



Bipolar fuzzy metric spaces with application

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Abstract

The concept of a bipolar metric space is presented in this article. In bipolar fuzzy metric space, the notions of continuous t -norms and bipolar continuous symmetry t_s -conorms employs important role. Requirements in metric space, triangular norms are used to extrapolate with the probability density function of triangle inequality. Dual processes of triangular norms are known as triangular conorms. Continuous t -norms and continuous symmetry t_s -conorms are used to define bipolar metric space in this paper. Besides, a number of topological and functional properties of the bipolar metric space have been studied. Afterwards, Baire Category and Uniform Convergence Theorems for bipolar metric spaces are presented. After that, an application on determining the appropriate type of vaccine in the treatment process of COVID-19 is given using similarity measure between bipolar fuzzy metric spaces. For vaccine selection and supply chain management, an advanced multi-attribute decision-making (MADM) algorithm is being developed. Finally, the validity of the proposed MADM method for selecting the best proper form of vaccine is also established.

Keywords Bipolar fuzzy metric · T_2 -space · Baire Category Theorem · Completeness · Multi-attribute decision-making

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1 Introduction

Being a member of elements in a set is determined in binary terms according to a bivalent condition in classical set theory, which means that an element either member of the set or does not member of the set. Fuzzy set theory, on the other hand, allows for a gradual assessment

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of the membership of elements in a set, which is defined using a membership function with a real unit interval $[0, 1]$. Since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1, fuzzy sets generalize classical sets. Crisp sets are widely used in fuzzy set theory to refer to classical bivalent sets. Many researchers have been interested in fuzzy sets, which were first advanced by Zadeh (1965) in 1965 to convey indefiniteness in data. A membership function $\mu : X \rightarrow [0, 1]$ defines the a fuzzy set description, which assigns a real number in the unit closed interval $[0, 1]$ to each object in the universe. The fuzzy set theory can be applied to a broad variety of domains, including bioinformatics, where knowledge is incomplete or imprecise. Chang (1968) developed the notion of fuzzy topological spaces. Atanassov (1984), Atanassov and Stoeva (1983) suggested the intuitionistic fuzzy set (IFS) as a direct extension of the fuzzy set, based on the definition of membership grade (MG) and non-membership grade (NMG), with the restriction that the number of MG and NMG not exceed 1. Çoker (1997) introduced the concept of the intuitionistic fuzzy topological space. Smarandache (2005) described the Neutrosophic Set (NS) as a modern version of the classical set definition. Neutrosophy has opened the way for many other modern mathematical ideas that extend both classical and fuzzy equivalents. Balasubramanian and Sundaram (1997) presented the definition of a generalized fuzzy closed set. Pythagorean fuzzy set and Pythagorean fuzzy membership grades were introduced by Yager (2014, 2017) and Yager and Abbasov (2013), respectively. Yager and Abbasov (2013), Park (2004), Yager (2014), Yager (2017) introduced Pythagorean fuzzy sets (PFSs), also known as intuitionistic fuzzy sets of type-2, as an extension of IFSs and presented Pythagorean membership grades with requirements, by changing the constraint on the parameters. In Gulistana et al. (2021) authors study complex bipolar fuzzy sets. The notion of rough bipolar fuzzy Γ -hyperideals in Γ -semihypergroups is introduced by Yaqoob et al. (2018). Besides, Yaqoob (2014) deal with on bipolar fuzzy sets. The concept of bipolar-valued fuzzification of ordered AG-groupoids is studied in Yousafzai et al. (2012). The definitions of probabilistic metric space and fuzzy sets are used to construct the fuzzy metric space (FMS) in Kramosil and Michàle (1975). Afterwards FMS was described by Kaleva and Seikkala (1984) as a non-negative fuzzy number that measures the distance between two points. Following that, FMS has been extended to a variety of fields, fixed point theory, image and signal processing, medical imaging, decision-making, and so on are some of the topics covered. After it was defined, the intuitionistic fuzzy set (IFS) was described and used in all areas where FS theory was explored. In addition these, the concept of intuitionistic fuzzy metric space was invented by Park (2004). He demonstrated that the topology created by the intuitionistic fuzzy metric (M, N) coincides with the topology generated by the fuzzy metric M for each intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$. A hesitant fuzzy linear regression model (HFLRM) is investigated in Sultan et al. (2021) to account for multicriteria decision-making (MCDM) problems in a hesitant environment. In Kizielewicz et al. (2021), a new modified TOPSIS technique for dealing with data uncertainty that is based on TFN similarity metrics is offered.

Based on the theory of NSs, a new metric space called Neutrosophic metric Spaces (NMS) was defined in Kirişçi and Şimşek (2020). Some properties of NMS are investigated in Kirişçi and Şimşek (2020), including open set, Hausdorff, neutrosophic bounded, compactness and completeness. For NMSs, they also include the Baire Category Theorem and the Uniform Convergence Theorem. As extensions of Partial Algebra, Smarandache (2005) generalized classical Algebraic Structures to NeutroAlgebraic Structures with partially real, partially indeterminate, and partially false operations and axioms. Recently, as a generalization of fuzzy sets, Zhang (1994, 1998) initiated the concept of bipolar fuzzy sets in 1994. For more information you can see (Abdullah et al. 2014; Hussain et al. 2019; Qurashi and Shabir

2018; Riaz et al. 2020). In this research, fundamental concepts about the subject was given in the second part. In addition, in this section, the concepts of bipolar continuous symmetry t_s -conorm and bipolar fuzzy metric space are introduced and relevant examples are discussed. In the third chapter, the topology of bipolar fuzzy sets has been constructed and some theorems and proofs are given. For example, it is showed that BFMS is T_2 -space. Additionally, in this part BF bounded, complete BFMS and convergence of a BFMS concepts are defined. It is demonstrated in Chapter fourth that a BFMS is complete. The Baire Category and Uniform Limit theorems, as well as their proofs, were discovered. In the fifth section, a new similarity measure method for bipolar fuzzy metric spaces is presented. A multiple attribute decision making (MADM) algorithm is also provided, which was designed to determine the best form of vaccine for COVID-19. In the future work, we plan to give definitions of bipolar normed space and statistical convergence on bipolar fuzzy sets. Besides, we will make an application about medicine image processing.

2 Preliminaries

This section contains some definitions for fuzziness, bipolar fuzziness, and bipolar fuzzy metric space.

Definition 1 (Zhang 1994). BFSs are a type of fuzzy set whose membership value range has been extended from $[0, 1]$ to $[-1, 1]$. In a bipolar fuzzy set, a membership value of 0 denotes that elements are irrelevant to the corresponding property, a membership value of $[-1, 0]$ denotes that elements satisfy the assumed counter-property, and a membership value of $(0, 1)$ denotes that elements satisfy the property. The accepted representation of a bipolar fuzzy set A on the domain X is as follows:

$$A = \{(x, (\mu_A^r(a), \mu_A^\ell(a))) : a \in X\}.$$

Here, the positive membership value $\mu_A^r(a) : X \rightarrow [0, 1]$ specifies the value of satisfaction of some element $a \in X$ to the property according to the bipolar fuzzy set A , and the negative membership value $\mu_A^\ell(a) : X \rightarrow [-1, 0]$ describes the value to which a certain element $a \in X$ satisfies some implied counter-property corresponding to A .

Now let's include some basic definitions defined on bipolar fuzzy sets.

