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On Generalized Fibonacci Lacunary Statistical Convergence of Double Sequences in Neutrosophic Normed Spaces

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Abstract: We investigate the concept for Fibonacci Lacunary statistical convergence of double sequences in neutrosophic normed spaces. We introduce Fibonacci Lacunary statistically Cauchy double sequences and establish some inclusion relations. Also, Fibonacci Lacunary statistical completeness and show that every neutrosophic normed space is Fibonacci Lacunary statistically complete are introduced.

Keywords: Lacunary sequence, Double sequence, Fibonacci Lacunary statistical convergence, Neutrosophic normed space.

AMS Subject Classification (2020): 40A30; 40G15

1. Introduction

The concept of 2-normed spaces was introduced and studied by Gähler [3]. This notion which is nothing but a two-dimensional analogue of a normed space got the attention of wider audience after the publication of a paper by Albert George, White Jr. [2] of USA in1969 entitled 2-Banach spaces.

Fuzziness has revolutionized many areas such as mathematics, science, engineering, medicine. This concept was given by Zadeh [14]. The concept of fuzziness are using by many researchers for cybernetics, Artificial Intelligence, Expert system and Fuzzy control, pattern recognition, Operation research, Decision making, Image analysis, Projectiles, Probability theory, Agriculture, weather forecasting. Recently, the fuzzy logic became an important area of research in several branches of mathematics like metric and topological spaces, theory of function etc.

Intuitionistic fuzzy set was examined by Atanassov [1] is appropriate for such situation. The notion of intuitionistic fuzzy metric space has been introduced by Park [10]. furthermore, the concept of intuitionistic fuzzy normed space is given by Saadati and Park [11].

The idea of neutrosophic sets was introduced by Smarandache [12] as an extension of the intuitionistic fuzzy set. For the situation when the aggregate of the components is 1, in the wake of satisfying the condition by applying the neutrosophic set operators, different outcomes can be acquired by applying the intuitionistic fuzzy operators, since the operators disregard the indeterminacy, while the neutrosophic operators are taken into the cognizance of the indeterminacy at a similar level as truth-membership and falsehood-nonmembership. Using the idea of neutrosophic sets, the notion of neutrosophic bipolar vague soft set and its application to decision making problems were defined. Further, Smarandache [12,13] investigated neutroalgebra which is a generalization of partial algebra, neutroalgebraic structures, and antialgebraic structures. Neutrosophic set is a more adaptable and effective tool because it handles, aside from autonomous components, additionally partially independent and dependent information. In 2019 Kirisci et al. [9] defined Neutrosophic Metric Space (NM-Space) as a generalization of Intuitionistic Fuzzy Metric Space and discussed some fixed point results in complete NM-space. In 2022, Jeyaraman, Ramachandran and Shakila [6] proved Approximate Fixed Point Theorems for Weak Contractions on neutrosophic normed spaces

The Fibonacci sequence was firstly used in the theory of sequence spaces by Kara and Başarir [7]. Afterward, Kara [8] defined the Fibonacci difference matrix $\hat{F}$ by using the Fibonacci sequence (f_n) for $n \in \{1,2,3,\dots\}$ and introduced the new sequence spaces related to the matrix domain of $\hat{F}$.

The present paper is to introduce and investigate the Fibonacci Lacunary Statistical Convergence of double sequence and Fibonacci Lacunary Statistically Cauchy double sequence on Neutrosophic Normed Space. We have developed all our results based on this new definition.

2. Preliminaries

Definition 2.1:

A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

i. $*$ is commutative and associative,
ii. $*$ is continuous,
iii. $p * 1 = p$,
iv. If $p \le r$ and $q \le s$, then $p * q \le r * s$, for all $p, q, r, s \in [0,1]$.

Definition 2.2:

A binary operation $\Diamond: [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if $\Diamond$ satisfies the following conditions:

i. $\Diamond$ is commutative and associative,
ii. $\Diamond$ is continuous,
iii. $p \Diamond 0 = p$ for all $p \in [0,1]$,
iv. If $p \le r$ and $q \le s$, then $p \Diamond q \le r \Diamond s$, for all $p, q, r, s \in [0,1]$.

Definition 2.3:

A binary operation $\odot: [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if $\odot$ satisfies the following conditions:

i. $\odot$ is commutative and associative,
ii. $\odot$ is continuous,
iii. $p \odot 0 = p$ for all $p \in [0,1]$,
iv. If $p \le r$ and $q \le s$, then $p \odot q \le r \odot s$, for all $p, q, r, s \in [0,1]$.

