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A new multi-criteria decision making algorithm for medical diagnosis and classification problems using divergence measure of picture fuzzy sets

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Abstract. A divergence measure plays an important part in distinguishing two probability distributions and drawing conclusions based on that discrimination. In this paper, we proposed the concept of divergence measure of picture fuzzy sets. We also built some formulas of the proposed divergence measure of picture fuzzy sets anddiscussed some basic properties of this measure.Based on the proposedmeasure, we developed a multi-criteria decision-making algorithm. Finally, we applied the proposed multi-criteria decision-making algorithm in the medical diagnosis problem and the classification problem.

Keywords: Picture fuzzy set, picture fuzzy divergence measure, medical diagnosis, classification problem

1. Introduction

The multi-criteria decision-making model has been applied to many practical problems. Approaches to building a decision-making model are also diverse and rich. The decision making depends on the information collected and the subjectivity of the decision maker. Information may be vague, inaccurate and uncertain. The subjectivity of decision makers can also be influenced by many factors such as information, psychology. Therefore, decision making in an uncertain environment has been of interest to many researchers, especially decision-making models in fuzzy environments [13–20, 24, 42, 51–55]. The ranking of objects is very important in the decision-making process. The ranking of objects can be done based on the aggregation operators [2, 14–19, 42], or cross-entropy operators [51]. The ranking can be based on measures such as the similarity measures, the distance measures or dissimilarity measures [13, 24, 42]. In addition, the decision-making based on the divergence measures of fuzzy sets and the intuitionistic fuzzy sets are also interested and developed by many researchers [9, 10, 12, 21–23, 25, 26, 32, 43, 44].

The picture fuzzy set [3] was first introduced by Cuong as an extension of the intuitionistic fuzzy set [1] and fuzzy set [50]. It is a useful mathematical tool

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for dealing with ambiguous and inaccurate problems. So far, many theoretical and applied results have been exploited on picture fuzzy sets [3–5, 7, 8, 11, 13, 24, 27, 29–31, 33, 34, 42, 45–47]. Cuong et al. continued to study the operators on picture fuzzy sets [4, 5]. Dinh et al. [7] studied the dissimilarity and distance measures of picture fuzzy sets and applied them in classification problems. The class of fuzzy clustering problems was illustrated by the authors Kumar et al. [11], Son et al. [29–31] and Thong et al. [38–40]. The aggregation operators of the picture fuzzy sets were also investigated and applied in multi-criterion decision-making problems as Garg [8], Wang et al. [45], and Wei et al. [46]. Wei et al. [47] studied the picture fuzzy cross-entropy for multiple attribute decision-making problems. Thao and Dinh [34] established the concept of rough picture fuzzy sets and studied the properties of them together picture fuzzy topologies on rough picture fuzzy sets. Dinh et al. studied the fuzzy picture database and applied it to searching for criminals [41]. Another operator of picture fuzzy sets is the correlation coefficient which was also published by Singh [27] and used in some classes of decision, clustering and classification problems with picture fuzzy information.

In the area of research on applications of fuzzy set theory, researchers are often very interested in measurements between fuzzy sets. Measures are often used to measure the degree of similarity or dissimilarity between objects. One of the dissimilarity measures of fuzzy sets/intuitionistic fuzzy sets was recently investigated by investigators as a measure of the divergence of fuzzy sets [22, 23, 25]. Divergence measures also have many applications in practical problem classes and give us interesting results. Some authors applied divergence measure to determine the relationship between the patient and the treatment regimen based on symptoms, thereby selecting the most appropriate treatment regimen for each patient [26]. Divergence measure was also used in multi-criteria decision problems [21, 32, 41]. The picture fuzzy set is considered an extension of the fuzzy set and the intuitionistic fuzzy set. Therefore, following the results of research on the measures on the fuzzy sets, the intuitionistic fuzzy sets for the picture fuzzy sets is also natural and necessary. That is the driving force for us to study the divergence measure of the picture fuzzy sets both in theory and its application.

In this paper, we introduce the concept of the divergence measure of picture fuzzy sets, called picture fuzzy divergence measure, a kind of dissimilarity measure. This is a new concept that has never been mentioned before. We also give some expressions that define the picture fuzzy divergence measuresand investigate the properties of them. After that, we develop a multi-criteria decision-making algorithm, apply it in the medical diagnosis problem and the classification problem and comparethe obtained results to the calculated results by the other measures. The results show thatourmeasure is really robust and effective.

The article is organized as follows: In Section 2, we recall the knowledge related to picture fuzzy sets. In Section 3, we introduce the concept of picture fuzzy divergence measure and investigate their properties. We show some applications of picture fuzzy divergence measures in Section 4. Section 5, we give conclusion on picture fuzzy divergence measures and some development direction of them.

2. Preliminary

Definition 1. Picture fuzzy set (PFS):

$$A = \{(x, \mu_A(x), \eta_A(x), \gamma_A(x))|x \in U\} \quad (1)$$

where $\mu_A(x) \in [0, 1]$ is a membership function, $\eta_A(x) \in [0, 1]$ is neutral membership function, $\gamma_A(x) \in [0, 1]$ is a non-membership function of A and $\mu_A(x) + \eta_A(x) + \gamma_A(x) \le 1$ for all $u \in S$.

We denote $PFS(U)$ is a collection of picture fuzzy set on U. In which $U = \{(u, 1, 0, 0)|u \in U\}$ and $\emptyset = \{(u, 0, 0, 1)|u \in U\}$.

The picture fuzzy set is a particular case of the neutrosophic set [28], where $\mu_A(x) + \eta_A(x) + \gamma_A(x) \le 3$ for all $u \in S$.

For two sets $A, B \in PFS(U)$ we have:

– Union of A and B:

$$A \cup B = \left\{ \begin{array}{l} (x, \max(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \\ \min(\gamma_A(x), \gamma_B(x)))|x \in U \end{array} \right\}$$

– Intersection of A and B:

$$A \cap B = \left\{ \begin{array}{l} (x, \min(\mu_A(x), \mu_B(x)), \min(\eta_A(x), \eta_B(x)), \\ \max(\gamma_A(x), \gamma_B(x)))|x \in U \end{array} \right\}$$

– Subset: $A \subseteq B$ iff $\mu_A(x) \le \mu_B(x), \eta_A(x) \le \eta_B(x)$

– Complement of A: $A^C = \{(x, \gamma_A(x), \eta_A(x), \mu_A(x))|x \in U\}$

Now, we define an operator called difference between picture fuzzy sets.

3. Divergence measure of picture fuzzy sets

Definition 2. Let A and B be two picture fuzzy sets on U. A function $D : PFS(U) \times PFS(U) \to R$ (R is the real number set) is a divergence measure of picture fuzzy sets if it satisfies the following conditions:

- Div1. $D(A, B) = D(B, A)$,
- Div2. $D(A, A) = 0$,
- Div3. $D(A \cap C, B \cap C) \leq D(A, B)$ for all $C \in PFS(U)$,
- Div4. $D(A \cup C, B \cup C) \leq D(A, B)$ for all $C \in PFS(U)$.

