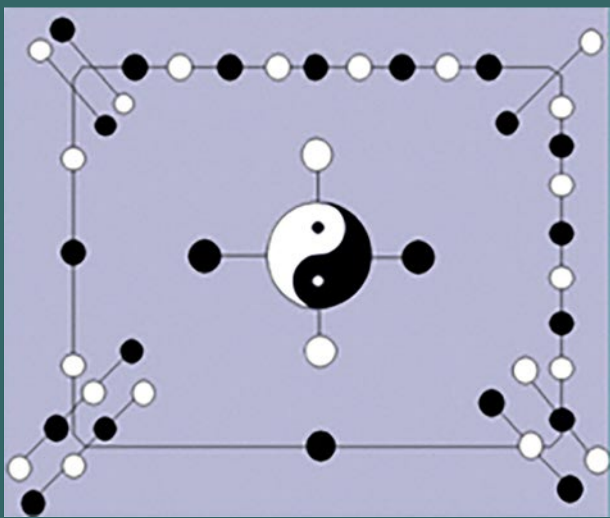
ISSN 1937 - 1055

VOLUME 2, 2022

INTERNATIONAL JOURNAL OF

MATHEMATICAL COMBINATORICS



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THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

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June, 2022

Vol.2, 2022 **ISSN 1937-1055**

International Journal of

Mathematical Combinatorics

(www.mathcombin.com)

Edited By

The Madis of Chinese Academy of Sciences and

Academy of Mathematical Combinatorics & Applications, USA

June, 2022

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I want to bring out the secrets of nature and apply them for the happiness of man. I don't know of any better service to offer for the short time we are in the world.

By Thomas Edison, an American inventor

International J.Math. Combin. Vol.2(2022), 1-7

Subclasses of Analytic Functions Associated with q-Derivative

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Abstract: In this paper, we define the classes $\mathcal{T}_q(A, B, \lambda)$ and $\mathcal{C}_q(A, B, \lambda)$ using Janowski class and q-derivative also we study coefficient estimates, extreme points and many more properties.

Key Words: Janowski class, extreme points, convex linear combination, q-derivative.

AMS(2010): 30C45.

§1. Introduction

Let $\mathcal{A}$ denote the family of analytic functions defined in the open unit disc

$$\mathcal{U} = \{z : |z| < 1\},$$

which are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{T}$ denote the subclass of $\mathcal{A}$ in $\mathcal{U}$, consisting of analytic functions whose non-zero coefficients from the second term onwards are negative. That is, an analytic function $f \in \mathcal{T}$ if it has a Taylor expansion of the form

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n \tag{1.2}$$

which are univalent in the open unit disc $\mathcal{U}$.

The q-shifted factorial is defined for α, $q \in \mathbb{C}$ as a product of n factors by

$$(\alpha, q)_n = \begin{cases} 1, & \text{n=0;} \\ (1-\alpha)(1-\alpha q)\cdots(1-\alpha q^{n-1}), & \text{n} \in \mathbb{N}, \end{cases} \tag{1.3}$$

[1]Received May 12, 2022, Accepted June 10, 2022.

and in terms of the basic analogue of the gamma function

$$(q^\alpha;q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \quad (n>0), \tag{1.4}$$

where the q-gamma functions [2], [3] is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty}, \quad (0<q<1). \tag{1.5}$$

Note that, if $|q|<1$, the q-shifted factorial (1.3), remains meaningful for $n=\infty$ as a convergent infinite product

$$(\alpha;q)_\infty = \prod_{m=0}^{\infty}(1-\alpha q^m).$$

Now recall the following q-analogue definitions given by Gasper and Rahman [2]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x), \text{ where } [x]_q = \frac{(1-q^x)}{(1-q)} \tag{1.6}$$

and called q-analogue of x.

Jackson's q-derivative and q-integral of a function f defined on a subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [2]

$$D_q f(z) = \frac{f(z)-f(zq)}{z(1-q)}, \quad (z\neq 0,\, q\neq 0). \tag{1.7}$$

$$\int_0^z f(t)d_q(t) = z(1-q)\sum_{m=0}^{\infty} q^m f(zq^m). \tag{1.8}$$

In view of the relation

$$\lim_{q\to 1^-} \frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n, \tag{1.9}$$

we observe that the q-shifted fractional (1.2) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n+1)$.

For $-1\le A<B\le 1, \mathcal{P}_1(A,B)$ [4] denotes the class of analytic functions in $\mathcal{U}$ which are of the form $\dfrac{1+A\omega(z)}{1+B\omega(z)}$, where ω is a bounded analytic function satisfying the conditions $\omega(0)=0$ and $|\omega(z)|<1$.

Now we define the subclass $\mathcal{T}_q(A,B,\lambda)$ consisting of functions $f\in\mathcal{T}$ such that

$$\frac{zD_q(f(z))}{\lambda z D_q(f(z)) + (1-\lambda)f(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}, \tag{1.10}$$

where, $-1\le A<B\le 1,\ 0<q<1,\ \lambda>0,\ z\in\mathcal{U}$.

Let $\mathcal{C}_q(A,B,\lambda)$ denote the class of functions $f\in\mathcal{T}$ such that $zf'\in\mathcal{T}_q(A,B,\lambda)$. For $\lambda=0$ and $q\to 1^-$ we get the well-known classes $\mathcal{T}^*(A,B)$ and $\mathcal{C}(A,B)$ studied by Ganesan in [1].

For parametric values $A = 2\alpha - 1$ and $B = 1$ and as $q \to 1^-$ we get the classes $\mathcal{T}(\lambda, \alpha)$ and $\mathcal{C}(\lambda, \alpha)$ studied by Mostafa [5]. In particular, if $q \to 1^-$ we get the classes studided by Ravikumar et al. [6].

In the next section we obtain the characterization properties for the classes $\mathcal{T}_q(A, B, \lambda)$ and $\mathcal{C}_q(A, B, \lambda)$.

§2. Main Results

Theorem 2.1 *A function* $f \in \mathcal{T}_q(A, B, \lambda)$ *if and only if*

$$\sum_{n=2}^{\infty} \{[n]_q(1+B) - (1+A)[\lambda([n]_q - 1) + 1]\} a_n \leq B - A \tag{2.1}$$

for $-1 \leq A < B \leq 1$, $0 < q < 1$, $\lambda > 0$ *and* $z \in \mathcal{U}$.

Proof Suppose $f \in \mathcal{T}_q(A, B, \lambda)$. Then

$$\Re\left\{\frac{zD_q(f(z))}{\lambda zD_q(f(z)) + (1-\lambda)f(z)}\right\} > \frac{1+A}{1+B},$$

$$\Re\left\{\frac{z - \sum_{n=2}^{\infty}[n]_q a_n z^n}{z - \sum_{n=2}^{\infty}[\lambda([n]_q - 1) + 1]a_n z^n}\right\} > \frac{1+A}{1+B}.$$

Letting $z \to 1$, then we get,

$$\left[1 - \sum_{n=2}^{\infty}[n]_q a_n z^n\right](1+B) > (1+A)\left[1 - \sum_{n=2}^{\infty}[\lambda([n]_q - 1) + 1]a_n\right].$$

Hence

$$\sum_{n=2}^{\infty} \{[n]_q(1+B) - (1+A)[\lambda([n]_q - 1) + 1]\} a_n \leq B - A.$$

Conversely, if (2.1) holds, it suffices to show that $|\omega(z)| < 1$. From (1.10), we have

$$\begin{aligned} |\omega(z)| &= \left|\frac{\sum_{n=2}^{\infty}[(\lambda - 1)([n]_q - 1)]a_n z^n}{(B-A)z - \sum_{n=2}^{\infty}[[n]_q B - A(\lambda - 1 - [n]_q\lambda)]a_n z^n}\right| \\ &\leq \frac{\sum_{n=2}^{\infty}[(\lambda - 1)([n]_q - 1)]a_n}{(B-A) - \sum_{n=2}^{\infty}[[n]_q B - A(\lambda - 1 - [n]_q\lambda)]a_n}. \end{aligned}$$

The last expression is bounded by 1 if

$$\sum_{n=2}^{\infty}[(\lambda-1)([n]_q-1)]a_n \leq (B-A)-\sum_{n=2}^{\infty}[[n]_qB-A(\lambda-1-[n]_q\lambda)]a_n$$

which is equivalent to (2.1). Hence the proof. □

Analogous to Theorem 2.1 we get the following result.

Theorem 2.2 *A function* $f \in \mathcal{C}_q(A,B,\lambda)$ *if and only if*

$$\sum_{n=2}^{\infty}[n]_q\left\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\right\}a_n \leq B-A. \tag{2.2}$$

Corollary 2.3 *If function* $f(z) \in \mathcal{T}_j$ *is in the class* $\mathcal{T}_q(A,B,\lambda)$ *then*

$$|a_n| \leq \frac{(B-A)}{\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}}$$

for some $-1 \leq A < B \leq 1$, $\lambda > 0$, $0 < q < 1$, *and* $z \in \mathcal{U}$.

Now we determine extreme points for the class $\mathcal{T}_q(A,B,\lambda)$.

Theorem 2.4 *Let* $f(z) \in \mathcal{T}_q(A,B,\lambda)$. *Define* $f_1(z) = z$ *and*

$$f_n(z) = z - \frac{B-A}{\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}}z^n, \quad n \geq 2$$

for some $-1 \leq A < B \leq 1$, $\lambda > 0$, $0 < q < 1$, *and* $z \in \mathcal{U}$. *Then* $f \in \mathcal{T}_q(A,B,\lambda)$ *if and only if* f *can be expressed as*

$$f(z) = \sum_{n=2}^{\infty}\mu_n f_n(z),$$

where $\mu_n \geq 0$ *and* $\sum_{n=1}^{\infty}\mu_n = 1$.

Proof If

$$f(z) = \sum_{n=2}^{\infty}\mu_n f_n(z) \text{ with } \sum_{n=1}^{\infty}\mu_n = 1, \quad \mu_n \geq 0,$$

then

$$\sum_{n=2}^{\infty}\frac{\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}}{\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}}\mu_n(B-A)$$
$$= \sum_{n=2}^{\infty}\mu_n(B-A) = (1-\mu_j)(B-A) \leq (B-A).$$

Hence, $f(z) \in \mathcal{T}_q(A,B,\lambda)$.

Conversely, let

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in \mathcal{T}_q(A, B, \lambda),$$

define

$$\mu_n = \frac{\{[n]_q(1+B) - (1+A)[\lambda([n]_q - 1) + 1]\}|a_k|}{(B-A)}, \quad n \geq 2$$

and

$$\mu_n = 1 - \sum_{n=2}^{\infty} \mu_n.$$

From Theorem 2.1, $\sum_{n=2}^{\infty} \mu_n \leq 1$ and hence $\mu_1 \geq 0$.

Since $\mu_n f_n(z) = \mu_n f(z) + a_n z^n$, we get that

$$\sum_{n=1}^{\infty} \mu_n f_n(z) = z - \sum_{n=2}^{\infty} a_n z^n = f(z).$$

□

Theorem 2.5 *The class* $\mathcal{T}_q(A, B, \lambda)$ *is closed under convex linear combination.*

Proof Let $f(z),\ g(z) \in \mathcal{T}_q(A, B, \lambda)$ and let

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = z - \sum_{n=2}^{\infty} b_n z^n.$$

For a number η such that $0 \leq \eta \leq 1$, it suffices to show that the function defined by $h(z) = (1-\eta)f(z) + \eta g(z)$, $z \in \mathcal{U}$ belongs to $\mathcal{T}_q(A, B, \lambda)$. Now

$$h(z) = z - \sum_{n=2}^{\infty} [(1-\eta)a_n + \eta b_n] z^n.$$

Applying Theorem 2.1 to $f(z),\ g(z) \in \mathcal{T}_q(A, B, \lambda)$, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \{[n]_q(1+B) - (1+A)[\lambda([n]_q - 1) + 1]\} [(1-\eta)a_n + \eta b_n] \\
&= (1-\eta) \sum_{n=2}^{\infty} \{[n]_q(1+B) - (1+A)[\lambda([n]_q - 1) + 1]\} a_n \\
&\quad + \eta \sum_{n=2}^{\infty} \{[n]_q(1+B) - (1+A)[\lambda([n]_q - 1) + 1]\} b_n \\
&\leq (1-\eta)(B-A) + \eta(B-A) = (B-A).
\end{aligned}$$

This implies that $h(z) \in \mathcal{T}_q(A, B, \lambda)$. □

Theorem 2.6 *For integers* $i = 1, 2, \cdots, n$, *let* $f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n \in \mathcal{T}_q(A, B, \lambda)$ *and*

$0<\beta_i<1$ *such that* $\sum\limits_{i=1}^{n}\beta_i=1,$ *then the function* $F(z)$ *defined by*

$$F(z)=\sum_{i=1}^{n}\beta_i f_i(z)$$

is also in $\mathcal{T}_q(A,B,\lambda)$.

Proof For each integer $i\in\{1,2,3,\cdots,n\}$, we obtain

$$\sum_{n=2}^{\infty}\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}\,|a_{n,i}|<(B-A).$$

Since

$$F(z)=\sum_{i=1}^{n}\beta_i(z-\sum_{n=2}^{\infty}a_{n,i}z^n)=z-\sum_{n=2}^{\infty}(\sum_{i=1}^{n}\beta_i a_{n,i})z^n$$

and

$$\begin{aligned}&\sum_{n=2}^{\infty}\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}\left[\sum_{i=1}^{n}\beta_i a_{n,i}\right]\\ =\ &\sum_{i=1}^{n}\beta_i\left[\sum_{n=2}^{\infty}\{[n]_q(1+B)-(1+A)[\lambda([n]_q-1)+1]\}\right]\\ <\ &\sum_{i=1}^{n}\beta_i(B-A)<(B-A),\end{aligned}$$

we therefore know that $F(z)\in\mathcal{T}_q(A,B,\lambda)$. □

Acknowledgement

The authors would like to thank the referee for his valuable comments which helped to improve the manuscript.

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International J.Math. Combin. Vol.2(2022), 8-20

A Study of
Kenmotsu Manifolds with Semi-Symmetric Metric Connection

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Abstract: The present paper aims to study semi-symmetric metric connection on Kenmotsu Manifolds. First section introduces us with the development of Kenmotsu manifolds. Next section gives us some preliminary ideas about the manifold. Here we have studied the necessary condition under which a vector field will be a strict-contact vector field. In the next section we have extended our study to generalized ϕ-recurrent $n = 2m+1$-dimensional Kenmotsu manifold with respect to semi-symmetric metric connection. Further we have studied this manifold satisfying the condition $L.S = 0$ w.r.t semi-symmetric connection. Lastly we have cited an example of Kenmotsu manifold with semi-symmetric metric connection.

Key Words: Kenmotsu manifolds, semi-symmetric metric connection, conharmonically curvature tensor, extended generalized $\phi-$recurrent Kenmotsu manifolds.

AMS(2010): 53C15, 53C25, 53C40.

§1. Introduction

In [24], S.Tanno classified the connected almost contact metric manifold whose automorphism group has maximum dimension, which are three classes following:

a) the homogeneous normal contact Riemannian manifolds with constant $\phi-$ holomorphic sectional curvature if the sectional curvature of the plain section containing ξ, say $C(X, \xi) > 0$.

b) the global Riemanian product of a line or a circle and a Käehlerian manifold with constant holomorphic sectional curvature, $C(X, \xi) = 0$.

c) a warped product space $RX_\lambda C^n$, if $C(X, \xi) < 0$.

The manifold of class (a) are characterized by some tensor equations, it has a Sasakian structure and manifolds of class (b) are characterized by a tensorial relation admitting a cosymplectic structure. In 1972 Kenmotsu has introduced a new class of almost contact Riemannian manifolds which are nowadays called Kenmotsu manifolds [11]. He obtained some tensorial equations to characterize manifolds of class (c).

[1]Received May 20, 2022, Accepted June 12, 2022.

Let (M, ϕ, ξ, η, g) be a $n = 2m + 1$ dimensional almost contact metric manifold. Then the product $\bar{M} = MXR$ has a natural almost complex structure J with the product metric G being Hermitian manifold $(\bar{M}, J, G)$. The notion of trans-sasakian manifolds was introduced by Oubina [15] in 1985. In general, a Trans-sasakian manifold $(M, \phi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type (α, β). Trans-Sasakian manifold of type $(0, \beta)$ is called β-Kenmotsu manifolds. In 1932, Hayden has given the notion of metric connection with torsion on Riemannian manifold [10]. The semi-symmetric connection on Riemannian manifold was studied by K.Yano [25] in 1970. The SemiC symmetric connections on Riemannian manifold was also studied by K.S. Amur [2], S.S. Pujar, C.S. Bagewadi [4] et.al in 1976.

The notion of local symmetry of a Riemannian manifolds has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi [22] introduced the notion of locally $\phi-$symmetry on a Sasakian manifolds. Generalizing the notion of $\phi-$symmetry one of the authors in [3] introduced the notion of $\phi-$recurrent Kenmotsu manifolds.

The notion of generalized recurrent manifolds has been introduced by Dubey [9] and studied by De and Guha [7]. Again, the notion of generalized Ricci-recurrent manifolds has been introduced and studied by De et al. [8]. A Riemannian manifold $(M^n, g), n > 2$, is called generalized recurrent [9], [7] if its curvature tensor R satisfies the condition

$$\nabla R = A \otimes R + B \otimes G, \tag{1.1}$$

where A and B are non-vanishing 1-forms defined by $A(\delta) = g(\delta, \rho 1), B(\delta) = g(\delta, \rho 2)$ and the tensor G is defined by

$$G(X, Y)Z = g(Y, Z)X - g(X, Z)Y \tag{1.2}$$

for all $X, Y, Z \in \chi(M)$; $\chi(M)$ being the Lie algebra of smooth vector fields on M and ∇ denotes the operator of covariant differentiation with respect to the metric g. The 1-forms A and B are called the associated 1-forms of the manifold. A Riemannian manifold $(M^n, g), n > 2$, is called generalized Ricci-recurrent [6] if its Ricci tensor S of type $(0, 2)$ satisfies the condition $\nabla S = A \otimes S + B \otimes g$, where A and B are non-vanishing 1-forms. In 2007, Özgür [16] studied generalized recurrent Kenmotsu manifolds. Recently Basari and Murathan [3] introduced the notion of generalized ϕ-recurrent Kenmotsu manifolds generalized the notion of Özgür. For extending the notion of Basari and Murathan in [3], A. Shaikh [20] introduce the notion of extended generalized ϕ-recurrent β-Kenmotsu manifolds. We in this paper have further studied and established few results on generalized ϕ-recurrent Kenmotsu manifolds.

§2. **Preliminaries**

Let $M^n(\phi, \xi, \eta, g)$ be an almost contact Riemannian manifold where ϕ is a tensor field of type $(1, 1)$, ξ is a vector field and η is a 1-form and g is the induced Riemannian metric on $\tilde{M}$ satisfying

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \tag{2.2}$$

$$g(X, \xi) = \eta(X), \tag{2.3}$$

$$g(\phi X, \phi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y) \tag{2.4}$$

for all vector fields X, Y on M. Now if

$$(\nabla_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi, \tag{2.5}$$

where ∇ is the Riemannian connection of g, then(M, ϕ, ξ, η, g) is called a Kenmotsu manifold. On Kenmotsu manifold M, we also have

$$\nabla_X \xi = X - \eta(X)\xi \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$.

Also we have the following relations on Kenmotsu manifolds

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.8}$$

$$S(X, \xi) = -(n-1)\eta(X). \tag{2.9}$$

Since $S(X, Y) = g(QX, Y)$ we can get

$$S(\phi X, \phi Y) = g(Q\phi X, \phi Y). \tag{2.10}$$

We have by using (2.1), (2.9), $Q\phi = \phi Q$ and $g(X, \phi Y) = -g(\phi X, Y)$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y). \tag{2.11}$$

Also, we have

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y) \tag{2.12}$$

and

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X). \tag{2.13}$$

Now we shall mention few definitions which are required to establish the theorems.

Definition 2.1 *A Kenmotsu manifold is said to be $\eta-$Einstein if its Ricci tensor S satisfies the form*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.14}$$

where a, b are smooth functions.

Definition 2.2 *A vector field X on a Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ is said to be Contact vector field if*

$$(\mathcal{L}_X \eta)(Y) = \sigma\eta(y), \tag{2.15}$$

where σ is a scalar function on M and $\mathcal{L}_X$ denote the lie derivative along X. X is called strict Contact vector field if $\sigma = 0$.

A relation between the curvature tensor R and $\bar{R}$ of type $(1,3)$ of the connections ∇ and $\bar{\nabla}$ respectively is given by [25]

$$\bar{R}(X,Y)Z = R(X,Y)Z + g(Y,Z)X - g(Z,X)Y. \tag{2.16}$$

Also Ricci tensor satisfies

$$\bar{S}(Y,Z) = S(Y,Z) - 2g(Y,Z) + 2\eta(Z)\eta(Y) + g(\phi Y,Z), \tag{2.17}$$

where $\bar{S}$ and S are Ricci tensor of M with respect to semi-symmetric metric connections $\bar{\nabla}$ and the Levi-Civita connection ∇, respectively. Also we have

$$\bar{S}(Y,Z) = S(Y,Z) + 2mg(Y,Z). \tag{2.18}$$

§3. Geometric Vector Fields on Kenmotsu Manifold with Respect to Semi-Symmetric Metric Connections

In this section we shall give the following proof on vector field.

Theorem 3.1 *Every contact vector field on a Kenmotsu manifold leaving the Ricci tensor with respect to semi-symmetric connection invariant is a strict contact vector field.*

Proof Let a Contact vector field X on a Kenmotsu manifolds leaves the Ricci tensor with respect to semi-symmetric metric connections invariant i.e

$$(\mathcal{L}_X\bar{S})(Y,Z) = 0. \tag{3.1}$$

From (3.1) we have $(\mathcal{L}_X\bar{S})(Y,Z) = \bar{S}(\mathcal{L}_XY,Z) + \bar{S}(Y,\mathcal{L}_XZ)$. Putting $Z = \xi$ we obtain

$$(\mathcal{L}_X\bar{S})(Y,\xi) = \bar{S}(\mathcal{L}_XY,\xi) + \bar{S}(Y,\mathcal{L}_X\xi). \tag{3.2}$$

Putting $Z = \xi$ in (2.16) we can get

$$\bar{S}(Y,\xi) = -(n-1)\eta(Y). \tag{3.3}$$

Taking Lie derivative on both the sides of the above equation and using definition (2.2) we can obtain

$$-(n-1)\sigma\eta(Y) = \bar{S}(Y,\mathcal{L}_X\xi). \tag{3.4}$$

Taking $Y = \xi$ in (3.4) we find

$$\eta(\mathcal{L}_X\xi) = \sigma. \tag{3.5}$$

Again from (2.14) snd using the definition for Lie derivative we can infer

$$-\eta(\mathcal{L}_X\xi) = \sigma. \tag{3.6}$$

Hence, combining (3.5) and (3.6) we conclude that $\sigma = 0$ and therefore the proof. □

§4. On Extended Generalized ϕ-Recurrent Kenmotsu Manifold with Respect to Semi-Symmetric Metric Connections

For this section we first define the following terms.

Definition 4.1 *A Kenmotsu manifolds with respect to semi-symmetric connection is said to be a ϕ-recurrent manifold if there exists a non-zero $1-$form B such that*

$$\phi^2((\bar{\nabla}_W R)(X,Y)Z) = B(W)R(X,Y)Z$$

for arbitrary vector fields X, Y, Z, W.

Again we define $\phi-$generalized recurrent Kenmotsu manifold.

Definition 4.2 *A Riemannian manifold (M^n, g) is called $\phi-$generalized recurrent [7], if its curvature tensor R satisfies the condition*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]$$

where A and B are two $1-$forms, B is non zero and these are defined by

$$g(W,\rho_1) = A(W), g(W,\rho_2) = B(W)$$

for all $W \in \chi(M)$. Here ρ_1 and ρ_2 being the vector fields associated to the $1-$form A and B respectively.

Lastly we define an extended generalized $\phi-$recurrent Kenmotsu manifolds.

Definition 4.3 *A Kenmotsu manifold is said to be an extended generalized $\phi-$recurrent Kenmotsu manifolds if its Curvature tensor R satisfies the relation*

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)\phi^2(R(X,Y)Z) + B(W)\phi^2[g(Y,Z)X - g(X,Z)Y]$$

for all $X, Y, Z, W \in \chi(M)$ where A, B are two non-vanishing 1-forms such that $g(W,\rho_1) = A(W)$ and $g(W,\rho_2) = B(W)$ for all $W \in \chi(M)$ with ρ_1 and ρ_2 being the vector fields associated 1-form A and B, respectively [16].

In this connection, we mention the works of Prakasha [17] on Sasakian manifolds. Then we can state the following theorem.

Theorem 4.4 *Suppose M^n is an $\eta-$Einstein Kenmotsu manifolds. If b and a are constant functions then either M^n is an Einstein manifold or M^n is an α Kenmotsu manifolds.*

Proof We first suppose that M^n is an $\eta-$Einstein Kenmotsu manifolds. Then the Ricci tensor satisfies the following relation

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{4.1}$$

where a, b are smooth functions on M^n. Putting $X = Y = \xi$ in (4.1) we get

$$S(\xi,\xi) = a + b.$$

Therefore from above we can calculate that

$$a + b = -(n+1). \tag{4.2}$$

In local coordinate (4.1) can be written as

$$R_{ij} = ag_{ij} + b\eta_i\eta_j. \tag{4.3}$$

On contraction of (4.3) with g^{ij} we get

$$r = 3a + b. \tag{4.4}$$

Taking Covariant derivative with respect to k from the equation (4.3) we obtain

$$R_{ij.k} = a_{.k}g_{ij} + b_{.k}\eta_i\eta_j + b\eta_{i.k}\eta_j + b\eta_i\eta_{j.k}. \tag{4.5}$$

Contracting (4.5) with g^{ik} we get

$$R^k_{j.k} = a_{.j} + b_{.k}\xi^k\eta_j + b\eta_{i.k}g^{ik}\eta_j + b\eta_i\eta_{j.k}g^{ik}. \tag{4.6}$$

We also know that

$$R^a_i = g^{aj}R_{ij}.$$

From Bianchi's Identity we have

$$R^a_{ijk.a} + R^a_{ika.j} + R^a_{iaj.k} = 0.$$

We can write from above equation

$$R^a_{ijk.a} + R_{ik,j} - R_{ij.k} = 0.$$

Multiplying above equation by g^{ij} we can obtain

$$g^{ij}R^a_{ijk.a} + g^{ij}R_{ik.j} - g^{ij}R_{ij.k} = 0.$$

Simplifying the above equation we have

$$2R^k_{j.k} = r_{.j}. \tag{4.7}$$

From (4.6) and (4.7) we can obtain

$$r_{.j} = 2R^k_{j.k} = 2[a_{.j} + b_{.k}\xi^k\eta_j + b\eta_{i.k}g^{ik}\eta_j + b\eta_i\eta_{j.k}g^{ik}]. \tag{4.8}$$

Solving above we get

$$\eta_{i.k}g^{ik} = n - \eta^k\eta_k = n - 1. \tag{4.9}$$

Therefore

$$r_{.j} = 2[a_{.j} + b_{.k}\xi^k\eta_j + (n-1)b\eta_j]. \tag{4.10}$$

Again taking Covariant derivative w.r.t k from equation (4.2)

$$a_{.k} + b_{.k} = 0. \tag{4.11}$$

Also taking Covariant derivative w.r.t j from equation (4.4) we can calculate

$$r_{.j} = 3a.j + b_{.j} = 2a.j + a.j + b_{.j} = 2a.j. \tag{4.12}$$

From equation (4.10) and (4.12) we have

$$(n-1)b = 0.$$

If $n > 1$ then above equation yields $b = 0$. Hence, we get the theorem. □

Theorem 4.5 *An extended generalized $\phi-$recurrent Kenmotsu manifold (M^n, g) with respect to semi-symmetric metric connection is an Einstein manifold and the $1-$forms A and B are related as $(n-1)A(W) - 2B(W) = 0$.*

Proof Consider an extended generalized $\phi-$recurrent Kenmotsu manifold $(M^n, \phi, \eta, \xi, g)$ with respect to semi-symmetric metric connection. The we have from definition (4.3)

$$\phi^2((\bar{\nabla}_W\bar{R})(X,Y)Z) = A(W)\phi^2(\bar{R}(X,Y)Z) + B(W)\phi^2[g(Y,Z)X - g(X,Z)Y]. \tag{4.13}$$

Using (1.2), (2.1) and (4.13) we can obtain

$$\begin{aligned} &-(\bar{\nabla}_W\bar{R})(X,Y)Z + \eta((\bar{\nabla}_W\bar{R})(X,Y)Z)\xi \\ &= A(W)[-\bar{R}(X,Y)Z + \eta(\bar{R}(X,Y)Z)\xi] + B(W)[-g(Y,Z)X + g(X,Z)Y \\ &\quad +\eta(X)g(Y,Z)\xi - g(X,Z)\eta(Y)\xi]. \end{aligned} \tag{4.14}$$

