmatrix}\begin{pmatrix}1&0\\3-\frac{7}{4}5P_1-\frac{1}{2}P_2&1\end{pmatrix}$$
$$=\begin{pmatrix}1+2P_1-P_2&0\\3+2P_1-P_2&2+5P_1+3P_2\end{pmatrix}=Y$$

4. The representation symbolic 2-plithogenic liner functions.

Definition.

Let $2-SP_V=\{x+yP_1+zP_2; x,y,z\in V\}$ be symbolic 2-plithogenic vector space, a function $f:2-SP_V\to 2-SP_V$ is called linear if and only if:

$f(X+Y)=f(X)+f(Y), f(A.X)=A.f(X)$ for all $X,Y\in 2-SP_V$ and $A\in 2-SP_F$ (the symbolic 2-plithogenic field which $2-SP_V$ defined over it).

Definition.

Let $Y=Y_0+Y_1P_1+Y_2P_2$ be a symbolic 2-plithogenic matrix, we say that Y is represented by the linear function $f:2-SP_V\to 2-SP_V$ if and only if:

$YT=f(T); T=t_0+t_1P_1+t_2P_2\in 2-SP_V$.

First, we characterize the structure of symbolic 2-plithogenic linear functions.

Theorem.

Let $f:2-SP_V\to 2-SP_V$ be a symbolic 2-plithogenic linear function, then there exist three classical linear transformations $f_0,f_1,f_2:V\to V$ such that:

$f(t_0+t_1P_1+t_2P_2)=f_0(t_0)+[(f_0+f_1)(t_0+t_1)-f_0(t_0)]P_1+[(f_0+f_1+f_2)(t_0+t_1+t_2)-(f_0+f_1)(t_0+t_1)]P_2$.

Proof.

First, we define $h:2-SP_V\to V\times V\times V$ such that:

$h(x_0+x_1P_1+x_2P_2)=(x_0,x_0+x_1,x_0+x_1+x_2)$, and $g:2-SP_F\to F\times F\times F$ such that:

$g(a_0+a_1P_1+a_2P_2)=(a_0,a_0+a_1,a_0+a_1+a_2)$.

It is known $F\times F\times F$ is a ring, $V\times V\times V$ is a module over the ring $F\times F\times F$.

We prove that (h) is a semi-module isomorphism.

g is well defined, if $a_0+a_1P_1+a_2P_2=b_0+b_1P_1+b_2P_2$, then $a_i=b_i; 0\le i\le 2$, thus:

$(a_0,a_0+a_1,a_0+a_1+a_2)=(b_0,b_0+b_1,b_0+b_1+b_2)$, hence g preserves addition:

DOI: https://doi.org/10.54216/IJNS.210109
Received: February 08, 2023 Revised: April 24, 2023 Accepted: May 06, 2023

$$g[(a_0+a_1P_1+a_2P_2)+(b_0+b_1P_1+b_2P_2)]=(a_0,a_0+a_1,a_0+a_1+a_2)+(b_0,b_0+b_1,b_0+b_1+b_2)$$
$$=g(a_0+a_1P_1+a_2P_2)+g(b_0+b_1P_1+b_2P_2)$$

g preserves multiplication:

$$g[(a_0+a_1P_1+a_2P_2)(b_0+b_1P_1+b_2P_2)]$$
$$=g[a_0b_0+(a_0b_1+a_1b_0+a_1b_1)P_1+(a_0b_2+a_2b_0+a_2b_1+a_1b_2+a_2b_2)P_2]$$
$$=(a_0b_0,a_0b_0+a_0b_1+a_1b_0+a_1b_1,a_0b_0+a_0b_1+a_1b_0+a_1b_1+a_0b_2+a_2b_0+a_2b_1+a_1b_2+a_2b_2)=g(a_0+a_1P_1+a_2P_2).g(b_0+b_1P_1+b_2P_2)$$

g is bijective:

$ker(g)=\{a_0+a_1P_1+a_2P_2\in 2-SP_F;(a_0,a_0+a_1,a_0+a_1+a_2)=(0,0,0)\}=\{0\}$

$Im(g)=\{(c_0,c_1,c_2)\in F\times F\times F;\exists a_0+a_1P_1+a_2P_2\in 2-SP_F;\ g(a_0+a_1P_1+a_2P_2)=(c_0,c_1,c_2)\}$
$=F\times F\times F$

Thus g is a ring isomorphism.

h is well defined.