Definition 2.4:

The seven tuple $(\mathrm{X}, \varphi, \omega, \psi, *, \Diamond, \odot)$ is named as Neutrosophic Normed Space (NNS) if X is a vector space, $*$ is a continuous t-norm, $\Diamond$ and $\odot$ are continuous t-conorm and φ, ω and ψ are fuzzy set on $\mathrm{X} \times (0, \infty)$ fulfilling the subsequent conditions: For every $a, b \in \mathrm{X}$ and $p, q > 0$:

i. $\varphi(a,q) + \omega(a,q) + \psi(a,q) \le 3$,
ii. $0 \le \varphi(a,q) \le 1, 0 \le \omega(a,q) \le 1, 0 \le \psi(a,q) \le 1$,
iii. $\varphi(a,q) > 0$,
iv. $\varphi(a,q) = 1$ iff $a = 0$,
v. $\varphi(ca,q) = \varphi\left(a, \frac{q}{|c|}\right)$ if $c \ne 0$,
vi. $\varphi(a,q) * \varphi(b,p) \le \varphi(a+b, q+p)$,
vii. $\varphi(a,.): (0,\infty) \to [0,1]$ is continuous in q,
viii. $\lim_{q\to\infty} \varphi(a,q) = 1$ and $\lim_{q\to 0} \varphi(a,q) = 0$,
ix. $\omega(a,q) < 1$,
x. $\omega(a,q) = 0$ iff $a = 0$,
xi. $\omega(ca,q) = \omega\left(a, \frac{q}{|c|}\right)$ if $c \ne 0$,
xii. $\omega(a,q) \Diamond \omega(b,p) \ge \omega(a+b, q+p)$,
xiii. $\omega(a,.): (0,\infty) \to [0,1]$ is continuous in q,
xiv. $\lim_{q\to\infty} \omega(a,q) = 0$ and $\lim_{q\to 0} \omega(a,q) = 1$,
xv. $\psi(a,q) < 1$,
xvi. $\psi(a,q) = 0$ iff $a = 0$,
xvii. $\psi(ca,q) = \psi\left(a, \frac{q}{|c|}\right)$ if $c \ne 0$
xviii. $\psi(a,q) \odot \psi(b,p) \ge \psi(a+b, q+p)$,
xix. $\psi(a,.): (0,\infty) \to [0,1]$ is continuous in q,
xx. $\lim_{q\to\infty} \psi(a,q) = 0$ and $\lim_{q\to 0} \psi(a,q) = 1$.

Definition 2.5

Let $(\mathrm{X}, \varphi, \omega, \psi, *, \Diamond, \odot)$ be NNS. A sequence $x = (x_k)$ is said to be convergent to $\xi \in X$ with respect to the NN (φ, ω, ψ) if for every $\varepsilon > 0$ and $t > 0$, there exists a positive integer n_0 such that $\varphi(x_k - \varepsilon, t) > 1 - \varepsilon$, $\omega(x_k - \varepsilon, t) < \varepsilon$ and $\psi(x_k - \varepsilon, t) < \varepsilon$, for all $k \ge n_0$.

In this case, we write $(\varphi, \omega, \psi) - \lim\ x = \xi$ or $x_k \xrightarrow{(\varphi,\omega,\psi)} \xi$ as $k \to \infty$.

Definition 2.6

Let $(\mathrm{X}, \varphi, \omega, \psi, *, \Diamond, \odot)$ be NNS. A sequence $x = (x_k)$ is said to be Cauchy with respect to the NN (φ, ω, ψ) if for every $\varepsilon > 0$ and $t > 0$, there exists a positive integer n_0 such that $\varphi(x_k - x_l, t) > 1-\varepsilon$, $\omega(x_k - x_l, t) < \varepsilon$ and $\psi(x_k - x_l, t) < \varepsilon$, for all $k, l \geq n_0$.

Definition 2.7

Let $(\mathrm{X}, \varphi, \omega, \psi, *, \diamondsuit, \odot)$ be NNS. A double sequence $x = (x_k)$ is said to be Fibonacci Statistical Convergent with regards to the NN (φ, ω, ψ), if for every $t > 0$ and $\varepsilon > 0$, there is $\xi \in X$ such that

$$K_\varepsilon(\hat{F}) = \left\{\begin{matrix} k \leq n: \varphi(\hat{F}x_k - \xi, t) \leq 1-\varepsilon \text{ or} \\ \omega(\hat{F}x_k - \xi, t) \geq \varepsilon \\ \text{and } \psi(\hat{F}x_k - \xi, t) \geq \varepsilon \end{matrix}\right\}$$

has natural density zero. i.e.,$d\left(K_\varepsilon(\hat{F})\right) = 0$. That is,

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{\begin{matrix} k \leq n: \varphi(\hat{F}x_k - \xi, t) \leq 1-\varepsilon \\ or\ \omega(\hat{F}x_k - \xi, t) \geq \varepsilon \\ \text{and } \psi(\hat{F}x_k - \xi, t) \geq \varepsilon \end{matrix}\right\}\right| = 0.$$

In this case, we write $d(\hat{F})_{NN} - \lim x_k = \xi$ or $x_k \to \xi\left(S(\hat{F})_{NN}\right)$

3. Main Results

In this section, we introduce the concept of Fibonacci Lacunary Statistically Convergent [$\check{\mathcal{F}}\mathcal{LSC}$] double sequence in NNS.