Definition 2 (Zhang 1994). Let's admit that $A = \{a, (\mu_A^\ell(a), \mu_A^r(a)) : a \in X\}$ and $B = \{a, (\mu_B^\ell(a), \mu_B^r(a)) : a \in X\}$ are two bipolar fuzzy sets defined on X . In this case, $A \subseteq B$, if

- $\mu_A^r(a) \leq \mu_B^r(a)$
- $\mu_A^\ell(a) \geq \mu_B^\ell(a)$ for all $a \in X$.

Additionally,

$$\begin{aligned} A \cup B &= \left\{ a, \left(\max\{\mu_A^r(a), \mu_B^r(a)\}, \min\{\mu_A^\ell(a), \mu_B^\ell(a)\} \right) : a \in X \right\} \text{ (Union),} \\ A \cap B &= \left\{ a, \left(\min\{\mu_A^r(a), \mu_B^r(a)\}, \max\{\mu_A^\ell(a), \mu_B^\ell(a)\} \right) : a \in X \right\} \text{ (Intersection),} \\ A^c &= \left\{ a, \left(1 - \mu_A^r(a), -1 - \mu_A^\ell(a) \right) : a \in X \right\} \text{ (Complement).} \end{aligned}$$

The positive membership degree $\mu_A^r(a)$, in BFSs, denotes the value to which the element a supplies the property of A , while the negative membership value $\mu_A^\ell(a)$ denotes the value

to which the element a supplies an implied counter-property of A . However, the membership value $\mu_A(a)$ denotes the value to which the element a satisfies the property A , while the membership value $\nu_A(a)$ denotes the value to which a satisfies the not-property of A , in IFSs. BFSs and IFSs are different extensions of FSs since a counter-property is not always equal to not-property. Their differences can be seen in how they view an element a with a membership value of $(0, 0)$. It is considered that the element a does not satisfy both the property A and its implicit counter-property by taking into account the perspective of bipolar fuzzy set A . It implies that it is unaffected by the property and its implicit counter-property. When this situation is considered in terms of NS A , the variable a is noted as not satisfying the property and its not-property. Menger (1942) established the use of triangular norms (t-norms). Menger discussed using probability distributions instead of numbers to solve the problem of calculating the distance between two elements in space. In metric space conditions, t-norms are used to make generalisations with the probability distribution of triangle inequality. Dual operations of TNs are triangular conorms (t-conorms).

Definition 3 (Menger 1942). Continuous t -norms are binary operations on the interval $[0, 1]$, it means that $\Delta : [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfies the following situations with neutral element 1:

1. Δ is commutative and associative
2. Δ is continuous
3. $a\Delta 1 = a$ for all $a \in [0, 1]$
4. $a\Delta b \leq c\Delta d$ where $a \leq c, b \leq d$ for all $a, b, c, d \in [0, 1]$.

Definition 4 A doubled arithmetic operation $\nabla : [-1, 0] \times [-1, 0] \rightarrow [-1, 1]$ is referred to as bipolar continuous symmetry t_s -conorm if ∇ applies the required specifications:

1. ∇ is commutative and associative
2. ∇ is continuous
3. $a\nabla 0 = a$ for all $a \in [-1, 0]$
4. $a\nabla b \leq c\nabla d$ where $a \leq c, b \leq d$ for all $a, b, c, d \in [-1, 0]$.

Definition 5 The tetra set $(A, \mathcal{B}, \Delta, \nabla)$ is called bipolar fuzzy metric space (BFMS) when the following conditions are satisfied for all $u, v, z \in A$, where A be arbitrary set, $\{< u, \mu_A^r(u), \mu_A^\ell(u) > : u \in A\}$ be BFS such that μ_A^r, μ_A^ℓ are defined as fuzzy sets on $A \times A \times (0, \infty)$, $\mathcal{B} : A \times A \times (0, \infty) \rightarrow [-1, 1]$, $\alpha, \gamma > 0$ and Δ, ∇ show the continuous t -norm and continuous symmetry t_s -conorm, respectively:

1. $0 \leq \mu_A^r(u, v, \gamma) \leq 1, -1 \leq \mu_A^\ell(u, v, \gamma) \leq 0$
2. $\mu_A^r(u, v, \gamma) + \mu_A^\ell(u, v, \gamma) \leq 1$
3. $\mu_A^r(u, v, \gamma) = 1 \Leftrightarrow u = v$
4. $\mu_A^r(u, v, \gamma) = \mu_A^r(v, u, \gamma)$
5. $\mu_A^r(u, v, \gamma)\Delta\mu_A^r(v, z, \alpha) \leq \mu_A^r(u, z, \gamma + \alpha)$
7. $\mu_A^r(u, v, \cdot) : [0, \infty) \rightarrow [-1, 1]$ is continuous
8. $\lim_k \mu_A^r(u, v, \gamma) = 1, (\forall \gamma > 0)$
9. $\mu_A^\ell(u, v, \gamma) = -1 \Leftrightarrow u = v$
10. $\mu_A^\ell(u, v, \gamma) = \mu_A^\ell(v, u, \gamma)$
11. $\mu_A^\ell(u, v, \gamma)\nabla\mu_A^\ell(v, z, \alpha) \geq \mu_A^\ell(u, z, \gamma + \alpha)$
12. $\mu_A^\ell(u, v, \cdot) : [0, \infty) \rightarrow [-1, 1]$ is continuous
13. $\lim_k \mu_A^\ell(u, v, \gamma) = -1, (\forall \gamma > 0)$.

The functions $\mu_A^r(u, v, \gamma)$ and $\mu_A^\ell(u, v, \gamma)$ represent the value of closeness and the value of distance between u and v with respect to γ , respectively.

Remark 1 In case of $\mu_A^r(u) = \mu_A^\ell(u) = 0$ for all $u \in A$ every fuzzy metric space is bipolar fuzzy metric space.

We can see from the definitions above that if we select $\delta_1, \delta_2 \in (0, 1)$ taken together $\delta_1 > \delta_2$, there is $\delta_3, \delta_4 \in (0, 1)$ such that $\delta_1 \Delta \delta_3 \geq \delta_2$ and $\delta_1 \geq \delta_4 \nabla \delta_2$. Also, in case of $\delta_5 \in (0, 1)$ there is $\delta_6, \delta_7 \in (0, 1)$ such that $\delta_6 \Delta \delta_6 \geq \delta_5$ and $\delta_7 \nabla \delta_7 \leq \delta_5$.

Example 1 Let (A, d) be a crisp metric space. Give the functions Δ and ∇ as default $u \Delta v = \min\{u, v\}$, $u \nabla v = \max\{u, v\}$. $\mu_A^r(u) = \frac{\gamma}{\gamma + d(u, v)}$, $\mu_A^\ell(u) = \frac{-\gamma}{\gamma + d(u, v)}$, for all $\gamma > 0$, $u, v \in A$. Then, $(A, \mathcal{B}, \Delta, \nabla)$ be bipolar fuzzy metric space by considering $\mathcal{B} : A \times A \times (0, \infty) \rightarrow [-1, 1]$.

Example 2 If we take $A = \mathbb{Z} - \{0\}$, $a \Delta b = \max\{0, a + b - 1\}$, $a \nabla b = \max\{0, a + b - 1\}$ for all $a, b \in [-1, 1]$ and define fuzzy sets μ_A^r, μ_A^ℓ on $A \times A \times (0, \infty)$ as given below:

$$\mu_A^r(u, v, \gamma) = \begin{cases} \frac{u}{v}, & u \leq v \\ \frac{v}{u}, & v \leq u \end{cases} \quad (1)$$

$$\mu_A^\ell(u, v, \gamma) = \begin{cases} -\frac{u}{v}, & u \leq v \\ -\frac{v}{u}, & v \leq u \end{cases} \quad (2)$$

for all $u, v \in A$ and $\gamma > 0$. Then, $(A, \mathcal{B}, \Delta, \nabla)$ is a bipolar fuzzy metric space.