We call the divergence measure in definition 2 is the picture fuzzy divergence measure.

We can easily see that a picture fuzzy divergence measure is not negative. Because, if we choose $C = \emptyset$ then from conditions Div2 and Div3 in definition 2 we have $D(A, B) \geq D(A \cap C, B \cap C) = D(\emptyset, \emptyset) = 0$.

Now we give some divergence measures of picture fuzzy sets and their properties.

Definition 3. Let $A$and B be two picture fuzzy sets on $U = \{u_1, u_2, ..., u_n\}$. A function $D : PFS(U) \times PFS(U) \to R$ is defined as follows:

$$D(A, B) = \frac{1}{n}\sum_{i=1}^{n}[D^i_\mu(A, B) + D^i_\eta(A, B) + D^i_\gamma(A, B)] \tag{2}$$

where

$$D^i_\mu(A, B) = \mu_A(u_i)\ln\frac{2\mu_A(u_i)}{\mu_A(u_i) + \mu_B(u_i)} + \mu_B(u_i)\ln\frac{2\mu_B(u_i)}{\mu_A(u_i) + \mu_B(u_i)} \tag{3}$$

$$D^i_\eta(A, B) = \eta_A(u_i)\ln\frac{2\eta_A(u_i)}{\eta_A(u_i) + \eta_B(u_i)} + \eta_B(u_i)\ln\frac{2\eta_B(u_i)}{\eta_A(u_i) + \eta_B(u_i)} \tag{4}$$

and

$$D^i_\gamma(A, B) = \gamma_A(u_i)\ln\frac{2\gamma_A(u_i)}{\gamma_A(u_i) + \gamma_B(u_i)} + \gamma_B(x_i)\ln\frac{2\gamma_B(u_i)}{\gamma_A(u_i) + \gamma_B(u_i)} \tag{5}$$

Example 1. Assume that there are two patterns denoted by picture fuzzy sets on$U = \{u_1, u_2, u_3\}$ as follows

$$A = \{(u_1, 0.2, 0.2, 0.2), (u_2, 0.3, 0.4, 0.1), (u_3, 0.2, 0.1, 0.6)\},$$

$$B = \{(u_1, 0.6, 0.1, 0.3), (u_2, 0.2, 0.1, 0.6), (u_3, 0.1, 0.2, 0.7)\}$$

By using Equations (2)–(5) in Definition 3,we have

$$D^1_\mu(A, B) = 0.10465,\ D^2_\mu(A, B) = 0.01007,$$
$$D^3_\mu(A, B) = 0.01699,\ D^1_\eta(A, B) = 0.01699,$$
$$D^2_\eta(A, B) = 0.09637,\ D^3_\eta(A, B) = 0.01699,$$
$$D^1_\gamma(A, B) = 0.01007,\ D^2_\gamma(A, B) = 0.19812,$$
$$D^3_\gamma(A, B) = 0.00385$$

and $D(A, B) = 0.15803$.

To proof that $D(A, B)$ is a divergence measure of picture fuzzy sets we need some following lemmas.

Lemma 1. *Given $a \in (0, 1]$. For all $z \in [0, 1 - a]$ then:*

$$f(z) = a\ln\frac{2a}{2a + z} + (a + z)\ln\frac{2a + 2z}{2a + z} \tag{6}$$

is a non-decreasing function and $f(z) \geq 0$.

Proof. We obtain $\frac{\partial f(z)}{\partial z} = \ln\frac{2a+2z}{2a+z} \geq 0$ for all $z \in [0, 1 - a]$.

Lemma 2. *Given $b \in (0, 1]$. For all $z \in (0, b]$ then*

$$f(z) = b\ln\frac{2b}{b + z} + z\ln\frac{2z}{b + z} \tag{7}$$

is a non-decreasing function and $f(z) \geq 0$.

Proof. We have $\frac{\partial f(z)}{\partial z} = \ln\frac{2z}{b+z} \leq 0$ for all $z \in (0, b]$.

Lemma 3. *Given $a \in (0, 1]$. For all $z \in [a, 1]$ then*

$$(z) = a\ln\frac{2a}{a + z} + z\ln\frac{2z}{a + z} \tag{8}$$

is a non-decreasing function and $f(z) \geq 0$.

Proof. We have $\frac{\partial f(z)}{\partial z} = \ln\frac{2z}{a+z} \leq 0$ for all $z \in [a, 1]$.

Theorem 1. *The function $D(A, B)$ defined by Equations (2)–(5) (in definition 3) is a divergence measure of two picture fuzzy sets.*

Proof. We check the conditions of the definition. For two picture fuzzy sets A and B on U, we have:

- Div1: $D(A, B) = D(B, A)$,
- Div2: If $B \equiv A$ we have $D^i_\mu(A, B) = D^i_\eta(A, B) = D^i_\gamma(A, B) = 0$. So that $D(A, B) = 0$.
- Div3. For all $C \in PFS(U)$and for all $u_i \in U, (i = 1, 2, ..., n)$. Because of the symmetry, we can consider the following cases we have:

– With positive degree, we have:

+If $\mu_A(u_i) \le \mu_B(u_i) \le \mu_C(u_i)$ then $\mu_{A\cap C}(u_i) = \mu_A(u_i)$ and $\mu_{B\cap C}(u_i) = \mu_B(u_i)$so that

$$\begin{aligned} & D^i_\mu(A \cap C, B \cap C) \\ & = \mu_{A\cap C}(u_i) \ln \frac{2\mu_{A\cap C}(u_i)}{\mu_{A\cap C}(u_i) + \mu_{B\cap C}(u_i)} \\ & +\mu_{B\cap C}(u_i) \ln \frac{2\mu_{B\cap C}(u_i)}{\mu_{A\cap C}(u_i) + \mu_{B\cap C}(u_i)} \\ & = \mu_A(u_i) \ln \frac{2\mu_A(u_i)}{\mu_A(u_i) + \mu_B(u_i)} \\ & +\mu_B(u_i) \ln \frac{2\mu_B(u_i)}{\mu_A(u_i) + \mu_B(u_i)} \\ & = D^i_\mu(A, B) \end{aligned}$$

+ If $\mu_A(u_i) \le \mu_C(u_i) \le \mu_B(u_i)$ then $\mu_{A\cup C}(u_i) = \mu_C(u_i)$ and $\mu_{B\cup C}(u_i) = \mu_B(u_i)$. So that, according the lemma 3 with $a = \mu_A(u_i)$ we have:

$$\begin{aligned} & D^i_\mu(A \cap C, B \cap C) \\ & = \mu_A(u_i) \ln \frac{2\mu_A(u_i)}{\mu_A(u_i) + \mu_C(u_i)} + \mu_C(u_i) \ln \frac{2\mu_C(u_i)}{\mu_A(u_i) + \mu_C(u_i)} \\ & \le \mu_A(u_i) \ln \frac{2\mu_A(u_i)}{\mu_A(u_i) + \mu_B(u_i)} + \mu_C(u_i) \ln \frac{2\mu_B(u_i)}{\mu_A(u_i) + \mu_B(u_i)} \\ & = D^i_\mu(A, B) \end{aligned}$$