Taking inner product of (4.14) with U and using (2.3) we can calculate

$$-g((\bar{\nabla}_W\bar{R})(X,Y)Z,U) + \eta((\bar{\nabla}_W\bar{R})(X,Y)Z)g(\xi,U)$$

$$= A(W)[-g(\bar{R}(X,Y)Z,U) + \eta(\bar{R}(X,Y)Z)g(\xi,U)] + B(W)[-g(Y,Z)g(X,U)$$
$$+g(X,Z)g(Y,U) + \eta(X)g(Y,Z)g(\xi,U) - g(X,Z)\eta(Y)g(\xi,U)] \quad (4.15)$$

Again applying (4.15) we have

$$-g((\bar{\nabla}_W\bar{R})(X,Y)Z,U) + \eta((\bar{\nabla}_W\bar{R})(X,Y)Z)\eta(U)$$
$$= A(W)[-g(\bar{R}(X,Y)Z,U) + \eta(\bar{R}(X,Y)Z)\eta(U)] + B(W)[-g(Y,Z)g(X,U)$$
$$+g(X,Z)g(Y,U) + \eta(X)g(Y,Z)\eta(U) - g(X,Z)\eta(Y)\eta(U)]. \quad (4.16)$$

Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal basis for the tangent space of M^n at a point $p \in M^n$. Putting $X = U = e_i$ in (4.16) and taking summation over i from 1 to n, we have

$$-(\bar{\nabla}_W\bar{S})(Y,Z) + \sum_{i=1}^{n}\eta((\bar{\nabla}_W\bar{R})(e_i,Y)Z)\eta(e_i)$$
$$= A(W)[-\bar{S}(Y,Z) + \eta(\bar{R}(\xi,Y)Z)] + B(W)[-g(Y,Z) - \eta(Y)\eta(Z)] \quad (4.17)$$

Putting $Z = \xi$ in (4.17) we get

$$-(\bar{\nabla}_W\bar{S})(Y,\xi) + \sum_{i=1}^{n}\eta((\bar{\nabla}_W\bar{R})(e_i,Y)\xi)\eta(e_i)$$
$$= A(W)[-\bar{S}(Y,\xi) + \eta(\bar{R}(\xi,Y)\xi)] + B(W)[-g(Y,\xi) - \eta(Y)\eta(\xi)] \quad (4.18)$$

On simplifying above equation we have

$$-(\bar{\nabla}_W\bar{S})(Y,\xi) + \sum_{i=1}^{n}\eta((\bar{\nabla}_W\bar{R})(e_i,Y)\xi)\eta(e_i) = -A(W)\bar{S}(Y,\xi) - 2B(W)\eta(Y). \quad (4.19)$$

Taking the second term of (4.19) we can calculate

$$\eta((\bar{\nabla}_W\bar{R})(e_i,Y)\xi)\eta(e_i) = g(\bar{\nabla}_W\bar{R}(e_i,Y)\xi,\xi) - g(\bar{R}(\bar{\nabla}_We_i,Y)\xi,\xi)$$
$$-g(\bar{R}(e_i,\bar{\nabla}_WY)\xi,\xi) - g(\bar{R}(e_i,Y)\bar{\nabla}_W\xi,\xi). \quad (4.20)$$

Let $p \in M^n$, since e_i is an orthonormal basis, so $\bar{\nabla}_We_i = 0$ at p. Also

$$g(\bar{R}(e_i,Y)\xi,\xi) = -g(\bar{R}(\xi,\xi)Y,e_i) = 0. \quad (4.21)$$

Since $\nabla_Wg = 0$, we have

$$g(\bar{\nabla}_W\bar{R}(e_i,Y)\xi,\xi) + g(\bar{R}(e_i,Y)\xi,\bar{\nabla}_W\xi) = 0. \quad (4.22)$$

From (4.20) and (4.22) we can obtain

$$g((\bar{\nabla}_W\bar{R})(e_i,Y)\xi,\xi) = -g(\bar{R}(e_i,Y)\xi,\bar{\nabla}_W\xi) - g(\bar{R}(\bar{\nabla}_We_i,Y)\xi,\xi)$$

$$-g(\bar{R}(e_i, \bar{\nabla}_W Y)\xi, \xi) - g(\bar{R}(e_i, Y)\bar{\nabla}_W \xi, \xi). \quad (4.23)$$

We also know

$$g(\bar{R}(e_i, \bar{\nabla}_W Y)\xi, \xi) = 0 = g(\bar{R}(\bar{\nabla}_W e_i, Y)\xi, \xi). \quad (4.24)$$

Now using (4.24) in (4.23) and using the fact that R is skew-symmetric we get

$$g((\bar{\nabla}_W \bar{R})(e_i, Y)\xi, \xi) = 0. \quad (4.25)$$

Therefore second term of (4.19) is zero, i.e.

$$\sum_{i=1}^{n} \eta((\bar{\nabla}_W \bar{R})(e_i, Y)\xi)\eta(e_i) = 0. \quad (4.26)$$

On using (4.26) in (4.19) we have

$$-(\bar{\nabla}_W \bar{S})(Y, \xi) = -A(W)\bar{S}(Y, \xi) - 2B(W)\eta(Y). \quad (4.27)$$

Now we know

$$(\bar{\nabla}_W \bar{S})(Y, \xi) = \bar{\nabla}_W \bar{S}(Y, \xi) - \bar{S}(\bar{\nabla}_W Y, \xi) - \bar{S}(Y, \bar{\nabla}_W \xi). \quad (4.28)$$

Using (2.6), (2.9) and (2.12) in (4.28) we can get

$$(\bar{\nabla}_W \bar{S})(Y, \xi) = -(n-1)g(Y, W) - S(Y, W). \quad (4.29)$$

From (4.27) and (4.29) we have

$$-(n-1)g(Y, W) - S(Y, W) = -A(W)\bar{S}(Y, \xi) - 2B(W)\eta(Y). \quad (4.30)$$

Putting $Y = \xi$ in (4.30) we get

$$(n-1)A(W) - 2B(W)\eta(Y) = 0. \quad (4.31)$$

Hence from (4.30) and (4.31) we can infer

$$S(Y, W) = -(n-1)g(Y, W), \quad (4.32)$$

where $a = -(n-1)$ and $b = 0$. Therefore M^n is an Einstein manifold. □

§5. Conharmonic Curvature Tensor on a Kenmotsu Manifolds with Respect to Semi-Symmetric Metric Connection

A conharmonic curvature tensor has been studied by Ozgur [16], M. Tarafdar and Bhattacharyya [23] in 2003. Further studies were carried in 2010 by Siddique and Ahsan [19]. In almost contact manifolds M of dimension $n \geq 3$, the conharmonic curvature tensor $\bar{L}$ with respect to semi-symmetric metric connection $\bar{\nabla}$ is given by

$$\bar{L}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-2}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] \quad (5.1)$$

for $X,Y,Z \in \chi(M)$ where $\bar{R},\bar{S},\bar{Q}$ are the Riemannian curvature tensor, Ricci tensor and the Ricci operator with respect to semi-symmetric connection $\bar{\nabla}$, respectively.

A conharmonic curvature tensor $\bar{L}$ with respect to semi-symmetric metric connection $\bar{\nabla}$ is said to be flat if it vanishes identically with respect to the Connection $\bar{\nabla}$. On the basis of above definitions we can state the following theorem.

Theorem 5.1 *If a $n(\geq 3)$ dimensional Kenmotsu manifolds with respect to semi-symmetric metric connection admitting a conharmonic curvature tensor and a non-zero Ricci tensor satisfies $\bar{L}(X,Y)\bar{S} = 0$, then the modulus of non-zero eigen values of the endomorphism $\bar{Q}$ of the tangent space corresponding to $\bar{S}$ is 0 where α,β are smooth functions on M^n.*

Proof We consider a $n(n \geq 3)$dimensional Kenmotsu manifolds with respect to semi-symmetric metric connection, satisfying the condition $\bar{L}(X,Y)\bar{S} = 0$. Then we have

$$\bar{S}(\bar{L}(X,Y)U,V) + \bar{S}(U,\bar{L}(X,Y)V) = 0 \quad (5.2)$$

for all $X,Y,U,V \in \chi(M)$. Substituting X by ξ in the above equation we can obtain

$$\bar{S}(\bar{L}(\xi,Y)U,V) + \bar{S}(U,\bar{L}(\xi,Y)V) = 0. \quad (5.3)$$

Let $\bar{\lambda}$ be the eigen values of the endomorphism $\bar{Q}$ corresponding to an eigenvector X, then

$$\bar{Q}X = \bar{\lambda}X. \quad (5.4)$$

We know $g(\bar{Q}X,Y) = \bar{S}(X,Y) = \bar{\lambda}g(X,Y)$. On using (2.16), (2.18), (5.1) and (5.3) we can calculate

$$\eta(U)\bar{S}(Y,V) - \eta(V)\bar{S}(U,Y) = 0. \quad (5.5)$$

Putting $U = \xi$ in (5.5) we get $\bar{S}(Y,V) = 0$. Hence, from (2.18) we know that

$$S(Y,V) = -2mg(Y,V). \quad (5.6)$$

On putting $Y = X = \xi$ in the relation $\bar{S}(X,Y) = \bar{\lambda}g(X,Y)$ we get $\bar{\lambda} = 0$. Therefore, we get the theorem. □

§6. Example of a Kenmotsu Manifold with Respect to Semi-Symmetric Metric Connections

Let $M = \{(x,y,z) \in \mathbf{R}^3 | (x,y,z) \neq (0,0,0)\}$ be a three-dimensional manifold [13]. The vector fields $e_1 = z\frac{\partial}{\partial x}, e_2 = z\frac{\partial}{\partial y}, \xi = e_3 = -z\frac{\partial}{\partial z}$ are linearly independent at each point of M. We define the Rimannian metric g by

$$g(e_i, e_i) = 1,\ g(e_i, e_j) = 0, \text{ where } i, j \in \{1, 2, 3\} \text{ and } i \neq j.$$

The (1,1) tensor field ϕ is defined as

$$\phi(e_1) = -e_2,\ \phi(e_2) = e_1,\ \phi(e_3) = 0.$$

If η is 1-form then $\eta(e_3) = g(e_3, e_3) = 1$. We can easily verify by the linearity of ϕ and g that (ϕ, ξ, η, g) is an almost contact metric structure on M.
Let ∇ be the Levi-Civita connection on $\mathbf{R}^3$. Then we have

$$[e_1, e_2] = 0,\ [e_1, e_3] = e_1,\ [e_2, e_3] = e_2.$$

By using Koszul's formula for the Riemannian metric g, we can find

$$\nabla_{e_1} e_1 = -e_3,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_3 = e_1,$$
$$\nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = -e_3,\ \nabla_{e_2} e_3 = e_2,$$
$$\nabla_{e_3} e_1 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = 0.$$

Using these we can verify $\nabla_X \xi = X - \eta(X)\xi$. Hence the manifold is a Kenmotsu manifold. We consider the linear connection $\tilde{\nabla}$ such that

$$\tilde{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + \eta(e_j) e_i - g(e_i, e_j) e_3.$$

From above relation we can calculate the non-zero components

$$\tilde{\nabla}_{e_1} e_1 = -2e_3,\ \tilde{\nabla}_{e_1} e_3 = 2e_1,\ \tilde{\nabla}_{e_1} e_2 = -e_3,\ \tilde{\nabla}_{e_2} e_3 = 2e_2.$$

Let $\bar{T}$ is the torsion tensor of metric connection $\bar{\nabla}$, then we have

$$\bar{T}(X, Y) = \eta(Y)X - \eta(X)Y.$$

On calculation we can see that

$$\bar{T}(X, Y) = 0.$$

We know that

$$(\nabla_X g)(Y, Z) = Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z)$$

and

$$(\bar{\nabla}_X g)(Y, Z) = Xg(Y, Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z).$$

Using above formula we can calculate

$$(\bar{\nabla}_{e_1} g)(e_2, e_3) = 0 = (\bar{\nabla}_{e_1} g)(e_3, e_2) = (\bar{\nabla}_{e_2} g)(e_1, e_3) = (\bar{\nabla}_{e_3} g)(e_2, e_1).$$

Therefore we can view that

$$(\bar{\nabla}_X g)(Y, Z) = 0$$

for all X, Y and $Z \in \chi(M)$. Hence $\bar{\nabla}$ is a semi-symmetric metric connection on M.

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International J.Math. Combin. Vol.2(2022), 21-32

Neighborhoods of Certain
Class of Analytic Functions Using Modified Sigmoid Function

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Abstract: By using an operator involving modified Sigmoid function we prove the neighborhoods problem.of more class for function $f_\gamma(z) \in T_\gamma$ in the unit disc $\mathfrak{U} = \{z : z \in \mathbb{C}$ and $|z| < 1\}$.

Key Words: Modified sigmoid function, $(m, \delta)-$neighborhood, analytic function, Al-Oboudi operator, Sălăgean Operator.

AMS(2010): 30C45.

§1. Introduction

A sigmoid function is a mathematical function having an "S" shape (sigmoid curve). The sigmoid function, also called the sigmoidal curve or logistic function is the function of the form $\gamma(s) = \frac{1}{1+e^{-s}}$; $S \in \mathbb{R}$. A sigmoid function is a bounded differentiable real function that is defined for all real input values and has a positive derivative at each point. It is useful in compressing, or squashing outputs. It is a monotone function The sigmoid function is the most popular of the three activation function in the hardware implementation of artificial neural network. The Sigmoid function is defined as

$$G(s) = \frac{1}{1+e^{-s}} = \frac{1}{2} + \frac{1}{4}s - \frac{1}{48}s^3 + \frac{1}{480}s^5 - \frac{17}{80640}s^7 + \cdots.$$

Let $\gamma(s)$ be a modified Sigmoid function, that is

$$\gamma(s) = \frac{2}{1+e^{-s}} = 1 + \frac{1}{2}s - \frac{1}{24}s^3 + \frac{1}{240}s^5 - \frac{17}{40320}s^7 + \cdots \quad s \geq 0 \tag{1.1}$$

with $\gamma(s) = 1$ for $s = 0$ be a modified Sigmoid function. For detail information on Sigmoid function see ([4], [3], [8], [7]).

Let $\mathcal{S}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.2}$$

[1]Received April 19, 2022, Accepted June 13, 2022.

which are analytic in the open unit disk $\mathfrak{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

The Hadmard product of two functions $f(z) \in \mathcal{S}$ and $g(z) \in \mathcal{S}$ which denoted by $(f * g)(z)$, that is, if $f(z)$ is given by (**??**) and $g(z)$ is given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad (z \in \mathfrak{U}),$$

then

$$(f * g)(z) = (g * f)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad (z \in \mathfrak{U}). \tag{1.3}$$

Also, we denote by T the subclass of $\mathcal{S}$ consisting of functions $f(z) \in \mathcal{S}$ which are analytic and univalent in $\mathfrak{U}$ and of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{1.4}$$

A function $f_\gamma(z) \in T_\gamma$ defined as

$$f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) a_k z^k, \quad a_k \geq 0 \tag{1.5}$$

where $\gamma(s)$ defined by (1.1). Furthermore, we define identity function for T_γ as $e_\gamma(z) = z$.

§2. Differential Operators

2.1. Sălăgean Differential Operator

Definition 2.1([10]) *For $f(z) \in \mathcal{S}$ and $n \in \mathbb{N}_0$, the Sălăgean differential operator $D^n : \mathcal{S} \to \mathcal{S}$ is defined by*

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= z f'(z) \\ &\vdots \\ D^{n+1} f(z) &= z(D^n f(z))'. \end{aligned}$$

Remark 2.1 If $f(z)$ is given by (1.2), then

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad z \in \mathfrak{U}. \tag{2.1}$$

2.2. Al-Oboudi Differential Operator

Definition 2.2 *For $f(z) \in \mathcal{S}$ and $n \in \mathbb{N}_0$, Al-Oboudi differential operator $D_\delta^n f(z)$ is defined as* (see [1])

$$\begin{aligned} D^0 f(z) &= f(z) \\ D_\delta f(z) &= D^1 f(z) = (1-\delta) f(z) + \delta z f^{'}(z), \ \ \delta \geq 0 \\ &\vdots \\ D_\delta^n f(z) &= D_\delta (D^{n-1} f(z)). \end{aligned}$$

Remark 2.2 D_δ^n is a linear operator for $f(z) \in \mathcal{S}$, we get

$$D_\delta^n f(z) = z + \sum_{k=2}^{\infty} \{1 + (k-1)\delta\}^n a_k z^k, \ \ z \in \mathfrak{U}, \delta \geq 0. \tag{2.2}$$

For $\delta = 1$, (2.2) becomes (2.1).

In [2], Darus and Ibrahim introduced a generalized differential operator

$$\begin{aligned} D^0 f(z) &= f(z) \\ D_{\alpha,\lambda}^1 f(z) &= (\alpha - \lambda) f(z) + (\lambda - \alpha + 1) z f^{'}(z) \\ D_{\alpha,\lambda}^n f(z) &= D_{\alpha,\lambda}^1 (D^{n-1} f(z)). \end{aligned}$$

Thus

$$D_{\alpha,\lambda}^n f(z) = z + \sum_{k=2}^{\infty} \{(k-1)(\lambda - \alpha) + k\}^n a_k z^k. \tag{2.3}$$

2.3. Differential Operator Involving Modified Sigmoid Function

In [4], Fadipe-Joseph et al. introduced Sălăgean differential operator involving modified sigmoid function which is defined as follows.

Let $f_\gamma(z) \in T_\gamma$, the Sălăgean differential operator $D^n f_\gamma(z)$ is defined by

$$\begin{aligned} D^0 f_\gamma(z) &= f_\gamma(z) \\ D^1 f(z) &= z\gamma(s) f_\gamma^{'}(z) \\ &\vdots \\ D^n f_\gamma(z) &= D(D^{n-1} f_\gamma(z)) = z\gamma(s)(D^{n-1} f_\gamma(z))^{'}. \end{aligned}$$

Hence

$$D^n f_\gamma(z) = \gamma^n(s) z + \sum_{k=2}^{\infty} \gamma^{n+1}(s) k^n a_k z^k. \tag{2.4}$$

2.4. Ruscheweyh Operator Involving Modified Sigmoid Function

Ruscheweyh differential operator involving the modified sigmoid function with $R^n : T_\gamma \to T_\gamma$ is defined as

$$R^n f_\gamma(z) = z - \sum_{k=2}^{\infty} \gamma(s) B_k(n) a_k z^k, \ a_k \geq 0, n \in \mathbb{N}_0, \tag{2.5}$$

where

$$\begin{aligned} B_k(n) &= B(n,k) = \begin{pmatrix} n+k-1 \\ n \end{pmatrix} \\ &= \frac{(n+1)(n+2)\dots(n+k-1)}{(k-1)!} = \frac{(n+1)_{k-1}}{(1)_{k-1}}. \end{aligned}$$

Hence

$$B(0,k) = \begin{pmatrix} k-1 \\ 0 \end{pmatrix} = \frac{(1)_{k-1}}{(1)_{k-1}} = 1.$$

See [4] and [9] for detail.

2.5. New Differential Operator Involving Modified Sigmoid Function

Definition 2.3 *Let $f_\gamma(z) \in T_\gamma$, then from (*2.3*) and (*2.4*) we obtain a generalized differential operator involving modified sigmoid function as follows*

$$D^n_{\lambda,\omega} f_\gamma(z) = \gamma^n(s) z - \sum_{k=2}^{\infty} \gamma^{n+1}(s) \left\{(k-1)(\lambda-\omega)+k\right\}^n a_k z^k, \tag{2.6}$$

for $\lambda, \omega \geq 0$ (see [4] and [2]).

2.6. Linear Combination of Generalized Sălăgean Differential Operator and Ruscheweyh Operator Involving Modified Sigmoid Function

We combine the generalised Sălăgean differential operator involving modified sigmoid function and the Ruscheweyh operator involving modified Sigmoid function to obtain a certain operator defined as

$$\begin{aligned} \Phi^n_{\lambda,\omega} f_\gamma(z) &= \mu D^n_{\lambda,\omega} f_\gamma(z) + (1-\mu) R^n f_\gamma(z) \\ &= \left[\mu \gamma^n(s) - \mu + 1\right] z \\ &\quad - \sum_{k=2}^{\infty} \gamma(s) \left\{\mu \left[\gamma^n(s)(k-1)(\lambda-\omega)+k\right]^n + (1-\mu) B_k(n)\right\} a_k z^k \end{aligned} \tag{2.7}$$

for $0 \leq \lambda, \mu \leq 1$ with special cases following:

(*i*) If $n = 0$ in (2.7), we get $f_\gamma(z)$ defined in (1.5);

(*ii*) If $\mu = 1$ in (2.7), then $\Phi^n_{\lambda,\omega} f_\gamma(z) = D^n_{\lambda,\omega} f_\gamma(z)$ defined in (2.6);

(*iii*) If $\mu = 0$ in (2.7), then $\Phi^n_{\lambda,\omega} f_\gamma(z) = R^n f_\gamma(z)$ defined in (2.5);

(*iv*) If $s = 0, \mu = 1$ and $\omega = 1$ in (2.7), then $\Phi^n_{\lambda,1} f_\gamma(z) = D^n_\delta f(z)$ defined in (2.2);

(*v*) If $s = 0, \mu = 1$ and $\lambda = \omega = 0$ in (2.7), then $\Phi^n_{0,0} f_\gamma(z) = D^n f(z)$ defined in (2.1).

Definition 2.4([8]) *For a function $f_\gamma(z) \in T_\gamma$ defined by (1.5) is in the class $T_\gamma\chi^n_\mu(\lambda,\omega,\alpha,p,\zeta,\eta)$ if*

$$\left|\frac{\left[\Phi^n_{\lambda,\omega}f_\gamma(z)\right]' - [\mu\gamma^n(s)-\mu+1]}{p\zeta\left[\left(\Phi^n_{\lambda,\omega}f_\gamma(z)\right)'-\alpha\right]-\left[\left(\Phi^n_{\lambda,\omega}f_\gamma(z)\right)'-[\mu\gamma^n(s)-\mu+1]\right]}\right| < \eta, \tag{2.8}$$

where $0 \le \alpha < \frac{1}{2}\zeta,\ 0 \le \mu \le 1,\ \lambda,\omega \ge 0,\ p \ge 2,\ \frac{1}{2} \le \zeta \le 1,\ 0 < \eta \le 1$ *and* $n \in \mathbb{N}_0$.

Note that:

(1) If $\alpha = 0,\ \mu = 1,\ p = 2$ and $\zeta = 1$, then

$$T_\gamma\chi^n(\lambda,\omega,0,2,1,\eta) = S\chi^*_\gamma(n,\eta,\lambda,\omega) = \left|\frac{\left[D^n_{\lambda,\omega}f_\gamma(z)\right]' - \gamma^n(s)}{\left[D^n_{\lambda,\omega}f_\gamma(z)\right]' + \gamma^n(s)}\right| < \eta.$$

(2) If $\alpha = 0,\ \mu = 1,\ p = 2$ and $\zeta = \frac{1}{2}$, then

$$T_\gamma\chi^n(\lambda,\omega,0,2,\frac{1}{2},\eta) = \chi^*_\gamma(n,\eta,\lambda,\omega) = \left|\frac{1}{\gamma^n(s)}\left[D^n_{\lambda,\omega}f_\gamma(z)\right]' - 1\right| < \eta.$$

(3) If $\mu = 0,\ p = 2$ and $\zeta = 1$, then

$$T_\gamma\chi^n_0(0,0,\alpha,2,1,\eta) = \chi^*_\gamma(n,\eta,\alpha) = \left|\frac{[R^n f_\gamma(z)]' - 1}{[R^n f_\gamma(z)]' - 2\alpha + 1}\right| < \eta.$$

(4) If $\alpha = 0,\ \mu = 0,\ p = 2$ and $\zeta = 1$, then

$$T_\gamma\chi^n_0(0,0,0,2,1,\eta) = S\chi^*_\gamma(n,\eta) = \left|\frac{[R^n f_\gamma(z)]' - 1}{[R^n f_\gamma(z)]' + 1}\right| < \eta.$$

(5) If $\alpha = 0,\ \mu = 0,\ p = 2$ and $\zeta = \frac{1}{2}$, then

$$T_\gamma\chi^n_0(0,0,0,2,\frac{1}{2},\eta) = \chi^*_\gamma(n,\eta) = \left|[R^n f_\gamma(z)]' - 1\right| < \eta.$$

(6) If $\eta = 1,\ \mu = 0,\ p = 2,\ n = 0$ and $\zeta = 1$, then

$$T_\gamma\chi^0_0(0,0,\alpha,2,1,1) = \chi^*_\gamma(\alpha) = \left|\frac{f'_\gamma(z) - 1}{f'_\gamma(z) - 2\alpha + 1}\right| < 1.$$

(7) If $\eta = 1,\ \alpha = 0,\ \mu = 0,\ p = 2,\ n = 0$ and $\zeta = 1$, then

$$T_\gamma\chi^0_0(0,0,0,2,1,1) = \chi^*(\gamma) = \left|\frac{f'_\gamma(z) - 1}{f'_\gamma(z) + 1}\right| < 1.$$

(8) If $\alpha = 0,\ \mu = 0,\ p = 2,\ n = 0$ and $\zeta = \frac{1}{2}$, then

$$T_\gamma\chi^0_0(0,0,0,2,\frac{1}{2},\eta) = \chi^*_\gamma(\eta) = \left|f'_\gamma(z) - 1\right| < \eta.$$

(9) If $\mu = 0,\ p = 2,\ s = 0$ and $\zeta = 1$, then

$$T\chi_0^n(0,0,\alpha,2,1,\eta) = \chi^*(n,\eta,\alpha) = \left| \frac{[R^n f(z)]' - 1}{[R^n f(z)]' - 2\alpha + 1} \right| < \eta.$$

(10) If $\alpha = 0,\ \mu = 0,\ p = 2,\ s = 0$ and $\zeta = 1$, then

$$T\chi_0^n(0,0,0,2,1,\eta) = S\chi^*(n,\eta) = \left| \frac{[R^n f(z)]' - 1}{[R^n f(z)]' + 1} \right| < \eta.$$

(11) If $\alpha = 0,\ \mu = 1,\ p = 2,\ s = 0$ and $\zeta = \frac{1}{2}$, then

$$T\chi^n(\lambda,\omega,0,2,\frac{1}{2},\eta) = \chi^*(n,\eta,\lambda,\omega) = \left| \left[D_{\lambda,\omega}^n f(z)\right]' - 1 \right| < \eta.$$

(12) If $\alpha = 0,\ s = 0,\ \mu = 0,\ p = 2$ and $\zeta = \frac{1}{2}$, then

$$T\chi_0^n(0,0,0,2,\frac{1}{2},\eta) = \chi^*(n,\eta) = \left| [R^n f(z)]' - 1 \right| < \eta.$$

(13) If $\mu = 0,\ n = 0,\ p = 2,\ s = 0$ and $\zeta = 1$, then

$$T\chi_0^0(0,0,\alpha,2,1,\eta) = \chi^*(\eta,\alpha) = \left| \frac{f'(z) - 1}{f'(z) - 2\alpha + 1} \right| < \eta \quad \text{(Juneja and Mogra [?])}.$$

(14) If $\alpha = 0,\ \mu = 0,\ n = 0,\ p = 2,\ s = 0$ and $\zeta = 1$, then

$$T\chi_0^0(0,0,0,2,1,\eta) = S\chi^*(\eta) = \left| \frac{f'(z) - 1}{f'(z) + 1} \right| < \eta, \quad \text{(Kim and Lee [?])}.$$

(15) If $\alpha = 0,\ n = 0,\ s = 0,\ \mu = 0,\ p = 2$ and $\zeta = \frac{1}{2}$, then

$$T\chi_0^0(0,0,0,2,\frac{1}{2},\eta) = \chi^*(\eta) = |f(z)' - 1| < \eta, \quad \text{(Kim and Lee [?])}.$$

We begin by proving the necessary and sufficient condition for a function belongs to the class $T_\gamma\chi_\mu^n(\lambda,\omega,\alpha,p,\zeta,\eta)$.