Definition 3.1

Let $(\mathrm{X}, \varphi, \omega, \psi, *, \diamondsuit, \odot)$ be NNS and θ be a Lacunary Sequence ($\hat{L}$S). Then, a sequence $x = (x_{kl})$ is said to be $\check{\mathcal{F}}\mathcal{LS}$- Convergent to $\xi \in X$ with regards to the NN (φ, ω, ψ), if for every $\varepsilon > 0$ and $t > 0$,

$$\delta_\theta\left(\left\{\begin{matrix} (k,l) \in \mathbb{N}\times\mathbb{N}: \varphi(\hat{F}x_{kl} - \xi, t) \leq 1-\varepsilon \\ \text{or } \omega(\hat{F}x_{kl} - \xi, t) \geq \varepsilon \\ and\ \psi(\hat{F}x_{kl} - \xi, t) \geq \varepsilon \end{matrix}\right\}\right) = 0 \tag{3.1.1}$$

or equivalently

$$\delta_\theta\left(\left\{\begin{matrix} (k,l) \in \mathbb{N}\times\mathbb{N}: \varphi(\hat{F}x_{kl} - \xi, t) > 1-\varepsilon \\ \text{or } \omega(\hat{F}x_{kl} - \xi, t) < \varepsilon \\ and\ \psi(\hat{F}x_{kl} - \xi, t) < \varepsilon \end{matrix}\right\}\right) = 1 \tag{3.1.2}$$

In this case, we write $S_\theta^{(\varphi,\omega,\psi)}(\hat{F}) - lim\, x = \xi$ or $x_{kl} \xrightarrow{(\varphi,\omega,\psi)} \xi\left(S_\theta(\hat{F})\right)$, where ξ is said to be $S_\theta^{(\varphi,\omega,\psi)}(\hat{F}) - lim\, x$ and we denote the set of all Fibonacci S_θ-convergent sequences with regards to NN (φ, ω, ψ) by $S_\theta^{(\varphi,\omega,\psi)}(\hat{F})$.

By using (3.1.1) and (3.1.2), we easily get the following lemma.

Lemma 3.2

Let $(\mathrm{X}, \varphi, \omega, \psi, *, \diamondsuit, \odot)$ be NNS and θ be a $\hat{L}$S. For every $\varepsilon > 0$ and $t > 0$, the following statements are equivalent:

a) $S_\theta^{(\varphi,\omega,\psi)} - lim\, x = \xi$;

b) $\delta_\theta(\{(k,l) \in \mathbb{N}\times\mathbb{N}: \varphi(\hat{F}x_{kl} - \xi, t) \leq 1-\varepsilon\})$
$= \delta_\theta(\{(k,l) \in \mathbb{N}\times\mathbb{N}: \omega(\hat{F}x_{kl} - \xi, t) \geq \varepsilon\})$
$= \delta_\theta(\{(k,l) \in \mathbb{N}\times\mathbb{N}: \psi(\hat{F}x_{kl} - \xi, t) \geq \varepsilon\}) = 0$;

c) $\delta_\theta\left(\left\{\begin{matrix} (k,l) \in \mathbb{N}\times\mathbb{N}: \varphi(\hat{F}x_{kl} - \xi, t) > 1-\varepsilon, \\ \omega(\hat{F}x_{kl} - \xi, t) < \varepsilon \\ \text{and } \psi(\hat{F}x_{kl} - \xi, t) < \varepsilon \end{matrix}\right\}\right)$

d) $\delta_0(\{(k,l) \in \mathbb{N}\times\mathbb{N}: \varphi(\hat{F}x_{kl} - \xi, t) > 1-\varepsilon\})$
$= \delta_0(\{(k,l) \in \mathbb{N}\times\mathbb{N}: \omega(\hat{F}x_{kl} - \xi, t) < \varepsilon\})$
$= \delta_0(\{(k,l) \in \mathbb{N}\times\mathbb{N}: \psi(\hat{F}x_{kl} - \xi, t) < \varepsilon\}) = 1$;

e) $S_\theta(\hat{F}) - lim\, \varphi(\hat{F}x_{kl} - \xi, t) = 1$,
$S_\theta(\hat{F}) - lim\, \omega(\hat{F}x_{kl} - \xi, t) = 0$ and
$S_\theta(\hat{F}) - lim\, \psi(\hat{F}x_{kl} - \xi, t) = 0$.

Theorem 3.3

Let $(\mathrm{X}, \varphi, \omega, \psi, *, \diamondsuit, \odot)$ be NNS and θ be a $\hat{L}$S. If a sequence $x = (x_{kl})$ is $\check{\mathcal{F}}\mathcal{LS}$ − Convergent to $\xi \in X$ with regards to the NN (φ, ω, ψ), then $S_\theta^{(\varphi,\omega,\psi)}(\hat{F})$ − limit is unique.

Proof:

Assume that there exist two distinct elements $\xi_1, \xi_2 \in X$ such that $S_\theta^{(\varphi,\omega,\psi)}(\hat{F}) - lim\, x_{kl} = \xi_1$ and $S_\theta^{(\varphi,\omega,\psi)}(\hat{F}) - lim\, x_{kl} = \xi_2$. Given $\varepsilon > 0$, choose $\gamma > 0$ such that
$(1-\gamma) * (1-\gamma) > 1-\varepsilon$, $\gamma \diamondsuit \gamma < \varepsilon$ and
$\gamma \odot \gamma < \varepsilon$.