+If $\mu_C(u_i) \le \mu_A(u_i) \le \mu_B(u_i)$ then $\mu_{A\cap C}(u_i) = \mu_{B\cap C}(u_i) = \mu_C(u_i)$ and $\mu_B(u_i) = \mu_C(u_i) + z$ with $z \in [0, 1 - \mu_A(u_i)]$ so that according the lemma 1 we have:

$$\begin{aligned} & D^i_\mu(A \cap C, B \cap C) \\ & = \mu_C(u_i) \ln \frac{2\mu_C(u_i)}{\mu_C(u_i) + \mu_C(u_i)} + \mu_C(u_i) \ln \frac{2\mu_C(u_i)}{\mu_C(u_i) + \mu_C(u_i)} = 0 \\ & \le \mu_A(u_i) \ln \frac{2\mu_A(u_i)}{2\mu_A(u_i) + z} + \mu_B(u_i) \ln \frac{2\mu_A(u_i) + 2z}{2\mu_A(u_i) + z} \\ & = \mu_A(u_i) \ln \frac{2\mu_A(u_i)}{\mu_A(u_i) + \mu_B(u_i)} + \mu_B(u_i) \ln \frac{2\mu_B(u_i)}{\mu_A(u_i) + \mu_B(u_i)} \\ & = D^i_\mu(A, B). \end{aligned}$$

– With neutral degree: We do the same with the case of positive degree.

– With negative degree, we have:

+If $\gamma_A(u_i) \le \gamma_B(u_i) \le \gamma_C(u_i)$ then $\gamma_{A\cap C}(u_i) = \gamma_C(u_i)$ and $\gamma_{B\cap C}(u_i) = \gamma_C(u_i)$ so that according lemma 1 we have:

$$\begin{aligned} & D^i_\gamma(A \cap C, B \cap C) \\ & = \gamma_{A\cap C}(u_i) \ln \frac{2\gamma_{A\cap C}(u_i)}{\gamma_{A\cap C}(u_i) + \gamma_{B\cap C}(u_i)} \\ & +\gamma_{B\cap C}(u_i) \ln \frac{2\gamma_{B\cap C}(u_i)}{\gamma_{A\cap C}(u_i) + \gamma_{B\cap C}(u_i)} \\ & = \gamma_C(u_i) \ln \frac{2\gamma_C(u_i)}{\gamma_C(u_i) + \gamma_C(u_i)} + \gamma_C(u_i) \ln \frac{2\gamma_C(u_i)}{\gamma_C(u_i) + \gamma_C(u_i)} = 0 \\ & \le \gamma_A(u_i) \ln \frac{2\gamma_A(u_i)}{\gamma_A(u_i) + \gamma_B(u_i)} + \gamma_B(u_i) \ln \frac{2\gamma_B(u_i)}{\gamma_A(u_i) + \gamma_B(u_i)} \\ & = D^i_\gamma(A, B) \end{aligned}$$

+If $\gamma_A(u_i) \le \gamma_C(u_i) \le \gamma_B(u_i)$ then $\gamma_{A\cup C}(u_i) = \gamma_C(u_i)$ and $\gamma_{B\cup C}(u_i) = \gamma_B(u_i)$. So that, according the lemma 2 with $b = \gamma_B(u_i)$ we have:

$$\begin{aligned} & D^i_\gamma(A \cap C, B \cap C) \\ & = \gamma_C(u_i) \ln \frac{2\gamma_C(u_i)}{\gamma_C(u_i) + \gamma_B(u_i)} + \gamma_B(u_i) \ln \frac{2\gamma_B(u_i)}{\gamma_C(u_i) + \gamma_B(u_i)} \\ & \le \gamma_A(u_i) \ln \frac{2\gamma_A(u_i)}{\gamma_A(u_i) + \gamma_B(u_i)} + \gamma_B(u_i) \ln \frac{2\gamma_B(u_i)}{\gamma_A(u_i) + \gamma_B(u_i)} \\ & = D^i_\gamma(A, B) \end{aligned}$$

+If $\gamma_C(u_i) \le \gamma_A(u_i) \le \gamma_B(u_i)$ then according the lemma 1 we have:

$$\begin{aligned} & D^i_\gamma(A \cap C, B \cap C) \\ & = \gamma_A(u_i) \ln \frac{2\gamma_A(u_i)}{\gamma_A(u_i) + \gamma_B(u_i)} + \gamma_B(u_i) \ln \frac{2\gamma_B(u_i)}{\gamma_A(u_i) + \gamma_B(u_i)} \\ & = D^i_\gamma(A, B). \end{aligned}$$

Now, we add that with respect to the respective components we have:

$$\begin{aligned} & D(A \cap C, B \cap C) \\ & = \frac{1}{n} \sum_{i=1}^{n} [D^i_\mu(A \cap C, B \cap C) + D^i_\eta(A \cap C, B \cap C) \\ & + D^i_\gamma(A \cap C, B \cap C)] \\ & \le \frac{1}{n} \sum_{i=1}^{n} [D^i_\mu(A, B) + D^i_\eta(A, B) + D^i_\gamma(A, B)] \\ & = D(A, B) \end{aligned}$$

- Div4. We perform as Div3.

Some properties of divergence measure defined by definition 3.

Theorem 2. *For all picture fuzzy sets $A, B \in PFS(U)$, we have:*

(D1) $D(A^C, B^C) = D(A, B)$,
(D2) $D(A^C, B) = D(A, B^C)$,
(D3) For all $A \subseteq B$, or $B \subseteq A$ we have $D(A \cap B, B) = D(A, A \cup B) \leq D(A, B)$,
(D4) $D(A \cap B, A \cup B) = D(A, B)$,
(D5) For all $A \subseteq B \subseteq C$ we have $D(A, B) \leq D(A, C)$,
(D6) For all $A \subseteq B \subseteq C$ we have $D(B, C) \leq D(A, C)$.