2.7. Coefficient Estimates

Theorem 2.1 *If a function $f_\gamma(z)$ belongs to the class $T_\gamma\chi_\mu^n(\lambda,\omega,\alpha,p,\zeta,\eta)$, then*

$$\sum_{k=2}^{\infty} k\gamma(s)\left[1 + \eta(p\zeta - 1)\right]\left\{\mu\gamma^n(s)\left[(k-1)(\lambda-\omega) + k\right]^n + (1-\mu)B_k(n)\right\} a_k$$

$$\leq \eta p\zeta\left[\mu\gamma^n(s) - \mu + 1 - \alpha\right]. \tag{2.9}$$

Proof Suppose $f_\gamma(z)$ belongs to the class $T_\gamma\chi_\mu^n(\lambda,\omega,\alpha,p,\zeta,\eta)$, by equation (2.7) and Definition 2.4, we have that

$$\left|-\sum_{k=2}^{\infty}k\gamma(s)\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}+(1-\mu)B_{k}(n)\right\}a_{k}z^{k-1}\right\}\right|$$
$$\leq\eta\left|p\zeta\left[\mu\gamma^{n}(s)-\mu+1-\alpha\right]-\sum_{k=2}^{\infty}(1-p\zeta)k\gamma(s)\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}\right.\right.$$
$$\left.\left.+(1-\mu)B_{k}(n)\right\}a_{k}z^{k-1}\right|,$$

$|z|\leq r$ and $r\to 1^{+}$, then

$$\sum_{k=2}^{\infty}k\gamma(s)\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}+(1-\mu)B_{k}(n)\right\}a_{k}$$
$$\leq\eta p\zeta\left[\mu\gamma^{n}(s)-\mu+1-\alpha\right]$$
$$+\sum_{k=2}^{\infty}\eta(1-p\zeta)k\gamma(s)\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}+(1-\mu)B_{k}(n)\right\}a_{k}$$

or

$$\sum_{k=2}^{\infty}k\gamma(s)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}+(1-\mu)B_{k}(n)\right\}a_{k}$$
$$\leq\eta p\zeta\left[\mu\gamma^{n}(s)-\mu+1-\alpha\right].$$

Hence,

$$\sum_{k=2}^{\infty}a_{k}\leq\frac{\eta p\zeta\left[\mu\gamma^{n}(s)-\mu+1-\alpha\right]}{k\gamma(s)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}+(1-\mu)B_{k}(n)\right\}}.\tag{2.10}$$

The result is sharp for

$$f(z)=z-\frac{\eta p\zeta\left[\mu\gamma^{n}(s)-\mu+1-\alpha\right]}{k\gamma(s)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^{n}(s)\left[(k-1)(\lambda-\omega)+k\right]^{n}+(1-\mu)B_{k}(n)\right\}}z^{k}$$

and $k\geq 2$. □

2.8. Neighborhoods for the Class $T_{\gamma}\chi_{\mu}^{n}(\lambda,\omega,\alpha,p,\zeta,\eta)$

Definition 2.5 *The* $(m,\delta)-$*neighborhood of the function* $f_{\gamma}(z)$ *belongs to the class* T_{γ} *by*

$$N_{m,\delta}(f_{\gamma})=\left\{g_{\gamma}:g_{\gamma}\in T_{\gamma},\ g_{\gamma}=z-\sum_{k=m+1}^{\infty}\gamma(s)b_{k}z^{k}\ \text{and}\ \sum_{k=m+1}^{\infty}k\gamma(s)\left|a_{k}-b_{k}\right|\leq\delta\right\}.\tag{2.11}$$

In particular, if identity function

$$e_{\gamma}(z)=z,$$

we immediately have

$$N_{m,\delta}(e_\gamma) = \left\{ g_\gamma : g_\gamma \in T_\gamma,\ g_\gamma = z - \sum_{k=m+1}^{\infty} \gamma(s) b_k z^k \ \textit{and} \ \sum_{k=m+1}^{\infty} k\gamma(s)\,|b_k| \le \delta \right\}. \tag{2.12}$$

Lemma 2.1 *Let the function $f_\gamma(z) \in T_\gamma$ be defined by*

$$f_\gamma(z) = z - \sum_{k=m+1}^{\infty} \gamma(s) a_k z^k, \quad a_k \ge 0. \tag{2.13}$$

Then, $f_\gamma(z)$ is in the class $T_\gamma \chi_\mu^n(\lambda, \omega, \alpha, p, \zeta, \eta)$ if and only if

$$\sum_{k=m+1}^{\infty} k\gamma(s)\left[1 + \eta(p\zeta - 1)\right]\left\{\mu\gamma^n(s)\left[(k-1)(\lambda - \omega) + k\right]^n + (1-\mu)B_k(n)\right\} a_k$$
$$\le \eta p\zeta\left[\mu\gamma^n(s) - \mu + 1 - \alpha\right]. \tag{2.14}$$

Theorem 2.2 *Let*

$$\delta = \frac{\eta p\zeta\left[\mu\gamma^n(s) - \mu + 1 - \alpha\right]}{\gamma(s)\left[1 + \eta(p\zeta - 1)\right]\left\{\mu\gamma^n(s)\left[m(\lambda - \omega) + m + 1\right]^n + (1-\mu)B_{m+1}(n)\right\}}. \tag{2.15}$$

If $\delta < 1$, then $T_\gamma \chi_\mu^n(\lambda, \omega, \alpha, p, \zeta, \eta) \subset N_{m,\delta}(e_\gamma)$.

Proof Let the function $f_\gamma(z)$ is in the class $T_\gamma \chi_\mu^n(\lambda, \omega, \alpha, p, \zeta, \eta)$, we have

$$(m+1)\gamma(s)\left[1 + \eta(p\zeta - 1)\right]\left\{\mu\gamma^n(s)\left[m(\lambda - \omega) + m + 1\right]^n + (1-\mu)B_{m+1}(n)\right\}$$
$$\sum_{k=m+1}^{\infty} a_k \le \eta p\zeta\left[\mu\gamma^n(s) - \mu + 1 - \alpha\right], \tag{2.16}$$

which readily yields

$$\sum_{k=m+1}^{\infty} a_k \le \frac{\eta p\zeta\left[\mu\gamma^n(s) - \mu + 1 - \alpha\right]}{(m+1)\gamma(s)\left[1 + \eta(p\zeta - 1)\right]\left\{\mu\gamma^n(s)\left[m(\lambda - \omega) + m + 1\right]^n + (1-\mu)B_{m+1}(n)\right\}}. \tag{2.17}$$

Applying (2.14) again, in conjunction with (2.17), we get

$$\gamma(s)\left[1 + \eta(p\zeta - 1)\right]\left\{\mu\gamma^n(s)\left[m(\lambda - \omega) + m + 1\right]^n + (1-\mu)B_{m+1}(n)\right\}$$
$$\sum_{k=m+1}^{\infty} k a_k \le \eta p\zeta\left[\mu\gamma^n(s) - \mu + 1 - \alpha\right].$$

So that

$$\sum_{k=m+1}^{\infty} ka_k \leq \frac{\eta p\zeta\left[\mu\gamma^n(s) - \mu + 1 - \alpha\right]}{\gamma(s)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^n(s)\left[m(\lambda-\omega)+m+1\right]^n + (1-\mu)B_{m+1}(n)\right\}} = \delta. \tag{2.18}$$

The proof is completed. □

Corollary 2.1 *Let*

$$\delta = 1 - \frac{\left[\gamma(s)\left(1+\eta\right)\left[m(\lambda-\omega)+m+1\right]^n - 2\eta\right]}{\gamma(s)\left(1+\eta\right)\left[m(\lambda-\omega)+m+1\right]^n}.$$

Then, $S\chi_\gamma^*(n,\eta,\lambda,\omega) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.2 *Let*

$$\delta = 1 - \frac{\left[\gamma(s)\left[m(\lambda-\omega)+m+1\right]^n - \eta\right]}{\gamma(s)\left[m(\lambda-\omega)+m+1\right]^n}.$$

Then, $\chi_\gamma^*(n,\eta,\lambda,\omega) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.3 *Let*

$$\delta = 1 - \frac{\left[\gamma(s)\left(1+\eta\right)B_{m+1}(n) - 2\eta(1-\alpha)\right]}{\gamma(s)\left(1+\eta\right)B_{m+1}(n)}.$$

Then, $\chi_\gamma^*(n,\eta,\alpha) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.4 *Let*

$$\delta = 1 - \frac{\left[\gamma(s)\left(1+\eta\right)B_{m+1}(n) - 2\eta\right]}{\gamma(s)\left(1+\eta\right)B_{m+1}(n)}.$$

Then, $S\chi_\gamma^*(n,\eta) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.5 *Let*

$$\delta = 1 - \frac{\left[\gamma(s)B_{m+1}(n) - \eta\right]}{\gamma(s)B_{m+1}(n)}.$$

Then, $\chi_\gamma^*(n,\eta) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.6 *Let*

$$\delta = 1 - \frac{\left[\gamma(s) - (1-\alpha)\right]}{\gamma(s)}.$$

Then, $\chi_\gamma^*(\alpha) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.7 *Let*

$$\delta = 1 - \frac{\left[\gamma(s) - 1\right]}{\gamma(s)}.$$

Then, $\chi^*(\gamma) \subset N_{m,\delta}(e_\gamma)$.

Corollary 2.8 *Let*

$$\delta = 1 - \frac{\left[\gamma(s) - \eta\right]}{\gamma(s)}.$$

Then, $\chi^*_\gamma(\eta) \subset N_{m,\delta}(e_\gamma)$.

Definition 2.6 *A function* $f_\gamma(z) \in T_\gamma$ *is said to be in the class* $T_\gamma\chi^n_\mu(\lambda,\omega,\alpha,p,\zeta,\eta)$ *if there exist a function* $h_\gamma(z) \in T_\gamma\chi^n_\mu(\lambda,\omega,\alpha,p,\zeta,\eta)$ *such that*

$$\left|\frac{f_\gamma(z)}{h_\gamma(z)} - 1\right| < 1-\rho, \quad (z \in \mathfrak{U}, 0 \le \rho < 1). \tag{2.19}$$

Theorem 2.3 *If* $h_\gamma(z) \in T_\gamma\chi^n_\mu(\lambda,\omega,\alpha,p,\zeta,\eta)$ *and*

$$\rho = 1 - \frac{A(\lambda,\omega,\alpha,p,\zeta,\eta)}{B(\lambda,\omega,\alpha,p,\zeta,\eta)}$$

where

$$\begin{aligned} A(\lambda,\omega,\alpha,p,\zeta,\eta) &= \delta(m+1)\gamma(s)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^n(s)\left[m(\lambda-\omega)+m+1\right]^n \right. \\ &\quad \left. +(1-\mu)B_{m+1}(n)\right\}, \\ B(\lambda,\omega,\alpha,p,\zeta,\eta) &= (m+1)^2\gamma(s)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^n(s)\left[m(\lambda-\omega)+m+1\right]^n \right. \\ &\quad \left. +(1-\mu)B_{m+1}(n)\right\} - (m+1)\eta p\zeta\left[\mu\gamma^n(s)-\mu+1-\alpha\right], \end{aligned}$$

then $N_{m,\delta}(h_\gamma) \subset T_\gamma\chi^n_\mu(\lambda,\omega,\alpha,p,\zeta,\eta)$.

Proof Let $f_\gamma(z) \in N_{m,\delta}(h_\gamma)$. We find from (2.11) that

$$\sum_{k=m+1}^{\infty} k\gamma(s)\left|a_k - b_k\right| \le \delta, \tag{2.20}$$

which ready implies that

$$\sum_{k=m+1}^{\infty} \gamma(s)\left|a_k - b_k\right| \le \frac{\delta}{m+1}. \tag{2.21}$$

Next, since $h_\gamma(z) \in T_\gamma\chi^n_\mu(\lambda,\omega,\alpha,p,\zeta,\eta)$, we have

$$\sum_{k=m+1}^{\infty} \gamma(s)b_k \le \frac{\eta p\zeta\left[\mu\gamma^n(s)-\mu+1-\alpha\right]}{(m+1)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^n(s)\left[m(\lambda-\omega)+m+1\right]^n +(1-\mu)B_{m+1}(n)\right\}},$$

so that

$$\left|\frac{f_\gamma(z)}{h_\gamma(z)} - 1\right| \le \frac{\sum\limits_{k=m+1}^{\infty} \gamma(s)\left|a_k-b_k\right|}{1-\sum\limits_{k=m+1}^{\infty}\gamma(s)\left|b_k\right|}$$

$$\leq \quad \frac{\delta}{m+1} \frac{\begin{array}{c}(m+1)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^{n}(s)\left[m(\lambda-\omega)+m+1\right]^{n}\right.\\ \left.+(1-\mu)B_{m+1}(n)\right\}\end{array}}{\begin{array}{c}(m+1)\left[1+\eta(p\zeta-1)\right]\left\{\mu\gamma^{n}(s)\left[m(\lambda-\omega)+m+1\right]^{n}\right.\\ \left.+(1-\mu)B_{m+1}(n)\right\}-\eta p\zeta\left[\mu\gamma^{n}(s)-\mu+1-\alpha\right]\end{array}}$$

$$= \quad \frac{A(\lambda,\omega,\alpha,p,\zeta,\eta)}{B(\lambda,\omega,\alpha,p,\zeta,\eta)} = 1-\rho.$$

This completes the proof. □

Corollary 2.9 *If* $h_\gamma(z) \in S\chi^*_\gamma(n,\eta,\lambda,\omega)$ *and*

$$\rho = 1 - \frac{\delta\gamma^{n+1}(s)\,(1+\eta)\left[m(\lambda-\omega)+m+1\right]^{n}}{(m+1)\gamma(s)\,(1+\eta)\left\{\gamma^{n}(s)\left[m(\lambda-\omega)+m+1\right]^{n}\right\}-2\eta\left(\gamma^{n}(s)-\alpha\right)}$$

then $N_{m,\delta}(h_\gamma) \subset S\chi^*_\gamma(n,\eta,\lambda,\omega)$.

Corollary 2.10 *If* $h_\gamma(z) \in \chi^*_\gamma(n,\eta,\lambda,\omega)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)\left[m(\lambda-\omega)+m+1\right]^{n}}{(m+1)\gamma(s)\left[m(\lambda-\omega)+m+1\right]^{n}-\eta}$$

then $N_{m,\delta}(h_\gamma) \subset \chi^*_\gamma(n,\eta,\lambda,\omega)$.

Corollary 2.11 *If* $h_\gamma(z) \in \chi^*_\gamma(n,\eta,\alpha)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)(1+\eta)B_{m+1}(n)}{(m+1)\gamma(s)(1+\eta)B_{m+1}(n)-2\eta(1-\alpha)}$$

then $N_{m,\delta}(h_\gamma) \subset \chi^*_\gamma(n,\eta,\alpha)$.

Corollary 2.12 *If* $h_\gamma(z) \in S\chi^*_\gamma(n,\eta)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)(1+\eta)B_{m+1}(n)}{(m+1)\gamma(s)(1+\eta)B_{m+1}(n)-2\eta}$$

then $N_{m,\delta}(h_\gamma) \subset S\chi^*_\gamma(n,\eta)$.

Corollary 2.13 *If* $h_\gamma(z) \in \chi^*_\gamma(n,\eta)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)B_{m+1}(n)}{(m+1)\gamma(s)B_{m+1}(n)-\eta}$$

then $N_{m,\delta}(h_\gamma) \subset \chi^*_\gamma(n,\eta)$.

Corollary 2.14 *If* $h_\gamma(z) \in \chi^*_\gamma(\alpha)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)}{(m+1)\gamma(s)-2(1-\alpha)}$$

then $N_{m,\delta}(h_\gamma) \subset \chi^*_\gamma(\alpha)$.

Corollary 2.15 *If* $h_\gamma(z) \in \chi^*(\gamma)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)}{(m+1)\gamma(s) - 2}$$

then $N_{m,\delta}(h_\gamma) \subset \chi^*(\gamma)$.

Corollary 2.16 *If* $h_\gamma(z) \in \chi^*_\gamma(\eta)$ *and*

$$\rho = 1 - \frac{\delta\gamma(s)}{(m+1)\gamma(s) - \eta}$$

then $N_{m,\delta}(h_\gamma) \subset \chi^*_\gamma(\eta)$.

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International J.Math. Combin. Vol.2(2022), 33-46

Cordial Labeling of Graphs Using Tribonacci Numbers

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Abstract: We introduce Tribonacci cordial labeling as an extension of Fibonacci cordial labeling, a well-known forms of vertex-labelings. A graph that admits Tribonacci cordial labeling is called Tribonacci cordial graph. In this paper we investigate whether some well-known graphs are Tribonacci cordial.

Key Words: Tribonacci cordial, generalized friendship graph, wheel graph, ring sum, joint sum, Smarandachely cordial k-labeling.

AMS(2010): 05C78.

§1. Introduction

Throughout the paper we assume that G is a simple connected graph of order n.

Definition 1.1 *A function $f : V(G) \to \{0,1\}$ is said to be cordial labeling if the induced function $f^* : E(G) \to \{0,1\}$ defined by*

$$f^*(uv) = |f(u) - f(v)|$$

satisfies the conditions $|v_f(0) - v_f(1)| \leqslant 1$, as well as $|e_f(0) - e_f(1)| \leqslant 1$, where,

$v_f(0) :=$ *number of vertices with label* 0,

$v_f(1) :=$ *number of vertices with label* 1,

$e_f(0) :=$ *number of edges with label* 0,

$e_f(1) :=$ *number of edges with label* 1.

Generally, if there are integers $k \in mathbbZ^+$ such that $|v_f(0) - v_f(1)| \leq k$ or $|e_f(0) - e_f(1)| \leq k$, f is called a Smarandachely vertex cordial k-labeling or Smarandachely edge cordial k-labeling, and G a Smarandache cordial k-labeling graph. Clearly, a Smarandache cordial 1-labeling is nothing else but the cordial labeling of graphs.

The concept of cordial labeling was introduced by Cahit [1] though a variety of vertex labeling. This was further extended to various labeling such as divisor cordial labeling, product cordial labeling, total product cordial labeling, prime cordial labeling etc (See [2] for a dynamic survey). Rokad and Ghodasara introduced Fibonacci cordial labeling [5] and provided a list

[1]Received May 15, 2022, Accepted June 15, 2022.

of families of graphs that are Fibonacci cordial. Later this labeling was explored for several other families of graph, (see [3], [4]). Motivated by their work, we investigate *Tribonacci cordial labeling*, which is an extension of the Fibonacci cordial labeling.

Definition 1.2 *The sequence T_n of Tribonacci numbers is defined by the third order linear recurrence relation (for $n \geqslant 0$):*

$$T_{n+3} = T_n + T_{n+1} + T_{n+2}; \quad T_0 = 0, T_1 = T_2 = 1,$$

Definition 1.3 *An injective function $f : V(G) \to \{T_0, T_1, \cdots, T_n\}$ is said to be Tribonacci cordial labeling if the induced function $f^* : E(G) \to \{0, 1\}$ defined by*

$$f^*(uv) = (f(u) + f(v)) \pmod 2$$

satisfies the condition $|e_f(0) - e_f(1)| \leqslant 1$. A graph which admits Tribonacci cordial labeling is called Tribonacci cordial graph.

In this paper we denote the total number of odd edges by $e(1)$ (analogously $e(0)$ for even edges) and $e(1) - e(0)$ will be denoted as $\tilde{e}$.

§2. Main Results

In this section we examine whether some of the trivial graphs like P_n, C_n, K_n are Tribonacci cordial. We can start with a simple observation that for any n, the sequence $\{T_0, T_1, \cdots T_n\}$ has m many evens, where

$$m = \begin{cases} k+1, & \text{if } n = 2k+1; \\ 2k+1, & \text{if } n = 4k \text{ or } 4k+2. \end{cases}$$

Theorem 2.1 *P_n is Tribonacci cordial.*

Proof Let $f : V(P_n) \to \{T_0, T_1, \cdots T_n\}$ be a labeling such that $f(v_i) = T_i$ for all $i = 1, 2, \cdots, n$. Clearly, it implies that the value of $\tilde{e}_{P_n}$ is 0 if n is even and 1 otherwise. □

Theorem 2.2 *For any two assigned Tribonacci labeling (injective) $f : V(C_n) \to \{T_0, T_1, \cdots, T_n\}$ and $g : V(C_n) \to \{T_0, T_1, \cdots, T_n\}$, $\tilde{e}_f - \tilde{e}_g \equiv 0 \pmod 4$.*

Proof Without loss of generality, suppose that f and g are the same Tribonacci labeling except at $v_0 \in V(C_n)$, i.e. $f(v_0) \neq g(v_0)$. Let us consider $e_f(0) = m$ and hence $e_f(1) = n - m$. Also consider v_L and v_R are two adjacent vertices of v_0.

If $f^*(v_L v_0) \equiv 1 \pmod 2$ and $f^*(v_0 v_R) \equiv 0 \pmod 2$ (or vice versa), then it is clear that

$$\tilde{e}_f = \tilde{e}_g.$$

Otherwise without loss of generality, we may assume that

$$f^*(v_L v_0) = f^*(v_0 v_R) \equiv 0 \pmod{2}$$

In this case, clearly $e_g(0)$ will be either m or $m-2$. Respectively, $e_g(1)$ will be either $n-m$ or $n-m+2$. Thus,

$$\tilde{e}_g - \tilde{e}_f = |e_g(0) - e_g(1)| - |e_f(0) - e_f(1)| \equiv 0 \pmod{4}. \qquad \Box$$

Corollary 2.3 *For any injective function $f: V(G) \to \{T_0, T_1, \cdots, T_{2m}\}$ on cyclic graph C_{2m}, if $|\tilde{e}| \equiv 2 \pmod 4$, then C_{2m} is not Tribonacci cordial.*

As it is clear from Corollary 2.3, if $n \equiv 2 \pmod 4$, C_n is not Tribonacci cordial under f, then any other function $g: V(G) \to \{T_0, T_1, \cdots, T_{2m}\}$ will not be able to generate $\tilde{e}_g \equiv 0 \pmod 4$. For $n \not\equiv 2 \pmod 4$, we can consider the labeling $f: V(C_n) \to \{T_0, T_1, \cdots T_n\}$ such that $f(v_i) = T_i$ for all $i = 1, 2, \cdots, n$. Clearly it produces odd and even edges alternatively. Thus we have the following theorem.

Theorem 2.4 *C_n is Tribonacci cordial, except $n \equiv 2 \pmod 4$.*

Lemma 2.5 *K_{2m+1} is Tribonacci cordial only for $m \leq 1$.*

Proof First note that the vertex labeling can be chosen from $T_0, T_1, \cdots, T_{2m+1}$, out of which $m+1$ labels are even and thus $m+1$ are odd. Since we only need $2m+1$ many labelings, we drop either an odd or an even Tribonacci number from the list. Without loss of generality, we use all of the Tribonacci numbers except an even one. As there are $m+1$ many odd and m many even vertex labels, $e(1) = m(m+1)$, and $e(0) = \binom{m+1}{2} + \binom{m}{2}$. Hence in order to be Tribonacci cordial we must have

$$|\tilde{e}| = \left| m(m+1) - \binom{m+1}{2} - \binom{m}{2} \right| \leqslant 1.$$

It simplifies to $|m| \leqslant 1$, which is only possible for $m = 0, 1$ and hence, $n = 1, 3$. $\Box$

Next we divide the case, n is even, into two categories, namely $n = 4m$, and $4m+2$ to get the following two lemmas.

Lemma 2.6 *K_{4m} is Tribonacci cordial only for $m = 1$.*

Lemma 2.7 *K_{4m+2} is Tribonacci cordial only for $m \leqslant 1$.*

The next theorem follows immediately from the previous lemmas, which provide the complete list of all complete graphs that are Tribonacci cordial.

Theorem 2.8 *A complete list of Tribonacci cordial complete graphs are $K_i, 1 \leq i \leq 4$ and K_6.*

A wheel graph W_n is a graph that contains a cycle of n many vertices such that every vertex of the cycle is connected with another vertex known as the hub, see Figure 1.

Theorem 2.9 *W_n is Tribonacci cordial.*

Proof We start by identifying the vertices of W_n as $V(W_n) = \{v\} \cup \{v_1, v_2, \cdots, v_n\}$ where v_i's are the vertices of the cycle in a clockwise manner and v is the hub of the cycle. Such as those shown in Figure 1.

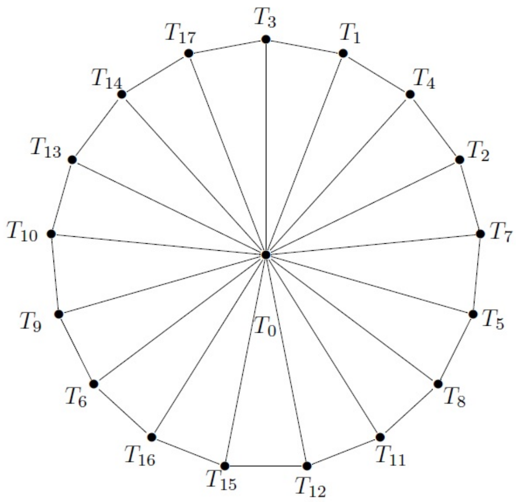

Figure 1. Tribonacci cordial labeling for W_{17} graph

Now, for $n = 4p + q$, where $0 \leqslant q \leqslant 3$, we define

$$p_1 = \begin{cases} p, & \text{if } q = 3; \\ p-1, & \text{otherwise} \end{cases}$$

and $p_2 = p_1 + 2$. Assigns Tribonacci labeling to vertices of the wheel graph W_n as follows:

For $1 \leqslant i \leqslant 2p_1$

$$f(v_i) = \begin{cases} T_{i-2}, & \text{if } i \equiv 0 \pmod 4; \\ T_{i+2}, & \text{if } i \equiv 1 \pmod 4; \\ T_{i-1}, & \text{if } i \equiv 2 \pmod 4; \\ T_{i+1}, & \text{if } i \equiv 3 \pmod 4. \end{cases}$$

For the vertices $2p_1 < i \leqslant n$, define $f(v_i) = T_k$, where,

$$k = \begin{cases} n + (i - 3p) - 2\left\lceil \frac{3p-i-2}{2} \right\rceil + 1 - q, & \text{for } 2p_1 + 1 \leqslant i \leqslant 2p_1 + p_2, \text{ and } q = 0,1,2,3; \\ i - 2\left\lfloor \frac{n-i-q}{2} \right\rfloor - 2, & \text{for } 2p_1 + p_2 < i \leqslant n, \text{ and } q = 0,1,2; \\ i - 2\left\lfloor \frac{n-i-2}{2} \right\rfloor - 2, & \text{for } 2p_1 + p_2 < i \leqslant n, \text{ and } q = 3. \end{cases}$$

Then, a simple calculation ensures the validity of the cordiality of the labeling. □

A shell graph is defined as a cycle C_n with $(n-3)$ chords sharing a common vertex, called the apex, see Figure 2. Shell graphs are denoted as $C_{(n,n-3)}$. The vertices of $C_{(n,n-3)}$ are denoted by $\{v_1, v_2, \cdots, v_n\}$, v_1 as the apex.

Theorem 2.10 *A shell graph $C_{(n,n-3)}$ is Tribonacci cordial for an integer $n \geq 4$.*

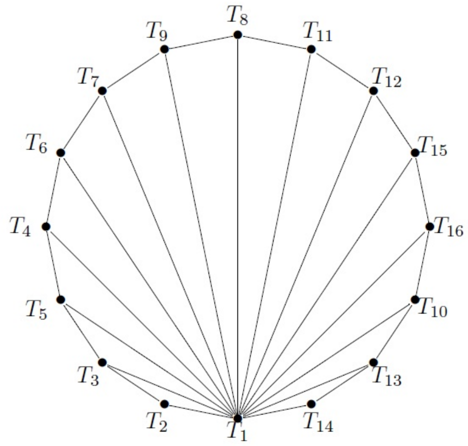

Figure 2. Tribonacci labeling for $C_{16,13}$ graph

Proof Let us rewrite n as $4p+q$, where $0 \leqslant q \leqslant 3$ and $0 \leqslant p \leqslant \lceil n/4 \rceil$. We define p_1 and p_2 as:

$$p_1 = \begin{cases} p, & \text{if } q = 0, 1; \\ p+1, & \text{if } q = 2, 3 \end{cases}$$

and $p_2 = 2p + 1 - p_1$. The function defined below assigns Tribonacci numbers to the vertices of the wheel graphs. Let $f(v_1) = T_1$, and

$$f(v_i) = \begin{cases} T_{i+1}, & \text{if } i \equiv 0 \pmod 4; \\ T_{i-1}, & \text{if } i \equiv 1 \pmod 4; \\ T_i, & \text{otherwise} \end{cases}$$

for $2 \leqslant i \leqslant 2p_1$. For the vertices $2p_1 < i \leqslant n$, let $f(v_i) = T_k$ where

$$k = \begin{cases} n + (i - 3p) - 2\left\lceil \frac{3p-i-2}{2} \right\rceil - 3 - q, & \text{for } 2p_1 + 1 \leqslant i \leqslant 2p_1 + p_2, \text{ and } q = 0,1,2,3; \\ j - 2\left\lfloor \frac{n-i-q}{2} \right\rfloor - 2, & \text{for } 2p_1 + p_2 < i \leqslant n, \text{ and } q = 0,1,2; \\ j - 2\left\lfloor \frac{n-i-2}{2} \right\rfloor - 2, & \text{for } 2p_1 + p_2 < i \leqslant n, \text{ and } q = 3. \end{cases}$$

A simple calculation generates $e_1 = n = e_0$, which ensures that the labeling on W_n is Tribonacci cordial for all integers n. □

The generalized friendship graph $F_{m,n}$ (see Figure 3 for an example) is a collection of n many cycles C_m, meeting at a common vertex. Clearly we can refer friendship graph F_n as $F_{3,n}$. For convenience, let us call the common vertex v the apex, and each cycle a blade of the graph. Consider $V(F_{m,n}) = \{v\} \cup \{v_{i,1}, v_{i,2}, \cdots, v_{i,m}\}_{i=1}^{n}$ where the i^{th} blade is $C_i = \{vv_{i,1}v_{i,2} \cdots v_{i,m}, v\}$. It is evident from the definition that the cardinality of the vertex and edge sets are given by $n(m-1)+1$ and mn respectively.