3 Investigation of topology induced by bipolar fuzzy metric

In this section, the topological structure of bipolar fuzzy sets is examined. In this way, we deal with the structural qualities of BFMS, such as open ball, open set, T_2 space, completeness, and nowhere dense.

Definition 6 Given $(A, \mathcal{B}, \Delta, \nabla)$ be a BFMS, $0 < \delta < 1$, $\gamma > 0$ and $u \in A$. The set $\phi(u, \delta, \gamma) = \{v \in A : \mu_A^r(u, v, \gamma) > 1 - \delta, \mu_A^\ell(u, v, \gamma) < \delta - 1\}$ is called an open ball because it has a middle point u and a radius δ in relation to γ .

Theorem 1 Any open ball, $\phi(u, \delta, \gamma)$ in $(A, \mathcal{B}, \Delta, \nabla)$ is an open set.

Proof Consider $\phi(u, \delta, \gamma)$ is a open ball together u as the center and δ as the radius. Let us take $v \in \phi(u, \delta, \gamma)$ after that $\mu_A^r(u, v, \gamma) > 1 - \delta$, $\mu_A^\ell(u, v, \gamma) < \delta - 1$. There exists $\gamma_0 \in (0, \gamma)$ so as $\mu_A^r(u, v, \gamma_0) > 1 - \delta$, $\mu_A^\ell(u, v, \gamma_0) < \delta - 1$ because of the fact that $\mu_A^r(u, v, \gamma) > 1 - \delta$. If we choose $\delta_0 = \mu_A^r(u, v, \gamma_0)$ then for $\delta_0 > 1 - \delta$, there is $\sigma \in (0, 1)$ such that $\delta_0 > 1 - \sigma > 1 - \delta$. Receive δ_0 and σ as $\delta_0 > 1 - \sigma$. After, $\delta_1, \delta_2 \in (0, 1)$ will exist such that $\delta_0 \Delta \delta_1 > 1 - \sigma$ and $(\delta_0 - 1) \nabla (\delta_2 - 1) \leq \sigma - 1$. And choose $\delta_3 = \max\{\delta_1, \delta_2\}$. Now, take into consideration the open ball $\phi(v, 1 - \delta_3, \gamma - \gamma_0)$. We will show that $\phi(v, 1 - \delta_3, \gamma - \gamma_0) \subset \phi(u, \delta, \gamma)$. If we get $z \in \phi(v, 1 - \delta_3, \gamma - \gamma_0)$ then, $\mu_A^r(v, z, \gamma - \gamma_0) > \delta_3$ and $\mu_A^\ell(v, z, \gamma - \gamma_0) < \delta_3$. Also, $\mu_A^r(u, z, \gamma) \geq \mu_A^r(u, v, \gamma_0) \Delta \mu_A^r(v, z, \gamma - \gamma_0) \geq \delta_0 \Delta \delta_3 \geq \delta_0 \Delta \delta_1 \geq 1 - \sigma > 1 - \delta$, $\mu_A^\ell(u, z, \gamma) \leq \mu_A^\ell(u, v, \gamma_0) \nabla \mu_A^\ell(v, z, \gamma - \gamma_0) \leq (\delta_0 - 1) \nabla (\delta_3 - 1) \leq (\delta_0 - 1) \nabla (\delta_2 - 1) \leq \sigma - 1 < \delta - 1$. It means that $z \in \phi(u, \delta, \gamma)$ and from here we deduced $\phi(v, 1 - \delta_3, \gamma - \gamma_0) \subset \phi(u, \delta, \gamma)$. The proof is now complete. \square

Remark 2 $(A, \mathcal{B}, \Delta, \nabla)$ be bipolar fuzzy metric space (BFMS). On A $\tau_{\mathcal{B}} = \{X \subset A : \gamma > 0 \text{ and } \delta \in (0, 1) \ni \phi(u, v, \gamma) \subset X \text{ for each } u \in X\}$ is a topology. Any BFM \mathcal{B} on A provides a topology $\tau_{\mathcal{B}}$ in that case.

Theorem 2 Any BFMS is T_2 space.

Proof Let us $(A, \mathcal{B}, \Delta, \nabla)$ be bipolar fuzzy metric space (BFMS) and u, v two discrete points in A . From here, $0 < \mu_A^r(u, v, \gamma) < 1, -1 < \mu_A^l(u, v, \gamma) < 0$. Take $\delta_1 = \mu_A^r(u, v, \gamma), \delta_2 = \mu_A^l(u, v, \gamma)$ and $\delta = \max\{\delta_1, -\delta_2\}$. If we take $\delta_0 \in (\delta, 1)$ then there exist δ_3 and δ_4 such that $\delta_3 \Delta \delta_3 \geq \delta_0$ and $(\delta_4 - 1) \nabla (\delta_4 - 1) \leq (\delta_0 - 1)$. Put $\delta_5 = \max\{\delta_3, \delta_4\}$ and consider the open balls $\phi(u, 1 - \delta_5, \frac{\gamma}{2})$ and $\phi(v, 1 - \delta_5, \frac{\gamma}{2})$. Clearly, $\phi(u, 1 - \delta_5, \frac{\gamma}{2}) \cap \phi(v, 1 - \delta_5, \frac{\gamma}{2}) = \emptyset$. If there exists $z \in \phi(u, 1 - \delta_5, \frac{\gamma}{2}) \cap \phi(v, 1 - \delta_5, \frac{\gamma}{2})$ then $\delta_1 = \mu_A^r(u, v, \gamma) \geq \mu_A^r(u, z, \frac{\gamma}{2}) \Delta \mu_A^r(z, v, \frac{\gamma}{2}) \geq \delta_5 \Delta \delta_5 \geq \delta_3 \Delta \delta_3 \geq \delta_0 > \delta_1$ and $\delta_2 = \mu_A^l(u, v, \gamma) \leq \mu_A^l(u, z, \frac{\gamma}{2}) \nabla \mu_A^l(z, v, \frac{\gamma}{2}) \leq (\delta_5 - 1) \nabla (\delta_5 - 1) \leq (\delta_4 - 1) \nabla (\delta_4 - 1) \leq (\delta_0 - 1) \leq \delta_2$. This leads in a contradiction. As a result $(A, \mathcal{B}, \Delta, \nabla)$ is T_2 space. \square

Theorem 3 Let $(A, \mathcal{B}, \Delta, \nabla)$ be a BFMS and $\tau_{\mathcal{B}}$ is topology on A . Then a sequence (u_k) of A is convergent to u if and only if $\mu_A^r(u_k, u, \gamma) \rightarrow 1$ and $\mu_A^l(u_k, u, \gamma) \rightarrow -1$ for $k \rightarrow \infty$.

Proof Let $\gamma > 0$ and $u_k \rightarrow u$. Then for $\delta \in (0, 1)$ there exists $k_0 \in \mathbb{N}$ such that $u_k \in \phi(u, \delta, \gamma)$ for all $k \geq k_0$. Then $\mu_A^r(u_k, u, \gamma) > 1 - \delta$ and $\mu_A^l(u_k, u, \gamma) > \delta - 1$. From here, we can deduced that $\mu_A^r(u_k, u, \gamma) \rightarrow 1$ and $\mu_A^l(u_k, u, \gamma) \rightarrow -1$ for $k \rightarrow \infty$.