Proof. (D1). We have:

$$\begin{aligned} D^i_\mu(A^C, B^C) &= \mu_{A^C}(u_i) \ln \frac{2\mu_{A^C}(u_i)}{\mu_{A^C}(u_i) + \mu_{B^C}(u_i)} \\ &\quad + \mu_{B^C}(u_i) \ln \frac{2\mu_{B^C}(u_i)}{\mu_{A^C}(u_i) + \mu_{B^C}(u_i)} \\ &= \gamma_A(u_i) \ln \frac{2\gamma_A(u_i)}{\gamma_A(u_i) + \gamma_B(u_i)} \\ &\quad + \gamma_B(u_i) \ln \frac{2\gamma_B(u_i)}{\gamma_A(u_i) + \gamma_B(u_i)} \\ &= D^i_\gamma(A, B), D^i_\eta(A^C, B^C) \\ &= D^i_\eta(A, B), D^i_\gamma(A^C, B^C) \\ &= D^i_\mu(A, B). \end{aligned}$$

(D2). We have:

$$\begin{aligned} D^i_\mu(A^C, B) &= \mu_{A^C}(u_i) \ln \frac{2\mu_{A^C}(u_i)}{\mu_{A^C}(u_i) + \mu_B(u_i)} \\ &\quad + \mu_B(u_i) \ln \frac{2\mu_B(u_i)}{\mu_{A^C}(u_i) + \mu_B(u_i)} \\ &= \gamma_A(u_i) \ln \frac{2\gamma_A(u_i)}{\gamma_A(u_i) + \mu_B(u_i)} \\ &\quad + \mu_B(u_i) \ln \frac{2\mu_B(u_i)}{\gamma_A(u_i) + \mu_B(u_i)} \\ &= D^i_\gamma(A, B^C), D^i_\eta(A^C, B) \\ &= D^i_\eta(A, B^C), D^i_\gamma(A^C, B) \\ &= D^i_\mu(A, B^C) \end{aligned}$$

So that

$$\begin{aligned} &D(A^C, B) \\ &= \frac{1}{n} \sum_{i=1}^n [D^i_\mu(A^C, B) + D^i_\eta(A^C, B) + D^i_\gamma(A^C, B)] \\ &= \frac{1}{n} \sum_{i=1}^n [D^i_\gamma(A, B^C) + D^i_\eta(A, B^C) + D^i_\mu(A, B^C)] \\ &= D(A, B^C). \end{aligned}$$

(D3). If $A \subseteq B$ then $D(A \cap B, B) = D(A, B)$, and $D(A, A \cup B) = D(A, B)$.

If $B \subseteq A$ then $D(A \cap B, B) = D(B, B) = 0$, and $D(A, A \cup B) = D(A, A) = 0$.

It means that if $A \subseteq B$, or $B \subseteq A$ we have $D(A \cap B, B) = D(A, A \cup B) \leq D(A, B)$.

(D4). Because of the symmetry of the divergence measure, we consider the cases:

– If $\mu_A(u_i) \leq \mu_B(u_i)$ then we have:

$$\begin{aligned} D^i_\mu(A \cup B, A \cap B) &= \mu_B(u_i) \ln \frac{2\mu_B(u_i)}{\mu_A(u_i) + \mu_B(u_i)} \\ &\quad + \mu_A(u_i) \ln \frac{2\mu_A(u_i)}{\mu_A(u_i) + \mu_B(u_i)} \\ &= D(A, B). \end{aligned}$$

– If $\mu_B(u_i) \leq \mu_A(u_i)$ then we have:

$$\begin{aligned} D^i_\mu(A \cup B, A \cap B) &= \mu_A(u_i) \ln \frac{2\mu_A(u_i)}{\mu_A(u_i) + \mu_B(u_i)} \\ &\quad + \mu_B(u_i) \ln \frac{2\mu_B(u_i)}{\mu_A(u_i) + \mu_B(u_i)} \\ &= D(A, B). \end{aligned}$$

By the same consideration for the neutral degree and negative degree, we obtain

$$D(A \cap B, A \cup B) = D(A, B).$$

(D5). For all $A \subseteq B \subseteq C$ and for all $u_i \in U$ we have:

– With the positive degree:

From condition $\mu_A(u_i) \leq \mu_B(u_i) \leq \mu_C(u_i)$ and lemma 2 we have:

$$\begin{aligned} &D^i_\mu(A, B) \\ &= \mu_A(u_i) \ln \frac{2\mu_A(u_i)}{\mu_A(u_i) + \mu_B(u_i)} + \mu_B(u_i) \ln \frac{2\mu_B(u_i)}{\mu_A(u_i) + \mu_B(u_i)} \\ &= \mu_A(u_i) \ln \frac{2\mu_A(u_i)}{\mu_A(u_i) + \mu_C(u_i)} + \mu_C(u_i) \ln \frac{2\mu_C(u_i)}{\mu_C(u_i) + \mu_A(u_i)} \\ &= D^i_\mu(A, C). \end{aligned}$$

– With the neutral degree:

By the same way as the positive we have $D^i_\eta(A, B) \leq D^i_\eta(A, C)$.

– With the negative degree:

From condition $\gamma_A(u_i) \geq \gamma_B(u_i) \geq \gamma_C(u_i)$ and lemma 3 we have:

$$\begin{aligned} D^i_\gamma(A,B) &= \gamma_A(u_i)\ln\frac{2\gamma_A(u_i)}{\gamma_A(u_i)+\gamma_B(u_i)} \\ &\quad +\gamma_B(u_i)\ln\frac{2\gamma_B(u_i)}{\gamma_A(u_i)+\gamma_B(u_i)} \\ &\leq \gamma_A(u_i)\ln\frac{2\gamma_A(u_i)}{\gamma_A(u_i)+\gamma_C(u_i)} \\ &\quad +\gamma_C(u_i)\ln\frac{2\gamma_C(u_i)}{\gamma_A(u_i)+\gamma_C(u_i)} \\ &= D^i_\gamma(A,C). \end{aligned}$$

So that, we obtain the result $D(A, B) \leq D(A, C)$.

(D6). By the same way as (D5) using lemma 1, lemma 2 and lemma 3, it is easy to derive these results when considering specific cases.

Definition 4. Let A and B be two picture fuzzy sets on $U = \{u_1, u_2, ..., u_n\}$. A function $D : PFS(U) \times PFS(U) \to R$ is defined as follows

$$\overline{D}(A,B) = \frac{1}{n}\sum_{i=1}^{n}[\overline{D}^i_\mu(A,B) + \overline{D}^i_\eta(A,B) + \overline{D}^i_\gamma(A,B)] \tag{9}$$

where

$$\overline{D}^i_\mu(A,B) = \mu_A(u_i)\ln\frac{\mu_A(u_i)}{\mu_B(u_i)} + \mu_B(u_i)\ln\frac{\mu_B(u_i)}{\mu_A(u_i)} \tag{10}$$

$$\overline{D}^i_\eta(A,B) = \eta_A(u_i)\ln\frac{\eta_A(u_i)}{\eta_B(u_i)} + \eta_B(u_i)\ln\frac{\eta_B(u_i)}{\eta_A(u_i)} \tag{11}$$

and

$$\overline{D}^i_\gamma(A,B) = \gamma_A(u_i)\ln\frac{\gamma_A(u_i)}{\gamma_B(u_i)} + \gamma_B(x_i)\ln\frac{\gamma_B(u_i)}{\gamma_A(u_i)} \tag{12}$$

for all $u_i \in U$.