Hence, for any $t > 0$. Define the following sets as:

$K_{\varphi,1}(\gamma, t) = \left\{(k,l) \in \mathbb{N}\times\mathbb{N}: \varphi\left(\hat{F}x_{kl} - \xi_1, \frac{t}{2}\right) \leq 1-\gamma\right\}$,
$K_{\varphi,2}(\gamma, t) = \left\{(k,l) \in \mathbb{N}\times\mathbb{N}: \varphi\left(\hat{F}x_{kl} - \xi_2, \frac{t}{2}\right) \leq 1-\gamma\right\}$,
$K_{\omega,1}(\gamma, t) = \left\{(k,l) \in \mathbb{N}\times\mathbb{N}: \omega\left(\hat{F}x_{kl} - \xi_2, \frac{t}{2}\right) \geq \gamma\right\}$,
$K_{\omega,2}(\gamma, t) = \left\{(k,l) \in \mathbb{N}\times\mathbb{N}: \omega\left(\hat{F}x_{kl} - \xi_2, \frac{t}{2}\right) \geq \gamma\right\}$,
$K_{\psi,1}(\gamma, t) = \left\{(k,l) \in \mathbb{N}\times\mathbb{N}: \psi\left(\hat{F}x_{kl} - \xi_1, \frac{t}{2}\right) \geq \gamma\right\}$,
$K_{\psi,2}(\gamma, t) = \left\{(k,l) \in \mathbb{N}\times\mathbb{N}: \psi\left(\hat{F}x_{kl} - \xi_2, \frac{t}{2}\right) \geq \gamma\right\}$.

Since $S_\theta^{(\varphi,\omega,\psi)}(\hat{F}) - lim\, x_{kl} = \xi_1$, we have by Lemma (3.2)

$$\delta_\theta\left(K_{\varphi,1}(\gamma,t)\right) = \delta_\theta\left(K_{\omega,1}(\gamma,t)\right) = \delta_\theta\left(K_{\psi,1}(\gamma,t)\right) = 0, \text{ for all } t > 0.$$

Furthermore, using $S_\theta^{(\varphi,\omega,\psi)}(\hat{F}) - lim\, x_{kl} = \xi_2$, we get

$$\delta_\theta\left(K_{\varphi,2}(\gamma,t)\right) = \delta_\theta\left(K_{\omega,2}(\gamma,t)\right) = \delta_\theta\left(K_{\psi,2}(\gamma,t)\right) = 0, \text{ for all } t > 0.$$

Now, let $K_{\varphi,\omega,\psi}(\gamma,t) = \left(K_{\varphi,1}(\gamma,t) \cup K_{\varphi,2}(\gamma,t)\right) \cap \left(K_{\omega,1}(\gamma,t) \cup K_{\omega,2}(\gamma,t)\right) \cap \left(K_{\psi,1}(\gamma,t) \cup K_{\psi,2}(\gamma,t)\right)$.

Then observe that $\delta_\theta\left(K_{\varphi,\omega,\psi}(\gamma,t)\right)=0$ which implies $\delta_\theta\left(\mathbb{N}\times\mathbb{N}\backslash K_{\varphi,\omega,\psi}(\gamma,t)\right)=1$.

If $\in\mathbb{N}\times\mathbb{N}\backslash K_{\varphi,\omega,\psi}(\gamma,t)$, then, we have three possible cases,

Case (i) $(k,l)\in\mathbb{N}\times\mathbb{N}\backslash\left(K_{\varphi,1}(\gamma,t)\cup K_{\varphi,2}(\gamma,t)\right)$,

Case (ii) $(k,l)\in\mathbb{N}\times\mathbb{N}\backslash\left(K_{\omega,1}(\gamma,t)\cup K_{\omega,2}(\gamma,t)\right)$ and

Case (iii) $(k,l)\in\mathbb{N}\times\mathbb{N}\backslash\left(K_{\psi,1}(\gamma,t)\cup K_{\psi,2}(\gamma,t)\right)$.

We first consider that $(k,l)\in\mathbb{N}\times\mathbb{N}\backslash\left(K_{\varphi,1}(\gamma,t)\cup K_{\varphi,2}(\gamma,t)\right)$. Then, we have

$$\varphi(\xi_1-\xi_2,t)\geq\ \varphi\left(\hat{F}x_{kl}-\xi_1,\tfrac{t}{2}\right)*\varphi\left(\hat{F}x_{kl}-\xi_2,\tfrac{t}{2}\right)$$
$$>(1-\gamma)*(1-\gamma)$$
$$>1-\varepsilon.$$

Similarly, if $(k,l)\in\mathbb{N}\times\mathbb{N}\backslash\left(K_{\omega,1}(\gamma,t)\cup K_{\omega,2}(\gamma,t)\right)$. Then, we may write

$$\omega(\xi_1-\xi_2,t)\leq\ \omega\left(\hat{F}x_{kl}-\xi_1,\frac{t}{2}\right)\Diamond\omega\left(\hat{F}x_{kl}-\xi_2,\frac{t}{2}\right)$$
$$<\ \gamma\Diamond\gamma<\varepsilon.$$

and if $(k,l)\in\mathbb{N}\times\mathbb{N}\backslash\left(K_{\psi,1}(\gamma,t)\cup K_{\psi,2}(\gamma,t)\right)$

$$\psi(\xi_1-\xi_2,t)\leq\ \psi\left(\hat{F}x_{kl}-\xi_1,\frac{t}{2}\right)\odot\psi\left(\hat{F}x_{kl}-\xi_2,\frac{t}{2}\right)$$
$$<\gamma\odot\gamma<\varepsilon.$$