Conversely, for each $\gamma > 0$, if $\mu_A^r(u_k, u, \gamma) \rightarrow 1$ and $\mu_A^l(u_k, u, \gamma) \rightarrow -1$ for $k \rightarrow \infty$, then for $\delta \in (0, 1)$ there exists $k_0 \in \mathbb{N}$ such that $1 - \mu_A^r(u_k, u, \gamma) < \delta$ and $-1 - \mu_A^l(u_k, u, \gamma) < \delta$ for all $k \geq k_0$. It follows that $\mu_A^r(u_k, u, \gamma) > 1 - \delta$ and $\mu_A^l(u_k, u, \gamma) < \delta - 1$ for all $k \geq k_0$. Thus, $u_k \in \phi(u, \delta, \gamma)$ for all $k \geq k_0$ and hence $u_k \rightarrow u$. \square

Definition 7 A BFMS is defined as $(A, \mathcal{B}, \Delta, \nabla)$. If there is one $M > 0$ and $\delta \in (0, 1)$ so that $\mu_A^r(u, v, M) > 1 - \delta$ and $\mu_A^l(u, v, M) < \delta - 1$ for all $u, v \in A$, the subset F of A is called bipolar fuzzy bounded (BFB).

Definition 8 Let $(A, \mathcal{B}, \Delta, \nabla)$ be a BFMS and (u_k) is a sequence of A . If for all $\delta > 0$ there exist $n_0 \in \mathbb{N} \ni$ for all $n, m \geq n_0$ $\mu_A^r(u, v, \gamma) > 1 - \delta$ and $\mu_A^l(u, v, \gamma) < \delta - 1$ then (u_k) is a Cauchy sequence. Additionally, if all Cauchy sequence is convergent in respect of $\tau_{\mathcal{B}}$, $(A, \mathcal{B}, \Delta, \nabla)$ be a complete bipolar fuzzy metric space.

Remark 3 Allow $(A, \mathcal{B}, \Delta, \nabla)$ to be a BFMS produced by a metric d on A . If and only if $X \subset A$ is bounded, it is bipolar fuzzy bounded.

Remark 4 Any compact set in bipolar fuzzy metric space is closed and bounded.

Theorem 4 Any bipolar fuzzy metric space $(A, \mathcal{B}, \Delta, \nabla)$ has a compact subset A that is bipolar fuzzy bounded.

Proof Assume that X is a compact subset of A , a bipolar fuzzy metric space. Let $\gamma > 0, \delta \in (0, 1)$ be the case. Take into account the $\{\phi(u, \delta, \gamma) : u \in X\}$ is a open cover. Because of the fact that X is compact, there exist $u_1, u_2, \dots, u_k \in X \ni X \subseteq \bigcup_{i=1}^k \phi(u_i, \delta, \gamma)$. Consider $u, n \in X$. After that $u \in \phi(u_i, \delta, \gamma)$ and $v \in \phi(u_\ell, \delta, \gamma)$ for some i, ℓ . It means that, we have $\mu_A^r(u, u_i, \gamma) > 1 - \varepsilon, \mu_A^l(u, u_i, \gamma) < \varepsilon - 1, \mu_A^r(v, u_\ell, \gamma) > 1 - \varepsilon, \mu_A^l(v, u_\ell, \gamma) < \varepsilon - 1$. Here, get $\lambda = \min\{\mu_A^r(u_i, u_\ell, \gamma) : 1 \leq i, \ell \leq k\}$ and $\lambda' = \max\{\mu_A^l(u_i, u_\ell, \gamma) : 1 \leq i, \ell \leq k\}$. Then, $\lambda \geq 0, \lambda' \leq 0$. Now we have

$\mu_A^r(u, v, 3\gamma) \geq \mu_A^r(u, u_i, 3\gamma) \Delta \mu_A^r(u_i, u_\ell, \gamma) \Delta \mu_A^r(u_\ell, v, \gamma) \geq (1-\delta)\Delta(1-\delta)\Delta\lambda > 1-\sigma'$ where $0 < \sigma' < 1$ and $\mu_A^\ell(u, v, 3\gamma) \leq \mu_A^\ell(u, u_i, 3\gamma) \nabla \mu_A^\ell(u_i, u_\ell, \gamma) \nabla \mu_A^\ell(u_\ell, v, \gamma) \leq (\delta-1)\nabla(\delta-1)\nabla\lambda' < 1-\sigma''$ where $-1 < \sigma'' < 0$. Considering $\sigma = \max\{\sigma', \sigma''\}$ and $\mu_A^r(u, v, 3\gamma) > 1-\sigma$, $\mu_A^\ell(u, v, 3\gamma) < \sigma-1$ for all $u, v \in X$. As a result, X is bipolar fuzzy bounded. \square

Theorem 5 Let $(A, \mathcal{B}, \Delta, \nabla)$ be a BFMS with a convergent subsequence for every Cauchy sequence in A . After that, $(A, \mathcal{B}, \Delta, \nabla)$ is complete.

Proof Let (u_k) be a Cauchy sequence and let (u_{k_n}) be a subsequence of (u_k) that converges to u . We must prove that $u_k \rightarrow u$. Let $\gamma > 0$ and $\varepsilon \in (0, 1)$. Choose $\delta \in (0, 1)$ such that $(1-\delta)\nabla(1-\delta) \geq 1-\varepsilon$ and $(\delta-1)\Delta(\delta-1) \leq \varepsilon-1$. Since (u_k) is Cauchy sequence, there is $k_0 \in \mathbb{N}$ such that $\mu_A^r(u_n, u_k, \frac{\gamma}{2}) > 1-\delta$ and $\mu_A^\ell(u_n, u_k, \frac{\gamma}{2}) > \delta-1$ for all $n, k \geq k_0$. Since $u_{k_n} \rightarrow u$, there is $n_i \in \mathbb{N}$ such that $n_i > n_0$, $\mu_A^r(u_{n_i}, u, \frac{\gamma}{2}) > 1-\delta$ and $\mu_A^\ell(u_{n_i}, u, \frac{\gamma}{2}) > \delta-1$. Then, if $k > k_0$, $\mu_A^r(u_k, u, \gamma) \geq \mu_A^r(u_k, u_{n_i}, \frac{\gamma}{2}) \Delta \mu_A^r(u_{n_i}, u, \frac{\gamma}{2}) > (1-\delta)\Delta(1-\delta) \geq 1-\varepsilon$ and $\mu_A^\ell(u_k, u, \gamma) \leq \mu_A^\ell(u_k, u_{n_i}, \frac{\gamma}{2}) \nabla \mu_A^\ell(u_{n_i}, u, \frac{\gamma}{2}) < (\delta-1)\nabla(\delta-1) \leq \varepsilon-1$. Therefore $u_k \rightarrow u$ and as a result $(A, \mathcal{B}, \Delta, \nabla)$ is complete. \square

Theorem 6 Let $(A, \mathcal{B}, \Delta, \nabla)$ be a BFMS and $X \subset A$ with the (μ_X^r, μ_X^ℓ) subspace bipolar fuzzy metric. If and only if X is a closed subset of A , then $(X, \mu_X^r, \mu_X^\ell, \Delta, \nabla)$ is complete.