Lemma 2.11 *If* $m \equiv 1 \pmod 2$ *and* $n \equiv 2 \pmod 4$, *then* $F_{m,n}$ *is not Tribonacci cordial.*

Proof Note that in any blade, any combination of even and odd Tribonacci labeling on the vertices of the cycle C_m including the apex vertex, generates only even values for e_1. Thus $e_1 \equiv 0 \pmod 2$. Now when $n = 4k+2$ and $m = 2p+1$, for some integer $k, p \geqslant 0$, $|E(F_{m,n})| = 8pk + 4(p+k) + 2$. In order to generate Tribonacci cordial labeling for $F_{m,n}$, e_1 must be $4pk + 2(p+k) + 1$, which clearly contradicts that e_1 is even. □

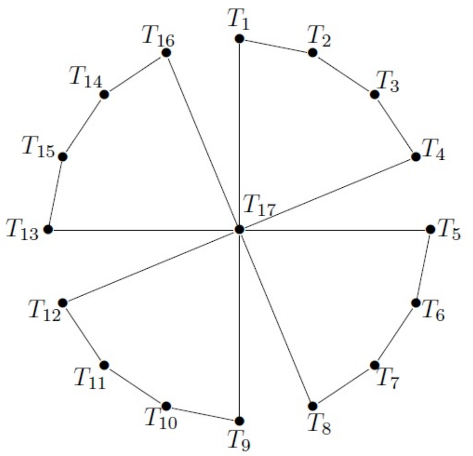

Figure 3. Tribonacci cordial labeling for Friendship graph $F_{5,4}$

Now we investigate whether $F_{m,n}$ is Tribonacci cordial for the remaining values, i.e., for $m \not\equiv 0 \pmod 2$ or $n \not\equiv 2 \pmod 4$. First, we look into the case when $m \equiv 1 \pmod 2$ and $n \not\equiv 2 \pmod 4$ to obtain the following.

Lemma 2.12 $F_{3,n}$ *is Tribonacci cordial if* $n \not\equiv 2 \pmod 4$.

Proof For convenience, we redefine vertex v_{ij} as v_k, where $k = 2(i-1)+j$, and

$$p = \begin{cases} n/2, & \text{if } n \equiv 0 \pmod 4; \\ (n+1)/2, & \text{if } n \equiv 1 \pmod 4; \\ (n-1)/2, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Now we provide the function that assigns Tribonacci numbers to the vertices of the $F_{3,n}$. For $n \equiv 0 \pmod 4$, we label v with T_{2n+1}, and label the rest of the vertices as follow:

$$f(v_k) = \begin{cases} T_{k+1}, & \text{for } 1 \leqslant k \leqslant p \text{ and } \ k \equiv 2 \pmod 4; \\ T_{k-1}, & \text{for } 1 \leqslant k \leqslant p \text{ and } \ k \equiv 3 \pmod 4; \\ T_k, & \text{otherwise .} \end{cases}$$

For $n \equiv 1, 3 \pmod 4$,

$$f(v_k) = \begin{cases} T_{k-1}, & \text{for } 1 \leqslant k \leqslant p; \\ T_k, & \text{for } p \leqslant k \leqslant 2n \end{cases}$$

and finally $f(v) = T_{n-1}$. □

Lemma 2.13 *For $n \not\equiv 2 \pmod 4$, $F_{5,n}$ is Tribonacci cordial.*

Proof Let us define

$$p = \begin{cases} 3n/4, & \text{if } n \equiv 0 \pmod 4; \\ (3n+1)/4, & \text{if } n \equiv 1 \pmod 4; \\ (3n-1)/4, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

The following function assigns Tribonacci labeling to the vertices of the graph $F_{5,n}$. For $1 \leqslant k \leqslant p$, $f(v_k) = T_k$, and for $p+1 \leqslant k \leqslant 4n$ we define

$$f(v_k) = \begin{cases} T_{k+1}, & \text{if } k \equiv 2 \pmod 4; \\ T_{k-1}, & \text{if } k \equiv 3 \pmod 4; \\ T_k, & \text{otherwise} \end{cases}$$

and finally $f(v) = T_{4n+1}$. It can be easily observed that for each blade, excluding the common vertex v, we label two categories of labelings in the following order: viz. p many even-even-odd-odd, and $n-p$ many even-odd-even-odd. Hence the value of $\tilde{e}$ in each blade, of former kind is -1, and 3 on the later (as the common vertex is being labeled by an odd Tribonacci number). Consequently we obtain

$$\tilde{e} = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 4; \\ -1, & \text{if } n \equiv 1 \pmod 4; \\ 1, & \text{if } n \equiv 3 \pmod 4. \end{cases}$$ □

We believe that the following holds.

Conjecture 2.14 *$F_{2k+1,n}$ is Tribonacci cordial for $n \not\equiv 2 \pmod 4$ and any positive integer $k \geqslant 3$.*

Theorem 2.15 *$F_{m,n}$ is Tribonacci cordial if $m \equiv 2 \pmod 4$ and $n \equiv 0 \pmod 2$ or $m \equiv 0 \pmod 4$ for any n.*

Proof Let $f : V(F_{m,n}) \to \{T_0, T_1, \cdots, T_{n(m-1)+1}\}$ such that $f(v_i) = T_{(m-1)n+2-i}$ and $f(v) = T_1$. It is clear for $m \equiv 0 \pmod 4$, each blade the vertices is getting the labels odd-even-even-odd-$\cdots$-even-odd, which with the odd labeled common vertex result in $\tilde{e} = 0$. Thus we have a Tribonacci cordial graph for any choice of n.

Now for $m \equiv 2 \pmod 4$ where $n \equiv 0 \pmod 2$, we can make the observation that the values of $\tilde{e}$ in each blade are to be $\{-2, 2, 2, -2, -2, 2, 2, -2 \cdots\}$. This clearly implies that that we have a Tribonacci cordial labeling ($\tilde{e} = 0$) when the number of blades are even, i.e., $n \equiv 0 \pmod 2$. This completes the proof. □

Theorem 2.16 *$K_{m,n}$ is Tribonacci cordial for all m, n.*

Proof Let us denote the vertices of $K_{m,n}$ by $\{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$. In order to assign Tribonacci labeling to the graph $K_{m,n}$ we consider the following cases.

Case 1. Assume that at least one of m, n is even. Without loss of generality we consider m to be even and use the following labeling: $f(u_i) = T_{i-1}, f(v_i) = T_{m+i-1}$. Note that, $m/2$ even and $m/2$ odd Tribonacci labels are used on one side, which result in $\tilde{e} = 0$, for any assignment of labels on the other side.

Case 2. Consider the case when both m and n are odd. Let $m = 2k_1 + 1$ and $n = 2k_2 + 1$, following the same pattern of labeling as the previous case. It can be easily noted that there are either k_1 and $k_2 + 1$, or $k_1 + 1$ and k_2 even labels used. The former case yields $\tilde{e} = k_1k_2 + (k_1 + 1)(k_2 + 1) - k_1(k_2 + 1) - (k_1 + 1)k_2 = 1$, whereas, later case implies $\tilde{e} = -1$, ensuring the cordiality of the labeling in either one. □

Bistar $B_{m,n}$ is the graph obtained by joining the apex vertices of two copies of star, viz. $K_{1,m}$ and $K_{1,n}$, by an edge. We identify the vertex set as $\{u_i : 1 \leqslant i \leqslant m\} \cup \{v_i : 1 \leqslant i \leqslant n\} \cup \{u, v\}$ where u, v are the apex vertices and u_i, v_i are the pendant vertices connected with u and v respectively. The vertex and edge set cardinalities are given by $m + n + 2$ and $m + n + 1$ respectively.

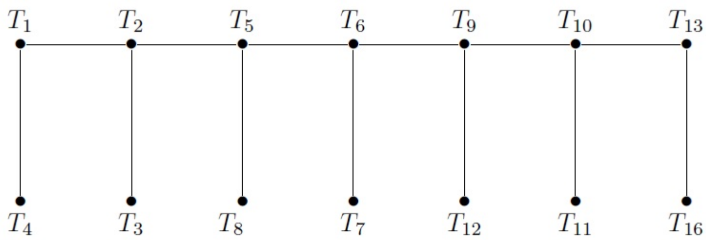

Figure 4. Tribonacci cordial labeling for $P_{\odot}K_1$ graph

Theorem 2.17 *Bistar graph $B_{m,n}$ are Tribonacci cordial.*

Proof Define the function $f : (B_{m,n}) \to \{T_0, T_1, \cdots, T_{m+n+2}\}$ as $f(u_i) = T_{i+1}, f(v_i) = T_{m+i+1}$ which assigns Tribonacci numbers to all pendant vertices. In order to label the apex vertices u, v, consider the following cases.

Case 1. Let at least one of m, n is even. Without loss of generality we can assume that m is even. Label $f(u) = T_0$ and $f(v) = T_1$ if $m+n \equiv 1 \pmod 4$, and vice-versa if $m+n \not\equiv 1 \pmod 4$. It is clear that $m/2$ many vertices are labeled with even (and odd) Tribonacci labels. Now if $m + n \equiv 1 \pmod 4$ then $(n-1)/2$ many even and $(n+1)/2$ many odd Tribonacci labels are used on other side. Thus $\tilde{e} = m/2 + 1 + (n-1)/2 - m/2 - (n+1)/2 = 0$.

On the other hand if $m+n \not\equiv 1 \pmod 4$, then there are $\lceil n/2 \rceil$ many even and $\lfloor n/2 \rfloor$ many odd Tribonacci labels used on the side of n pendant vertices. Hence $|\tilde{e}| = |m/2 - m/2 + 1 + \lfloor n/2 \rfloor - \lceil n/2 \rceil| \leqslant 1$.

Case 2. Consider both m, n are odd. Once again without loss of generality we will consider three cases: $m \equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$; $m \equiv 3 \pmod 4$ and $n \equiv 1 \pmod 4$; $m \equiv 3 \pmod 4$ and $n \equiv 3 \pmod 4$. For the first case $f(u) = T_1$ and $f(v) = T_0$ and vice-versa on last the two cases. It is very similar to the previous case to verify that this assigns a Tribonacci labeling to the graph $B_{m,n}$. □

Theorem 2.18 *Complete binary trees are Tribonacci cordial.*

Proof Let T be the complete binary tree. We denote the vertices as $\{v_i^j : 1 \leqslant i \leqslant 2^j, 0 \leqslant j \leqslant \ell\}$, where ℓ denoted the levels, and v_i^ℓ is connected to $v_{2i-1}^{\ell+1}$ and $v_{2i}^{\ell+1}$. We define the labeling $f : V(T) \to \{T_0, T_1, \cdots, T_{2n}\}$ as $f(v_i^j) = T_{2^j+i-1}$. If we denote e_k as the edge connecting the vertices the parent $v_{\lceil i/2 \rceil}^{j-1}$ and the child vertex v_i^j, where $k = 2^j + i - 2$, then we observe the following:

$$f^*(e_k) = \begin{cases} 1 \pmod 2, & \text{if } k \equiv 1, 4, 6, 7 \pmod 8; \\ 0 \pmod 2, & \text{if } k \equiv 0, 2, 3, 5 \pmod 8. \end{cases}$$

As the maximum possible value for k is $2(2^\ell - 1)$, we have $e_1 = e_0 = 2^\ell - 1$, which implies $\tilde{e} = 0$. Thus T is Tribonacci cordial. □

Theorem 2.19 P_n^2 *is Tribonacci cordial.*

Proof Let us identify the vertices of the graph as $V(P_n^2) = \{v_1, v_2, \cdots, v_n\}$. We define labeling $f : V(P_n^2) \to \{T_0, T_1, T_2, \cdots, T_n\}$ as follows:

$$f(v_i) = \begin{cases} T_{i-2}, & \text{if } q = 0; \\ T_{i-1}, & \text{if } q = 1, 2; \\ T_i, & \text{if } q = 3, \end{cases}$$

where, $i = 4p + q$. Clearly this labeling generates $e_1 = n$ and $e_0 = n - 1$, hence $\tilde{e} = 1$, which proves the theorem. □

Theorem 2.20 C_n^2 *is Tribonacci cordial only for integers* $n \equiv 0 \pmod 2$.

Proof First we note that, for any two Tribonacci labeling $f : V(C_n^2) \to \{T_0, T_1, \cdots, T_n\}$ and $g : V(C_n^2) \to \{T_0, T_1, \cdots, T_n\}$, $\tilde{e}_f - \tilde{e}_g \equiv 0 \pmod 4$. We omit the proof as it very similar to Theorem 2.2. Now observe that when n is odd, i.e., $n = 2k + 1$, $|E(C_n^2)| = 4k + 2$, thus $\tilde{e}_{C_n^2} \equiv 2 \pmod 4$. For the case n being even, let $v_1, v_2, \cdots, v_n$ be the vertices of the graph C_n^2. We define the following function that assign labelings to the vertices.

$$f(v_i) = \begin{cases} T_{i-2}, & \text{if } q = 0; \\ T_{i-1}, & \text{if } q = 1, 2; \\ T_i, & \text{if } q = 3, \end{cases}$$

where, $i = 4p + q$. Clearly this labeling generates $e_1 = e_0 = n$, hence $\tilde{e} = 0$, which proves the theorem. □

Theorem 2.21 *For an integer $n \geq 1$, the ladder graph $L_n = P_n \times P_2$ is Tribonacci cordial.*

Proof We start by identifying the vertex set $V(L_n) = \{u_i, v_i : 1 \leqslant i \leqslant n\}$, and the edge set $E(L_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leqslant i \leqslant n-1\} \cup \{u_iv_i : 1 \leqslant i \leqslant n\}$. The cardinality of the vertex and edges sets are $2n$ and $3n-2$ respectively. As a base case consider $n = 2$. Note that $L_2 = C_4$, which is Tribonacci cordial by Theorem 2.4. Choose the labeling $f(u_1) = T_0, f(u_2) = T_3, f(v_i) = T_i$ for $i = 1, 2$, which gives us $\tilde{e}_{L_2} = 0$. Now generate a Tribonacci cordial labeling for L_3, by assigning $f(u_3) = T_4$ and $f(v_3) = T_5$, which lead us to $\tilde{e}_{L_3} = -1$. Finally, for $n = 4$, label (and relabel) as follows: $f(u_3) = T_6, f(u_4) = T_7$ and $f(u_4) = T_4$, we again get $\tilde{e}_{L_4} = 0$.

Continue labeling the vertices of L_{n+1} from L_n in this fashion, where u_{n+1} and v_{n+1} get the even and odd labeling from $\{T_{2n}, T_{2n+1}\}$ respectively if $n \not\equiv 3 \pmod 4$. When $n \equiv 3 \pmod 4$ in addition to the previous step, we switch the labeling of u_n and u_{n+1}.

Now we show that the above style of labeling always provides Tribonacci cordial labeling for L_n. It can be easily verified that $\tilde{e}_{n+1} = \tilde{e}_n - 1$ for $n \equiv 1, 2 \pmod 4$ and $\tilde{e}_{n+1} = \tilde{e}_n + 1$ for $n \equiv 0, 3 \pmod 4$. Thus L_n is Tribonacci cordial for all n. □

Theorem 2.22 *For any integer $n \geq 1$, the comb graphs $P_n \odot K_1$ are Tribonacci cordial.*

Proof Identify the vertices of a comb graph as $V(G) = \{u_1, u_2, \cdots, u_n\} \cup \{v_1, v_2, \cdots, v_n\}$ where $u_0, u_1, \cdots, u_n$ are the vertices of the path P_n and $v_1, v_2, \cdots, v_n$ are the attached pendant vertices (see Figure 4). The following function assign Tribonacci cordial labeling to the vertices of the comb graph.

$$f(u_i) = \begin{cases} T_{2i-1}, & \text{if } i \equiv 1 \pmod 2; \\ T_{2i-2}, & \text{if } i \equiv 0 \pmod 2, \end{cases}$$

$$f(v_i) = \begin{cases} T_{2i}, & \text{if } i \equiv 0 \pmod 2 \text{ and } n \equiv 0 \pmod 2; \\ T_{2i+1}, & \text{if } i \equiv 1 \pmod 2 \text{ and } n \equiv 0 \pmod 2; \\ T_{2i-1}, & \text{if } i \equiv 0 \pmod 2 \text{ and } n \equiv 1 \pmod 2; \\ T_{2i+2}, & \text{if } i \equiv 1 \pmod 2 \text{ and } n \equiv 1 \pmod 2) \end{cases}$$

for $i \in \{1, 2, \cdots, n\}$. It can be easily verified that $f^*(u_iu_{i+1}) = 0$ and $f^*(u_iv_i) = 1$ for all i. Thus, for any value of n, $\tilde{e} = e_1 - e_0 = n - (n-1) = 1$. □

Theorem 2.23 *$C_n \odot K_1$ are Tribonacci cordial.*

Proof In the graph $G = C_n \odot K_1$, let $V = \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$, where v_i's denote the vertices of C_n and u_i's represent the pendant vertices adjacent to v_i's. The following function assigns Tribonacci cordial labeling to the graph.

$$f(v_i) = \begin{cases} T_{2i-1}, & \text{if } i \equiv 0 \pmod 2; \\ T_{2i-2}, & \text{if } i \equiv 1 \pmod 2, \end{cases}$$

$$f(u_i) = \begin{cases} T_{2i-2}, & \text{if } i \equiv 0 \pmod 2; \\ T_{2i-1}, & \text{if } i \equiv 1 \pmod 2). \end{cases}$$

This completes the proof. □

Theorem 2.24 *Petersen graph is Tribonacci cordial.*

Proof Let v_1, v_2, v_3, v_4, v_5 be the internal vertices and $v_6, v_7, v_8, v_9, v_{10}$ be the external vertices of Petersen graph such that each v_i is adjacent to v_{i+5}, $1 \leqslant i \leqslant 5$. The following function assigns Tribonacci cordial labeling to the vertices of the Petersen graph.

$$f(v_i) = \begin{cases} T_0, & \text{if } i = 10; \\ T_{i+2}, & \text{if } i = 3, 4; \\ T_{i-2}, & \text{if } i = 5, 6; \\ T_i, & \text{otherwise}. \end{cases}$$

This completes the proof. □

Definition 2.25 *the ring sum $G_1 \oplus G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \oplus G_2 = (V_1 \cup V_2, (E1 \cup E_2) - (E_1 \cap E_2))$.*

We now construct a family of graph $G = C_n \oplus K_{1,n}$ where the apex vertex of the star graph $K_{1,n}$ is a member of the graph C_n.

Theorem 2.26 *The graph $C_n \oplus K_{1,n}$ is Tribonacci cordial, $n \geqslant 3$.*

Proof Start by identifying the vertices of the graph G. Let $V(G) = V_1 \cup V_2$, where $V_1 = \{v_1, v_2, \cdots, v_n\}$ is the vertex set of the C_n and $V_2 = \{u = v_1, u_1, u_2, \cdots, u_n\}$ be the vertex set of $K_{1,n}$. Assign Tribonacci numbers to the vertices in the following fashion: $f(v_i) = T_{i-1}$ for $i \in \{1, 2, \cdots, n\}$. However in order to label the pendant vertices, if $n \not\equiv 2 \pmod 4$ then $f(u_i) = T_{n+i-1}$ for $i \in \{1, 2, \cdots, n\}$, whereas

$$f(u_i) = \begin{cases} T_{n+i-1}, & \text{for } 1 \leqslant i \leqslant n-2; \\ T_{n+i}, & \text{for } n-1 \leqslant i \leqslant n \end{cases}$$

if $n \equiv 2 \pmod 4$. It can be easily verified that the above function provides Tribonacci cordial labeling for the graph $C_n \oplus K_{1,n}$. □

The jellyfish graph $J(n, n)$ is obtained from a 4-cycle u_1, u_2, u_3, u_4 by joining u_1 and u_3 with a chord and appending $\{v_i : 1 \leqslant j \leqslant n\}$ and $\{w_i : 1 \leqslant j \leqslant n\}$ pendent vertices from u_4 and u_2 respectively. Consider the following vertex labeling function: $f(u_i) = T_{i-1}$ for$1 \leqslant i \leqslant 4$, $f(w_i) = T_{2i+2}$ for $1 \leqslant i \leqslant n$ and $f(v_i) = T_{2i+n+2}$ for $1 \leqslant i \leqslant n$. It can be easily verified that this function provides a Tribonacci cordial labeling for $J(n, n)$, thus we have the following result.

Theorem 2.27 *$J(n,n)$ is Tribonacci cordial for all n.*

Definition 2.28 *the Joint sum $G_1 \boxplus G_2$ of two graphs G_1 and G_2 is the graph obtained by connecting a vertex of G_1 with a vertex of G_2.*

Theorem 2.29 *The joint sum graph $C_m \boxplus P_n$ is Tribonacci cordial for all values of m, n.*

Proof Define the graph $C_m \boxplus P_n$ as follows:

$V(C_m \boxplus P_n) = \{u_1, u_2, \cdots, u_m\} \cup \{v_1, v_2, \cdots, v_n\}$, where $\{u_1, u_2, \cdots, u_m\}$ are the vertices of the cycle and $\{v_1, v_2, \cdots, v_n\}$ are the vertices of the path. The edges of the cycle are given by $u_i u_{i+1 (\text{mod } m)}$ for $1 \leqslant i \leqslant m$ and $v_j v_{j+1}$ for $1 \leqslant i \leqslant n-1$ construct the edges of the path and the edge $u_m v_1$ connects the cycle and the path, see Figure 5 for details.

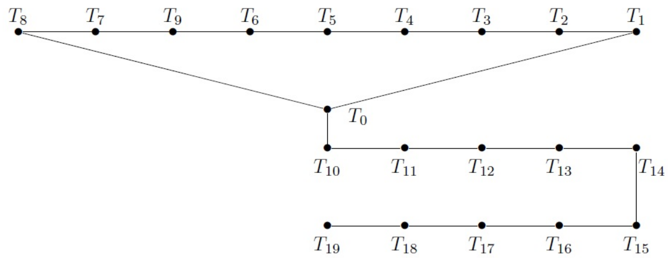

Figure 5. Tribonacci cordial labeling of the joint graph $C_{10} \boxplus P_{10}$

The following vertex labeling generates the Tribonacci Cordial Labeling $f(v_j) = T_{j+m-1}$ for $1 \leqslant j \leqslant n$.

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant m-1; \\ T_0, & \text{for } j = m \end{cases}$$

for all m when $m \not\equiv 2(\text{mod } 4)$. Otherwise $m \equiv 2(\text{mod } 4)$

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant m-4; \\ T_{m-1}, & \text{for } j = m-3; \\ T_{i-1}, & \text{for } m-2 \leqslant i \leqslant m-1; \\ T_0, & \text{for } j = m. \end{cases}$$

This completes the proof. □

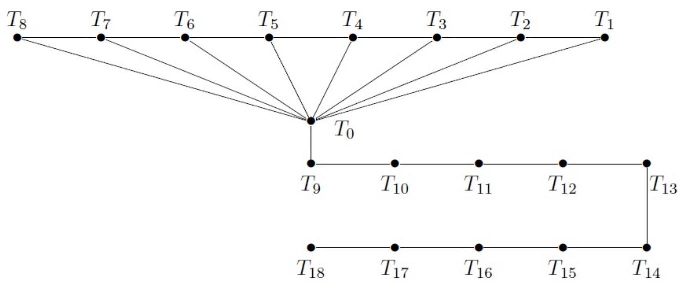

Figure 6. Tribonacci cordial labeling of the joint graph $F_8 \boxplus P_{10}$

Theorem 2.30 *The graph $F_m \boxplus P_n$ is Tribonacci cordial for all m, n.*

Proof We begin identifying the vertices of the graph $F_m \boxplus P_n$. Let $V(F_m \boxplus P_n) = V_1 \cup V_2$, where $V_1 = \{u\} \cup \{u_1, u_2, \cdots, u_m\}$ is the vertex set of F_m (u is hub vertex, which is connected with $u_i, 1 \leqslant i \leqslant m$) and $V_2 = \{v_1, v_2, \cdots, v_n\}$ be the vertex set of P_n, see Figure 6.

First, label the vertices of the path in following manner:

$f(u) = T_0$ and $f(v_i) = T_{i+m}$ for all $1 \leqslant i \leqslant n$, and for the cycle when $m \equiv 1 (\text{mod } 4)$ we have

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant m-3; \\ T_m, & \text{for } j = m-2; \\ T_{i-1}, & \text{for } m-1 \leqslant i \leqslant m. \end{cases}$$

Otherwise, when $m \not\equiv 1(\text{mod } 4)$,$f(u_i) = T_i$ for all $1 \leqslant i \leqslant m$. The vertex labeling defined above generates Tribonacci cordial labeling for the given graph. □

Theorem 2.31 *The graph $C_m \boxplus K_{1,n}$ is Tribonacci cordial for all values of m, n.*

Proof The vertex set of $C_m \boxplus K_{1,n}$ is defined as $V(C_m \boxplus K_{1,n}) = \{u_1, u_2, \cdots, u_m\} \cup \{v, v_1, v_2, \cdots, v_n\}$, where $\{u_1, u_2, \cdots, u_m\}$ are the vertices of the cycle and $\{v_1, v_2, \cdots, v_n\}$ are the pendant vertices of the star graph whereas v is the apex vertex star garph $K_{1,n}$. The edge set of $C_m \boxplus K_{1,n}$ is given by the collection of edges $u_iu_{i+1(\text{mod } m)}$ for $1 \leqslant i \leqslant m$, u_mv and vv_j for $1 \leqslant i \leqslant n$.

The following vertex label assignment ensures that $C_m \boxplus K_{1,n}$ satisfies all the conditions of being a Tribonacci cordial graph.

$f(v_j) = T_{j+m}$ for $1 \leqslant j \leqslant n$, $f(v) = T_m$.

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant m-1; \\ T_0, & \text{for } j = m \end{cases}$$

for all m when $m \not\equiv 2(\text{mod } 4)$. Otherwise for $m \equiv 2(\text{mod } 4)$ $f(v) = T_m$ and

$$f(v_j) = \begin{cases} T_4, & \text{for } j = 1; \\ T_{j+m}, & \text{for } 2 \leqslant j \leqslant n \end{cases}$$

and

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant 3; \\ T_{i+1}, & \text{for } 4 \leqslant i \leqslant m-1; \\ T_0, & \text{for } i = m. \end{cases}$$

This completes the proof. □

Definition 2.32 *The ring sum $G_1 \oplus G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \oplus G_2 = (V_1 \bigcup V_2, (E_1 \bigcup E_2 - E_1 \bigcap E_2))$.*

Theorem 2.33 *The graph $C_m \oplus K_{1,n}$ is Tribonacci cordial for all values of m, n.*

Proof Define the graph $C_m \oplus K_{1,n}$ constituting the vertex set as $V(C_m \oplus K_{1,n}) = \{u_1, u_2, \cdots, u_m\} \cup \{v_1, v_2, \cdots, v_n\}$, where $\{u_1, u_2, \cdots, u_m\}$ and $\{v_1, v_2, \cdots, v_n\}$ are respectively the vertices of the cycle and the star graph. The edges of the cycle are given by $u_i u_{i+1 (\text{mod } m)}$ for $1 \leqslant i \leqslant m$ and $u_m v_j$ for $1 \leqslant i \leqslant n$ define the edges of the star graph, see Figure 7.

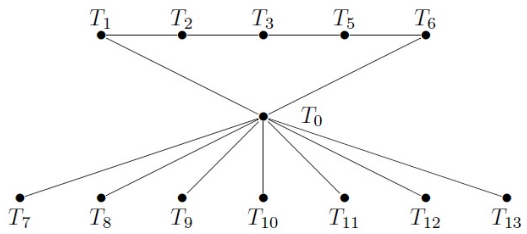

Figure 7. Tribonacci cordial labeling of the ring sum graph $C_6 \oplus K_{1,7}$

The following vertex labeling generates the Tribonacci Cordial Labeling for the given graph. $f(v_j) = T_{j+m-1}$ for $1 \leqslant j \leqslant n$, and

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant m-1; \\ T_0, & \text{for } i = m \end{cases}$$

for all m when $m \not\equiv 2(\text{mod } 4)$. Otherwise (i.e. when $m \equiv 2 \pmod 4$)

$$f(v_j) = \begin{cases} T_4, & \text{for } j = 1; \\ T_{j+m-1}, & \text{for } 2 \leqslant j \leqslant n \end{cases}$$

and

$$f(u_i) = \begin{cases} T_i, & \text{for } 1 \leqslant i \leqslant 3; \\ T_{i+1}, & \text{for } 4 \leqslant i \leqslant m-1; \\ T_0, & \text{for } j = m. \end{cases}$$

This completes the proof. □

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International J.Math. Combin. Vol.2(2022), 47-55

Semi-invariant Sub-manifolds of Generalized Sasakian-Space-Forms

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Abstract: In this paper, we study the decomposition of basic equation of generalized Sasakian space-forms is taken out into horizontal and vertical projections and also we discuss the integrability of distributions D, $D \oplus [\xi]$ and $D^{\perp}$ totally geodesic of semi-invariant sub-manifolds of generalized Sasakian-space-forms.

Key Words: Sub-manifold, semi-invariant sub-manifold, generalized Sasakian-space-forms, totally umbilical(geodesic), integrability condition of distribution.

AMS(2010): 53C15, 53C25, 53C40, 53C50.