Example 2. With two picture fuzzy sets A and B in Example 1, using Equations (9)–(12) in Definition 4, we have

$$\overline{D}^1_\mu(A,B) = 0.43944, \overline{D}^2_\mu(A,B) = 0.04055,$$

$$\overline{D}^3_\mu(A,B) = 0.06931,$$

$$\overline{D}^1_\eta(A,B) = 0.06931, \overline{D}^2_\eta(A,B) = 0.41589,$$

$$\overline{D}^3_\eta(A,B) = 0.06931,$$

$$\overline{D}^1_\gamma(A,B) = 0.04055, \overline{D}^2_\gamma(A,B) = 0.89588,$$

$$\overline{D}^3_\gamma(A,B) = 0.01542$$

and $\overline{D}(A, B) = 0.68522$.

To proof that $D(A, B)$ is a divergence measure of picture fuzzy sets we need some following lemmas.

Now we consider the function:

$$f(x,y) = x\ln\frac{x}{y} + y\ln\frac{y}{x}, \forall x, y \in (0, 1] \tag{13}$$

Transforms the expression of function $f(x, y)$ we have $f(x, y) = (x - y)(\ln x - \ln y) \geq 0, \forall x, y \in (0, 1]$.

Besides, we also have:
$\frac{\partial f}{\partial x}(x, y) = \frac{y-x}{y} + \ln y - \ln x \geq 0$ for all $y \geq x$, and
$\frac{\partial f}{\partial y}(x, y) = \frac{x-y}{y} + \ln x - \ln y \leq 0$ for all $x \leq y$.

So that, we have result that is stated in lemma 4.

Lemma 4. *Function $f(x, y) = x\ln\frac{x}{y} + y\ln\frac{y}{x}$ is*

a. $f(x, y), \forall x, y \in (0, 1]$.
b. Increasing with variable x such that $y \geq x$,
c. Decreasing with variable y such that $x \leq y$.

Theorem 3. *The function $D(A, B)$ defined by using Equations (9)–(12) (in definition 4) is a divergence measure of two picture fuzzy sets.*

Proof. We consider divergence measure conditions for function $\overline{D}(A, B)$ defined by Equation (9).

Div1: $\overline{D}(A, B) = \overline{D}(B, A)$ is obvious.

Div2: If $A = B$ then $\mu_A(u_i) = \mu_B(u_i)$, $\eta_A(u_i) = \eta_B(u_i)$, $\gamma_A(u_i) = \gamma_B(u_i)$. So that, when we replace them into the Equations (10, 11, 12) we obtain $\overline{D}(A, B) = 0$.

Div3: $\overline{D}(A \cap C, B \cap C) \leq \overline{D}(A, B)\frac{1}{2}$ for all $C \in PFS(U)$. Indeed, we consider the cases for each degree of picture fuzzy sets:

– With the positive degree:

+If $\mu_A(u_i) \leq \mu_B(u_i) \leq \mu_C(u_i)$ then $\mu_{A\cap C}(u_i) = \mu_A(u_i)$ and $\mu_{B\cap C}(u_i) = \mu_B(u_i)$ so that

$$\begin{aligned} &\overline{D}^i_\mu(A\cap C, B\cap C) \\ &= \mu_{A\cap C}(u_i)\ln\frac{\mu_{A\cap C}(u_i)}{\mu_{B\cap C}(u_i)} + \mu_{B\cap C}(u_i)\ln\frac{\mu_{B\cap C}(u_i)}{\mu_{A\cap C}(u_i)} \\ &= \mu_A(u_i)\ln\frac{\mu_A(u_i)}{\mu_B(u_i)} + \mu_B(u_i)\ln\frac{\mu_B(u_i)}{\mu_A(u_i)} \\ &= \overline{D}^i_\mu(A,B). \end{aligned}$$

+If $\mu_A(u_i) \leq \mu_C(u_i) \leq \mu_B(u_i)$ then $\mu_{A\cup C}(u_i) = \mu_C(u_i)$ and $\mu_{B\cup C}(u_i) = \mu_B(u_i)$. So that, according the lemma 4.b with $x = \mu_A(u_i)$ we have:

$$\begin{aligned}
&\overline{D}^i_\mu(A \cap C, B \cap C)\\
&= \mu_A(u_i) \ln \frac{\mu_A(u_i)}{\mu_C(u_i)} + \mu_C(u_i) \ln \frac{\mu_C(u_i)}{\mu_A(u_i)}\\
&\leq \mu_A(u_i) \ln \frac{\mu_A(u_i)}{\mu_B(u_i)} + \mu_C(u_i) \ln \frac{\mu_B(u_i)}{\mu_A(u_i)}\\
&= \overline{D}^i_\mu(A, B).
\end{aligned}$$

+If $\mu_C(u_i) \leq \mu_A(u_i) \leq \mu_B(u_i)$ then $\mu_{A\cap C}(u_i) = \mu_{B\cap C}(u_i) = \mu_C(u_i)$ so that according the lemma 4.a we have:

$$\begin{aligned}
&\overline{D}^i_\mu(A \cap C, B \cap C)\\
&= \mu_C(u_i) \ln \frac{\mu_C(u_i)}{\mu_C(u_i)} + \mu_C(u_i) \ln \frac{\mu_C(u_i)}{\mu_C(u_i)} = 0\\
&\leq \mu_A(u_i) \ln \frac{\mu_A(u_i)}{\mu_B(u_i)} + \mu_B(u_i) \ln \frac{\mu_B(u_i)}{\mu_A(u_i)}\\
&= \overline{D}^i_\mu(A, B).
\end{aligned}$$

with neutral degree, we do the same with the case of positive degree.

– With negative degree, we have:

+If $\gamma_A(u_i) \leq \gamma_B(u_i) \leq \gamma_C(u_i)$ then $\gamma_{A\cap C}(u_i) = \gamma_C(u_i)$ and $\gamma_{B\cap C}(u_i) = \gamma_C(u_i)$ so that according lemma 1 we have:

$$\begin{aligned}
&\overline{D}^i_\gamma(A \cap C, B \cap C)\\
&= \gamma_{A\cap C}(u_i) \ln \frac{\gamma_{A\cap C}(u_i)}{\gamma_{B\cap C}(u_i)} + \gamma_{B\cap C}(u_i) \ln \frac{\gamma_{B\cap C}(u_i)}{\gamma_{A\cap C}(u_i)}\\
&= \gamma_C(u_i) \ln \frac{\gamma_C(u_i)}{\gamma_C(u_i)} + \gamma_C(u_i) \ln \frac{\gamma_C(u_i)}{\gamma_C(u_i)} = 0\\
&\leq \gamma_A(u_i) \ln \frac{\gamma_A(u_i)}{\gamma_B(u_i)} + \gamma_B(u_i) \ln \frac{\gamma_B(u_i)}{\gamma_A(u_i)}\\
&= \overline{D}^i_\gamma(A, B).
\end{aligned}$$