Now, using the fact that $(1-\gamma)*(1-\gamma)>\ 1-\varepsilon$, $\gamma\Diamond\gamma<\varepsilon$ and $\gamma\odot\gamma<\varepsilon$, we se that $\varphi(\xi_1-\xi_2,t)>1-\varepsilon$, $\omega(\xi_1-\xi_2,t)<\varepsilon$ and $\psi(\xi_1-\xi_2,t)<\varepsilon$. Since arbitrary $\varepsilon>0$, we get $\varphi(\xi_1-\xi_2,t)=\ 1$, $\omega(\xi_1-\xi_2,t)=\ 0$ and $\psi(\xi_1-\xi_2,t)=\ 0$, for each $t>0$,which gives that $\xi_1=\xi_2$. So, we conclude that $S_\theta^{(\varphi,\omega,\psi)}(\hat{F})-$ limit is unique.

Definition 3.4

A sequence $x=(x_{kl})$ is said to Fibonacci convergent to $\xi\in X$ with respect to NN (φ,ω,ψ), if for every $\varepsilon>0$ and $t>0$, there exists a positive integer k_0,l_0 such that $\varphi\left(\hat{F}x_{kl}-\xi,t\right)>1-\varepsilon$, $\omega\left(\hat{F}x_{kl}-\xi,t\right)<\varepsilon$ and $\psi\left(\hat{F}x_{kl}-\xi,t\right)<\varepsilon$, for all $k\geq k_0$, $l\geq l_0$.

In this case, we write $(\varphi,\omega,\psi)_{\hat{F}}-lim\,x_{kl}=\xi$ or $x_{kl}\xrightarrow{(\varphi,\omega,\psi)}\xi\left(\hat{F}\right)$ as $k,l\rightarrow\infty$.

Theorem 3.5

Let $(\mathrm{X},\varphi,\omega,\psi,*,\Diamond,\odot)$ be NNS and θ be a $\hat{L}$S. If $(\varphi,\omega,\psi)_{(\hat{F})}-lim\,x=\xi$, then $S_\theta^{(\varphi,\omega,\psi)}(\hat{F})-lim\,x=\xi$.

Proof:

Let $(\varphi,\omega,\psi)_{(\hat{F})}-lim\,x=\xi$. Then, for every $\varepsilon>0$ and $t>0$, there exists a positive integer k_0,l_0 such that $\varphi\left(\hat{F}x_{kl}-\xi,t\right)>1-\varepsilon$, $\omega\left(\hat{F}x_{kl}-\xi,t\right)<\varepsilon$ and $\psi\left(\hat{F}x_{kl}-\xi,t\right)<\varepsilon$, for all $k\geq k_0$, $l\geq l_0$.

Hence, the set

$$\left\{\begin{matrix}(k,l)\in\mathbb{N}\times\mathbb{N}:\varphi\left(\hat{F}x_{kl}-\xi,t\right)\leq 1-\epsilon\\ \text{or }\omega\left(\hat{F}x_{kl}-\xi,t\right)\geq\epsilon\\ \text{and }\psi\left(\hat{F}x_{kl}-\xi,t\right)\geq\epsilon\end{matrix}\right\}$$

has finite number of terms. Since every finite subset of $\mathbb{N}\times\mathbb{N}$ has density zero and hence

$$\delta_\theta\left(\left\{\begin{matrix}(k,l)\in\mathbb{N}\times\mathbb{N}:\varphi\left(\hat{F}x_{kl}-\xi,t\right)\leq 1-\epsilon\\ \text{or }\omega\left(\hat{F}x_{kl}-\xi,t\right)\geq\epsilon\text{ and}\\ \psi\left(\hat{F}x_{kl}-\xi,t\right)\geq\epsilon\end{matrix}\right\}\right)=0.$$

That is, $S_\theta^{(\varphi,\omega,\psi)}(\hat{F})-lim\,x=\xi$.

Theorem 3.6

Let $(\mathrm{X},\varphi,\omega,\psi,*,\Diamond,\odot)$ be NNS and θ be a $\hat{L}$S. Then, for any $\hat{L}$S θ, $S_\theta^{(\varphi,\omega,\psi)}(\hat{F})-lim\,x=\xi$ if and only if there exists a subset $k=\{(k,l)\}\subseteq\mathbb{N}\times\mathbb{N}$, $k,l=1,2,\dots$ such that $\delta_\theta(k)=1$ and $(\varphi,\omega,\psi)_{(\hat{F})}-\lim_{k,l\rightarrow\infty}x=\xi$.

Proof:

Necessity, assume that $S_\theta^{(\varphi,\omega,\psi)}(\hat{F})-lim\,x=\xi$.