It can be proved by a similar discussion of g.

h preserves addition:

It can be proved by a similar discussion of g.

h is bijective.

It can be proved by a similar discussion of g.

h has the property $h(A.X)=g(A)h(X)$.

$A.X=a_0x_0+(a_0x_1+a_1x_0+a_1x_1)P_1+(a_0x_2+a_2x_0+a_2x_1+a_1x_2+a_2x_2)P_2.$

$$h(A.X)=(a_0x_0,a_0x_0+a_0x_1+a_1x_0+a_1x_1,a_0x_0+a_0x_1+a_1x_0+a_1x_1+a_0x_2+a_2x_0+a_2x_1+a_1x_2+a_2x_2)=(a_0,a_0+a_1,a_0+a_1+a_2)(x_0,x_0+x_1,x_0+x_1+x_2)=g(A)h(X)$$

thus h is semi-module isomorphism.

Now, assume that $f:2-SP_V\to 2-SP_V$ is a linear function, then $g\circ f:2-SP_V\to V\times V\times V$ is a well-defined function.

Let $X=x_0+x_1P_1+x_2P_2\in 2-SP_V$, then:

$\acute{X}=g(X)=(x_0,x_0+x_1,x_0+x_1+x_2)\in V\times V\times V.$

Let $L_0,L_1,L_2:V\to V$ be three linear transformations then $L(x,y,z)=\big(L_0(x),L_1(y),L_2(z)\big)$ is module homomorphism, thus:

$g^{-1}\circ L\big(\acute{X}\big)=L_0(x_0)+[L_1(x_0+x_1)-L_0(x_0)]P_1+[L_2(x_0+x_1+x_2)-L_1(x_0+x_2)]P_2$ is a linear function.

We have $g^{-1}\circ L\big(\acute{X}\big)=g^{-1}\circ L\circ g(X):2-SP_V\to 2-SP_V$, which means that for every linear function $f:2-SP_V\to 2-SP_V$, there exists $L_0,L_1,L_2:V\to V$ such that:

$f(x_0+x_1P_1+x_2P_2)=L_0(x_0)+[L_1(x_0+x_1)-L_0(x_0)]P_1+[L_2(x_0+x_1+x_2)-L_1(x_0+x_1)]P_2$

Theorem.

Let $A=A_0+A_1P_1+A_2P_2$ be a symbolic 2-plithogenic matrix, and $X=X_0+X_1P_1+X_2P_2\in 2-SP_V$, then there exists a linear function $f:2-SP_V\to 2-SP_V$ such that $f(X)=A.X$.

Proof.

$$A.X=A_0X_0+[(A_0+A_1)(X_0+X_1)-A_0X_0]P_1$$
$$+[(A_0+A_1+A_2)(X_0+X_1+X_2)-(A_0+A_1)(X_0+X_1)]P_2$$
$$=L_0(x_0)+[L_1(x_0+x_1)-L_0(x_0)]P_1+[L_2(x_0+x_1+x_2)-L_1(x_0+x_1)]P_2$$

$$\text{Where}\begin{cases}L_0:V\to V,L_1:V\to V,L_2:V\to V\\ L_0(x_0)=A_0X_0\\ L_1(x_0+x_1)=(A_0+A_1)(X_0+X_1)\\ L_2(x_0+x_1+x_2)=(A_0+A_1+A_2)(X_0+X_1+X_2)\end{cases}$$

This implies that $A.X=f(X)$, where $f:2-SP_V\to 2-SP_V$ is a linear function according to the previous theorem.