Proof Let (u_k) be a Cauchy sequence in $(X, \mu_X^r, \mu_X^\ell, \Delta, \nabla)$, and $X \subset A$ be a closed. And so, (u_k) in A is a Cauchy sequence, and there is a point u in A where $u_k \rightarrow u$. As a result, $u \in \overline{X} = X$, and (u_k) converges in X . From here, $(X, \mu_X^r, \mu_X^\ell, \Delta, \nabla)$ is complete.

Consider the case, where $(X, \mu_X^r, \mu_X^\ell, \Delta, \nabla)$ is complete but X is not closed. Take $u \in \overline{X} \setminus X$. Then there will be a sequence of points in X called (u_k) that converges to u , and (u_k) is a Cauchy sequence. It means that, for each $\delta \in (0, 1)$ and $\gamma > 0$, there is $k_0 \in \mathbb{N} \ni \mu_A^r(u_k, u_\ell, \gamma) > 1-\delta$ and $\mu_A^\ell(u_k, u_\ell, \gamma) > \delta-1$ for all $k, \ell \geq k_0$. So, (u_k) is a sequence in X , $\mu_A^r(u_k, u_\ell, \gamma) = \mu_X^r(u_k, u_\ell, \gamma)$ and $\mu_A^\ell(u_k, u_\ell, \gamma) = \mu_X^\ell(u_k, u_\ell, \gamma)$. As a result, in X , (u_k) is a Cauchy sequence. There is a $v \in X \ni u_k \rightarrow v$ because $(X, \mu_X^r, \mu_X^\ell, \Delta, \nabla)$ is complete. For all $k \geq k_0$, there is $k_0 \in \mathbb{N} \ni \mu_X^r(v, u_k, \gamma) > 1-\delta$ and $\mu_X^\ell(v, u_k, \gamma) < \delta-1$ for all $k \geq k_0$ for each $\delta \in (0, 1)$ and each $\gamma > 0$. However, because (u_k) is a sequence in X and $v \in X$, $\mu_A^r(v, u_k, \gamma) = \mu_X^r(v, u_k, \gamma)$ and $\mu_A^\ell(v, u_k, \gamma) = \mu_X^\ell(v, u_k, \gamma)$ are valid. As a result, (u_k) converges to both u and v in $(A, \mathcal{B}, \Delta, \nabla)$. Because of the fact that $u \notin X$ and $v \in X$, $u \neq v$. This is an inconsistency. \square

Lemma 1 A BFMS is defined as $(A, \mathcal{B}, \Delta, \nabla)$. In the case of $\gamma > 0$, $\delta, \varepsilon \in (0, 1)$ and $(1-\varepsilon)\Delta(1-\varepsilon) \geq 1-\delta$ and $(\varepsilon-1)\nabla(\varepsilon-1) \leq \delta-1$, so $\overline{\phi(u, \varepsilon, \frac{\gamma}{2})} \subset \phi(u, \delta, \gamma)$ is valid.

Proof Let us take $v \in \overline{\phi(u, \varepsilon, \frac{\gamma}{2})}$ and $\phi(v, \varepsilon, \frac{\gamma}{2})$ be an open ball with center u and radius ε . Because of the fact that $\phi(v, \varepsilon, \frac{\gamma}{2}) \cap \phi(u, \varepsilon, \frac{\gamma}{2}) \neq \emptyset$, there exist a $z \in \phi(v, \varepsilon, \frac{\gamma}{2}) \cap \phi(u, \varepsilon, \frac{\gamma}{2})$. Next, there's $\mu_A^r(u, v, \gamma) \geq \mu_A^r(u, z, \frac{\gamma}{2}) \Delta \mu_A^r(v, z, \frac{\gamma}{2}) > (1-\varepsilon)\Delta(1-\varepsilon) \geq 1-\delta$ and $\mu_A^\ell(u, v, \gamma) \leq \mu_A^\ell(u, z, \frac{\gamma}{2}) \nabla \mu_A^\ell(v, z, \frac{\gamma}{2}) < (\varepsilon-1)\nabla(\varepsilon-1) \leq \delta-1$. Hence $z \in \phi(u, \delta, \gamma)$ and thus $\overline{\phi(u, \varepsilon, \frac{\gamma}{2})} \subset \phi(u, \delta, \gamma)$. \square

Theorem 7 If and only if a open set in $A \neq \emptyset$ includes an open ball which closure is disjoint from X , then a subset X of a bipolar fuzzy metric space $(A, \mathcal{B}, \Delta, \nabla)$ is dense in nowhere.

Proof Let $\emptyset \neq W \subset X$ be the case. Then there is a open set $Z \neq \emptyset$ in which $Z \subset W$ and $Z \cap \overline{X} \neq \emptyset$ are valid. Let $u \in Z$. After that, there's $\delta \in (0, 1)$ and $\gamma > 0 \ni \phi(u, \delta, \gamma) \subset Z$.

Consider $\varepsilon \in (0, 1)$ so that $(1 - \varepsilon)\Delta(1 - \varepsilon) \geq 1 - \delta$ and $(\varepsilon - 1)\nabla(\varepsilon - 1) \leq \delta - 1$. By taking into account Lemma 1 $\phi(u, \varepsilon, \frac{\gamma}{2}) \subset \phi(u, \delta, \gamma)$. It means that, $\phi(u, \varepsilon, \frac{\gamma}{2}) \subset W$ and $\phi(u, \varepsilon, \frac{\gamma}{2}) \cap X = \emptyset$.

Assume, on the other hand, that X is not dense anywhere. Then, $\text{int}(\bar{X}) \neq \emptyset$, implying that there is a nonempty open set W such that $W \subset \bar{X}$. Let $\phi(u, \delta, \gamma)$ be an open ball with the property that $\phi(u, \delta, \gamma)$ is a subset of W . Then $\phi(u, \delta, \gamma) \cap X \neq \emptyset$ comes into play. This is an inconsistency. \square

For bipolar fuzzy metric space, we can prove Baire's theorem.

Theorem 8 Allow $\{W_k : k \in \mathbb{N}\}$ represent a sequence of dense open subsets of a complete bipolar fuzzy metric space $(A, \mathcal{B}, \Delta, \nabla)$. And after that $\bigcap_{k \in \mathbb{N}} W_k$ is dense in A .