§1. Introduction

The notion of semi-invariant sub-manifold is a generalization of invariant and anti-variant sub-manifolds of almost contact metric manifolds. Many authors [6, 8, 9, 20] have obtained the decomposition of basic equations of Kenmotsu, LP-Sasakian, (k, μ)-contact, LP-Cosymplectic manifolds into horizontal and vertical components and also they have studied the integrability of horizontal and vertical distributions. Further, the analysis of totally umbilical and totally geodesics of sub-manifolds of (k, μ)-contact manifolds is done by the author [6]. In [10, 19], the authors studied totally geodesics of sub-manifolds of (ϵ, δ)-trans-Sasakian manifolds. As a generalization of Sasakian space-form, Alegre et al. [1] introduced and studied the notion of generalized Sasakian-space-form with the existence of such notions with various examples.

§2. Preliminaries

An n-dimensional generalized Sasakian-space-forms $\overline{M}$ is a smooth connected manifold with a metric g, that is, $\overline{M}$ admits a smooth symmetric tensor field g of type $(0, 2)$ such that for

[1]Received May 18, 2022, Accepted June 16, 2022.

each point the tensor $g_p : T_P\overline{M} \times T_P\overline{M} \to R$ is a non-degenerate bilinear form of signature $(-,+,\cdots,+)$, where $T_P\overline{M}$ denotes the tangent vector space of $\overline{M}$ at p and R is the real number space, which satisfies

$$\phi^2(X_1) = -X_1 + \eta(X_1)\xi, \;\; \phi\xi = 0, \;\; \eta(\phi X_1) = 0, \tag{2.1}$$

$$g(\phi X_1, \phi Y_1) = g(X_1, Y_1) - \eta(X_1)\eta(Y_1), \;\; g(X_1, \xi) = \eta(X_1). \tag{2.2}$$

for any $X_1, Y_1 \in T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$. An almost contact metric manifold is called a generalized Sasakian-space-form if

$$(\overline{\nabla}_{X_1}\phi)(Y_1) = (f_1 - f_3)(g(X_1, Y_1)\xi - \eta(Y_1)X_1), \tag{2.3}$$

$$\overline{\nabla}_{X_1}\xi = -(f_1 - f_3)\phi X_1, \tag{2.4}$$

$$(\overline{\nabla}_{X_1}\eta)(Y_1) = g(\overline{\nabla}_{X_1}\xi, Y_1), \tag{2.5}$$

$$g(X_1, \phi Y_1) = -g(\phi X_1, Y_1), \tag{2.6}$$

where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$.

The sub-manifold M of the generalized Sasakian-space-form $\overline{M}$ is said to be semi-invariant if it is endowed with the pair of orthogonal distribution $(D, D^{\perp})$ satisfying the conditions

(*i*) $TM = D \oplus D^{\perp} \oplus [\xi]$;

(*ii*) the distribution D is invariant under ϕ, that is, $\phi D_x = D_x$, for each $x \in M$;

(*iii*) the distribution $D^{\perp}$ is anti-invariant under ϕ, that is, $\phi D_x^{\perp} \subset T_x M^{\perp}$ for each $x \in M$,

where D and $D^{\perp}$ are the horizontal and vertical distribution respectively. A semi-invariant sub-manifold M is said to be invariant if we have $D_x^{\perp} = 0$ and is said to be anti-invariant if $D_x = 0$ for each $x \in M$. We denote the projection morphisms of TM to D and $D^{\perp}$ by P and Q respectively. For any $X_1 \in \Gamma(TM)$ and $N \in \Gamma(TM^{\perp})$, we have

$$X_1 = PX_1 + QX_1 + \eta(X_1)\xi, \tag{2.7}$$

$$\phi N = BN + CN, \tag{2.8}$$

where BN and CN denotes the tangential and normal components of ϕN.

The equations of Gauss and Weingarten for the immersion of M in $\overline{M}$ are given by

$$\overline{\nabla}_{X_1}Y_1 = \nabla_{X_1}Y_1 + h(X_1, Y_1), \tag{2.9}$$

$$\overline{\nabla}_{X_1}N = -A_N X_1 + \nabla^{\perp}_{X_1}N, \tag{2.10}$$

for any $X_1, Y_1 \in \Gamma(TM)$ and $N \in TM^{\perp}$, where ∇ is the Levi-Civita connection on M, $\nabla^{\perp}$ is the linear connection induced by $\overline{\nabla}$ on the normal bundle $TM^{\perp}$, h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section

N. Also, we have

$$g(h(X_1, Y_1), N) = g(A_N, Y_1), \tag{2.11}$$

for any $X_1, Y_1 \in \Gamma(TM), N \in \Gamma(TM^{\perp})$.

For readers unfamiliar with terminology, notations, recent overviews and introductions, we suggest the authors to refer the papers [2, 3, 4, 5, 7, 11, 11, 12, 13, 14, 15, 16, 17, 18].

§3. Decomposition of Basic Equations

For $X_1, Y_1 \in \Gamma(TM)$, we take

$$u(X_1, Y_1) = \nabla_{X_1} \phi P Y_1 - A_{\phi Q Y_1} X_1. \tag{3.1}$$

Lemma 3.1 *Let M be a semi-invariant sub-manifold of generalized Sasakian-space-form $\overline{M}$. Then, we have*

$$P(u(X_1, Y_1)) = (f_1 - f_3) g(X_1, Y_1) P\xi - (f_1 - f_3)\eta(Y_1) P X_1 + \phi P(\nabla_{X_1} Y_1), \tag{3.2}$$

$$Q(u(X_1, Y_1)) = (f_1 - f_3) g(X_1, Y_1) Q\xi - (f_1 - f_3)\eta(Y_1) Q X_1 + Bh(X_1, Y_1), \tag{3.3}$$

$$h(X_1, \phi P Y_1)) = -\nabla^{\perp}_{X_1} \phi Q Y_1 + \phi Q(\nabla_{X_1} Y_1) + Ch(X_1, Y_1), \tag{3.4}$$

$$\eta(u(X_1, Y_1)) = (f_1 - f_3) g(\phi X_1, \phi Y_1). \tag{3.5}$$

for all $X_1, Y_1 \in TM$.

Proof In the view of (2.3) and (2.7), we have

$$\begin{aligned}(\overline{\nabla}_{X_1}\phi)(Y_1) = (f_1 - f_3)[&g(X_1, Y_1)P\xi + g(X_1, Y_1)Q\xi + g(X_1, Y_1)\xi \\ &- \eta(Y_1)PX_1 - \eta(Y_1)QX_1 - \eta(X_1)\eta(Y_1)\xi].\end{aligned} \tag{3.6}$$

Now, decompose the LHS of (2.3) and by using (2.8), (2.9), (2.10), we get:

$$\begin{aligned}(\overline{\nabla}_{X_1}\phi)Y_1 &= \overline{\nabla}_{X_1}\phi P Y_1 + \overline{\nabla}_{X_1}\phi Q Y_1 - \phi(\nabla_{X_1} Y_1) - \phi h(X_1, Y_1) \\ &= \nabla_{X_1}\phi P Y_1 + h(X_1, \phi P Y_1) - A_{\phi Q Y_1} + \nabla^{\perp}_{X_1}\phi Q Y_1 - \phi P(\nabla_{X_1} Y_1) \\ &\quad - \phi Q(\nabla_{X_1} Y_1) - Bh(X_1, Y_1) - Ch(X_1, Y_1).\end{aligned} \tag{3.7}$$

Now using (3.1) in above equation, we get

$$\begin{aligned}(\overline{\nabla}_{X_1}\phi)Y_1 = u(X_1, Y_1) &+ h(X_1, \phi P Y_1) + \nabla^{\perp}_{X_1}\phi Q Y_1 \\ &- \phi P(\nabla_{X_1} Y_1) - \phi Q(\nabla_{X_1} Y_1) - Bh(X_1, Y_1) - Ch(X_1, Y_1).\end{aligned} \tag{3.8}$$

Again using (2.7) in above equation, we have

$$\begin{aligned}(\overline{\nabla}_{X_1}\phi)Y_1 &= Pu(X_1,Y_1)+Qu(X_1,Y_1)+\eta(u(X_1,Y_1)\xi)\\&+h(X_1,\phi PY_1)+\nabla^{\perp}_{X_1}\phi QY_1-\phi P(\nabla_{X_1}Y_1)-\phi Q(\nabla_{X_1}Y_1)\\&-Bh(X_1,Y_1)-Ch(X_1,Y_1).\end{aligned}\tag{3.9}$$

Now on comparing (3.6) and (3.9) and equating the horizontal and vertical components, we obtain (3.2), (3.3), (3.4) and (3.5), respectively. □

Lemma 3.2 *Let M be a semi-invariant sub-manifold of generalized Sasakian-space-form M. Then we have*

$$\nabla_{X_1}\xi=-(f_1-f_3)\phi X_1,\quad h(X_1,\xi)=0,\quad for\ any\ X_1\in\Gamma(D);\tag{3.10}$$

$$\nabla_{Y_1}\xi=0\quad h(Y_1,\xi)=-(f_1-f_3)\phi QY_1,\quad for\ any\ Y_1\in\Gamma(D^{\perp});\tag{3.11}$$

$$\nabla_{\xi}\xi=0\quad h(\xi,\xi)=0.\tag{3.12}$$

Proof In consequence of (2.4) and (2.9), we get

$$-(f_1-f_3)\phi X_1=\nabla_{X_1}\xi+h(X_1,\xi).\tag{3.13}$$

Using (2.7) in the above equation, we have

$$\nabla_{X_1}\xi+h(X_1,\xi)=-(f_1-f_3)(\phi PX_1+\phi QY_1).\tag{3.14}$$

After equating tangential and normal parts, we get (3.10), (3.11) and (3.12). □

Lemma 3.3 *Let M be a semi-invariant sub-manifold of generalized Sasakian-space-forms $\overline{M}$, then we find:*

$$\begin{aligned}&\nabla_{\xi}X_2\in\Gamma(D);\quad for\ any\ X_2\in\Gamma(D),\\&\nabla_{\xi}Y_2\in\Gamma(D^{\perp});\quad for\ any\ Y_2\in\Gamma(D^{\perp}).\end{aligned}\tag{3.15}$$

Proof The above follow from $g(\xi,X_2)=0, g(\xi,Y_2)=0$ and (3.12) and covariant differentiation. □

Lemma 3.4 *Let M be a semi-invariant sub-manifold of generalized Sasakian-space-form $\overline{M}$, then we have*

$$[X_1,\xi]\in\Gamma(D)\quad for\ any\ X_1\in\Gamma(D),\tag{3.16}$$

$$[Y_1,\xi]\in\Gamma(D^{\perp})\quad for\ any\ Y_1\in\Gamma(D^{\perp}).\tag{3.17}$$

Proof The proof follows from Lemma 3.3. □

§4. Integrability of Invariant and Anti-Invariant Sub-Manifolds

In this section, we study the integrability of $D, D \oplus [\xi]$ and $D^{\perp}$ of semi-invariant sub-manifolds of generalized Sasakian-space-forms.

Proposition *Let M be a semi-invariant sub-manifold such that ξ is tangent to $\overline{M}$. Then the invariant distribution D is integrable provided $f_1 = f_3$.*

Proof We have for $X_1, Y_1 \in D$ and $\xi \in [\xi]$

$$g([X_1, Y_1], \xi) = g(\nabla_{X_1} Y_1 - \nabla_{Y_1} X_1, \xi) \tag{4.1}$$

using (2.9) in above equation, we have

$$\begin{aligned} g([X_1, Y_1], \xi) = g(&\overline{\nabla}_{X_1} Y_1 - h(X_1, Y_1) - \overline{\nabla}_{Y_1} X_1 + h(Y_1, X_1), \xi) \\ &+ g(\overline{\nabla}_{X_1} Y_1, \xi) - g(\overline{\nabla}_{Y_1} X_1, \xi). \end{aligned} \tag{4.2}$$

Taking the covariant differentiation for the above equation, we get

$$\begin{aligned} g([X_1, Y_1], \xi) = \overline{\nabla}_{X_1} g(Y_1, \xi) - g(Y_1, \overline{\nabla}_{X_1} \xi) \\ - \overline{\nabla}_{Y_1} g(X_1, \xi) + g(X_1, \overline{\nabla}_{Y_1} \xi). \end{aligned} \tag{4.3}$$

Now by the definition of semi-invariant sub-manifold, we have

$$g([X_1, Y_1], \xi) = -g(Y_1, \overline{\nabla}_{X_1} \xi) + g(X_1, \overline{\nabla}_{Y_1} \xi). \tag{4.4}$$

Now by taking (2.4) in the above equation, we get

$$g([X_1, Y_1], \xi) = (f_1 - f_3) g(Y_1, \phi X_1) - (f_1 - f_3) g(X_1, \phi Y_1). \tag{4.5}$$

Now with reference to (2.6), we have

$$g([X_1, Y_1], \xi) = 2(f_1 - f_3) g(Y_1, \phi X_1). \tag{4.6}$$

Thus, if $X_1, Y_1 \in D$, then $[X_1, Y_1] \in D$, that is, the invariant distribution D is integrable, provided $f_1 = f_3$. □

Theorem 4.1 *Let M be a semi-invariant sub-manifold in a generalized Sasakian-space-form $\overline{M}$. Then the distribution D is integrable if and only if the second fundamental form h satisfies*

$$h(X_1, \phi Y_1) = h(\phi X_1, Y_1) \quad for\, X_1, Y_1 \in D. \tag{4.7}$$

Proof For $X_1, Y_1 \in D \oplus [\xi]$ and $Y_2 \in T^{\perp}M$ then by the virtue of (2.9), we have

$$\begin{aligned} g(\phi[X_1, Y_1], Y_2) &= g(\phi(\nabla_{X_1} Y_1 - \nabla_{Y_1} X_1), Y_2) \\ &= g(\phi(\overline{\nabla}_{X_1} Y_1) - h(X_1, Y_1) - \overline{\nabla}_{Y_1} X_1) + h(Y_1, X_1), Y_2 \\ &= g(\phi(\overline{\nabla}_{X_1} Y_1), Y_2) - g(\phi(\overline{\nabla}_{Y_1} X_1), Y_2). \end{aligned} \tag{4.8}$$

Now by the covariant differentiation and using (2.3), (2.9), we have

$$\begin{aligned} g(\phi[X_1, Y_1], Y_2) = {} & g(\nabla_{X_1} \phi Y_1, Y_2) + g(h(X_1, \phi Y_1), Y_2) \\ & + (f_1 - f_3)[g(X_1, Y_2)\eta Y_1 - g(Y_1, Y_2)\eta X_1] - g(\nabla_{Y_1} \phi X_1, Y_2) \\ & - g(h(Y_1, \phi X_1), Y_2). \end{aligned} \tag{4.9}$$

By (2.1) and (2.6) in the above equation, we get

$$g(\phi[X_1, Y_1], Y_2) = g(h(X_1, \phi Y_1) - h(Y_1, \phi X_1), Y_2). \tag{4.10}$$

Therefore,

$$\phi[X_1, Y_1] = h(X_1, \phi Y_1) - h(Y_1, \phi X_1). \tag{4.11}$$

Thus, the distribution D is integrable if and only if the second fundamental form h satisfies

$$h(X_1, \phi Y_1) = h(Y_1, \phi X_1). \tag{4.12}$$

This completes the proof. □

Theorem 4.2 *Let M be a semi-invariant sub-manifold of generalized Sasakian-space-form $\overline{M}$ such that ξ is tangent to $\overline{M}$ and $D^{\perp}$ be the anti-invariant subspace of TM. Then the anti-invariant distribution $D^{\perp}$ is always integrable provided $f_1 = f_3$.*

Proof By the definition of covariant differentiation, we have

$$\begin{aligned} g(\phi[Z_1, Z_2], X_1) &= g(\phi(\nabla_{Z_1} Z_2 - \nabla_{Z_2} Z_1), X_1) \\ &= g(\phi(\overline{\nabla}_{Z_1} Z_2 - \phi h(Z_1, Z_2) - \phi(\nabla_{Z_2} Z_1) + \phi h(Z_2, Z_1), X_1). \end{aligned} \tag{4.13}$$

Now using (2.3) and (2.10) in above equation, we have

$$\begin{aligned} g(\phi[Z_1, Z_2], X_1) &= g((\overline{\nabla}_{Z_1} \phi Z_2) - (\overline{\nabla}_{Z_1} \phi) Z_2 - (\overline{\nabla}_{Z_2} \phi Z_1) + (\overline{\nabla}_{Z_2} \phi) Z_1, X_1) \\ &= g(-A_{\phi Z_2} Z_1 + \nabla^{\perp}_{Z_1} \phi Z_2 + A_{\phi Z_1} Z_2 - \nabla^{\perp}_{Z_2} \phi Z_1, X_1) \\ &\quad + (f_1 - f_3) g[\eta Z_2 Z_1 - \eta Z_1 Z_2, X_1] \\ &= g(-A_{\phi Z_2} Z_1 + \nabla^{\perp}_{Z_1} \phi Z_2 + A_{\phi Z_1} Z_2 - \nabla^{\perp}_{Z_2} \phi Z_1, X_1). \end{aligned} \tag{4.14}$$

Since, $A_{\phi Z_1}Z_2 - A_{\phi Z_2}Z_1$ is tangential to M and $\nabla^{\perp}_{Z_1}\phi Z_2 - \nabla^{\perp}_{Z_2}\phi Z_1$ is normal to M.

$$g(\phi[Z_1, Z_2], X_1) = g(-A_{\phi Z_2}Z_1 + A_{\phi Z_1}Z_2, X_1). \tag{4.15}$$

Hence,

$$\phi[Z_1, Z_2] = -A_{\phi Z_2}Z_1 + A_{\phi Z_1}Z_2. \tag{4.16}$$

Therefore, it follows that $[Z_1, Z_2] \in D^{\perp}$ for any $Z_1, Z_2 \in D^{\perp}$ if and only if

$$A_{\phi Z_2}Z_1 = A_{\phi Z_1}Z_2 \quad \text{for any } Z_1, Z_2 \in D^{\perp} \tag{4.17}$$

and

$$g([Z_1, Z_2], \xi) = 0 \quad \text{for any } Z_1, Z_2 \in D^{\perp} \ \textit{and} \ \xi \in [\xi]. \tag{4.18}$$

Conversely, using (2.9) and (2.11) for any $Z_1, Z_2 \in D^{\perp}$ and $X_1 \in TM$, we have

$$\begin{aligned} g(A_{\phi Z_1}Z_2, X_1) &= g(h(Z_2, X_1), \phi Z_1) = g(\overline{\nabla}_{X_1}Z_2, \phi Z_1) \\ &= -g(\phi\overline{\nabla}_{X_1}Z_2, Z_1) \\ &= -g(\overline{\nabla}_{X_1}\phi Z_2 - (\overline{\nabla}_{X_1}\phi)Z_2, Z_1) \\ &= g(-\overline{\nabla}_{X_1}\phi Z_2 + (f_1 - f_3)(g(X_1, Z_2)\xi - \eta(Z_2)X_1, Z_1)) \\ &= -g(-A_{\phi Z_2}X_1 + \nabla^{\perp}_{X_1}\phi Z_2, Z_1) \\ &= g(A_{\phi Z_2}X_1, Z_1) \\ &= g(A_{\phi Z_2}Z_1, X_1). \end{aligned} \tag{4.19}$$

Thus, $A_{\phi Z_1}Z_2 = A_{\phi Z_2}Z_1$ holds.

By using (2.4), we have

$$\begin{aligned} g([Z_1, Z_2], \xi) &= g(\overline{\nabla}_{Z_1}Z_2 - \overline{\nabla}_{Z_2}Z_1, \xi) \\ &= g(Z_2, \overline{\nabla}_{Z_1}\xi) - g(Z_1, \overline{\nabla}_{Z_2}\xi) \\ &= (f_1 - f_3)(g(Z_1, \phi Z_2) - g(Z_2, \phi Z_1)) \\ &= 2(f_1 - f_3)(g(Z_1, \phi Z_2). \end{aligned} \tag{4.20}$$

Hence, (4.17) and (4.18) hold when $f_1 = f_3$ then $g([Z_1, Z_2], \xi) = 0$. □

§5. Totally Umbilical and Totally Geodesic Sub-Manifolds

Here we consider totally umbilical sub-manifolds of generalized Sasakian-space-forms by proving following Lemmas.

Lemma 5.1 *Let D be a distribution on sub-manifold M of a generalized Sasakian-space-form* *such that $\xi \in D$. If M is D-totally umbilical, then M is D-totally geodesic.*

Proof If M is D-totally umbilical, then by $X_1, Y_1 \in D$ we have

$$h(X_1, Y_1) = g(X_1, Y_1)H, \tag{5.1}$$

where H is the mean curvature. With reference to (3.10) and (3.12), we get

$$H = g(\xi, \xi)H = h(\xi, \xi) = 0. \tag{5.2}$$

Hence $H = 0$ and therefore M is D-totally geodesic. □

Lemma 5.1 *Let $D^{\perp}$ be a distribution on sub-manifold M of a generalized Sasakian-space-form such that $\xi \in D^{\perp}$. If M is $D^{\perp}$-totally umbilical, then M is $D^{\perp}$-totally geodesic provided $\phi Q = Q\phi$.*

Proof If M is $D^{\perp}$-totally umbilical, then by $X_1, Y_1 \in D$, we have:

$$h(X_1, Y_1) = g(X_1, Y_1)K, \tag{5.3}$$

where K is the mean curvature. Now with reference to (3.11), we have:

$$K = g(\xi, \xi)K = h(\xi, \xi) = -(f_1 - f_3)\phi Q\xi. \tag{5.4}$$

Suppose $\phi Q = Q\phi$, hence $K = -(f_1 - f_3)Q\phi\xi = 0$. Therefore, M is D-totally geodesic. □

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International J.Math. Combin. Vol.2(2022), 56-71

4−Remainder Cordial of Some Tree Related Graphs

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Abstract: Let G be a (p, q) graph. Let f be a map from $V(G)$ to the set $\{1, 2, \cdots, k\}$ where k is an integer $2 < k \leq |V(G)|$. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \geq f(v)$ or $f(v) \geq f(u)$. The function f is called a k-remainder cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i, j \in \{1, \cdots, k\}$ where $v_f(x)$ denote the number of vertices labelled with x and $|\eta_o - \eta_e| \leq 1$ where η_e and η_o respectively denote the number of edges labeled with even integers and number of edges labelled with odd integers. A graph with a k-remainder cordial labeling is called a k-remainder cordial graph. In this paper we investigate the 4−remainder cordial labeling behavior of banana tree, coconut tree, bamboo tree.

Key Words: Tree, banana tree, coconut tree, double coconut tree, bamboo tree, k-remainder cordial labeling, Smarandachely k-remainder cordial labeling.

AMS(2010): 05C78.

§1. Introduction

All graphs in this paper are finite, undirected and simple. The vertex set and edge set of a graph are denoted by $V(G)$ and $E(G)$ respectively. Ponraj et al. defined the k−remainder cordial labeling of a graph in [3]. k-Remainder cordial labeling behavior of path, cycle, star, complete graph, wheel, comb etc have been investigated in [3]. Here we investigate the 4-Remainder cordial labeling behavior of Banana tree,Coconut tree,Double coconut tree,Bamboo tree,caterpillar tree.

§2. 4- Remainder Cordial Labeling

Definition 2.1 *Let G be a (p, q) graph. Let f be a map from $V(G)$ to the set $\{1, 2, \cdots, k\}$ where k is an integer $2 < k \leq |V(G)|$. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \geq f(v)$ or $f(v) \geq f(u)$. The function f is called a k-remainder cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$,*

[1]Received May 8, 2020, Accepted December 18, 2020.

$i, j \in \{1, \cdots, k\}$ where $v_f(x)$ denote the number of vertices labelled with x and $|\eta_e - \eta_o| \le 1$ where η_e and η_o respectively denote the number of edges labeled with even integers and number of edges labelled with odd integers. A graph with a k-remainder cordial labeling is called a k-remainder cordial graph.

Conversely, the function f is called a Smarandachely k-remainder cordial labeling of G if there an integer pair $\{i, j\} \subset \{1, 2 \cdots, k\}$ with $|v_f(i) - v_f(j)| > 1$ and $|\eta_e - \eta_o| \le 1$. Such a graph with a Smarandachely k-remainder cordial labeling is called a Smarandachely k-remainder cordial graph.

Definition 2.2 *The banana tree $B(m, n)$ is a graph obtained by connecting one leaf of each of $m-$ copies of the star $K_{1,n}$ with a single root vertex that is distinct from all the stars.*

Definition 2.3 *The coconut tree $CT(m, n)$ is a graph obtained from the path P_n by appending m new pendent edges at an end vertex of P_n.*

Definition 2.4 *The double Coconut tree $DCT(m, n, r)$ is a tree obtained by attaching $m > 1$ pendent vertices to one end of the path P_n and $r > 1$ pendent vertices to the other end of P_n.*

Definition 2.5 *The fire cracker $F_{n,k}$ is obtained by the concantenation of $n-$ copies of $k-$ stars by linking one leaf from each.*

Definition 2.6 *The bamboo tree $BT(n, m, k)$ is a tree obtained from k copies of paths P_n of length $n-1$ and $K_{1.m}$ stars.Identify one of the two pendant vertices of the j^{th} path with centre of the j^{th} star.Identify the other pendant vertex of a each path with a single vertex u_0.*

Definition 2.7 *The caterpillar $S_n(m_1, m_2, \ldots, m_n)$ is a tree obtained from the path $u_1u_2 \cdots u_n$ by identifying the centre of the star $K_{1,m_i}, (1 \le i \le n)$ with u_i. And let the vertex set and edge set of star K_{1,m_i} be*

$$V(K_{1,m_i}) = \{v_i, u_{i,j} : 1 \le i \le n, 1 \le j \le m\}, \quad E(K_{1,m_i}) = \{v_i u_{i,j} : 1 \le i \le n, 1 \le j \le m\},$$

identifying u_i with v_i $(1 \le i \le n)$.

§3. Main Results

Theorem 3.1 *The banana tree $B(m, n)$ is 4−remainder cordial for $m \equiv 0 \pmod 4$ and n is any positive integer.*

Proof Let the vertex set and edge set of $B(m, n)$ be

$$\begin{aligned} V(B(m,n)) &= \{a_1, a_{i,1}, a_{i,2}, \cdots, a_{i,m} : 1 \le i \le n\}, \\ E(B(m,n)) &= \{a_1 a_{1,i}, a_{1,i} a_{2,i} : 1 \le i \le m\} \cup \{a_{2,j} a_{i,j} : 3 \le i \le n, 1 \le j \le m\}. \end{aligned}$$

First, assign the label 3 to the vertex a_1 which has degree n.Next assign the labels $1, 2, 3, 4$ respectively to the vertices $a_{1,i}$ $(1 \le i \le 4)$ which has degree 2 and assign the labels $1, 2, 3, 4$

respectively to the vertices $a_{1,i}\,(5 \leq i \leq 8)$ which has degree 2.Proceeding like this until we reach the vertex $a_{1,m}$.

Next, move to the vertices which has degree $n-1$. Define

$$f\left(a_{2,i}\right)=\begin{cases}2 & \text{if } f\left(a_{1,i}\right)=1,\\ 3 & \text{if } f\left(a_{1,i}\right)=2,\\ 4 & \text{if } f\left(a_{1,i}\right)=3,\\ 1 & \text{if } f\left(a_{1,i}\right)=4.\end{cases}$$

Finally, move to the pendant vertices. Assign the labels to the vertices $f\left(a_{i,j}\right),((1 \leq j \leq m)$ and $(3 \leq i \leq n))$ by

$$f\left(a_{i,j}\right)=\begin{cases}1 & \text{if } f\left(a_{1,i}\right)=1,\\ 2 & \text{if } f\left(a_{1,i}\right)=3,\\ 3 & \text{if } f\left(a_{1,i}\right)=2,\\ 4 & \text{if } f\left(a_{1,i}\right)=4.\end{cases}$$

Thus, this vertex labeling shows that f is the 4-remainder cordial labeling of $B\left(m,n\right)$ for $m \equiv 0 \pmod 4$. Since $v_f(1)=v_f(2)=v_f(4)=\frac{mn}{4}$, $v_f(3)=\frac{mn+4}{4}$ and $\eta_e=\eta_o=\frac{mn}{2}$. This completes the proof. □

A 4−remainder cordial labeling of $B\left(4,5\right)$ is given in Figure 1.

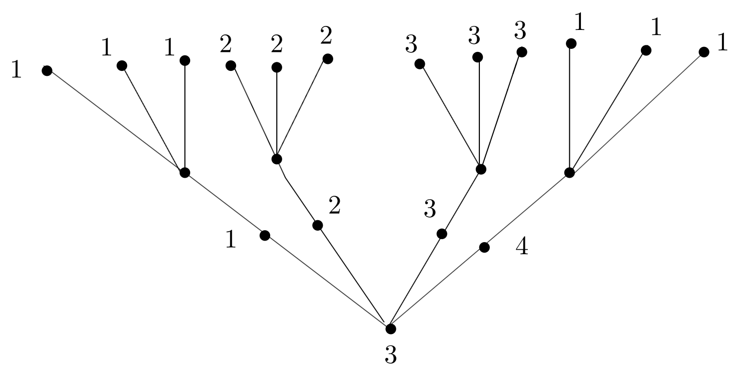

Figure 1

Theorem 3.2 *The coconut tree $CT\left(n,n\right)$ is 4−remainder cordial for all values of n.*

Proof Let P_n be the path $v_1v_2\cdots v_n$ and let $V\left(K_{1,n}\right)=\{w,w_i : 1 \leq i \leq n\}$ are the vertex set of $CT\left(n,n\right)$ and edge set $E\left(CT(n,n)\right)=E\left(P_n\right)\cup E\left(K_{1,n}\right)$ identifying the w with v_n. The proof of this theorem this proved in the following four cases.