Example.

Let's find the linear representation of the following symbolic 2-plithogenic matrix.

$$A=\begin{pmatrix}1+P_1+P_2 & 1-3P_1\\ 2-P_2 & 1+P_1+5P_2\end{pmatrix}=\begin{pmatrix}1&1\\2&1\end{pmatrix}+\begin{pmatrix}1&-3\\0&1\end{pmatrix}P_1+\begin{pmatrix}1&0\\-1&5\end{pmatrix}P_2=A_0+A_1P_1+A_2P_2$$

We have:

$$A_0=\begin{pmatrix}1&1\\2&1\end{pmatrix},A_0+A_1=\begin{pmatrix}2&-2\\2&2\end{pmatrix},A_0+A_1+A_2=\begin{pmatrix}3&-2\\1&7\end{pmatrix}$$

$V=R^2.$

We have: $L_0(X_0)=A_0X_0;\ \ X_0=(\acute{x_0},\ x_0'')$, thus:

DOI: https://doi.org/10.54216/IJNS.210109

Received: February 08, 2023 Revised: April 24, 2023 Accepted: May 06, 2023

$L_0(X_0) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_0' \\ x_0'' \end{pmatrix} = (x_0' + x_0'', 2x_0' + x_0'')$
$L_1(X_1) = (A_0 + A_1)X_1;\ X_1 = (x_1', x_1'')$, thus:
$L_1(X_1) = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1' \\ x_1'' \end{pmatrix} = (2x_1' - 2x_1'', 2x_1' + 2x_1'')$
$L_2(X_2) = (A_0 + A_1 + A_2)X_2;\ X_2 = (x_2', x_2'')$, thus:
$L_2(X_2) = \begin{pmatrix} 3 & -2 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} x_2' \\ x_2'' \end{pmatrix} = (3x_2' - 2x_2'', x_2' + 7x_2'')$
Thus $L(X) = L_0(X_0) + [(L_0 + L_1)(X_0 + X_1) - L_0(X_0)]P_1 + [(L_0 + L_1 + L_2)(X_0 + X_1 + X_2) - (L_0 + L_1)(X_0 + X_1)]P_2$
We have:

$$\begin{aligned}(L_0 + L_1)(X_0 + X_1) &= L_0(x_0' + x_1', x_0'' + x_1'') + L_1(x_0' + x_1', x_0'' + x_1'') \\ &= (x_0' + x_1' + x_0'' + x_1'', 2x_0' + 2x_1' + x_0'' + x_1'') \\ &\quad + \left(2x_0' + 2x_1' - 2x_0'' - 2x_1'', 2x_0' + 2x_1' + 2x_0'' + 2x_1''\right) \\ &= \left(3x_0' + 3x_1' - x_0'' - x_1'', 4x_0' + 4x_1' + 3x_0'' + 3x_1''\right)\end{aligned}$$

Thus $(L_0 + L_1)(X_0 + X_1) - L_0(X_0) = \left(2x_0' + 3x_1' - 2x_0'' - x_1'', 2x_0' + 4x_1' + 2x_0'' + 3x_1''\right)$
In addition $(L_0 + L_1 + L_2)(X_0 + X_1 + X_2) = L_0(X_0 + X_1 + X_2) + L_1(X_0 + X_1 + X_2) + L_2(X_0 + X_1 + X_2)$