Proof Let's get this part with $Z \neq \emptyset$ be a open set of A . Because W_1 is dense in A , $Z \cap W_1 \neq \emptyset$. Take $u_1 \in Z \cap W_1$. Since $Z \cap W_1$ is open, there is $\delta_1 \in (0, 1)$ and $\gamma_1 > 0$ so that $\phi(u_1, \delta_1, \gamma_1) \subset Z \cap W_1$. Consider $\delta'_1 < \delta$ and $\gamma'_1 = \min(\gamma_1, 1)$ such that $\phi(u_1, \delta'_1, \gamma'_1) \subset Z \cap W_1$. Since W_2 is dense in A , $\phi(u_1, \delta'_1, \gamma'_1) \cap W_2 \neq \emptyset$. Let $u_2 \in \phi(u_1, \delta'_1, \gamma'_1) \cap W_2$. Since $\phi(u_1, \delta'_1, \gamma'_1) \cap W_2$ is open there exist $\delta_2 \in (0, \frac{1}{2})$ and $\gamma_2 > 0$ $\overline{\phi(u_2, \delta_2, \gamma_2)} \subset \phi(u_1, \delta'_1, \gamma'_1) \cap W_2$. Consider $\delta'_2 < \delta_2$ and $\gamma'_2 = \min(\gamma_2, \frac{1}{2})$ such that $\phi(u_2, \delta'_2, \gamma'_2) \subset \phi(u_1, \delta'_1, \gamma'_1) \cap W_2$. If we keep going this way, we obtain a sequence (u_k) in A and a sequence (γ'_k) such that $0 < \gamma'_k < \frac{1}{k}$ and $\phi(u_k, \delta'_k, \gamma'_k) \subset \phi(u_{k-1}, \delta'_{k-1}, \gamma'_{k-1}) \cap W_k$. Now we will take into account that (u_k) is a Cauchy sequence. For $\gamma > 0$ and $\varepsilon > 0$, select $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \gamma$ and $\frac{1}{k_0} < \varepsilon$. Then for $k \geq k_0$ and $\ell \geq k$, $\mu_A^r(u_k, u_\ell, \gamma) \geq \mu_A^r(u_k, u_\ell, \frac{1}{k}) \geq 1 - \frac{1}{k} > 1 - \varepsilon$, $\mu_A^\ell(u_k, u_\ell, \gamma) \leq \mu_A^\ell(u_k, u_\ell, \frac{1}{k}) \leq \frac{1}{k} - 1 < \varepsilon - 1$. As a result, the sequence (u_k) is a Cauchy sequence. Because of the fact that A is complete, $u \in A$ arises such that $u_k \rightarrow u$. We offer $u \in \phi(u_k, \delta'_k, \gamma'_k)$ for $n \geq k$ because $u_n \in \phi(u_k, \delta'_k, \gamma'_k)$. Consequently, $u \in \phi(u_k, \delta'_k, \gamma'_k) \subset \phi(u_{k-1}, \delta'_{k-1}, \gamma'_{k-1}) \cap W_k$ for all k . Therefore, $Z \cap (\bigcap_{k \in \mathbb{N}} W_k) \neq \emptyset$. from here, $\bigcap_{k \in \mathbb{N}} W_k$ is dense in A . \square

Note 1 The first group does not include any complete bipolar fuzzy metric space that cannot be described as the union of a sequence of nowhere dense sets. As a result, any complete bipolar fuzzy metric space belongs to the second group.

Remark 5 Since every metric induces a bipolar fuzzy metric, and bipolar fuzzy metric is a general statement of fuzzy metric, Baire's theorem for complete metric space (Zhang 1998) and Baire's theorem for complete fuzzy metric space (Qurashi and Shabir 2018) are special cases of the above theorem.

4 Some properties of complete bipolar fuzzy metric spaces

The topological structure of bipolar fuzzy sets is investigated in further depth in this section, as well as the Uniform Limit Theorem.

Definition 9 A BFMS is defined as $(A, \mathcal{B}, \Delta, \nabla)$. A set $[D_k]_{k \in \mathbb{N}}$ is said to have bipolar fuzzy diameter zero if there exists $k_0 \in \mathbb{N}$ for each $\delta \in (0, 1)$ and each $\gamma > 0$ so that $\mu_A^r(u, v, \gamma) > 1 - \varepsilon$ and $\mu_A^\ell(u, v, \gamma) < \varepsilon - 1$ for all $u, v \in D_{k_0}$.

Remark 6 If and only if D is a singleton set, a nonempty subset D of a bipolar fuzzy metric space A has bipolar fuzzy diameter zero.

Theorem 9 *If and only if every nested sequence $[D_k]_{k \in \mathbb{N}}$ of nonempty closed sets with bipolar fuzzy diameter zero has nonempty intersection, a bipolar fuzzy metric space $(A, \mathcal{B}, \Delta, \nabla)$ is complete.*

Proof Assume that the specified condition exists first. $(A, \mathcal{B}, \Delta, \nabla)$, we imply, is complete. In A , consider (u_k) to be a Cauchy sequence. The set $N_k = \{u_n : n \geq k\}$ and $D_k = \overline{N_k}$ then $[D_k]$ is said to have a bipolar fuzzy diameter zero. For taken $\varepsilon \in (0, 1)$ and $\gamma > 0$, we select $\delta \in (0, 1)$ as a result $(1 - \delta)\Delta(1 - \delta)\Delta(1 - \delta) > 1 - \varepsilon$ and $(\delta - 1)\nabla(\delta - 1)\nabla(\delta - 1) < \varepsilon - 1$. Because (u_k) is a Cauchy sequence, there will be $k_0 \in \mathbb{N}$ such that $\mu_A^r(u_k, u_\ell, \frac{\gamma}{3}) > 1 - \varepsilon$ and $\mu_A^\ell(u_k, u_\ell, \frac{\gamma}{3}) < \varepsilon - 1$ are valid for all $\ell, k \geq k_0$. As a result, $\mu_A^r(u, v, \frac{\gamma}{3}) > 1 - \varepsilon$ and $\mu_A^\ell(u, v, \frac{\gamma}{3}) > \varepsilon - 1$ for all $u, v \in N_{k_0}$. Let $u, v \in D_{k_0}$. Then in N_{k_0} , there are sequences (u'_k) and (v'_k) so that $u'_k \rightarrow u$ and $v'_k \rightarrow v$. Hence $u'_k \in \phi(u, \delta, \frac{\gamma}{3})$ and $v'_k \in \phi(v, \delta, \frac{\gamma}{3})$ for an enough large k . Now we have $\mu_A^r(u, v, \gamma) \geq \mu_A^r(u, u'_k, \frac{\gamma}{3})\Delta\mu_A^r(u'_k, v'_k, \frac{\gamma}{3})\Delta\mu_A^r(v'_k, v, \frac{\gamma}{3}) > (1 - \delta)\Delta(1 - \delta)\Delta(1 - \delta) > 1 - \varepsilon$ and $\mu_A^\ell(u, v, \gamma) \leq \mu_A^\ell(u, u'_k, \frac{\gamma}{3})\nabla\mu_A^\ell(u'_k, v'_k, \frac{\gamma}{3})\nabla\mu_A^\ell(v'_k, v, \frac{\gamma}{3}) < (\delta - 1)\nabla(\delta - 1)\nabla(\delta - 1) < \varepsilon - 1$. After that, $\mu_A^r(u, v, \gamma) > 1 - \varepsilon$ and $\mu_A^\ell(u, v, \gamma) < \varepsilon - 1$ for all $u, v \in D_{k_0}$. Eventually, $[D_k]$ has bipolar fuzzy diameter zero and as a result of hypothesis the variable $\bigcap_{k \in \mathbb{N}} D_k$ is nonempty.

Take $u \in \bigcap_{k \in \mathbb{N}} D_k$. We show that $u_k \rightarrow u$. Then for $\delta \in (0, 1)$ and $\gamma > 0$, there exists $k_1 \in \mathbb{N}$ such that $\mu_A^r(u_k, u, \gamma) > 1 - \delta$ and $\mu_A^\ell(u_k, u, \gamma) < \delta - 1$ for all $k \geq k_1$. Therefore, for each $\gamma > 0$ $\mu_A^r(u_k, u, \gamma) \rightarrow 1$ and $\mu_A^\ell(u_k, u, \gamma) \rightarrow -1$ as $k \rightarrow \infty$ and hence $u_k \rightarrow u$. Therefore, $(A, \mathcal{B}, \Delta, \nabla)$ is complete.