Let for any $t>0$ and $s=1,2,\dots$

$$T_{(\varphi,\omega,\psi)}(s,t)=\left\{\begin{matrix}(k,l)\in\mathbb{N}\times\mathbb{N}:\varphi\left(\hat{F}x_{kl}-\xi,t\right)>1-\frac{1}{s},\\ \omega\left(\hat{F}x_{kl}-\xi,t\right)<\frac{1}{s}\text{ and}\\ \psi\left(\hat{F}x_{kl}-\xi,t\right)<\frac{1}{s}\end{matrix}\right\}$$

and

$$R_{(\varphi,\omega,\psi)}(s,t)=\left\{\begin{matrix}(k,l)\in\mathbb{N}\times\mathbb{N}:\varphi\left(\hat{F}x_{kl}-\xi,t\right)\leq 1-\frac{1}{s},\\ \omega\left(\hat{F}x_{kl}-\xi,t\right)\geq\frac{1}{s}\text{ and}\\ \psi\left(\hat{F}x_{kl}-\xi,t\right)\geq\frac{1}{s}\end{matrix}\right\}.$$

Then $\delta_\theta\left(R_{(\varphi,\omega,\psi)}(s,t)\right)=0$.

Since $S_\theta^{(\varphi,\omega,\psi)}(\hat{F})-lim\,x=\xi$.

Also, $T_{(\varphi,\omega,\psi)}(s,t)\supset\ T_{(\varphi,\omega,\psi)}(s+1,t)$ (3.6.3)

and

$$\delta_\theta\left(T_{(\varphi,\omega,\psi)}(s,t)\right)=1,\ \text{for } t>0\text{ and } s=1,2,\dots \qquad (3.6.4)$$

Now, we have to show that for $(k,l)\in T_{(\varphi,\omega,\psi)}(s,t)$, $x_{kl}\xrightarrow{(\varphi,\omega,\psi)}\xi\left(\hat{F}\right)$.

Suppose that for some $(k,l)\in T_{(\varphi,\omega,\psi)}(s,t)$, $x_{kl}\nrightarrow\xi\left(\hat{F}\right)$.

Therefore, there is $\sigma>0$ and a positive integer k_0,l_0 such that $\varphi\left(\hat{F}x_{kl}-\xi,t\right)\leq 1-\sigma$,

$\omega(\hat{F}x_{kl}-\xi,t)\geq\sigma$ and $\psi(\hat{F}x_{kl}-\xi,t)\geq\sigma$, for all $k\geq k_0$, $l\geq l_0$.
Let $\varphi(\hat{F}x_{kl}-\xi,t)>1-\sigma$, $\omega(\hat{F}x_{kl}-\xi,t)<\sigma$ and $\psi(\hat{F}x_{kl}-\xi,t)<\sigma$, for all $k\geq k_0$, $l\geq l_0$.
Then, we get

$$\delta_\theta\left(\left\{\begin{matrix}(k,l)\in\mathbb{N}\times\mathbb{N}:\varphi(\hat{F}x_{kl}-\xi,t)>1-\sigma\\ \text{or }\omega(\hat{F}x_{kl}-\xi,t)<\sigma\text{ and}\\ \psi(\hat{F}x_{kl}-\xi,t)<\sigma\end{matrix}\right\}\right)=0.$$

Since $\sigma>\frac{1}{s}$, we have $\delta_\theta\left(T_{(\varphi,\omega,\psi)}(s,t)\right)=0$, which contradicts (3.6.4).
Therefore $x_{kl}\xrightarrow{(\varphi,\omega,\psi)}\xi(\hat{F})$.
Sufficiency, suppose that there exists a subset $k=\{(k,l)\}\in\mathbb{N}\times\mathbb{N}$ such that $\delta_0(k)=1$ and $(\varphi,\omega,\psi)_{(\hat{F})}-\lim_{n\to\infty}x_{kl}=\xi$, i.e., there exists $N\in\mathbb{N}$ such that for every $\sigma>0$ and $t>0$,
$\varphi(\hat{F}x_{kl}-\xi,t)>1-\sigma$,
$\omega(\hat{F}x_{kl}-\xi,t)<\sigma$ and $\psi(\hat{F}x_{kl}-\xi,t)<\sigma$.
Now,

$$R_{(\varphi,\omega,\psi)}(\sigma,t)=\left\{\begin{matrix}(k,l)\in\mathbb{N}\times\mathbb{N}:\varphi(\hat{F}x_{kl}-\xi,t)\leq1-\sigma\\ \text{or }\omega(\hat{F}x_{kl}-\xi,t)\geq\sigma\\ \text{and }\psi(\hat{F}x_{kl}-\xi,t)\geq\sigma\end{matrix}\right\}$$

$$\subseteq\mathbb{N}\times\mathbb{N}-\{(k_{N+1},l_{N+1}),(k_{N+2},l_{N+2}),\dots\}.$$

Therefore $\delta_\theta R_{(\varphi,\omega,\psi)}(\sigma,t)\leq1-1=0$.
Hence $S_\theta^{(\varphi,\omega,\psi)}(\hat{F})-\lim x=\xi$.
This completes the proof.
Now, we define $\check{\mathcal{F}}\mathcal{LS}-$ Cauchy sequences with respect to a NNS and introduce a new concept of $\check{\mathcal{F}}\mathcal{LS}-$ Completeness.