$$\begin{aligned}L_0(X_0 + X_1 + X_2) &= L_0(x_0' + x_1' + x_2', x_0'' + x_1'' + x_2'') \\ &= (x_0' + x_1' + x_2' + x_0'' + x_1'' + x_2'', 2x_0' + 2x_1' + 2x_2' + x_0'' + x_1'' + x_2'') \\ L_1(X_0 + X_1 + X_2) &= L_1(x_0' + x_1' + x_2', x_0'' + x_1'' + x_2'') \\ &= (2x_0' + 2x_1' + 2x_2' - 2x_0'' - 2x_1'' - 2x_2'', 2x_0' + 2x_1' + 2x_2' + 2x_0'' + 2x_1'' + 2x_2'') \\ L_2(X_0 + X_1 + X_2) &= L_2(x_0' + x_1' + x_2', x_0'' + x_1'' + x_2'') \\ &= (3x_0' + 3x_1' + 3x_2' - 2x_0'' - 2x_1'' - 2x_2'', x_0' + x_1' + x_2' + 7x_0'' + 7x_1'' + 7x_2'')\end{aligned}$$

This implies that:

$$\begin{aligned}&(L_0 + L_1 + L_2)(X_0 + X_1 + X_2) \\ &\quad = (6x_0' + 6x_1' + 6x_2' - 3x_0'' - 3x_1'' - 3x_2'', 5x_0' + 5x_1' + 5x_2' + 10x_0'' + 10x_1'' \\ &\quad + 10x_2'')\end{aligned}$$

Now, we have:

$$\begin{aligned}&(L_0 + L_1 + L_2)(X_0 + X_1 + X_2) - (L_0 + L_1)(X_0 + X_1) \\ &\quad = (3x_0' + 3x_1' + 6x_2' - 4x_0'' - 4x_1'' - 3x_2'', x_0' + x_1' + 5x_2' + 7x_0'' + 7x_1'' + 10x_2'')\end{aligned}$$

Thus:

$$\begin{aligned}L(X) &= (x_0' + x_0'', 2x_0' + x_0'') + (2x_0' + 3x_1' - 2x_0'' - x_1'', 2x_0' + 4x_1' + 2x_0'' + 3x_1'')P_1 \\ &\quad + (3x_0' + 3x_1' + 6x_2' - 4x_0'' - 4x_1'' - 3x_2'', x_0' + x_1' + 5x_2' + 7x_0'' + 7x_1'' + 10x_2'')P_2\end{aligned}$$

5. The exponent of a symbolic 2-plithogenic matrix.

It is known that if A is a matrix, then:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

Theorem.
Let $S = S_0 + S_1P_1 + S_2P_2 \in 2 - SP_M$, then:
$e^S = e^{S_0} + [e^{S_0+S_1} - e^{S_0}]P_1 + [e^{S_0+S_1+S_2} - e^{S_0+S_1}]P_2$
Proof.
$e^S = \sum_{n=0}^{\infty} \frac{S^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!}[S_0{}^n + [(S_0 + S_1)^n - S_0{}^n]P_1 + [(S_0 + S_1 + S_2)^n - (S_0 + S_1)^n]P_2] = \sum_{n=0}^{\infty} \frac{S_0{}^n}{n!} + P_1\left[\sum_{n=0}^{\infty} \frac{(S_0+S_1)^n}{n!} - \sum_{n=0}^{\infty} \frac{S_0{}^n}{n!}\right] + P_2\left[\sum_{n=0}^{\infty} \frac{(S_0+S_1+S_2)^n}{n!} - \sum_{n=0}^{\infty} \frac{(S_0+S_1)^n}{n!}\right] = e^{S_0} + [e^{S_0+S_1} - e^{S_0}]P_1 + [e^{S_0+S_1+S_2} - e^{S_0+S_1}]P_2.$

6. Conclusion

In this paper, we have studied for the first time the matrices with symbolic 2-plithogenic entries from many algebraic sides. We discussed the problem of diagonalization and provided an easy algorithm for solving it. As well, eigen values and vectors are proved with computing matrix exponents and determinants. In addition, the representation of this class of matrices by linear functions has been founded and provided with many related examples.

Acknowledgments " This study is supported via funding from Prince sattam bin Abdulaziz University project number (PSAU/2023/R/1444) ".