In the opposite case, assume $(A, \mathcal{B}, \Delta, \nabla)$ is complete and $[D_k]_{k \in \mathbb{N}}$ is a nested sequence of nonempty closed sets with bipolar fuzzy diameter zero. Get a point $u_k \in D_k$ for each $k \in \mathbb{N}$. A Cauchy sequence is defined as (u_k) . Because of the fact that $[D_k]$ has bipolar fuzzy diameter zero, for $\gamma > 0$ and $\delta \in (0, 1)$, there is $k_0 \in \mathbb{N}$ so that $\mu_A^r(u, v, \gamma) > 1 - \delta$ and $\mu_A^\ell(u, v, \gamma) < \delta - 1$ for all $u, v \in D_{k_0}$. Since, $[D_k]$ is nested sequence $\mu_A^r(u_k, u_\ell, \gamma) > 1 - \delta$ and $\mu_A^\ell(u_k, u_\ell, \gamma) < \delta - 1$ for all $\ell, k \geq k_0$. As a consequence, the sequence (u_k) is a Cauchy sequence. Since $(A, \mathcal{B}, \Delta, \nabla)$ is complete, $u_k \rightarrow u$ for some $u \in A$. As a result, for every $k, u \in \overline{D_k} = D_k$ and $u \in \bigcap_{k \in \mathbb{N}} D_k$. The proof is now finished. \square

Remark 7 The element $u \in \bigcap_{k \in \mathbb{N}} D_k$ is singular. If two elements are present as $u, v \in \bigcap_{k \in \mathbb{N}} D_k$, since $[D_k]$ has bipolar fuzzy diameter zero, for each fixed $\gamma > 0$, $\mu_A^r(u, v, \gamma) > 1 - \frac{1}{k}$ and $\mu_A^\ell(u, v, \gamma) < \frac{1}{k} - 1$ for each $k \in \mathbb{N}$. This denotes $\mu_A^r(u, v, \gamma) = 1$ and $\mu_A^\ell(u, v, \gamma) = -1$ and consequently $u = -v$.

It's worth noting that the regular bipolar fuzzy metric and the corresponding metric produce the same topologies. But here's what we've got:

Corollary 1 *Any nested sequence $[D_k]_{k \in \mathbb{N}}$ of nonempty closed sets with diameter tending to zero has nonempty intersection if and only if a metric space (X, d) is complete.*

Theorem 10 *Every separable bipolar fuzzy metric space is second countable.*

Proof The provided separable bipolar fuzzy metric space is $(A, \mathcal{B}, \Delta, \nabla)$. A countable dense subset of A will be $X = \{u_k : k \in \mathbb{N}\}$. Consider the case of the family $\mathcal{X} = \{\phi(u_j, \frac{1}{n}, \frac{1}{n}) : j, n \in \mathbb{N}\}$. Then \mathcal{X} can be counted. We conclude that \mathcal{X} is a base for the family of all open sets in A . Take O be any open set in A and $u \in O$. Then there is $\gamma > 0$ and $\delta \in (0, 1)$ such that $\phi(u, \delta, \gamma) \subset O$. Because of the fact that $\delta \in (0, 1)$, $\varepsilon \in (0, 1)$ can be selected such that

$(1 - \varepsilon)\Delta(1 - \varepsilon) > 1 - \delta$ and $(\varepsilon - 1)\nabla(\varepsilon - 1) < \delta - 1$. Take $t \in \mathbb{N}$ such that $\frac{1}{t} < \min\{\varepsilon, \frac{\gamma}{2}\}$. Since X is dense in A , there exists $u_j \in X$ such that $u_j \in \phi(u, \frac{1}{t}, \frac{1}{t})$. Now, if $v \in \phi(u_j, \frac{1}{t}, \frac{1}{t})$, then $\mu_A^r(u, v, \gamma) \geq \mu_A^r(u, u_j, \frac{\gamma}{2})\Delta\mu_A^r(v, u_j, \frac{\gamma}{2}) \geq \mu_A^r(u, u_j, \frac{1}{t})\Delta\mu_A^r(v, u_j, \frac{1}{t}) \geq (1 - \frac{1}{t})\Delta(1 - \frac{1}{t}) \geq (1 - \varepsilon)\Delta(1 - \varepsilon) > 1 - \delta$ and $\mu_A^\ell(u, v, \gamma) \leq \mu_A^\ell(u, u_j, \frac{\gamma}{2})\nabla\mu_A^\ell(v, u_j, \frac{\gamma}{2}) \leq \mu_A^\ell(u, u_j, \frac{1}{t})\nabla\mu_A^\ell(v, u_j, \frac{1}{t}) \leq (\frac{1}{t} - 1)\nabla(\frac{1}{t} - 1) \leq (\varepsilon - 1)\nabla(\varepsilon - 1) < \delta - 1$. Finally, $v \in \phi(u, \delta, \gamma) \subset O$ and as a result \mathcal{X} is a base. \square

Remark 8 Since second countability is an inherited property and second countability means separability, we get: A separable bipolar fuzzy metric space has separable subspaces.

Definition 10 Let A be a set that is not empty and $(B, \mathcal{B}, \Delta, \nabla)$ be an bipolar fuzzy metric space. If we take $\gamma > 0$ and $\delta \in (0, 1)$, then a sequence (g_k) of functions from A to B converges uniformly to a function g from A to B and $k_0 \in \mathbb{N}$ arises so that $\mu_A^r(g_k(u), g(u), \gamma) > 1 - \delta$ and $\mu_A^\ell(g_k(u), g(u), \gamma) > \delta - 1$ for all $k \geq k_0$ and $u \in A$.

Theorem 11 (Uniform limit theorem) Let $g_k : A \rightarrow B$ denote a sequence of continuous functions connecting a topological space A to a bipolar fuzzy metric space $(B, \mathcal{B}, \Delta, \nabla)$. Then, g is continuous, if (g_k) converges uniformly to $g : A \rightarrow B$.

Proof Assume Z is an open set of B and allow $u_0 \in g^{-1}(Z)$. We want to explore a W neighborhood of u_0 where $g(W) \subset Z$. Because Z is open, $\gamma > 0$ and $\delta \in (0, 1)$ exist, allowing $\phi(g(u_0), \delta, \gamma) \subset Z$. Because $\delta \in (0, 1)$, we select a $\varepsilon \in (0, 1)$ so that $(1 - \varepsilon)\Delta(1 - \varepsilon)\Delta(1 - \varepsilon) > 1 - \delta$ and $(\varepsilon - 1)\nabla(\varepsilon - 1)\nabla(\varepsilon - 1) < \delta - 1$. Because (g_k) converges to g uniformly, in the case of $\gamma > 0$ and $\varepsilon \in (0, 1)$, $k_0 \in \mathbb{N}$ exists in a form that $\mu_A^r(g_k(u), g(u), \frac{\gamma}{3}) > 1 - \varepsilon$ and $\mu_A^\ell(g_k(u), g(u), \frac{\gamma}{3}) < \varepsilon - 1$ for all $k \geq k_0$ and $u \in A$. While g_k is continuous for all $k \in \mathbb{N}$ there is a neighborhood W of u_0 in which $g_k(W) \subset \phi(g_k(u_0), \varepsilon, \frac{\gamma}{3})$. Consequently, $\mu_A^r(g_k(u), g_k(u_0), \frac{\gamma}{3}) > 1 - \varepsilon$ and $\mu_A^\ell(g_k(u), g_k(u_0), \frac{\gamma}{3}) < \varepsilon - 1$ for all $u \in W$. Also, $\mu_A^r(g(u), g(u_0), \gamma) \geq \mu_A^r(g(u), g_k(u), \frac{\gamma}{3})\Delta\mu_A^r(g_k(u), g_k(u_0), \frac{\gamma}{3})\Delta\mu_A^r(g_k(u_0), g(u_0), \frac{\gamma}{3}) \geq (1 - \varepsilon)\Delta(1 - \varepsilon)\Delta(1 - \varepsilon) > 1 - \delta$ and $\mu_A^\ell(g(u), g(u_0), \gamma) \leq \mu_A^\ell(g(u), g_k(u), \frac{\gamma}{3})\nabla\mu_A^\ell(g_k(u), g_k(u_0), \frac{\gamma}{3})\nabla\mu_A^\ell(g_k(u_0), g(u_0), \frac{\gamma}{3}) \leq (\varepsilon - 1)\nabla(\varepsilon - 1)\nabla(\varepsilon - 1) < \delta - 1$. Thus $g(u) \in \phi(g(u_0), \delta, \gamma) \subset Z$ for all $u \in W$. From here, $g(W) \subset Z$ and so g is continuous. \square