Case 1. $n \equiv 0 \pmod 4$.

First, assign the label to the vertices of the path P_n. Assign the labels $1,2,3,4$ respectively to the vertices v_1,v_2,v_3,v_4 and assign the labels $1,2,3,4$ respectively to the vertices v_5,v_6,v_7,v_8.Next assign the labels $1,2,3,4$ respectively to the vertices v_9,v_{10},v_{11},v_{12}.Proceeding like this until we reach the vertex v_{n-4} and assign the labels $1, 1, 2, 3$ respectively to the vertices $v_{n-3}, v_{n-2}, v_{n-1}, v_n$.

Next, move to the vertices of the star $K_{1,n}$. Assign the label 1 to the vertices $w_1, w_5, w_9, \cdots,$ w_{n-7} and assign the label 2 to the vertices $w_2, w_6, w_{10}, \cdots, w_{n-6}$. Secondly assign the label 3 to the vertices $w_3, w_7, w_{11}, w_{n-5}$ and assign the label 4 to the vertices $w_4, w_8, w_{12}, \cdots, w_{n-4}$. Finally assign the labels $2, 3, 4, 4$ respectively to the vertices $w_{n-3}, w_{n-2}, w_{n-1}, w_n$.

Case 2. $n \equiv 1 \pmod 4$.

First, assign the label to the vertices of the path P_n. Assign the label 1 to the vertices $v_1, v_5, v_9, \cdots, v_{n-4}$ and assign the label 2 to the vertices $v_2, v_6, v_{10}, \cdots, v_{n-3}$. Secondly assign the label 3 to the vertices $v_3, v_7, v_{11}, \cdots, v_{n-2}$ and assign the label 4 to the vertices $v_4, v_8, v_{12}, \cdots, v_{n-1}$. Finally assign the labels 3 to the vertex v_n.

Next, move to the vertices of the star $K_{1,n}$. Assign the label 1 to the vertices $w_4, w_8, w_{12}, \cdots,$ w_{n-5} and assign the label 2 to the vertices $w_5, w_9, w_{13}, \cdots, w_{n-4}$. Secondly assign the label 3 to the vertices $w_6, w_{10}, w_{14}, \cdots, w_{n-3}$ and assign the label 4 to the vertices $w_7, w_{11}, w_{15}, \cdots, w_{n-2}$. Finally assign the label 1 to the vertices w_1 and w_{n-1} and assign the label 3 to the vertex w_2 then assign the label 4 to the vertex w_3 and assign the label 2 to the vertex w_n.

Case 3. $n \equiv 2 \pmod 4$.

First, assign the label to the vertices of the path P_n. Assign the label 1 to the vertices $v_1, v_5, v_9, \cdots, v_{n-5}$ and assign the label 2 to the vertices $v_2, v_6, v_{10}, \cdots, v_{n-4}$. Secondly assign the label 3 to the vertices $v_3, v_7, v_{11}, \cdots, v_{n-3}$ and assign the label 4 to the vertices $v_4, v_8, v_{12}, \cdots, v_{n-2}$. Finally assign the labels $1, 3$ respectively to the vertices v_{n-1}, v_n.

Next, assign the labels to the vertices of the star $K_{1,n}$. Assign the label 1 to the vertices $w_4, w_8, w_{12}, \cdots, w_{n-6}$ and assign the label 2 to the vertices $w_5, w_9, w_{13}, \cdots, w_{n-5}$. Secondly, assign the label 3 to the vertices $w_6, w_{10}, w_{14}, \cdots, w_{n-4}$ and assign the label 4 to the vertices $w_7, w_{11}, w_{15}, \cdots, w_{n-3}$. Finally, assign the label 1 to the vertex w_1 and assign the label 2 to the vertices w_2 and w_{n-2}, then assign the label 3 to the vertex w_{n-1} and assign the labels 4 to the vertices w_3 and w_n.

Case 4. $n \equiv 3 \pmod 4$.

First, assign the labels to the vertices of the path P_n. Assign the label 1 to the vertices $v_1, v_5, v_9, \cdots, v_{n-2}$ and assign the label 2 to the vertices $v_2, v_6, v_{10}, \cdots, v_{n-1}$. Secondly, assign the label 3 to the vertices $v_3, v_7, v_{11}, \cdots, v_n$ and assign the label 4 to the vertices $v_4, v_8, v_{12}, \cdots, v_{n-3}$.

Next, assign the labels to the vertices of the star $K_{1,n}$. Assign the label 1 to the vertices $w_2, w_6, w_{10}, \cdots, w_{n-1}$ and assign the label 2 to the vertices $w_3, w_7, w_{11}, \cdots, w_n$. Secondly assign the label 3 to the vertices $w_4, w_8, w_{12}, \cdots, w_{n-3}$ and assign the label 4 to the vertices $w_5, w_9, w_{13}, \cdots, w_{n-2}$. Finally, assign the labels 4 to the vertex w_1 and assign the labels $1, 2$ to the vertices w_{n-1}, w_n.

Thus, the Table 1 given below shows that coconut tree graph admits the 4−remainder cordial labeling.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$n-1$	n
$n \equiv 1 \pmod 4$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$n-1$	n
$n \equiv 2 \pmod 4$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$n-1$	n
$n \equiv 3 \pmod 4$	$\frac{n+1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n-1}{2}$	$n-1$	n

Table 1.

This completes the proof. □

A 4−remainder cordial labeling of $CT(6,6)$ is given in Figure 2.

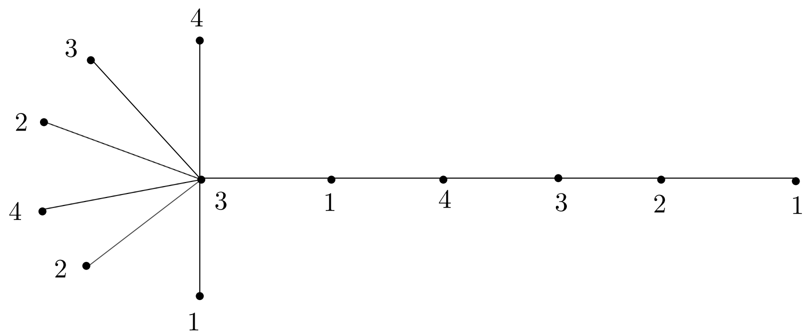

Figure 2

Theorem 3.3 *A double coconut tree $DCT(n,n,n)$ is 4− remainder cordial for all values of n.*

Proof Let P_n be the path $w_1w_2\cdots w_n$ and $V(DCT(n,n,n)) = V(P_n) \cup \{u_i, v_i : 1 \le i \le n\}$ and $E(DCT(n,n,n)) = E(P_n) \cup \{v_iw_1 : 1 \le i \le n\} \cup \{u_iw_n : 1 \le i \le n\}$. Clearly $DCT(n,n,n)$ has $3n$ vertices and $3n-1$ edges.

Now we describe the vertex labeling as follows.There are four cases arises.

Case 1. $n \equiv 0 \pmod 4$.

First, assign the label to the vertices of the path P_n. Assign the label $3,2$ to the vertices w_1, w_2 and assign the label $2,3$ to the vertices w_{n-1}, w_n. Next assign the label $1,2,3,4$ to the vertices w_3, w_4, w_5, w_6 and assign the label $1,2,3,4$ to the vertices w_7, w_8, w_9, w_{10}. Proceeding like this until we reach w_{n-2}.

Next, assign the labels for the pendent vertices. First, assign the label 3 to the vertex u_{n-3} and then assign the labels 4 to the vertices u_{n-2}, u_{n-1}, u_n. Secondly assign the label 2 to the vertex v_{n-3} and then assign the labels 1 to the vertices v_{n-2}, v_{n-1}, v_n. Next assign the labels for the remaining vertices.Assign the labels $1,2,3,4$ to the vertices u_i and v_i $(3 \le i \le 6)$. Then assign the labels $1,2,3,4$ to the vertices u_i and v_i $(7 \le i \le 10)$. Proceeding like this until we reach u_{n-4} and v_{n-4}.

Case 2. $n \equiv 1 \pmod 4$.

First, assign the label to the vertices of the path P_n. Assign the label $3,2$ to the vertices

w_1, w_2 and assign the label $4, 2, 3$ to the vertices w_{n-2}, w_{n-1}, w_n. Next assign the label $1, 2, 3, 4$ to the vertices w_3, w_4, w_5, w_6 and assign the label $1, 2, 3, 4$ to the vertices w_7, w_8, w_9, w_{10}. Proceeding like this until we reach w_{n-3}.

Next, assign the labels for the pendent vertices.First assign the label 3 to the vertex u_{n-4}, u_{n-3} and then assign the label 4 to the vertices u_{n-2}, u_{n-1}, u_n. Secondly assign the label 2 to the vertex v_{n-4}, v_{n-3} and then assign the labels 1 to the vertices v_{n-2}, v_{n-1}, v_n. Next assign the labels for the remaining vertices.Assign the labels $1, 2, 3, 4$ to the vertices u_i and $v_i\,(3 \le i \le 6)$. Then assign the labels $1, 2, 3, 4$ to the vertices u_i and $v_i\,(7 \le i \le 10)$. Proceeding like this until we reach u_{n-5} and v_{n-5}.

Case 3. $n \equiv 2 \pmod 4$.

First, assign the label to the vertices of the path P_n. Assign the label $3, 2$ to the vertices w_1, w_2. Next assign the label $1, 2, 3, 4$ to the vertices w_3, w_4, w_5, w_6 and assign the label $1, 2, 3, 4$ to the vertices w_7, w_8, w_9, w_{10}. Proceeding like this until we reach w_n.

Next, assign the labels for the pendent vertices.First assign the label 3 to the vertices $u_i, \left(i = 1, 2, \ldots, \frac{n}{2}\right)$ which is adjacent with the vertex w_n and assign the label 2 to the vertices $u_i, \left(i = \frac{n+2}{2}, \frac{n+4}{2}, \ldots, n\right)$ which is adjacent with vertex w_n. Next assign the labels 4 to the vertices $v_i, \left(i = 1, 2, \ldots, \frac{n}{2}\right)$ which is adjacent with the vertex w_1 and then assign the label 1 to the vertices $v_i, \left(i = \frac{n+2}{2}, \frac{n+4}{2}, \ldots, n\right)$ which is adjacent with vertex w_1.

Case 4. $n \equiv 3 \pmod 4$.

First, assign the label to the vertices of the path P_n. Assign the label $3, 2$ to the vertices w_1, w_2. Next assign the label $1, 2, 3, 4$ to the vertices w_3, w_4, w_5, w_6 and assign the label $1, 2, 3, 4$ to the vertices w_7, w_8, w_9, w_{10}. Proceeding like this until we reach w_{n-1}.Next assign the label 4 to the vertex w_n.

Next, assign the labels for the pendent vertices. First assign the label 3 to the vertices $u_i, \left(i = 1, 2, \ldots, \frac{n+1}{2}\right)$ which is adjacent with the vertex w_n and assign the label 2 to the vertices $u_i, \left(i = \frac{n+3}{2}, \frac{n+5}{2}, \cdots, n\right)$ which is adjacent with vertex w_n.Next assign the labels 1 to the vertices $v_i, \left(i = 1, 2, \ldots, \frac{n+1}{2}\right)$ which is adjacent with the vertex w_1 and then assign the label 4 to the vertices $v_i, \left(i = \frac{n+3}{2}, \frac{n+5}{2}, \cdots, n\right)$ which is adjacent with vertex w_1.

Thus, the Table 2 given below shows that the double coconut tree graph admits the 4− remainder cordial.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n-2}{2}$	$\frac{3n}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n-3}{4}$	$\frac{3n+1}{4}$	$\frac{3n+1}{4}$	$\frac{3n+1}{4}$	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{3n-2}{4}$	$\frac{3n+2}{4}$	$\frac{3n+2}{4}$	$\frac{3n-2}{4}$	$\frac{3n-2}{2}$	$\frac{3n}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$\frac{3n+3}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$

Table 2

This competes the proof. □

Theorem 3.4 *The fire cracker $F_{n,k}$ is 4− remainder cordial for all values of n.*

Proof Let $V(F_{n,k}) = \{v_o^i, v_1^i, v_2^i : 1 \le i \le n\}$ where v_o^i be the apex vertices of the star and $E(F_{n,k}) = \left\{v_o^i v_1^j : 1 \le i \le n, 1 \le j \le 2\right\} \cup \{v_o^i v_o^{i+1} : 1 \le i \le n-1\}$. Clearly G has $3n$ vertices and $3n-1$ edges.

Now we describe the vertex labeling.There are four cases arises.

Case 1. $n \equiv 0 \pmod 4$.

First, assign the labels for the vertices v_o^i. Assign the labels $1,2,3,4$ respectively to the vertices $v_o^1, v_o^2, v_o^3, v_o^4$ and assign the labels $1,2,3,4$ respectively to the vertices $v_o^5, v_o^6, v_o^7, v_o^8$. Next assign the labels $1,2,3,4$ respectively to the vertices $v_o^9, v_o^{10}, v_o^{11}, v_o^{12}$. Proceeding like this until we reach v_o^n. In similar way assign the labels for $v_2^i, (1 \le i \le n)$. Next assign the labels for v_1^i, assign the labels $4,3,2,1$ respectively to the vertices $v_1^1, v_1^2, v_1^3, v_1^4$ and assign the labels $4,3,2,1$ respectively to the vertices $v_1^5 v_1^6, v_1^7, v_1^8$ and then assign the labels $4,3,2,1$ respectively to the vertices $v_1^9, v_1^{10}, v_1^{11}, v_1^{12}$. Proceeding like this until we reach v_1^n.

Case 2. $n \equiv 1 \pmod 4$.

As in Case 1, assign the labels for the vertices $v_o^i, v_1^i, v_2^i \, (1 \le i \le n-1)$. Finally assign the label 1 to the vertex v_o^n and assign the label 2 to the vertex v_1^n, then assign the label 3 to the vertex v_2^n.

Case 3. $n \equiv 2 \pmod 4$.

As in Case 1, assign the labels for the vertices $v_o^i, v_1^i, v_2^i \, (1 \le i \le n-2)$. Finally assign the label 1 to the vertex v_o^{n-1} and assign the label 2 to the vertex v_o^n, then assign the label 2 to the vertex v_1^{n-1} and assign the label 3 to the vertex v_1^n. Next assign the label 3 to the vertex v_2^{n-1} and assign the label 4 to the vertex v_2^n.

Case 4. $n \equiv 3 \pmod 4$.

As in Case 1, assign the labels for the vertices $v_o^i, v_1^i \, (1 \le i \le n)$ and $v_2^i \, (1 \le i \le n-1)$. Note that in this process the vertex v_o^{n-2} and v_2^{n-2} gets the label 1, and the vertex v_o^{n-1} and v_2^{n-1} gets the label 2, the vertex v_o^n gets the label 3 and the vertex v_1^{n-2} get the label 4.Next the vertex v_1^{n-1} get the label 3 and to the vertex v_1^n get the label 2. Finally assign the label 4 to the vertex v_2^n.

Thus, the Table 3 below shows that this vertex labeling gives the $4-$remainder cordial labeling of fire cracker.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n-2}{2}$	$\frac{3n}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n+1}{4}$	$\frac{3n+1}{4}$	$\frac{3n+1}{4}$	$\frac{3n-3}{4}$	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{3n-2}{4}$	$\frac{3n+2}{4}$	$\frac{3n+2}{4}$	$\frac{3n-2}{4}$	$\frac{3n-2}{2}$	$\frac{3n}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n-1}{4}$	$\frac{3n+3}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{2}$	$\frac{3n-1}{2}$

Table 3

This completes the proof. □

Theorem 3.5 *The bamboo tree $BT(n,n,n)$ is 4− remainder cordial for $n \equiv 0, 2 \pmod 4$.*

Proof Let

$$\begin{aligned} V(BT(n,n,n)) &= \{u_o, u_{i,j}, w_{i,i} : 1 \le i \le n, 1 \le j \le n-1\}, \\ E(BT(n,n,n)) &= \{u_o u_{i,1}, u_{i,n-1} w_{i,i} : 1 \le i \le n\} \cup \{u_{i,i+1} : 1 \le i \le n-2\} \end{aligned}$$

We prove this theorem in two cases.

Case 1. $n \equiv 0 \pmod 4$.

First, assign the label 3 to the vertex u_o. Let $f : V \to \{1,2,3,4\}$ defined by

$$f(u_{i,1}) = \begin{cases} 1 & \text{for all } i = 1, 5, 9, \ldots i+4, \ldots, n-3, \\ 2 & \text{for all } i = 2, 6, 10, \ldots i+4, \ldots, n-2, \\ 3 & \text{for all } i = 3, 7, 11, \ldots i+4, \ldots, n-1, \\ 4 & \text{for all } i = 4, 8, 12, \ldots i+4, \ldots, n. \end{cases}$$

Next, assign the labels for the vertices of the paths. Consider the path $P_1, P_5, \cdots, P_{n-3}$. Assign the labels $4,1,4,1,\cdots,4,1$ consecutively to the vertices of this paths. Next consider the paths $P_2, P_6, \cdots, P_{n-2}$. Assign the labels $3,2,3,2,\cdots,3,2$ consecutively to the vertices of these paths. Next consider the paths $P_3, P_7, \cdots, P_{n-1}$. Assign the labels $2,3,2,3,\cdots,2,3$ consecutively to the vertices of these paths, lastly consider the paths $P_4, P_8, \cdots, P_n$. Assign the labels $1,4,1,4,\cdots,1,4$ consecutively to the vertices of these paths.

And then, move to the vertices of the star. Assign the label 1 to the vertices $w_{i,i}, (1 \le i \le n)$ if $u_{i,n-1}, (1 \le i \le n)$ gets label 1. Secondly assign the label 3 to the vertices $w_{i,i}, (1 \le i \le n)$ if $u_{i,n-1}, (1 \le i \le n)$ gets label 2. Then assign the label 2 to the vertices $w_{i,i}, (1 \le i \le n)$ if $u_{i,n-1}, (1 \le i \le n)$ gets label 3. Finally assign the label 4 to the vertices $w_{i,i}, (1 \le i \le n)$ if $u_{i,n-1}, (1 \le i \le n)$ gets label 4.

Case 2. $n \equiv 2 (mod 4)$.

First assign the label 3 to the vertex u_o. Define

$$f(u_{i,1}) = \begin{cases} 1 & \text{for all } i = 1, 5, 9, \ldots i+4, \ldots, n-1, \\ 2 & \text{for all } i = 2, 6, 10, \ldots i+4, \ldots, n, \\ 3 & \text{for all } i = 3, 7, 11, \ldots i+4, \ldots, n-3, \\ 4 & \text{for all } i = 4, 8, 12, \ldots i+4, \ldots, n-2. \end{cases}$$

As in Case 1 assign the labels to the vertices of paths $u_{i,i+1}, (1 \le i \le n)$ and the vertices of the stars $w_{i,i}, (1 \le i \le n-2)$. Finally assign the label 1 to the vertices $w_{n-1,i}, 1 \le i \le \frac{n}{2}$ if $f(u_{i,n-1}) = 2$ and assign the label 2 to the vertices $w_{n-1,i}, \frac{n+2}{2} \le i \le n$ if $f(u_{i,n-1}) = 1$, then assign the label 4 to the vertices $w_{n,i}, 1 \le i \le \frac{n}{2}$ if $f(u_{i,n-1}) = 2$ and assign the label 3 to the vertices $w_{n,i}, \frac{n+2}{2} \le i \le n$ if $f(u_{i,n-1}) = 2$.

Thus, the Table 4 below shows that this vertex labeling gives the 4−remainder cordial labeling of bamboo tree.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{2n^2-n}{4}$	$\frac{2n^2-n}{4}$	$\frac{2n^2-n+2}{4}$	$\frac{2n^2-n}{4}$	$\frac{2n^2-n}{2}$	$\frac{2n^2-n}{2}$
$n \equiv 2 \pmod 4$	$\frac{2n^2-n+2}{4}$	$\frac{2n^2-n+2}{4}$	$\frac{2n^2-n+2}{4}$	$\frac{2n^2-n-2}{4}$	$\frac{2n^2-n}{2}$	$\frac{2n^2-n}{2}$

Table 4

This completes the proof. □

A 4−remainder cordial labeling of bamboo tree is given in Figure 3.

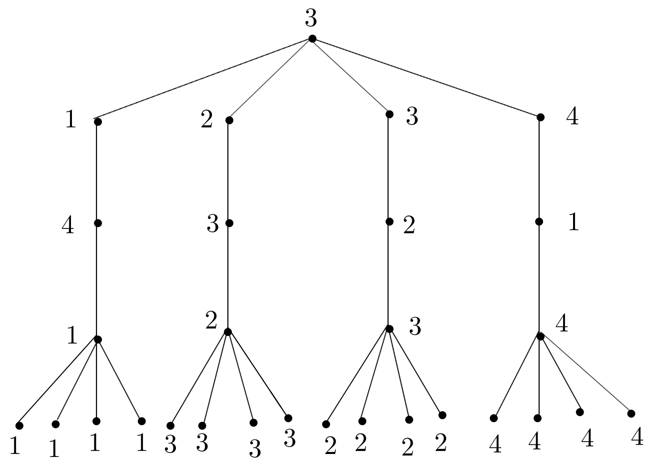

Figure 3

Theorem 3.6 *The caterpillar $S_n(m, m, \ldots, m)$ is 4- remainder cordial for all values of n.*

Proof Taking the vertex set and edge set is of $S_n(m, m, \cdots, m)$ as in Definition 2.8. We prove this theorem in two cases.

Case 1. $m = n$.

Subcase 1.1 $n \equiv 0 \pmod 4$.

First, assign the labels $1, 2, 3, 4$ respectively to the vertices u_1, u_2, u_3, u_4 and assign the labels $1, 2, 3, 4$ respectively to the vertices u_5, u_6, u_7, u_8.Next assign the labels $1, 2, 3, 4$ respectively to the vertices $u_9, u_{10}, u_{11}, u_{12}$.Proceeding like this until we reach the vertices $u_{n-3}, u_{n-2}, u_{n-1}, u_n$.

Next, assign the label to the pendent vertices $u_{i,j}$ $(1 \le i, j \le n)$ by

$$f(u_{i,j}) = \begin{cases} 1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \cdots, n-3, \\ 2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \cdots, n-2, \\ 3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \cdots, n-1, \\ 4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \cdots, n. \end{cases}$$

Subcase 1.1 $n \equiv 1 \pmod 4$.

First, assign the labels $1,2,3,4$ respectively to the vertices u_1,u_2,u_3,u_4 and assign the labels $1,2,3,4$ respectively to the vertices u_5,u_6,u_7,u_8. Next assign the labels $1,2,3,4$ respectively to the vertices u_9,u_{10},u_{11},u_{12}. Proceeding like this until we assign the labels $1,2,3,4$ to the vertices $u_{n-4},u_{n-3},u_{n-2},u_{n-1}$ and then assign the label 3 to the vertex u_n.

Next, assign the label to the pendent vertices $u_{i,j}$ $(1 \le i,j \le n)$ by

$$f(u_{i,j}) = \begin{cases} 1 & \text{if } u_i \text{ gets label } 1, i = 1,5,9,\cdots,n-4, \\ 2 & \text{if } u_i \text{ gets label } 3, i = 2,6,10,\cdots,n-3, \\ 3 & \text{if } u_i \text{ gets label } 2, i = 3,7,11,\cdots,n-2, \\ 4 & \text{if } u_i \text{ gets label } 4, i = 4,8,12,\cdots,n-1. \end{cases}$$

Finally, assign the label $1,2,3,4$ respectively to the vertices $u_{n,1},u_{n,2},u_{n,3},u_{n,4}$ and assign the label $1,2,3,4$ respectively to the vertices $u_{n,5},u_{n,6},u_{n,7},u_{n,8}$.Proceeding like this until we reach the vertex $u_{n,(n-1)}$ then assign the label 1 to the vertex $u_{n,n}$.

Subcase 1.3 $n \equiv 2 \pmod 4$.

First, assign the labels $1,2,3,4$ respectively to the vertices u_1,u_2,u_3,u_4 and assign the labels $1,2,3,4$ respectively to the vertices u_5,u_6,u_7,u_8. Next assign the labels $1,2,3,4$ respectively to the vertices u_9,u_{10},u_{11},u_{12}. Proceeding like this until we assign the labels $1,2,3,4$ to the vertices $u_{n-5},u_{n-4},u_{n-3},u_{n-2}$ and then assign the label $2,3$ to the vertices u_{n-1},u_n.

Next, assign the label to the pendent vertices$u_{i,j}$ $(1 \le i,j \le n)$ by

$$f(u_{i,j}) = \begin{cases} 1 & \text{if } u_i \text{ gets label } 1, i = 1,5,9,\cdots,n-5, \\ 2 & \text{if } u_i \text{ gets label } 3, i = 2,6,10,\cdots,n-4, \\ 3 & \text{if } u_i \text{ gets label } 2, i = 3,7,11,\cdots,n-3, \\ 4 & \text{if } u_i \text{ gets label } 4, i = 4,8,12,\cdots,n-2. \end{cases}$$

Finally, assign the label 1 to the vertices $u_{(n-1),i}, (i = 1,2,\cdots,\frac{n}{2})$ and assign the label 2 to the vertices $u_{(n-1),i}, (i = \frac{n+2}{2},\frac{n+4}{2},\cdots,n)$.Then assign the label 3 to the vertices $u_{n,i}, (i = 1,2,\cdots,\frac{n}{2})$ and then assign the label 4 to the vertices $u_{n,i}, (i = \frac{n+2}{2},\frac{n+4}{2},\cdots,n)$.

Subcase 1.4 $n \equiv 3 \pmod 4$.

First, assign the labels $1,2,3,4$ respectively to the vertices u_1,u_2,u_3,u_4 and assign the labels $1,2,3,4$ respectively to the vertices u_5,u_6,u_7,u_8.Next assign the labels $1,2,3,4$ respectively to the vertices u_9,u_{10},u_{11},u_{12}.Proceeding like this until we reach u_{n-3}, then assign the label 1,2,3 to the vertices u_{n-2},u_{n-1},u_n.

Next, assign the label to the pendent vertices $u_{i,j}$ $(1 \le i,j \le n)$ by

$$f(u_{i,j}) = \begin{cases} 1 & \text{if } u_i \text{ gets label } 1, i = 1,5,9,\cdots,n-6, \\ 2 & \text{if } u_i \text{ gets label } 3, i = 2,6,10,\cdots,n-5, \\ 3 & \text{if } u_i \text{ gets label } 2, i = 3,7,11,\cdots,n-4, \\ 4 & \text{if } u_i \text{ gets label } 4, i = 4,8,12,\cdots,n-3. \end{cases}$$

Finally, assign the label 1 to the vertices $u_{(n-2),i},(i=1,2,\cdots,\frac{3n-1}{4})$ and assign the label 2 to the vertices $u_{(n-2),i},(i=\frac{3n+3}{4},\frac{3n+7}{4},\cdots,n)$ and $u_{(n-1),i},(i=1,2,\cdots,\frac{n-1}{2})$.Then assign the label 3 to the vertices $u_{(n-1),i},(i=\frac{n+1}{2},\frac{n+3}{2},\cdots,n)$ and $u_{n,i},(i=1,2,\cdots,\frac{n-3}{4})$ and then assign the label 4 to the vertices $u_{n,i},(i=\frac{n+1}{4},\frac{n+5}{4},\cdots,n)$.

Thus, the Table 5 below shows that this vertex labeling gives the 4−remainder cordial labeling of caterpillar.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n\equiv 0 \pmod 4$	$\frac{n^2+n}{4}$	$\frac{n^2+n}{4}$	$\frac{n^2+n}{4}$	$\frac{n^2+n}{4}$	$\frac{n^2+n}{2}$	$\frac{n^2+n-2}{2}$
$n\equiv 1 \pmod 4$	$\frac{n^2+n+2}{4}$	$\frac{n^2+n-2}{4}$	$\frac{n^2+n+2}{4}$	$\frac{n^2+n-2}{4}$	$\frac{n^2+n-2}{2}$	$\frac{n^2+n}{2}$
$n\equiv 2 \pmod 4$	$\frac{n^2+n-2}{4}$	$\frac{n^2+n+2}{4}$	$\frac{n^2+n+2}{4}$	$\frac{n^2+n-2}{4}$	$\frac{n^2+n-2}{2}$	$\frac{n^2+n}{2}$
$n\equiv 3 \pmod 4$	$\frac{n^2+n}{4}$	$\frac{n^2+n}{4}$	$\frac{n^2+n}{4}$	$\frac{n^2+n}{4}$	$\frac{n^2+n-2}{2}$	$\frac{n^2+n}{2}$

Table 5

Case 2. $m\neq n$.

Subcase 2.1 $n\equiv 0 \pmod 4, m\equiv 0 \pmod 4$.

As in Case 1 assign the label to the vertices u_i and $u_{i,j}$.

Subcase 2.2 $n\equiv 1 \pmod 4, m\equiv 0 \pmod 4$.