Definition 3.7

Let $(X,\varphi,\omega,\psi,*,\Diamond,\odot)$ be NNS and θ be a $\hat{L}$S. Then, a sequence $x=(x_{kl})$ is said to be Fibonacci Lacunary Statistically- Cauchy $[\hat{F}S_\theta-\text{Cauchy}]$ with respect to the NN (φ,ω,ψ) if for every $\varepsilon>0$ and $t>0$, there exists $N=N(\varepsilon)$ and $M=M(\varepsilon)$ such that

$$\delta_\theta\left(\left\{\begin{matrix}(k,l)\in\mathbb{N}\times\mathbb{N}:\varphi(\hat{F}x_{kl}-\hat{F}x_{mn},t)\leq1-\epsilon\\ \text{or }\omega(\hat{F}x_{kl}-\hat{F}x_{mn},t)\geq\epsilon\\ \text{and }\psi(\hat{F}x_{kl}-\hat{F}x_{mn},t)\geq\epsilon\end{matrix}\right\}\right)=0.$$

Theorem 3.8

Let $(X,\varphi,\omega,\psi,*,\Diamond,\odot)$ be NNS and θ be a LS. A sequence $x=(x_{kl})$ is $\check{\mathcal{F}}\mathcal{LS}$ - Convergent if and only if it is $\check{\mathcal{F}}\mathcal{LS}$ - Cauchy with respect to $\text{NN}(\varphi,\omega,\psi)$.

Proof:

Let $x_{kl}\to\xi\left(S_\theta^{(\varphi,\omega,\psi)}(\hat{F})\right)$. Then, for any $t>0$, we have

$$\delta_\theta\left(\left\{\begin{matrix}(k,l)\in\mathbb{N}\times\mathbb{N}:\varphi\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)\leq1-\varepsilon\\ \text{or }\omega\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)\geq\varepsilon\\ \text{and }\psi\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)\geq\varepsilon\end{matrix}\right\}\right)=0.$$

In particular, for $k=N$, $l=M$

$$\delta_\theta\left(\left\{\begin{matrix}(k,l)\in\mathbb{N}\times\mathbb{N}:\varphi\left(\hat{F}x_{NM}-\xi,\frac{t}{2}\right)\leq1-\varepsilon\\ \text{or }\omega\left(\hat{F}x_{MN}-\xi,\frac{t}{2}\right)\geq\varepsilon\\ \text{and }\psi\left(\hat{F}x_{NM}-\xi,\frac{t}{2}\right)\geq\varepsilon\end{matrix}\right\}\right)=0.$$

Since

$$\varphi(\hat{F}x_{kl}-\hat{F}x_{NM},t)=\varphi\left(\hat{F}x_{kl}-\xi-\hat{F}x_{NM}+\xi,\frac{t}{2}+\frac{t}{2}\right)\geq\varphi\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)*\varphi\left(\hat{F}x_{NM}-\xi,\frac{t}{2}\right)$$

and since

$$\omega(\hat{F}x_{kl}-\hat{F}x_{NM},t)=\omega\left(\hat{F}x_{kl}-\xi-\hat{F}x_{NM}+\xi,\frac{t}{2}+\frac{t}{2}\right)\leq\omega\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)\Diamond\omega\left(\hat{F}x_{NM}-\xi,\frac{t}{2}\right)$$

and

$$\psi(\hat{F}x_{kl}-\hat{F}x_{NM},t)=\psi\left(\hat{F}x_{kl}-\xi-\hat{F}x_{NM}+\xi,\frac{t}{2}+\frac{t}{2}\right)\leq\psi\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)\odot\psi\left(\hat{F}x_{NM}-\xi,\frac{t}{2}\right).$$

We have

$$\delta_\theta\left(\left\{\begin{matrix}(k,l)\in\mathbb{N}\times\mathbb{N}:\varphi(\hat{F}x_{kl}-\hat{F}x_{MN},t)\leq1-\varepsilon\\ \text{or }\omega(\hat{F}x_{kl}-\hat{F}x_{MN},t)\geq\varepsilon\\ \text{and }\psi(\hat{F}x_{kl}-\hat{F}x_{MN},t)\geq\varepsilon\end{matrix}\right\}\right)=0.$$

That is, x is a $\check{\mathcal{F}}\mathcal{LS}$ - Cauchy with respect to $\text{NN}(\varphi,\omega,\psi)$.
Conversely, let x be a $\check{\mathcal{F}}\mathcal{LS}$ -Cauchy but not $\check{\mathcal{F}}\mathcal{LS}$ -Convergent with respect to $\text{NN}(\varphi,\omega,\psi)$.
Then, there exists N,M such that

$$\delta_\theta(A(\varepsilon,t))=0\qquad(3.8.5)$$

$$\delta_\theta(B(\varepsilon,t))=0,\text{ i.e., }\delta_\theta(B^c(\varepsilon,t))=1;\qquad(3.8.6)$$

where

$$A(\varepsilon,t)=\left\{\begin{matrix}(k,l)\in\mathbb{N}\times\mathbb{N}:\varphi(\hat{F}x_{kl}-\hat{F}x_{MN},t)\leq1-\varepsilon\\ \text{or }\omega(\hat{F}x_{kl}-\hat{F}x_{MN},t)\geq\varepsilon\\ \text{and }\psi(\hat{F}x_{kl}-\hat{F}x_{MN},t)\geq\varepsilon\end{matrix}\right\}$$

$$B(\varepsilon,t)=\left\{\begin{matrix}(k,l)\in\mathbb{N}\times\mathbb{N}:\varphi\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)>1-\frac{\varepsilon}{2}\\ \text{or }\omega\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)<\frac{\varepsilon}{2}\\ \text{and }\psi\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)<\frac{\varepsilon}{2}\end{matrix}\right\}.$$