References

[1] Nader Mahmoud Taffach , Ahmed Hatip., "A Review on Symbolic 2-Plithogenic Algebraic Structures " Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

[2] Nader Mahmoud Taffach , Ahmed Hatip.," A Brief Review on The Symbolic 2-Plithogenic Number Theory and Algebraic Equations ", Galoitica Journal Of Mathematical Structures and Applications, Vol.5, 2023.

[3] Merkepci, H., and Abobala, M., " On The Symbolic 2-Plithogenic Rings", International Journal of Neutrosophic Science, 2023.

[4] Smarandache, F., " Introduction to the Symbolic Plithogenic Algebraic Structures (revisited)", Neutrosophic Sets and Systems, vol. 53, 2023.

[5] Taffach, N., " An Introduction to Symbolic 2-Plithogenic Vector Spaces Generated from The Fusion of Symbolic Plithogenic Sets and Vector Spaces", Neutrosophic Sets and Systems, Vol 54, 2023.

[6] Taffach, N., and Ben Othman, K., " An Introduction to Symbolic 2-Plithogenic Modules Over Symbolic 2-Plithogenic Rings", Neutrosophic Sets and Systems, Vol 54, 2023.

[7] Merkepci, H., and Rawashdeh, A., " On The Symbolic 2-Plithogenic Number Theory and Integers ", Neutrosophic Sets and Systems, Vol 54, 2023.

[8] Albasheer, O., Hajjari., A., and Dalla., R., " On The Symbolic 3-Plithogenic Rings and TheirAlgebraic Properties", Neutrosophic Sets and Systems, Vol 54, 2023.

[9] Abobala, M., On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations, Journal Of Mathematics, Hindawi, 2021

[10] Olgun, N., Hatip, A., Bal, M., and Abobala, M., " A Novel Approach To Necessary and Sufficient Conditions For The Diagonalization of Refined Neutrosophic Matrices", International Journal of neutrosophic Science, Vol. 16, pp. 72-79, 2021.

[11] Abobala, M., Hatip, A., and Bal, M., " A Review On Recent Advantages In Algebraic Theory Of Neutrosophic Matrices", International Journal of Neutrosophic Science, Vol.17, 2021.

[12] Abobala, M., Ziena, B,M., Doewes, R., and Hussein, Z., "The Representation Of Refined Neutrosophic Matrices By Refined Neutrosophic Linear Functions", International Journal Of Neutrosophic Science, 2022.

[13] Merkepci, H., and Abobala, M., " The Application of AH-isometry In The Study Of Neutrosophic Conic Sections", Galoitica Journal Of Mathematical Structures And Applications, Vol.2, 2022.

[14] Abobala, M., and Zeina, M.B., " A Study Of Neutrosophic Real Analysis By Using One Dimensional Geometric AH-Isometry", Galoitica Journal Of Mathematical Structures And Applications, Vol.3, 2023.

[15] Khaldi, A., " A Study On Split-Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.

[16] Ahmad, K., " On Some Split-Complex Diophantine Equations", Neoma Journal Of Mathematics and Computer Science, 2023.

[17] Ali, R., " On The Weak Fuzzy Complex Inner Products On Weak Fuzzy Complex Vector Spaces", Neoma Journal Of Mathematics and Computer Science, 2023.

[18] Von Shtawzen, O., " Conjectures For Invertible Diophantine Equations Of 3-Cyclic and 4-Cyclic Refined Integers", Journal Of Neutrosophic And Fuzzy Systems, Vol.3, 2022.

[19] Basheer, A., Ahmad, K., and Ali, R., " A Short Contribution To Von Shtawzen's Abelian Group In n-Cyclic Refined Neutrosophic Rings", Journal Of Neutrosophic And Fuzzy Systems,, 2022

DOI: https://doi.org/10.54216/IJNS.210109
Received: February 08, 2023 Revised: April 24, 2023 Accepted: May 06, 2023