As in Case 1 assign the label to the vertices $u_i, 1\leq i\leq n-5$ and assign the label $1,3,2,4,3$ respectively to the vertices $u_{n-4},u_{n-3},u_{n-2},u_{n-1},u_n$.Next assign the labels to the vertices $u_{i,j}(1\leq i\leq n-1, 1\leq j\leq m)$ by

$$f(u_{i,j})=\begin{cases}1 & \text{if } u_i \text{ gets label } 1, i=1,5,9,\cdots,n-4,\\ 2 & \text{if } u_i \text{ gets label } 3, i=2,6,10,\cdots,n-3,\\ 3 & \text{if } u_i \text{ gets label } 2, i=3,7,11,\cdots,n-2,\\ 4 & \text{if } u_i \text{ gets label } 4, i=4,8,12,\cdots,n-1.\end{cases}$$

Finally, assign the label $1,2,3,4$ respectively to the vertices $u_{n,1},u_{n,2},u_{n,3},u_{n,4}$ and assign the label $1,2,3,4$ respectively to the vertices $u_{n,5},u_{n,6},u_{n,7},u_{n,8}$. Proceeding like this until we reach the vertex $u_{n,m}$.

Subcase 2.3 $n\equiv 2 \pmod 4, m\equiv 0 \pmod 4$.

First, assign the labels $1,2,3,4$ respectively to the vertices u_1,u_2,u_3,u_4 and assign the labels $1,2,3,4$ respectively to the vertices u_5,u_6,u_7,u_8. Next assign the labels $1,2,3,4$ respectively to the vertices u_9,u_{10},u_{11},u_{12}. Proceeding like this until we assign the labels $1,2,3,4$ to the vertices $u_{n-5},u_{n-4},u_{n-3},u_{n-2}$ and then assign the label $1,3$ to the vetices u_{n-1},u_n.

Next assign the label to the pendent vertices $u_{i,j}$ $(1 \le i \le n-2, 1 \le j \le m)$ by

$$f(u_{i,j}) = \begin{cases} 1 & \text{if } u_i \text{ gets label } 1, i = 1,5,9,\cdots,n-5, \\ 2 & \text{if } u_i \text{ gets label } 3, i = 2,6,10,\cdots,n-4, \\ 3 & \text{if } u_i \text{ gets label } 2, i = 3,7,11,\cdots,n-3, \\ 4 & \text{if } u_i \text{ gets label } 4, i = 4,8,12,\cdots,n-2. \end{cases}$$

Finally, assign the label 1 to the vertices $u_{(n-1),i}, (i = 1,2,\cdots,\frac{m}{2})$ and assign the label 3 to the vertices $u_{(n-1),i}, (i = \frac{m+2}{2}, \frac{m+4}{2}, \cdots, m)$.Then assign the label 2 to the vertices $u_{n,i}, (i = 1,2,\cdots,\frac{m}{2})$ and then assign the label 4 to the vertices $u_{n,i}, (i = \frac{m+2}{2}, \frac{m+4}{2}, \cdots, m)$.

Subcase 2.4 $n \equiv 3 \pmod 4, m \equiv 0 \pmod 4$.

As in Case 1 assign the label to the vertices $u_i, 1 \le i \le n$ and then assign the label to the pendent vertices $u_{i,j}$ $(1 \le i \le n-3, 1 \le j \le m)$ by

$$f(u_{i,j}) = \begin{cases} 1 & \text{if } u_i \text{ gets label } 1, i = 1,5,9,\cdots,n-6, \\ 2 & \text{if } u_i \text{ gets label } 3, i = 2,6,10,\cdots,n-5, \\ 3 & \text{if } u_i \text{ gets label } 2, i = 3,7,11,\cdots,n-4, \\ 4 & \text{if } u_i \text{ gets label } 4, i = 4,8,12,\cdots,n-3. \end{cases}$$

Finally, assign the label 1 to the vertices $u_{(n-2),i}, (i = 1,2,\cdots,\frac{3m}{4})$ and assign the label 3 to the vertices $u_{(n-2),i}, (i = \frac{3m+4}{4}, \frac{3m+8}{4}, \cdots, m)$ and $u_{(n-1),i}, (i = 1,2,\cdots,\frac{m}{2})$.Then assign the label 2 to the vertices $u_{(n-1),i}, (i = \frac{m+2}{2}, \frac{m+4}{2}, \cdots, m)$ and $u_{n,i}, (i = 1,2,\cdots,\frac{m}{4})$ and then assign the label 4 to the vertices $u_{n,i}, (i = \frac{m+4}{4}, \frac{m+8}{4}, \cdots, m)$.

Thus, the Table 6 below shows that this vertex labeling gives the 4−remainder cordial labeling of caterpillar.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n-2}{2}$	$\frac{nm+n}{2}$
$n \equiv 1 \pmod 4$	$\frac{nm+n-1}{4}$	$\frac{nm+n+3}{4}$	$\frac{nm+n-1}{4}$	$\frac{nm+n-1}{4}$	$\frac{nm+n-2}{2}$	$\frac{nm+n}{2}$
$n \equiv 2 \pmod 4$	$\frac{nm+n+2}{4}$	$\frac{nm+n-2}{4}$	$\frac{nm+n+2}{4}$	$\frac{nm+n-2}{4}$	$\frac{nm+n-1}{2}$	$\frac{nm+n-1}{2}$
$n \equiv 3 \pmod 4$	$\frac{nm+n+1}{4}$	$\frac{nm+n+1}{4}$	$\frac{nm+n+1}{4}$	$\frac{nm+n-3}{4}$	$\frac{nm+n-1}{2}$	$\frac{nm+n-1}{2}$

Table 6

Subcase 2.5 $n \equiv 0 \pmod 4, m \equiv 1 \pmod 4$.

As in Case 1 assign the label to the vertices u_i and $u_{i,j}$.

Subcase 2.6 $n \equiv 1 \pmod 4, m \equiv 1 \pmod 4$.

As in Case 1 assign the label to the vertices u_i and $u_{i,j}$.

Subcase 2.7 $n \equiv 2 \pmod 4, m \equiv 1 \pmod 4$.

First, assign the labels $1,2,3,4$ respectively to the vertices u_1, u_2, u_3, u_4 and assign the labels $1,2,3,4$ respectively to the vertices u_5,u_6,u_7,u_8. Next assign the labels $1,2,3,4$ respectively to the vertices u_9,u_{10},u_{11},u_{12}. Proceeding like this until we assign the labels $1,2,3,4$ to the vertices $u_{n-5},u_{n-4},u_{n-3},u_{n-2}$ and then assign the label $2,3$ to the vertices u_{n-1},u_n.

Next, assign the label to the pendent vertices$u_{i,j}$ $(1 \leq i \leq n-2, 1 \leq j \leq m)$ by

$$f(u_{i,j}) = \begin{cases} 1 & \text{if } u_i \text{ gets label } 1, i = 1,5,9,\cdots,n-5, \\ 2 & \text{if } u_i \text{ gets label } 3, i = 2,6,10,\cdots,n-4, \\ 3 & \text{if } u_i \text{ gets label } 2, i = 3,7,11,\cdots,n-3, \\ 4 & \text{if } u_i \text{ gets label } 4, i = 4,8,12,\cdots,n-2. \end{cases}$$

Finally, assign the label 1 to the vertices $u_{(n-1),i},(i=1,2,\cdots,\frac{m+1}{2})$ and assign the label 3 to the vertices $u_{(n-1),i},(i=\frac{m+3}{2},\frac{m+5}{2},\ldots,m)$. Then assign the label 2 to the vertices $u_{n,i},(i=1,2,\ldots,\frac{m-1}{2})$ and then assign the label 4 to the vertices $u_{n,i},(i=\frac{m+1}{2},\frac{m+3}{2},\cdots,m)$.

Subcase 2.8 $n \equiv 3 \pmod 4, m \equiv 1 \pmod 4$.

As in Case 1 assign the label to the vertices $u_i, 1 \leq i \leq n$ by

$$f(u_{i,j}) = \begin{cases} 1 & \text{if } u_i \text{ gets label } 1, i = 1,5,9,\cdots,n-6, \\ 2 & \text{if } u_i \text{ gets label } 3, i = 2,6,10,\cdots,n-5, \\ 3 & \text{if } u_i \text{ gets label } 2, i = 3,7,11,\cdots,n-4, \\ 4 & \text{if } u_i \text{ gets label } 4, i = 4,8,12,\cdots,n-3. \end{cases}$$

Finally, assign the label 1 to the vertices $u_{(n-2),i},(i=1,2,\cdots,\frac{m+1}{2})$ and assign the label 2 to the vertices $u_{(n-2),i},(i=\frac{m+3}{2},\frac{m+5}{2},\cdots,m)$ and $u_{(n-1),i},(i=1,2,\cdots,\frac{m-1}{2})$.Then assign the label 3 to the vertices $u_{(n-1),i},(i=\frac{m+1}{2},\frac{m+3}{2},\cdots,m)$ and $u_{n,i},(i=1,2,\cdots,\frac{m-1}{4})$ and then assign the label 4 to the vertices $u_{n,i},(i=\frac{m+3}{4},\frac{m+7}{4},\cdots,m)$.

Thus, the Table 7 below shows that this vertex labeling gives the 4−remainder cordial labeling of caterpillar.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{2}$	$\frac{nm+n-2}{2}$
$n \equiv 1 \pmod 4$	$\frac{nm+n+2}{4}$	$\frac{nm+n-2}{4}$	$\frac{nm+n+2}{4}$	$\frac{nm+n-2}{4}$	$\frac{nm+n-2}{2}$	$\frac{nm+n}{2}$
$n \equiv 2 \pmod 4$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n-1}{2}$	$\frac{nm+n-1}{2}$
$n \equiv 3 \pmod 4$	$\frac{nm+n-2}{4}$	$\frac{nm+n+2}{4}$	$\frac{nm+n+2}{4}$	$\frac{nm+n-2}{4}$	$\frac{nm+n+2}{2}$	$\frac{nm+n-2}{2}$

Table 7

Subcase 2.9 $n \equiv 0 \pmod 4, m \equiv 2 \pmod 4$.

As in Case 1 assign the label to the vertices u_i and $u_{i,j}$.

Subcase 2.10 $n \equiv 1 \pmod 4, m \equiv 2 \pmod 4$.

As in Case 1 assign the label to the vertices $u_i, 1 \leq i \leq n-5$ and assign the label $1,3,2,4,3$

respectively to the vertices $u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n$.Next assign the labels to the vertices $u_{i,j}$ by

$$f(u_{i,j}) = \begin{cases} 1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \cdots, n-4, \\ 2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \cdots, n-2, \\ 3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \cdots, n-3, \\ 4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \cdots, n-1. \end{cases}$$

Finally, assign the label $1, 2, 3, 4$ respectively to the vertices $u_{n,1}, u_{n,2}, u_{n,3}, u_{n,4}$ and assign the label $1, 2, 3, 4$ respectively to the vertices $u_{n,5}, u_{n,6}, u_{n,7}, u_{n,8}$.Proceeding like this until we reach the vertex $u_{n,(m-2)}$ then finally assign the label $1, 2$ to the vertices $u_{n,(m-1)}, u_{n,m}$.

Subcase 2.11 $n \equiv 2 \pmod 4, m \equiv 2 \pmod 4$.

As in Case 1 assign the label to the vertices u_i and $u_{i,j}$.

Subcase 2.12 $n \equiv 3 \pmod 4, m \equiv 2 \pmod 4$.

As in Case 1 assign the label to the vertices $u_i, 1 \le i \le n$ by

$$f(u_{i,j}) = \begin{cases} 1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \cdots, n-6, \\ 2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \cdots, n-5, \\ 3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \cdots, n-4, \\ 4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \cdots, n-3. \end{cases}$$

Finally, assign the label 1 to the vertices $u_{(n-2),i}, (i = 1, 2, \cdots, \frac{3m+2}{4})$ and assign the label 3 to the vertices $u_{(n-2),i}, (i = \frac{3m+6}{4}, \frac{3n+10}{4}, \cdots, m)$ and $u_{(n-1),i}, (i = 1, 2, \cdots, \frac{m}{2})$.Then assign the label 2 to the vertices $u_{(n-1),i}, (i = \frac{m+2}{2}, \frac{m+4}{2}, \cdots, m)$ and $u_{n,i}, (i = 1, 2, \cdots, \frac{m+2}{4})$ and then assign the label 4 to the vertices $u_{n,i}, (i = \frac{m+6}{4}, \frac{m+10}{4}, \cdots, m)$.

Thus, the Table 8 below shows that this vertex labeling gives the 4−remainder cordial labeling of caterpillar.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n-2}{2}$	$\frac{nm+n}{2}$
$n \equiv 1 \pmod 4$	$\frac{nm+n+1}{4}$	$\frac{nm+n+1}{4}$	$\frac{nm+n+1}{4}$	$\frac{nm+n-3}{4}$	$\frac{nm+n-1}{2}$	$\frac{nm+n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{nm+n-2}{4}$	$\frac{nm+n+2}{4}$	$\frac{nm+n+2}{4}$	$\frac{nm+n-2}{4}$	$\frac{nm+n-2}{2}$	$\frac{nm+n}{2}$
$n \equiv 3 \pmod 4$	$\frac{nm+n+3}{4}$	$\frac{nm+n-1}{4}$	$\frac{nm+n-1}{4}$	$\frac{nm+n-1}{4}$	$\frac{nm+n-1}{2}$	$\frac{nm+n-1}{2}$

Table 8

Subcase 2.13 $n \equiv 0 \pmod 4, m \equiv 3 \pmod 4$.

As in Case 1 assign the label to the vertices u_i and $u_{i,j}$.

Subcase 2.14 $n \equiv 1 \pmod 4, m \equiv 3 \pmod 4$.

As in Case 1 assign the label to the vertices $u_i, 1 \le i \le n-5$ and assign the label $1, 3, 2, 4, 3$ respectively to the vertices $u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n$.Next assign the labels to the vertices $u_{i,j}$

by

$$f(u_{i,j}) = \begin{cases} 1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \cdots, n-4, \\ 2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \cdots, n-2, \\ 3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \cdots, n-3, \\ 4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \cdots, n-1. \end{cases}$$

Finally, assign the label $1, 2, 3, 4$ respectively to the vertices $u_{n,1}, u_{n,2}, u_{n,3}, u_{n,4}$ and assign the label $1, 2, 3, 4$ respectively to the vertices $u_{n,5}, u_{n,6}, u_{n,7}, u_{n,8}$. Proceeding like this until we reach the vertex $u_{n,(m-3)}$ then finally assign the label $1, 2, 4$ to the vertices $u_{n,(m-2)}, u_{n,(m-1)}, u_{n,m}$.

Subcase 2.15 $n \equiv 2 \pmod 4, m \equiv 3 \pmod 4$.

As in Case 1 assign the label to the vertices $u_i, 1 \leq i \leq n-6$ and assign the label $1, 3, 2, 4, 2, 3$ respectively to the vertices $u_{n-5}, u_{n-4}, u_{n-3}, u_{n-2}, u_{n-1}, u_n$. Next assign the labels to the vertices $u_{i,j}$ by

$$f(u_{i,j}) = \begin{cases} 1 & \text{if } u_i \text{ gets label } 1, i = 1, 5, 9, \cdots, n-5, \\ 2 & \text{if } u_i \text{ gets label } 3, i = 2, 6, 10, \cdots, n-3, \\ 3 & \text{if } u_i \text{ gets label } 2, i = 3, 7, 11, \cdots, n-4, \\ 4 & \text{if } u_i \text{ gets label } 4, i = 4, 8, 12, \cdots, n-2. \end{cases}$$

Finally, assign the label $1, 2, 3, 4$ respectively to the vertices $u_{(n-1),1}, u_{(n-1),2}, u_{(n-1),3}, u_{(n-1),4}$ and assign the label $1, 2, 3, 4$ respectively to the vertices $u_{(n-1),5}, u_{(n-1),6}, u_{(n-1),7}, u_{(n-1),8}$. Proceeding like this until we reach the vertex $u_{(n-1),(m-3)}$ then finally assign the label $1, 2, 4$ to the vertices $u_{(n-1),(m-2)}, u_{(n-1),(m-1)}, u_{(n-1),m}$. Finally assign the label $1, 2, 3, 4$ respectively to the vertices $u_{n,1}, u_{n,2}, u_{n,3}, u_{n,4}$ and assign the label $1, 2, 3, 4$ respectively to the vertices $u_{n,5}, u_{n,6}, u_{n,7}, u_{n,8}$. Proceeding like this until we reach the vertex $u_{n,(m-3)}$ then finally assign the label $2, 4, 4$ to the vertices $u_{n,(m-2)}, u_{n,(m-1)}, u_{n,m}$.

Subcase 2.16 $n \equiv 3 \pmod 4, m \equiv 3 \pmod 4$.

As in Case 1 assign the label to the vertices u_i and $u_{i,j}$.

Thus, the Table 9 below shows that this vertex labeling gives the $4-$ remainder cordial labeling of caterpillar.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n-2}{2}$	$\frac{nm+n}{2}$
$n \equiv 1 \pmod 4$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n-2}{2}$	$\frac{nm+n}{2}$
$n \equiv 2 \pmod 4$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{2}$	$\frac{nm+n-2}{2}$
$n \equiv 3 \pmod 4$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n}{4}$	$\frac{nm+n-2}{2}$	$\frac{nm+n+2}{2}$

Table 9

This completes the proof. □

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International J.Math. Combin. Vol.2(2022), 72-80

k-Product Cordial Labeling of Cone Graphs

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Abstract: Let f be a map from $V(G)$ to $\{0, 1, \cdots, k-1\}$ where k is an integer, $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(mod\ k)$. f is called a k-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, \cdots, k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x ($x = 0, 1, \cdots, k-1$). In this paper, we prove that the graphs such as 1-cone $C_n + K_1$ and double cone DC_n admit 5-product cordial labeling. Also, we show that the double cone DC_n does not admit 4-product cordial labeling.

Key Words: Cordial labeling, product cordial labeling, k-product cordial labeling, 4-product cordial graph, 5-product cordial graph, Smarandachely k-product cordial labeling.

AMS(2010): 05C78.

§1. Introduction

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [4]. While studying graph theory, one that has gained a lot of popularity during the last 60 years is the concept of labelings of graphs due to its wide range of applications. Labeling is a function that allocates the elements of a graph to real numbers, usually positive integers. In 1967, Rosa [15] published a pioneering paper on graph labeling problems. Thereafter many types of graph labeling techniques have been studied by several authors. Gallian [3] in his survey beautifully classified them into graceful labeling and harmonious labelings, variations of graceful labelings, variations of harmonious labelings, magic type labelings, anti-magic type labelings and miscellaneous labelings. Cordial labeling is a weaker version of graceful and harmonious labeling was introduced by Cahit [2]. Let f be a

[1]Received May 20, 2020, Accepted June 19, 2022.

function from the vertices of G to $\{0,1\}$ and for each edge xy assign the label $|f(x)-f(y)|$. f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1.

In 2004, Sundaram et al. [17] extended the concept of cordial labeling and defined product cordial labeling as follows: Let f be a function from $V(G)$ to $\{0,1\}$. For each edge uv, assign the label $f(u)f(v)$. Then f is called product cordial labeling if $|v_f(0)-v_f(1)|\leq 1$ and $|e_f(0)-e_f(1)|\leq 1$ where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges respectively labeled with $i(i=0,1)$. Sundaram et al. [17] proved that the wheels are not product cordial. Many researchers have shown interest on this topic and showed that several classes of graphs admit product cordial labeling. An interested reader can refer to [3].

Followed by this, Ponraj et al. [14] further extended the concept of product cordial labeling and introduced a new labeling called k-product cordial labeling [14]. Let f be a map from $V(G)$ to $\{0,1,\cdots,k-1\}$ where k is an integer, $1\leq k\leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(mod\ k)$. f is called a k-product cordial labeling if $|v_f(i)-v_f(j)|\leq 1$, and $|e_f(i)-e_f(j)|\leq 1$, $i,j\in\{0,1,\cdots,k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x ($x=0,1,\cdots,k-1$), and otherwise, f is called a Smarandachely k-product cordial labeling if there is an integer pair $\{i,j\}\subset\{0,1,\cdots,k-1\}$ such that $|v_f(i)-v_f(j)|>1$ or $|e_f(i)-e_f(j)|>1$. For k-product cordial labeling, they proved that k-product cordial labeling of stars and bistars further they studied the 4-product cordial labeling behavior of paths, complete graphs and combs. Jeyanthi and Maheswari [12] proved that W_n if $n\equiv 1(mod\ 3)$ is 3-product cordial graph. In [13] Ponraj et al. proved that wheel $W_n=C_n+K_1$ is 4-Product Cordial if and only if n=5 or 9. For further results on 3-product and 4-product cordial labeling one can refer to [3]. Inspired by the concept of k-product cordial labeling and the results in [14], we further studied on k-product cordial labeling and established that the following graphs admit k-product cordial labeling: union of graphs [5]; fan and double fan graphs [6]; powers of paths [7]; the maximum number of edges in a 4-product cordial graph of order p is $4\lceil\frac{p-1}{4}\rceil\lfloor\frac{p-1}{4}\rfloor+3$ [8]; Napier bridge graphs [9]; paths [10] and product of graphs [11]. In this paper we find some new results on k-product cordial labeling.

We recall the following definitions to prove our main results.

Definition 1.1([1]) *The graph C_n+K_1 is called as 1-cone. It is also called as wheel.*

Definition 1.2([16]) *The graph $C_n+\overline{K_2}$ is called as a double cone denoted by DC_n.*

§2. Main Results

In this section, we exhibit that the graphs 1-cone C_n+K_1 and double cone DC_n admit 5-product cordial labeling. Also we show that the double cone DC_n does not admit 4-product cordial labeling.

Theorem 2.1 *The 1-cone C_n+K_1 is a 5-product cordial graph if and only if $n\equiv 1,2$ or $3(mod\ 5)$ for $n>3$.*

Proof Let the vertex set and the edge set of $C_n + K_1$ be $V(C_n + K_1) = \{v, v_i; 1 \leq i \leq n\}$ and $E(C_n + K_1) = \{(v, v_i); 1 \leq i \leq n\} \bigcup \{(v_i, v_{i+1}); 1 \leq i \leq n-1\} \bigcup \{(v_n, v_1)\}$ respectively. Let us consider the following six cases. Define $f : V(C_n + K_1) \to \{0, 1, 2, 3, 4\}$ as follows:

Case 1. If $n \equiv 1(mod\ 5)$ for $n > 3$, then $f(v) = 4,\ f(v_n) = 1,\ f(v_i) = 0;\ 1 \leq i \leq \left\lfloor \frac{n}{5} \right\rfloor$.

Subcase 1.1 If n is even, let $i = \left\lfloor \frac{n}{5} \right\rfloor + j\ ,\ 1 \leq j \leq 4 \left\lfloor \frac{n}{5} \right\rfloor$.

$$f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1,\ 6(mod\ 8), \\ 1 & \text{if } j \equiv 2,\ 5(mod\ 8), \\ 2 & \text{if } j \equiv 3,\ 7(mod\ 8), \\ 3 & \text{if } j \equiv 4,\ 0(mod\ 8). \end{cases}$$

Subcase 1.2 If n is odd, let $i = \left\lfloor \frac{n}{5} \right\rfloor + j\ ,\ 1 \leq j \leq 4 \left\lfloor \frac{n}{5} \right\rfloor$.

For $n = 11,\ f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1,\ 6(mod\ 8), \\ 4 & \text{if } j \equiv 2,\ 5(mod\ 8), \\ 2 & \text{if } j \equiv 3,\ 7(mod\ 8), \\ 3 & \text{if } j \equiv 4,\ 0(mod\ 8). \end{cases}$

For $n > 11,\ f(v_i) = \begin{cases} 2 & \text{if } j \equiv 3(mod\ 4), \\ 3 & \text{if } j \equiv 0(mod\ 4). \end{cases}$

For $1 \leq j \leq 8,\ f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1(mod\ 4), \\ 4 & \text{if } j \equiv 2(mod\ 4). \end{cases}$

For $4 \left\lfloor \frac{n}{5} \right\rfloor - 7 \leq j \leq 4 \left\lfloor \frac{n}{5} \right\rfloor,\ f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1(mod\ 4), \\ 1 & \text{if } j \equiv 2(mod\ 4). \end{cases}$

For $9 \leq j \leq 4 \left\lfloor \frac{n}{5} \right\rfloor - 8,\ f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1,\ 6(mod\ 8), \\ 1 & \text{if } j \equiv 2,\ 5(mod\ 8). \end{cases}$

From the above cases we get

$$v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) + 1 = v_f(4) = \left\lfloor \frac{n}{5} \right\rfloor + 1,$$
$$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) + 1 = e_f(4) = 2 \left\lfloor \frac{n}{5} \right\rfloor + 1.$$

Case 2. $n \equiv 2(mod\ 5)$ for $n > 3$.

Subcase 2.1 n is odd.

We assign labels to the vertices v and v_i $(1 \leq i \leq n-1)$ as in Subcase 1.1, then assign 2 to v_n.

Subcase 2.2 n is even.

We assign labels to the vertices v and v_i $(1 \leq i \leq n-1)$ as in Subcase 1.2, then assign 2 to v_n. From this label we get

$$v_f(0)+1 = v_f(1) = v_f(2) = v_f(3)+1 = v_f(4) = \left\lfloor \frac{n}{5} \right\rfloor + 1,$$
$$e_f(0) = e_f(1)+1 = e_f(2) = e_f(3) = e_f(4) = 2\left\lfloor \frac{n}{5} \right\rfloor + 1.$$

Case 3. If $n \equiv 3(mod\ 5)$ for $n > 3$, then $f(v) = 4$, $f(v_{n-2}) = 1$, $f(v_{n-1}) = 2$, $f(v_n) = 3$, $f(v_i) = 0$; $1 \leq i \leq \lfloor \frac{n}{5} \rfloor - 1$, $f(v_{\lfloor \frac{n}{5} \rfloor + 2}) = 2$, $f(v_{\lfloor \frac{n}{5} \rfloor + 3}) = 3$, $f(v_{\lfloor \frac{n}{5} \rfloor + 4}) = 0$ and for $i = \left\lfloor \frac{n}{5} \right\rfloor + 4 + j$, $1 \leq j \leq 4\left\lfloor \frac{n}{5} \right\rfloor - 4$,

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1,\ 6(mod\ 8), \\ 4 & \text{if } j \equiv 2,\ 5(mod\ 8), \\ 2 & \text{if } j \equiv 3,\ 7(mod\ 8), \\ 3 & \text{if } j \equiv 4,\ 0(mod\ 8). \end{cases}$$

If n is even,

$$f(v_{\lfloor \frac{n}{5} \rfloor}) = 1, \quad f(v_{\lfloor \frac{n}{5} \rfloor + 1}) = 4;$$

if n is odd,

$$f(v_{\lfloor \frac{n}{5} \rfloor}) = 4, \quad f(v_{\lfloor \frac{n}{5} \rfloor + 1}) = 1.$$

Then, we have

$$v_f(0)+1 = v_f(1) = v_f(2) = v_f(3) = v_f(4) = \left\lfloor \frac{n}{5} \right\rfloor + 1$$

and for $n = 8$,

$$e_f(0)+1 = e_f(1)+1 = e_f(2)+1 = e_f(3) = e_f(4)+1 = 2\lfloor \frac{n}{5} \rfloor + 2,$$

for $n > 8$,

$$e_f(0) = e_f(1)+1 = e_f(2)+1 = e_f(3)+1 = e_f(4)+1 = 2\left\lfloor \frac{n}{5} \right\rfloor + 2.$$

Case 4. $n \equiv 4(mod\ 5)$ for $n > 3$.

Let $n = 5t+4$, then $|V(C_n+K_1)| = 5t+5$ and $|E(C_n+K_1)| = 10t+8$. Thus, $v_f(i) = t+1$ $(i = 0,1,2,3,4)$ and $e_f(i) = 2t+1$ *or* $2t+2$ $(i = 0,1,2,3,4)$. Clearly, $f(v) \neq 0$. If $v_f(0) = t+1$, then $e_f(0) > 2t+2$ for $t \geq 0$. Therefore, $|e_f(0) - e_f(j)| > 1$ for some $j = 1,2,3,4$. Hence, $C_n + K_1$ is not a 5-product cordial graph if $n \equiv 4(mod\ 5)$.

Case 5. $n \equiv 0(mod\ 5)$ for $n > 3$.

Let $n = 5t$, then $|V(C_n+K_1)| = 5t+1$ and $|E(C_n+K_1)| = 10t$. Thus, $v_f(i) = t$ *or* $t+1$ $(i = 0,1,2,3,4)$ and $e_f(i) = 2t$ $(i = 0,1,2,3,4)$. Clearly, $f(v) \neq 0$. If $v_f(0) = t$ *or* $t+1$, then $e_f(0) > 2t$ for $t \geq 1$. Therefore, $|e_f(0) - e_f(j)| > 1$ for some $j = 1,2,3,4$. Hence, $C_n + K_1$ is not a 5-product cordial graph if $n \equiv 0(mod\ 5)$.

Case 6. If $n = 3$, then $|V(C_3+K_1)| = 4$ and $|E(C_3+K_1)| = 6$. Thus, $v_f(i) = 0$ *or* 1

$(i = 0, 1, 2, 3, 4)$ and $e_f(i) = 1$ *or* 2 $(i = 0, 1, 2, 3, 4)$. If $v_f(0) = 0$, then $e_f(0) = 0$. If $v_f(0) = 1$, then $e_f(0) = 3$. Therefore, $|e_f(0) - e_f(j)| > 1$ for some $j = 1, 2, 3, 4$. Hence, $C_3 + K_1$ is not a 5-product cordial graph. □

An example of 5-product cordial labeling of $C_{11} + K_1$ is shown in Figure 1.