+If $\gamma_A(u_i) \leq \gamma_C(u_i) \leq \gamma_B(u_i)$ then $\gamma_{A\cup C}(u_i) = \gamma_C(u_i)$ and $\gamma_{B\cup C}(u_i) = \gamma_B(u_i)$. So that, according the lemma 4.c with $y = \gamma_B(u_i)$ we have:

$$\begin{aligned}
&\overline{D}^i_\gamma(A \cap C, B \cap C)\\
&= \gamma_C(u_i) \ln \frac{\mu_C(u_i)}{\mu_B(u_i)} + \gamma_B(u_i) \ln \frac{\mu_B(u_i)}{\mu_C(u_i)}\\
&\leq \gamma_A(u_i) \ln \frac{\gamma_A(u_i)}{\gamma_B(u_i)} + \gamma_B(u_i) \ln \frac{\gamma_B(u_i)}{\gamma_A(u_i)}\\
&= \overline{D}^i_\gamma(A, B).
\end{aligned}$$

+If $\gamma_C(u_i) \leq \gamma_A(u_i) \leq \gamma_B(u_i)$ then according the lemma 4.a we have:

$$\begin{aligned}
&\overline{D}^i_\gamma(A \cap C, B \cap C)\\
&= \gamma_A(u_i) \ln \frac{\gamma_A(u_i)}{\gamma_B(u_i)} + \gamma_B(u_i) \ln \frac{\gamma_B(u_i)}{\gamma_A(u_i)}\\
&= \overline{D}^i_\gamma(A, B).
\end{aligned}$$

Now, we add that with respect to the respective components we have:

$$\begin{aligned}
&\overline{D}(A \cap C, B \cap C)\\
&= \tfrac{1}{n} \sum_{i=1}^{n} [\overline{D}^i_\mu(A \cap C, B \cap C) + \overline{D}^i_\eta(A \cap C, B \cap C)\\
&\quad + \overline{D}^i_\gamma(A \cap C, B \cap C)]\\
&\leq \tfrac{1}{n} \sum_{i=1}^{n} [\overline{D}^i_\mu(A, B) + \overline{D}^i_\eta(A, B) + \overline{D}^i_\gamma(A, B)]\\
&= \overline{D}(A, B).
\end{aligned}$$

Div4. We perform as Div3.

Some properties of divergence measure can be defined by theorem 4.

Theorem 4. *For all picture fuzzy set $A, B \in PFS(U)$.*

(D1) $\overline{D}(A^C, B^C) = \overline{D}(A, B)$,
(D2) $\overline{D}(A^C, B) = \overline{D}(A, B^C)$,
(D3) For all $A \subseteq B$, or $B \subseteq A$ we have $\overline{D}(A \cap B, B) = \overline{D}(A, A \cup B) \leq \overline{D}(A, B)$,
(D4) $\overline{D}(A \cap B, A \cup B) = \overline{D}(A, B)$,
(D5) For all $A \subseteq B \subseteq C$, or we have $\overline{D}(A, B) \leq \overline{D}(A, C)$,
(D6) For all $A \subseteq B \subseteq C$, or we have $\overline{D}(B, C) \leq \overline{D}(A, C)$
(D7) $\overline{D}(A, A^C) = 0$ if only if $\mu_A(u_i) = \gamma_A(u_i)$, and $\eta_A(u_i) \in [0, 1 - 2\mu_A(u_i)]$.

Proof. The results (D1), (D2), (D3), (D4), (D5), (D6) are proved similar to theorem 2 by using lemma 4.

(D7). We have:

$$\begin{aligned}
&\overline{D}^i_\eta(A, A^C)\\
&= \eta_A(u_i) \ln \frac{\eta_A(u_i)}{\eta_{A^C}(u_i)} + \eta_{A^C}(u_i) \ln \frac{\eta_{A^C}(u_i)}{\eta_A(u_i)}\\
&= \eta_A(u_i) \ln \frac{\eta_A(u_i)}{\eta_A(u_i)} + \eta_A(u_i) \ln \frac{\eta_A(u_i)}{\eta_A(u_i)} = 0.
\end{aligned}$$

– Let $\overline{D}(A, A^C) = 0$. Because $\overline{D}(A, A^C) \geq 0$ and we must have $\overline{D}^i_\mu(A, A^C) = 0$, $\overline{D}^i_\gamma(A, A^C) = 0$. We consider:

$$\begin{aligned}
&\overline{D}^i_\mu(A, A^C)\\
&= \mu_A(u_i)\ln\frac{\mu_A(u_i)}{\mu_{A^C}(u_i)} + \mu_{A^C}(u_i)\ln\frac{\mu_{A^C}(u_i)}{\mu_A(u_i)}\\
&= (\mu_A(u_i) - \mu_{A^C}(u_i))(\ln\mu_A(u_i) - \ln\mu_{A^C}(u_i)) = 0.
\end{aligned}$$

This implies that $\mu_A(u_i) - \mu_{A^C}(u_i) = 0$. It means $\mu_A(u_i) = \mu_{A^C}(u_i) = \gamma_A(u_i)$.

– In contrast, assume that $\mu_A(u_i) = \gamma_A(u_i)$, we have:

$$\begin{aligned}
&\overline{D}^i_\mu(A, A^C)\\
&= \mu_A(u_i)\ln\frac{\mu_A(u_i)}{\mu_{A^C}(u_i)} + \mu_{A^C}(u_i)\ln\frac{\mu_{A^C}(u_i)}{\mu_A(u_i)}\\
&= (\mu_A(u_i) - \mu_{A^C}(u_i))(\ln\mu_A(u_i) - \ln\mu_{A^C}(u_i))\\
&= (\mu_A(u_i) - \gamma_A(u_i))(\ln\mu_A(u_i) - \ln\gamma_A(u_i)) = 0.
\end{aligned}$$

and

$$\begin{aligned}
&\overline{D}^i_\gamma(A, A^C)\\
&= \gamma_A(u_i)\ln\frac{\gamma_A(u_i)}{\gamma_{A^C}(u_i)} + \gamma_{A^C}(u_i)\ln\frac{\gamma_{A^C}(u_i)}{\gamma_A(u_i)}\\
&= \gamma_A(u_i)\ln\frac{\gamma_A(u_i)}{\mu_A(u_i)} + \mu_A(u_i)\ln\frac{\mu_A(u_i)}{\gamma_A(u_i)} = 0.
\end{aligned}$$

So that $\overline{D}(A, A^C) = \frac{1}{n}\sum_{i=1}^{n}\{\overline{D}^i_\mu(A, A^C) + \overline{D}^i_\eta(A, A^C) + \overline{D}^i_\gamma(A, A^C)\} = 0.$

4. Applications of divergence measure of picture fuzzy set.

In this section, we apply the picture fuzzy divergence measures in the medical diagnosis and classification problems.