Since $\varphi(\hat{F}x_{kl}-\hat{F}x_{MN},t)\geq2\varphi\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)>1-\varepsilon$,
$\omega(\hat{F}x_{kl}-\hat{F}x_{MN},t)\leq2\omega\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)<\varepsilon$ and
$\psi(\hat{F}x_{kl}-\hat{F}x_{MN},t)\leq2\psi\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)<\varepsilon$.
If $\varphi\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)>1-\frac{\varepsilon}{2}$,
$\omega\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)<\frac{\varepsilon}{2}$ and $\psi\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)<\frac{\varepsilon}{2}$.
Therefore

$$\delta_\theta\left(\left\{\begin{array}{c}(k,l)\in\mathbb{N}\times\mathbb{N}:\varphi\big(\hat{F}x_{kl}-\hat{F}x_{MN},t\big)>1-\varepsilon\\ \text{or }\omega\big(\hat{F}x_{kl}-\hat{F}x_{MN},t\big)<\varepsilon\\ \text{and }\psi\big(\hat{F}x_{kl}-\hat{F}x_{MN},t\big)<\varepsilon\end{array}\right\}\right)=0,$$

that is, $\delta_\theta\big(A(\varepsilon,t)\big)=1$, which contradicts (3.8.5). Since x was $\check{\mathcal{F}}\mathcal{LS}$ - Cauchy with respect to NN (φ,ω,ψ). Hence x must be $\check{\mathcal{F}}\mathcal{LS}$ - Convergent with respect to NN (φ,ω,ψ).

Definition 3.9

A NNS $(X,\varphi,\omega,\psi,*,\diamondsuit,\odot)$ is said to be $\hat{F}S_\theta$-Complete if every $\hat{F}S_\theta$- Cauchy sequence with respect to the NN(φ,ω,ψ).

Theorem 3.10

Let θ be a $\hat{L}$S. Then every NNS$(X,\varphi,\omega,\psi,*,\diamondsuit,\odot)$ is $\hat{F}S_\theta$- Complete.

Proof:

Let $x=(x_{kl})$ be $\hat{F}S_\theta$- Cauchy but not $\hat{F}S_\theta$-convergent with respect to the NN(φ,ω,ψ). For given $\varepsilon>0$, choose $\gamma>0$ such that $(1-\gamma)*(1-\gamma)>1-\varepsilon,\gamma\diamondsuit\gamma<\varepsilon$ and $\gamma\odot\gamma<\varepsilon$.

Now,

$$\varphi\big(\hat{F}x_{kl}-\hat{F}x_{MN},t\big)$$
$$\geq\varphi\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)*\varphi\left(\hat{F}x_{NM}-\xi,\frac{t}{2}\right)$$
$$>(1-\gamma)*(1-\gamma)>1-\varepsilon,$$
$$\omega\big(\hat{F}x_{kl}-\hat{F}x_{MN},t\big)$$
$$\leq\omega\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)\diamondsuit\omega\left(\hat{F}x_{NM}-\xi,\frac{t}{2}\right)$$
$$<\gamma\diamondsuit\gamma<\varepsilon \text{ and}$$
$$\psi\big(\hat{F}x_{kl}-\hat{F}x_{MN},t\big)$$
$$\leq\psi\left(\hat{F}x_{kl}-\xi,\frac{t}{2}\right)\odot\psi\left(\hat{F}x_{NM}-\xi,\frac{t}{2}\right)<\gamma\odot\gamma<\varepsilon.$$

Since x is not $\hat{F}S_\theta^{(\varphi,\omega,\psi)}$- Convergent.

Therefore $\delta_\theta\big(B^c(\varepsilon,t)\big)=0$,

where

$$B(\varepsilon,t)=\left\{\begin{array}{c}(k,l)\in\mathbb{N}\times\mathbb{N}:\omega_{x_{kl}-x_{NM}}\leq1-r\\ \text{and }\psi_{x_{kl}-x_{NM}}\leq1-r\end{array}\right\} \text{ and}$$

so, $\delta_\theta\big(B(\varepsilon,t)\big)=1$, which is contradiction, since x was $\hat{F}S_\theta$- Cauchy with respect to the NN (φ,ω,ψ). So, x must be $\hat{F}S_\theta$-Convergent with respect to the NN (φ,ω,ψ).

Hence, every NNS is $\hat{F}S_\theta$- Complete.

4. Conclusion

In this paper, we have introduced the concept of $\check{\mathcal{F}}\mathcal{LS}$ convergence of double sequences in NNS. Also, we proved some basic results in this space. Firstly, the definition of $\check{\mathcal{F}}\mathcal{LS}$ convergence with respect to NNS is discussed. After, the definition of $\check{\mathcal{F}}\mathcal{LS}$- Cauchy with respect to NNS and $\hat{F}S_\theta$-Complete are given. It shows that every NNS is $\hat{F}S_\theta$- Complete.

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