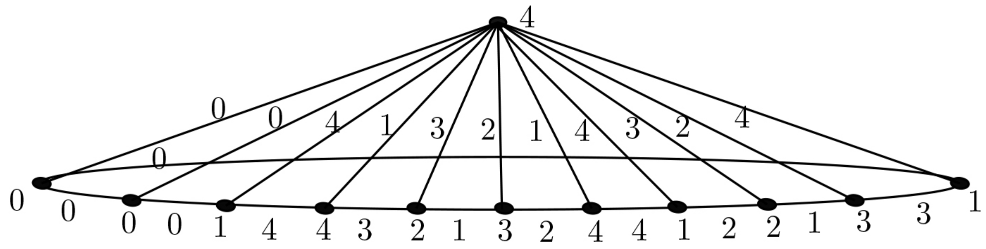

Figure 1. 5-product cordial labeling of $C_{11} + K_1$

Theorem 2.2 *The double cone DC_n is not a 4-product cordial graph for all $n \geq 3$.*

Proof Let the vertex set and the edge set of DC_n be $V(DC_n) = \{u, v, v_i; 1 \leq i \leq n\}$ and $E(DC_n) = \{(u, v_i), (v, v_i); 1 \leq i \leq n\} \bigcup \{(v_i, v_{i+1}); 1 \leq i \leq n-1\} \bigcup \{(v_1, v_n)\}$ respectively. We assume that DC_n is a 4-product cordial graph with a 4-product cordial labeling f on DC_n. Let us consider the following four cases.

Case 1. If $n \equiv 0(mod\ 4)$ for $n > 3$, let $n = 4t$, then $|V(DC_n)| = 4t + 2$ and $|E(DC_n)| = 12t$. Thus, $v_f(i) = t$ *or* $t + 1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 3t$ $(i = 0, 1, 2, 3)$. Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. If $v_f(0) = t$ *or* $t + 1$, then $e_f(0) > 3t$ for $t \geq 1$. We get a contradiction to f is a 4-product cordial labeling. Hence, DC_n is not a 4-product cordial graph if $n \equiv 0(mod\ 4)$.

Case 2. If $n \equiv 1(mod\ 4)$ for $n > 3$, let $n = 4t + 1$, then $|V(DC_n)| = 4t + 3$ and $|E(DC_n)| = 12t + 3$. Thus, $v_f(i) = t$ *or* $t + 1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 3t$ *or* $3t + 1$ $(i = 0, 1, 2, 3)$. Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. Obviously, $v_f(0) = t$. Otherwise $e_f(0) > 3t + 1$ is not possible. We assign 0 to the vertices of the cycle in such a way that $e_f(0) = 3t + 1$. Then, $v_f(2) = t + 1$. Clearly, $f(v) \neq 2$, $f(u) \neq 2$ and 2 must be assigned inconsecutively. Otherwise $e_f(0) > 3t + 1$ is not possible. Then, $4t + 2 \leq e_f(2) \leq 4t + 4$ for $t \geq 1$. We get a contradiction to f is a 4-product cordial labeling. Hence, DC_n is not a 4-product cordial graph if $n \equiv 1(mod\ 4)$ for $n > 3$.

Case 3. If $n \equiv 2(mod\ 4)$ for $n > 3$, let $n = 4t + 2$, then $|V(DC_n)| = 4t + 4$ and $|E(DC_n)| = 12t + 6$. Thus, $v_f(i) = t + 1$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 3t + 1$ *or* $3t + 2$ $(i = 0, 1, 2, 3)$. Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. Obviously, $v_f(0) = t + 1$ then $e_f(0) > 3t + 2$ for $t \geq 1$. We get a contradiction to f is a 4-product cordial labeling. Hence, DC_n is not a 4-product cordial graph if $n \equiv 2(mod\ 4)$ for $n > 3$.

Case 4. If $n \equiv 3(mod\ 4)$ for $n \geq 3$, let $n = 4t + 3$, then $|V(DC_n)| = 4t + 5$ and $|E(DC_n)| = 12t + 9$. Thus, $v_f(i) = t + 1$ *or* $t + 2$ $(i = 0, 1, 2, 3)$ and $e_f(i) = 3t + 2$ *or* $3t + 3$ $(i = 0, 1, 2, 3)$. Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. If $v_f(0) = t + 1$ *or* $t + 2$, then $e_f(0) > 3t + 3$ for $t \geq 0$. We get a contradiction to f is a 4-product cordial labeling. Hence, DC_n is not a 4-product cordial graph if $n \equiv 3(mod\ 4)$ for $n \geq 3$. □

Theorem 2.3 *The double cone DC_n is a 5-product cordial graph if and only if $n \equiv 1$ or $2 (mod\ 5)$ for $n \geq 3$.*

Proof Let the vertex set and the edge set of DC_n be $V(DC_n) = \{u, v, v_i; 1 \leq i \leq n\}$ and $E(DC_n) = \{(u, v_i), (v, v_i); 1 \leq i \leq n\} \bigcup \{(v_i, v_{i+1}); 1 \leq i \leq n-1\} \bigcup \{(v_1, v_n)\}$ respectively. Let us consider the following five cases.

Define $f : V(DC_n) \to \{0, 1, 2, 3, 4\}$ as follows:

Case 1. If $n \equiv 1(mod\ 5)$ for $n > 3$, then $f(u) = 3$, $f(v) = 4$, $f(v_i) = 0$; $1 \leq i \leq \lfloor \frac{n}{5} \rfloor$.

Subcase 1.1 If n is even, then $f(v_{n-4}) = 4$, $f(v_{n-3}) = 1$, $f(v_{n-2}) = 2$, $f(v_{n-1}) = 3$, $f(v_n) = 1$.

Let $i = \left\lfloor \frac{n}{5} \right\rfloor + j$, $1 \leq j \leq 4 \left\lfloor \frac{n}{5} \right\rfloor - 4$.

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1,\ 6(mod\ 8), \\ 4 & \text{if } j \equiv 2,\ 5(mod\ 8), \\ 2 & \text{if } j \equiv 3,\ 7(mod\ 8), \\ 3 & \text{if } j \equiv 4,\ 0(mod\ 8). \end{cases}$$

From this label we get

$$v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) = v_f(4) = \left\lfloor \frac{n}{5} \right\rfloor + 1.$$

For $n = 6$,

$$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) = e_f(4) = 3 \left\lfloor \frac{n}{5} \right\rfloor + 1.$$

For $n > 6$,

$$e_f(0) = e_f(1) + 1 = e_f(2) = e_f(3) + 1 = e_f(4) = 3 \left\lfloor \frac{n}{5} \right\rfloor + 1.$$

Subcase 1.2 If n is odd, let $i = \left\lfloor \frac{n}{5} \right\rfloor + j$, $1 \leq j \leq 4 \left\lfloor \frac{n}{5} \right\rfloor + 1$,

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1,\ 6(mod\ 8) \\ 4 & \text{if } j \equiv 2,\ 5(mod\ 8) \\ 2 & \text{if } j \equiv 3,\ 7(mod\ 8) \\ 3 & \text{if } j \equiv 4,\ 0(mod\ 8). \end{cases}$$

Then, we have

$$v_f(0) + 1 = v_f(1) = v_f(2) + 1 = v_f(3) = v_f(4) = \left\lfloor \frac{n}{5} \right\rfloor + 1,$$

$$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) = e_f(4) = 3 \left\lfloor \frac{n}{5} \right\rfloor + 1.$$

Case 2. If $n \equiv 2(mod\ 5)$ for $n > 3$, then $f(u) = 3$, $f(v) = 4$, $f(v_{n-1}) = 1$, $f(v_n) = 2$, $f(v_i) = 0$; $1 \leq i \leq \lfloor \frac{n}{5} \rfloor - 1$ and $f(v_{\lfloor \frac{n}{5} \rfloor}) = 4$, $f(v_{\lfloor \frac{n}{5} \rfloor+1}) = 1$, $f(v_{\lfloor \frac{n}{5} \rfloor+2}) = 2$, $f(v_{\lfloor \frac{n}{5} \rfloor+3}) = 3$, $f(v_{\lfloor \frac{n}{5} \rfloor+4}) = 0$.

Subcase 2.1 If n is odd, let $i = \left\lfloor \frac{n}{5} \right\rfloor + 4 + j$, $1 \leq j \leq 4 \left\lfloor \frac{n}{5} \right\rfloor - 4$ and

$$f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1,\ 6(mod\ 8), \\ 4 & \text{if } j \equiv 2,\ 5(mod\ 8), \\ 2 & \text{if } j \equiv 3,\ 7(mod\ 8), \\ 3 & \text{if } j \equiv 4,\ 0(mod\ 8). \end{cases}$$

Subcase 2.2 If n is even, let $i = \left\lfloor \frac{n}{5} \right\rfloor + 4 + j$; $1 \leq j \leq 4 \left\lfloor \frac{n}{5} \right\rfloor - 4$,

$$f(v_i) = \begin{cases} 4 & \text{if } j \equiv 1,\ 6(mod\ 8), \\ 1 & \text{if } j \equiv 2,\ 5(mod\ 8), \\ 2 & \text{if } j \equiv 3,\ 7(mod\ 8), \\ 3 & \text{if } j \equiv 4,\ 0(mod\ 8). \end{cases}$$

Then, we have

$$v_f(0) + 1 = v_f(1) = v_f(2) = v_f(3) = v_f(4) = \left\lfloor \frac{n}{5} \right\rfloor + 1.$$

For $n = 7$,

$$e_f(0) + 1 = e_f(1) + 1 = e_f(2) + 1 = e_f(3) = e_f(4) + 1 = 3 \left\lfloor \frac{n}{5} \right\rfloor + 2.$$

For $n > 7$,

$$e_f(0) = e_f(1) + 1 = e_f(2) + 1 = e_f(3) + 1 = e_f(4) + 1 = 3 \left\lfloor \frac{n}{5} \right\rfloor + 2.$$

Case 3. If $n \equiv 3(mod\ 5)$ for $n \geq 3$, let $n = 5t + 3$, then $|V(DC_n)| = 5t + 5$ and $|E(DC_n)| = 15t + 9$. Thus, $v_f(i) = t + 1$ $(i = 0, 1, 2, 3, 4)$ and $e_f(i) = 3t + 1$ *or* $3t + 2$ $(i = 0, 1, 2, 3, 4)$. Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. If $v_f(0) = t + 1$, then $e_f(0) > 3t + 2$ for $t \geq 0$. Hence, DC_n is not a 5-product cordial graph if $n \equiv 3(mod\ 5)$ for $n \geq 3$.

Case 4. If $n \equiv 4(mod\ 5)$ for $n > 3$, let $n = 5t + 4$, then $|V(DC_n)| = 5t + 6$ and $|E(DC_n)| = 15t+12$. Thus, $v_f(i) = t+1$ *or* $t+2$ $(i = 0, 1, 2, 3, 4)$ and $e_f(i) = 3t+2$ *or* $3t+3$ $(i = 0, 1, 2, 3, 4)$. Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. If $v_f(0) = t + 1$ *or* $t + 2$, then $e_f(0) > 3t + 3$ for $t \geq 0$. Hence, DC_n is not a 5-product cordial graph if $n \equiv 4(mod\ 5)$.

Case 5. If $n \equiv 0(mod\ 5)$ for $n > 3$, let $n = 5t$, then $|V(DC_n)| = 5t + 2$ and $|E(DC_n)| = 15t$. Thus, $v_f(i) = t$ *or* $t + 1$ $(i = 0, 1, 2, 3, 4)$ and $e_f(i) = 3t$ $(i = 0, 1, 2, 3, 4)$. Clearly, $f(v) \neq 0$ and $f(u) \neq 0$. If $v_f(0) = t$ *or* $t + 1$, then $e_f(0) > 3t$ for $t > 0$. Hence, DC_n is not a 5-product cordial graph if $n \equiv 0(mod\ 5)$ for $n > 3$. □

An example of 5-product cordial labeling of DC_{12} is shown in Figure 2.

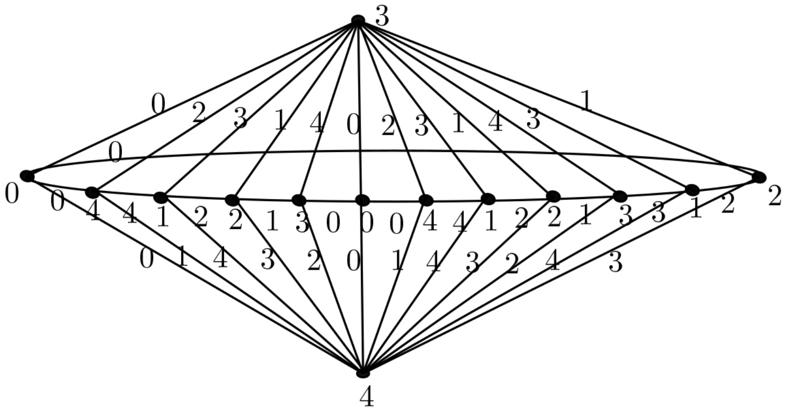

Figure 2. 5-product cordial labeling of DC_{12}

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International J.Math. Combin. Vol.2(2022), 81-85

A Note on Fixed Point Theorem in Complex Valued Intuitionistic Fuzzy Metric Space

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Abstract: We show several common fixed point theorems for contraction condition satisfying certain requirements in complex valued intuitionistic fuzzy metric spaces in this study.

Key Words: Common fixed point, intuitionistic fuzzy set, complex valued, continuous t-norm.

AMS(2010): 47H10.

§1. Introduction

In 1965, Zadeh [12] proposed the concept of fuzzy sets. Fuzzy set theory is a useful tool for describing situations involving imprecise or ambiguous data. Fuzzy sets deal with situations like these by assigning a degree of belonging to a set to each object. Since then, it has become a burgeoning field of study in engineering, medicine, social science, graph theory, metric space theory, and complex analysis, among other fields. Kramosil and Michalek [6] introduced fuzzy metric spaces in a variety of ways in 1975. With the help of continuous t-norms, George and Veermani [4] improved the concept of fuzzy metric spaces in 1994.

Buckley [3] was the one who originally established the concept of fuzzy complex numbers and fuzzy complex analysis. 1987. Some authors were influenced by Buckley's work. Re-examination of fuzzy complex numbers continues. The year was 2002, and Fuzzy sets were extended to complicated fuzzy sets by Ramot et al. [8]. as though it were a blanket statement Ramot et al. claim that A membership function defines a sophisticated fuzzy set. function with a range that extends beyond [0, 1]the complicated plane's unit circle Singh was born in the year 2016. The concept of complex valued fuzzy was introduced by et al.[10]. Using complex valued continuous to create metric spaces t -norm as well as the concept of convergent convergence. in a complex valued fuzzy sequence, Cauchy sequence in complex valued fuzzy metric spaces.By introducing the concept of non-membership grade to fuzzy set theory, Atanassov [1] created a stir in 1983.In this paper, we generalise the results of Jeyaraman, Shakila [13]

In the complex valued intuitionistic fuzzy metric spaces, this work gives some common fixed point theorems for pairs of occasionally weakly compatible mappings satisfying various requirements.

[1]Received March 12, 2022, Accepted June 20, 2022.

§2. Preliminaries

Definition 2.1 *A binary operation* $* : r_s(\cos\theta + i\sin\theta) \times r_s(\cos\theta + i\sin\theta) \to r_s(\cos\theta + i\sin\theta)$, *where* $r_s \in [0, 1]$ *and a fix* $\theta \in \left[0, \frac{\pi}{2}\right]$, *is called complex valued continuous t-norm if it satisfies the followings:*

(1) $*$ *is associative and commutative;*

(2) $*$ *is continuous;*

(3) $a * e^{i\theta} = a, \forall a \in r_s(\cos\theta + i\sin\theta)$*;*

(4) $\mathrm{a} * \mathrm{b} \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d, \forall \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} \in r_s(\cos\theta + i\sin\theta)$.

Definition 2.2 *A binary operation* $r_s(\cos\theta + i\sin\theta) \times r_s(\cos\theta + i\sin\theta) \to r_s(\cos\theta + i\sin\theta)$*,where* $r_s \in [0, 1]$ *and a fix* $\theta \in \left[0, \frac{\pi}{2}\right]$, *is called complex valued continuous t-co norm if it satisfies the followings:*

(1) *is associative and commutative;*

(2) *is continuous;*

(3) $\mathrm{a} \diamond 0 = a, \forall a \in r_s(\cos\theta + i\sin\theta)$*;*

(4) $\mathrm{a} \diamond \mathrm{b} \leq \mathrm{c} \diamond \mathrm{d}$ *whenever* $a \leq c$ *and* $b \leq d, \forall \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} \in r_s(\cos\theta + i\sin\theta$ *).*

Example 2.3 The following are examples for complex valued continuous t-norm.

(i) $a * b = \min\{a, b\}, \forall a, b \in r_s(\cos\theta + i\sin\theta)$ and a fix $\theta \in \left[0, \frac{\pi}{2}\right]$

(ii) $a * b = \max(a + b - (\cos\theta + i\sin\theta), 0)$, for all $a, b \in r_s(\cos\theta + i\sin\theta)$ and a fix $\theta \in \left[0, \frac{\pi}{2}\right]$.

Example 2.4 The following are examples for complex valued continuous t-conorm.

(i) $\mathrm{a} \diamond b = \max\{a, b\}, \forall a, b \in r_s(\cos\theta + i\sin\theta)$ and a fix $\theta \in \left[0, \frac{\pi}{2}\right]$;

(ii) $\mathrm{a}\diamond b = \min(a + b, 1)$, for all $a, b \in r_s(\cos\theta + i\sin\theta)$ and a fix $\theta \in \left[0, \frac{\pi}{2}\right]$.

Definition 2.5 *The 5-triplet (* $X, \mathrm{M}, \mathrm{N}, *, \diamond$*) is said to be complex valued intuitionistic fuzzy metric space if* X *is an arbitrary non empty set,* $*$ *is a complex valued continuous* t*-norm,* $\diamond$ *is a complex valued continuous t-conorm and* $M, N : X \times X \times (0, \infty) \to r_s(\cos\theta + i\sin\theta)$ *are complex valued fuzzy sets, where* $r_s \in [0, 1], r_s(\cos\theta + i\sin\theta)$ *are complex valued fuzzy sets, where* $r_s \in$ *and* $\theta \in \left[0, \frac{\pi}{2}\right]$, *satisfying the following conditions:*

For all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in X; \mathrm{t}, \mathrm{s} \in (0, \infty); r_s \in [0, 1]$ *and* $\theta \in \left[0, \frac{\pi}{2}\right]$,

$(cf1)$ $M(a, b, p) + M(a, b, p) \leq (\cos\theta + i\sin\theta)$;

$(cf2)$ $M(a, b, p) > 0$;

$(cf3)$ $M(a, b, p) = (\cos\theta + i\sin\theta)$, for all $p \in (0, \infty)$ if and only if $\mathrm{a} = \mathrm{b}$;

$(cf4)$ $M(a, b, p) = M(b, a, p)$;

(cff) $M(a, b, p + s) \geq M(a, c, p) * M(c, b, s)$;

$(cf6)$ $M(a, b, p) : (0, \infty) \to r_s(\cos\theta + i\sin\theta)$ *is continuous;*

$(cf7)$ $N(a, b, p) < (\cos\theta + i\sin\theta)$;

$(cf8)$ $N(a, b, p) = 0$, *for all* $p \in (0, \infty)$ *if and only if* $\mathrm{a} = \mathrm{b}$*;*

$(cf9)$ $N(a, b, p) = N(b, a, p)$;

$(cf10)$ $N(a, b, p + s) \leq N(a, c, p) \diamond N(c, b, s)$;

$(cf11)$ *$N(a,b,p) : (0,\infty) \to r_s(\cos\theta + i\sin\theta)$ is continuous.*

The pair (M,N) is called a complex valued intuitionistic fuzzy metric space. The functions $M(a,b,p)$ and $N(a,b,p)$ denotes the degree of nearness and non-nearness between a and b with respect to t. It is noted that if we take $\theta = 0$, then complex valued intuitionistic fuzzy metric simply goes to real valued intuitionistic fuzzy metric.

§3. Main Results

Theorem 3.1 *Let (X, M, N, $*$, $\diamond$) be a complex valued intuitionistic fuzzy metric space with $\lim_{t\to\infty} M(a,b,p) = (\cos\theta + i\sin\theta)$ and $\lim_{t\to\infty} N(a,b,p) = 0$, for all $a,b \in X, p > 0$ and let A and B be self mappings on X. If there exists $d \in (0,1)$ such that*

$$M(Aa,Bb,dp) \geq M(a,b,p), \quad N(Aa,Bb,dp) \leq N(a,b,p) \; for \; all a,b \in X \; and dp > 0, \tag{3.1}$$

then A and B have a unique common fixed point in X.

Proof Let $a_0 \in X$ be an arbitrary point and we define the sequence $\{a_n\}$ by $a_{2n+1} = Aa_{2n}$ and $a_{2n+2} = Ba_{2n+1}; n = 0,1,2,\cdots$. Now, for $d \in (0,1)$ and for all $p > 0$, then from (3.1) we have

$$\begin{aligned} M\left(a_{2n+1}, a_{2n+2}, dp\right) &= M\left(Aa_{2n}, Ba_{2n+1}, dp\right) \\ &\geq M\left(a_{2n}, a_{2n+1}, p\right) \\ M\left(a_{2n}, a_{2n+1}, dp\right) &= M\left(Aa_{2n-1}, Ba_{2n}, dp\right) \\ &\geq M\left(a_{2n-1}, a_{2n}, p\right), \text{ and} \\ N\left(a_{2n+1}, a_{2n+2}, dp\right) &= N\left(Aa_{2n}, Ba_{2n+1}, dp\right) \\ &\leq N\left(a_{2n}, a_{2n+1}, p\right), \\ N\left(a_{2n}, a_{2n+1}, dp\right) &= N\left(Aa_{2n-1}, Ba_{2n}, dp\right) \\ &\leq N\left(a_{2n-1}, a_{2n}, p\right). \end{aligned}$$

In general, we have

$$M\left(a_{n+1}, a_{n+2}, dp\right) \geq M\left(a_n, a_{n+1}, p\right), \quad N\left(a_{n+1}, a_{n+2}, dp\right) \leq N\left(a_n, a_{n+1}, p\right)$$

for for all $p > 0$ and $d \in (0,1); n = 0,1,2,\cdots$ but $\{a_n\}$ be a sequence in a complex valued intuitionistic fuzzy metric space (X, M, N, $*$, $\diamond$), with $\lim_{p\to\infty} M(a,b,p) = \cos\theta + \mathrm{i}\sin\theta$ and $\lim_{p\to\infty} N(a,b,p) = 0, \forall a,b \in X$. If $\lim_{p\to 0} N(a,b,p) = 0$, there exists $d \in (0,1)$ such that $M\left(a_{n+1}, a_{n+2}, dp\right) \geqslant M\left(a_n, a_{n+1}, p\right)$ and $N\left(a_{n+1}, a_{n+2}, dp\right) \leqslant \left(a_n, a_{n+1} p\right)$, for all $p > 0$, then $\{a_n\}$ is a cauchy sequence in X.Since X is Complete then there exist $V \in X$ such that $a_n \to v$ as $n \to \infty$ and $\{a_{2n}\}$ and $\langle a_{2n+1}\}$ are subsequences of the same point $v \in X$, i.e.

$$a_{2n} \to v, a_{2n+1} \to v, \text{as } n \to \infty.$$

Now from (3.1) we have,

$$\begin{aligned}M(Av,v,dp) &= M\left(Av,v,\frac{dp}{2}+\frac{dp}{2}\right)\\ &\geqslant M\left(Au,a_{2n+2}\frac{dp}{2}\right)*M\left(a_{2n+2},v,\frac{dp}{2}\right)\\ &= M\left(Au,Ba_{2n+1},\frac{dp}{2}\right)*M\left(a_{2n+2},v,\frac{dp}{2}\right)\\ &\geqslant M\left(v,a_{2n+1},\frac{p}{2}\right)*M\left(a_{2n+2},v,\frac{dp}{2}\right)\end{aligned}$$

$$\begin{aligned}N(Av,v,dp) &= N\left(Av,v,\frac{dp}{2}+\frac{dp}{2}\right)\\ &\leq N\left(Av,a_{2n+2},\frac{dp}{2}\right)\diamond N\left(a_{2n+2},v,\frac{dp}{2}\right)\\ &= N\left(Av,Ba_{2n+1},\frac{dp}{2}\right)\diamond \mathrm{N}\left(q_{2n+2},v,\frac{dp}{2}\right)\\ &\leq N\left(v,a_{2n+1},\frac{p}{2}\right)\diamond\left(a_{2n+2},v,\frac{dp}{2}\right)\end{aligned}$$

On taking limit $n\to\infty$,

$$\begin{aligned}M(Av,v,dp) &\geqslant (\cos\theta+i\sin\theta)*(\cos\theta+i\sin\theta)\\ &= \cos\theta+i\sin\theta\end{aligned}$$

$$N(Av,v,dp) \quad \leqslant 0\diamond 0=0.$$

So $Av=v$ again, and

$$\begin{aligned}M(Av,v,dp) &= M\left(v,\ \mathrm{B}v,\frac{dp}{2}+\frac{dp}{2}\right)\\ &\geqslant M\left(v,a_{2n+1},\frac{dp}{2}\right)*M\left(a_{2n+1},\ \mathrm{B}v,\frac{dp}{2}\right)\\ &= M\left(v,a_{2n+1},\frac{dp}{2}\right)*M\left(Aa_{2n},\ \mathrm{B}v,\frac{dp}{2}\right)\\ &\geqslant M\left(v,a_{2n+1},\frac{p}{2}\right)*M\left(a_{2n},v,\frac{p}{2}\right)\\ N(Av,v,dp) &= N\left(v,\ \mathrm{B}v,\frac{dp}{2}+\frac{dp}{2}\right)\\ &\leq N\left(v,a_{2n+1},\frac{dp}{2}\right)\diamond N\left(a_{2n+1},\ \mathrm{B}v,\frac{dp}{2}\right)\\ &= N\left(v,a_{2n+1},\frac{dp}{2}\right)\diamond N\left(Aa_{2n},\ \mathrm{B}v,\frac{dp}{2}\right)\\ &\leq N\left(v,a_{2n+1},\frac{p}{2}\right)\diamond N\left(a_{2n},v,\frac{p}{2}\right)\end{aligned}$$

On taking limit $n \to \infty$,

$$M(Av, v, dp) \geqslant (\cos\theta + i\sin\theta) * (\cos\theta + i\sin\theta) = \cos\theta + i\sin\theta$$

$$N(Av, v, dp) \quad \leqslant 0 \diamond 0 = 0.$$

So $Bv = v$, and $Av = Bv = v$. Hence v is a common fixed point of A and B. For uniqueness let c be any another fixed point of A and B. Now from (3.1),

$$M(v, c, dp) = M(Av, Bc, dp) \geq M(v, c, p), \quad N(v, c, dp) = N(Av, Bc, dp) \leq N(v, c, p),$$

we know that when $(X, \mathrm{M}, \mathrm{N}, *, \diamond)$ be a complex valued intuitionistic fuzzy metric space such that $\lim_{p\to\infty} M(a, b, p) = \cos\theta + \mathrm{i}\sin\theta$ and $\lim_{p\to\infty} N(a, b, p) = 0, \forall a, b \in X$. If $M(a, b, dp) \geq M(a, b, p)$ and $N(a, b, dp) \leq N(a, b, p)$ for some $0 < d < 1$, for all $a, b \in X, p \in (0, \infty)$, then $a = b$. Hence $v = c$. □

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Famous Words

Einstein was wrong when he said "*god does not play dice*". Consideration of black holes suggests, not only that god does play dice, but that he sometimes confuses us by throwing them where they cant be seen.

By Stephen William Hawking, a British theoretical physicist

Author Information

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Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.
[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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June 2022

Contents

Subclasses of Analytic Functions Associated with q-Derivative

By N. Ravikumar and P. Siva Kota Reddy 01

A Study of Kenmotsu Manifolds with Semi-Symmetric Metric Connection

By B. Laha and A. Mallick 08

Neighborhoods of Certain Class of Analytic Functions Using Modified Sigmoid Function

By Gehan Abo Elyazyd, Hanan Darwish and Awatif Shahin 21

Cordial Labeling of Graphs Using Tribonacci Numbers

By Sarbari Mitra and Soumya Bhoumik 33

Semi-invariant Sub-manifolds of Generalized Sasakian-Space-Forms

By G. Somashekhara, N. Pavani and P. S. K. Reddy 47

4-Remainder Cordial of Some Tree Related Graphs

By R. Ponraj, A. Gayathri and S. Somasundaram 56

k-Product Cordial Labeling of Cone Graphs

By K. Jeya Daisy, R. Santrin Sabibha, P. Jeyanthi and Maged Z. Youssef 72

A Note on Fixed Point Theorem in Complex Valued Intuitionistic Fuzzy Metric Space

By Ram Milan Singh 81

An International Journal on Mathematical Combinatorics

ISSN 1937-1055