4.1. In the medical diagnosis

Input:

– Diagnosis, in which each Diagnosis is a picture fuzzy set on a universal set of symptoms characteristics.
– Patients in which each pattern is a picture fuzzy set on a universal set of symptoms characteristics.

Output: What diagnosis is best for each patient?

Algorithm:

– **Step 1**. We compute the divergence measure of each patient for all diagnosis by using Equations (2) or (9).
– **Step 2**. For each patient, we choose the smallest value of the divergence measure in Step 1. This will give us the best diagnosis for each patient.

Now, we applied the picture fuzzy divergence measure for obtaining a proper diagnosis for the data given in Tables 1 and 2. This data was modified from the data introduced in [6]. When using the divergence measure we compute all the divergence measure between each patient and each diagnosis. After that, we chose the smallest value of them. This will be give us the best diagnosis for each patient (Tables 3 and 4). The results show that the optimization will be Al (Typhoid), Bob (Stomach Problem), Joe (Viral fever), and Ted (Typhoid).

• Compare with using other measures

+Using the dissimilarity measure of picture fuzzy sets of Le et al., in [13].

$$\mathrm{DM}_1(A, B) = \frac{1}{n}\sum_{i=1}^{n} DM^i_1(A, B) \tag{14}$$

where
$DM^i_1(A, B) = (|(\mu_A(u_i) - \gamma_A(u_i)) - ((\mu_B(u_i) - \gamma_B(u_i))| + |(\eta_A(u_i) - \gamma_A(u_i)) - ((\eta_B(u_i) - \gamma_B(u_i))|)$ for all $u_i \in U$ and $i = 1, 2, ..., n$.

When using the dissimilarity measure, we compute all the dissimilarity measures between each patient and each diagnosis. After that, we choose the smallest value of them. This measure also gives us the best diagnosis for each patient (Table 5). The results also show that the optimization will be Al (Typhoid), Bob (Stomach Problem), Joe (Viral fever), and Ted (Typhoid).

+Using the dissimilarity measure of picture fuzzy sets of Thao in [24].

$$\mathrm{DM}_T(A, B) = \frac{1}{n}\sum_{i=1}^{n} DM^i_T(A, B) \tag{15}$$

where
$DM^i_T(A, B) = (|(\mu_A(u_i) + \eta_A(u_i) - \gamma_A(u_i)) - (\mu_B(u_i) + \eta_B(u_i) - \gamma_B(u_i))| + |(\eta_A(u_i) - \gamma_A(u_i)) - (\eta_B(u_i) - \gamma_B(u_i))|)$ for all $u_i \in U$ and $i = 1, 2, ..., n$.

Table 1
Symptoms characteristics for the diagnosis

	Viral fever (V)	Malaria (M)	Typhoid (T)	Stomach problem (S)	Chest problem (C)
Temperature	(0.4,0.5,0.1)	(0.7,0.2,0.1)	(0.3,0.4,0.2)	(0.1,0.2,0.7)	(0.1,0.1,0.8)
Headache	(0.3,0.2,0.4)	(0.2,0.2,.0.5)	(0.6,0.1,0.2)	(0.2,0.4,0.3)	(0.05,0.2,0.7)
Stomach pain	(0.8,0.1,0.1)	(0.01,0.9,0.05)	(0.2,0.1,0.5)	(0.7,0.2,0.1)	(0.2,0.1,0.6)
Cough	(0.45,0.3,0.1)	(0.7,0.2,0.1)	(0.2,0.2,0.5)	(0.2,0.1,0.65)	(0.2,0.1,0.6)
Chest pain	(0.1,0.6,0.2)	(0.1,0.1,0.8)	(0.1,0.05,0.8)	(0.2,0.1,0.6)	(0.8,0.1,0.1)

Table 2
Symptoms characteristics for the patients

	Temperature	Headache	Stomach pain	Cough	Chest pain
Al	(0.7,0.1,0.15)	(0.6,0.3,0.05)	(0.25,0.45,0.25)	(0.2,0.25,0.5)	(0.1,0.2,0.6)
Bob	(0.2,0.3,0.45)	(0.05,0.5,0.4)	(0.6,0.15,0.25)	(0.25,0.4,0.35)	(0.02,0.25,0.65)
Joe	(0.75,0.05,0.05)	(0.02,0.85,0.1)	(0.3,0.2,0.4)	(0.7,0.25,0.05)	(0.25,0.4,0.4)
Ted	(0.4,0.2,0.3)	(0.7,0.2,0.1)	(0.2,0.2,0.5)	(0.2,0.1,0.65)	(0.1,0.5,0.25)

Table 3
Diagnosis results for the divergence measure using Equation (2)

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	0.223922	0.206541	**0.102753**	0.188912	0.282006
Bob	0.147244	0.315669	0.166453	**0.115887**	0.255222
Joe	**0.224068**	0.260287	0.310353	0.365106	0.380268
Ted	0.197407	0.350241	**0.099601**	0.282984	0.165600

Table 4
Diagnosis results for the divergence measure using Equation (9)

	Viral fever	Malaria	Typhoid	Stomach Problem	Chest Problem
Al	0.3231	0.3159	**0.1456**	0.2746	0.3387
Bob	0.1858	0.5115	0.1998	**0.1312**	0.1806
Joe	**0.3337**	0.4017	0.4980	0.5257	0.3938
Ted	0.2850	0.5302	**0.1477**	0.2871	0.1990

When using the dissimilarity measure we compute all the dissimilarity measure between each patient and each diagnosis. After that, we choose the smallest value of them. This measure also gives us the best diagnosis for each patient (Table 6). The results also show that the optimization will be Al (Typhoid), Bob (Stomach Problem), Joe (Viral fever), and Ted (Typhoid).

+Using the similarity measure of picture fuzzy sets of Weiin [45].

$$C(A, B) = \frac{1}{n} \sum_{i=1}^{n} C^i(A, B) \qquad (16)$$

where

$$C^i(A, B) = \frac{(\mu_A(u_i)\mu_B(u_i) + \eta_A(u_i)\eta_B(u_i) + \gamma_A(u_i)\gamma_B(u_i))}{sqrt(\mu_A^2(u_i) + \eta_A^2(u_i) + \gamma_A^2(u_i)) \times sqrt(\mu_B^2(u_i) + \eta_B^2(u_i) + \gamma_B^2(u_i))}$$

for all $u_i \in U$ and $i = 1, 2, ..., n$.

Table 5
Diagnosis results for the dissimilarity measure using Equation (14)

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	0.2500	0.2110	**0.1350**	0.1875	0.3750
Bob	0.2015	0.2320	0.1785	**0.1340**	0.3090
Joe	**0.1810**	0.2010	0.2590	0.2885	0.3785
Ted	0.2375	0.3545	**0.1375**	0.2050	0.2250

Table 6
Diagnosis results for the dissimilarity measure using Equation (15)

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	0.2550	0.2420	**0.1575**	0.1675	0.4025
Bob	0.2065	0.2080	0.1816	**0.1440**	0.3490
Joe	**0.2085**	0.2705	0.2510	0.3210	0.4160
Ted	0.2675	0.4145	**0.1850**	0.2150	0.2450

Table 7
Diagnosis results for the similarity measure using Equation (16)

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Al	0.6736	0.7876	**0.8627**	0.7384	0.5212
Bob	0.8052	0.6482	0.7648	**0.9162**	0.6792
Joe	**0.7875**	0.7725	0.6257	0.6322	0.6193
Ted	0.8088	0.6379	**0.8693**	0.7029	0.8559

When using the similarity measure, we compute all the similarity measures between each patient and each diagnosis. After that, we choose the largest value of them. This measure also gives us the best

Table 8
The most suitable relationship between patients and diagnosis when using some other measures (T: Typhoid, S: Stomach Problem, V: Viral fever)

	Proposed method in Equation (2)	Proposed method in Equation (9)	Method in [13]	Method in [24]	Method in [48]
Al	T	T	T	T	T
Bob	S	S	S	S	S
Joe	V	V	V	V	V
Ted	T	T	T	T	T

diagnosis for each patient (Table 7). The results also show that the optimization will be Al (Typhoid), Bob (Stomach Problem), Joe (Viral fever), and Ted (Typhoid).

The combined results of using different methods for solving the above problem are shown in Table 8. It gives us the most relevant results of the relationship between the patient and the diagnosis when using different measures. From Table 8, we find that the results of using the proposed method are also consistent with the results when using other existing measures on the picture fuzzy sets.

4.2. In the classification problem

Assume that, we have m pattern $\{A_1, A_2, ..., A_m\}$, in which each pattern is a picture fuzzy set on universal set $U = \{u_1, u_2, ..., u_n\}$. Suppose that, we have a sample B with the given feature information. Our goal is to classify sample B into which sample. To solve this, we calculate the divergence measure of B with each pattern $A_i (i = 1, 2, ..., m)$. Then we choose the smallest value. It gives us the class that B belongs to.

Input:

- m pattern $\{A_1, A_2, ..., A_m\}$, in which each pattern is a picture fuzzy set on universal set $U = \{u_1, u_2, ..., u_n\}$.
- Sample B is a picture fuzzy set on universal set $U = \{u_1, u_2, ..., u_n\}$.

Output: What classification is B belong to?

Algorithm:

- **Step 1**. We compute the divergence measures $D(A_i, B), i = 1, 2, ..., m$ by using Equations (2) or (9).
- **Step 2**. Put B belongs to the class of A_{i^*} in which $D(A_{i^*}, B) = \min\{D(A_i, B)|i = 1, 2, ..., m\}$.

Example 3. Assume that there are three picture fuzzy patterns in $U = \{u_1, u_2, u_3\}$ as following

$$A_1 = \{(\mathrm{u}_1, 0.4, 0.4, 0.1), (\mathrm{u}_2, 0.7, 0.15, 0.1), (\mathrm{u}_3, 0.3, 0.3, 0.2)\}$$

$$A_2 = \{(\mathrm{u}_1, 0.5, 0.3, 0.1), (\mathrm{u}_2, 0.7, 0.2, 0.05), (\mathrm{u}_3, 0.5, 0.3, 0.1)\}$$

$$A_3 = \{(\mathrm{u}_1, 0.4, 0.5, 0.1), (\mathrm{u}_2, 0.7, 0.1, 0.1), (\mathrm{u}_3, 0.4, 0.3, 0.2)\}$$

Assume that a sample

$$B = \{(\mathrm{u}_1, 0.1, 0.1, 0.4), (\mathrm{u}_2, 0.8, 0.05, 0.05), (\mathrm{u}_3, 0.05, 0.8, 0.05)\}$$

Using the divergence measure in Equation (2) we have $D(A_1, B) = 0.11845$, $D(A_2, B) = 0.13717$, $D(A_3, B) = 0.13593$. So that we can classifies that B belongs to the class of A_1.

If using the divergence measure in Equation (9) then we have $\overline{D}(A_1, B) = 0.51037$, $\overline{D}(A_2, B) = 0.61238$, $\overline{D}(A_3, B) = 0.59688$. So that we can also classifies that B belongs to the class of A_1.

- Compare with using other measures

+Using the dissimilarity measure of picture fuzzy sets in [13] or [24].

Step 1. We compute the dissimilarity measures $DM(A_i, B)$, $i = 1, 2, ..., m$ by using Equations (14) or (15).

Step 2. Put B belongs to the class of A_{i^*} in which $DM(A_{i^*}, B) = \min\{DM(A_i, B)|i = 1, 2, ..., m\}$.

Using the dissimilarity measure in Equation (14) we have $DM_1(A_1, B) = 0.1792$, $DM_2(A_2, B) = 0.2$, $DM_1(A_3, B) = 0.1917$. So that we can classifies that B belongs to the class of A_1.

If using the dissimilarity measure in Equation (15) then we have $DM_T(A_1, B) = 0.2208$, $DM_T(A_2, B) = 0.2375$, $DM_T(A_3, B) = 0.2360$. So that we can also classifies that B belongs to the class of A_1.

+Using the similarity measure of picture fuzzy sets in [48].

Step 1. We compute the similarity measures $C(A_i, B), i = 1, 2, ..., m$ by using Equation (16).

Step 2. Put B belongs to the class of A_{i^*} in which $C(A_{i^*}, B) = m\text{ax}\{C(A_i, B)|i = 1, 2, ..., m\}$.

If using the dissimilarity measure in Equation (16) then we have $C(A_1, B) = 0.7273$, $C(A_2, B) = 0.6744$, $C(A_3, B) = 0.6970$. So that we can also classifies that B belongs to the class of A_1.

We find that, when using the new measures, the classification results also give the same results as the previous measures.

5. Conclusions

There are many theoretical and applied results on picture fuzzy sets that are built and developed. In this paper, we study the divergence measure of picture fuzzy sets. Along with that, we offer some divergence formulas on picture fuzzy sets and give some properties of these measures. Finally, we apply the proposed measures in some cases. We also compared the results using the proposed new measure with other measures. The results obtained using the proposed new measure also yield the same results with some of the measures proposed on the picture fuzzy set. A divergence measure plays an important part in distinguishing two probability distributions and making conclusions based on that discrimination. This idea should also be generalized to the picture fuzzy sets to distinguish two picture fuzzy sets and to draw conclusions based on that discrimination. In the future, we will continue to study this measure and offer some of their applications in other areas such as image segmentation, clustering, the multi-criteria decision making problems or studying the relationship of this measure with other types of measures, such as [31, 35–37, 49], on the picture fuzzy sets andapply to the practical problems.

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