5 Application to vaccine selection in COVID-19

The COVID-19 pandemic has emerged as a serious public health emergency and prompted a process to be addressed. As of March 11, 2020, the World Health Organization declared this event as a public health emergency in accordance with the International Health Regulations. It is important to choose the most appropriate vaccine for the groups to be administered COVID-19 vaccine by evaluating the risks of exposure to the disease, the risks of spreading and transmitting the disease, and the negative impact of the disease on the functioning of social life. Until now, serious side effects have not been encountered in both clinical trials conducted for COVID-19 vaccines and current vaccination practices. Post-vaccination side effects are often mild.

These; mild side effects such as fatigue, headache, fever, chills, muscle pain, vomiting, diarrhea, pain, redness, swelling in the area where the vaccine was applied.

No vaccine can be approved or put on the market without conducting phase studies and without transparently sharing the results of these studies with the relevant authorities. The

organization conducting vaccine studies must comply with international quality standards during the development and production process. These standards are Good Laboratory Practices (GLP), Good Clinical Practices (GCP) and Good Manufacturing Practices (GMP). In addition, during the phase studies, when a situation that is thought to be related to the vaccine and significantly affects human health is detected, the studies are stopped. Studies are carried out only if it is ensured that the problem is not related to the vaccine, and if vaccination is performed, it is continued from where it left off. However, it is still important to choose the most suitable vaccine for the person.

Several different types of vaccines are being developed for COVID-19. All of these vaccines are designed to teach the body's immune system to safely introduce and destroy the virus that causes COVID-19. We evaluate these vaccines in five categories.

1. Vaccines that do not cause disease but contain inactivated virus (Inactivated vaccines) that produce an immune response. (L_1)
2. Vaccines containing attenuated virus that do not cause disease but produce an immune response (Live attenuated vaccines). (L_2)
3. Protein-based vaccines that use protein fragments that mimic the structure of the COVID-19 virus to safely induce an immune response. (L_3)
4. Viral vector vaccines using non-pathogenic viruses that carry RNA particles of the COVID-19 virus to create a safe immune response. (L_4)
5. M-RNA and DNA vaccines, a state-of-the-art approach uses genetically engineered RNA and DNA fragments to produce proteins that induce a safe immune response on its own. (L_5)

In this section, we have included an application that shows which of these 5 vaccine types is more suitable. Appropriate measurements of similarity were used in this application that allow us to make a comparison between selected bipolar fuzzy sets.

Now, let's start by giving a definition of similarity measure on BFS.

Definition 11 (Abdullah et al. 2014). Let U be the universe of discourse and E be the set of parameters and $B \subseteq E$. Let $\psi : B \rightarrow BF^U$ be a mapping, then a bipolar fuzzy soft set (BFSS) (ψ, B) or ψ_B is defined by

$$(\psi, B) = \left\{ \left(e, \{ \ell, \mu_A^r(\ell), \mu_A^\ell(\ell) \} \right) : e \in B, \ell \in U \right\}.$$

If $U = \{\ell_1, \dots, \ell_m\}$, $B = \{e_1, \dots, e_n\}$, then BFSS ψ_B in the tabular form is expressed as in the following:

Definition 12 Let $U = \{\ell_i : i = 1, 2, \dots, m\}$ be a classical set and $P = \{P_j : j = 1, 2, \dots, n\}$ is the sum of attributes. Then

$$\psi_1 = (\psi_1, B) = \begin{pmatrix} (\mu_{11}^r, \mu_{11}^\ell)_{\psi_1} & (\mu_{12}^r, \mu_{12}^\ell)_{\psi_1} & \cdots & (\mu_{1n}^r, \mu_{1n}^\ell)_{\psi_1} \\ (\mu_{21}^r, \mu_{21}^\ell)_{\psi_1} & (\mu_{22}^r, \mu_{22}^\ell)_{\psi_1} & \cdots & (\mu_{2n}^r, \mu_{2n}^\ell)_{\psi_1} \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1}^r, \mu_{m1}^\ell)_{\psi_1} & (\mu_{m2}^r, \mu_{m2}^\ell)_{\psi_1} & \cdots & (\mu_{mn}^r, \mu_{mn}^\ell)_{\psi_1} \end{pmatrix}$$

and

$$\psi_2 = (\psi_2, B) = \begin{pmatrix} (\mu_{11}^r, \mu_{11}^\ell)_{\psi_2} & (\mu_{12}^r, \mu_{12}^\ell)_{\psi_2} & \cdots & (\mu_{1n}^+, \mu_{1n}^\ell)_{\psi_2} \\ (\mu_{21}^r, \mu_{21}^\ell)_{\psi_2} & (\mu_{22}^r, \mu_{22}^\ell)_{\psi_2} & \cdots & (\mu_{2n}^r, \mu_{2n}^\ell)_{\psi_2} \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{m1}^r, \mu_{m1}^\ell)_{\psi_2} & (\mu_{m2}^r, \mu_{m2}^\ell)_{\psi_2} & \cdots & (\mu_{mn}^r, \mu_{mn}^\ell)_{\psi_2} \end{pmatrix}$$

are BFS-matrices (ψ_1, P) and (ψ_2, P) , then similarity measure between (ψ_1, P) and (ψ_2, P) is given as below:

$$\text{Sim}(\psi_1, \psi_2) = \frac{\langle \psi_1, \psi_2 \rangle}{\|\psi_1\| \|\psi_2\|}.$$

Here

$$\begin{aligned} \langle \psi_1, \psi_2 \rangle &= \sum_{i,j} (\mu_{ij}^r, \mu_{ij}^l)_{\psi_1} \cdot (\mu_{ij}^r, \mu_{ij}^l)_{\psi_2}, \\ \|\psi_1\| &= \sqrt{\langle \psi_1, \psi_1 \rangle}. \end{aligned}$$

Proposition 1 *The similarity measure defined in Definition 12 satisfies the following situations:*

1. $0 \leq \text{Sim}(\psi_1, \psi_2) \leq 1$
2. $\text{Sim}(\psi_1, \psi_2) = 1 \Leftrightarrow \psi_1 = \psi_2$
3. $\text{Sim}(\psi_1, \psi_2) = \text{Sim}(\psi_2, \psi_1)$.

Definition 13 Let ψ_1 and ψ_2 be two BFSSs. Suppose that the weights of attributes p_j are $\vartheta_j \in [0, 1]$ for $j = 1, 2, \dots, n$. The weighted similarity measure Sim_W between ψ_1 and ψ_2 is defined by

$$\text{Sim}_W(\psi_1, \psi_2) = \frac{\langle \psi_1, \psi_2 \rangle}{\|\psi_1\| \|\psi_2\|}$$

where

$$\begin{aligned} \langle \psi_1, \psi_2 \rangle &= \frac{\sum_{i,j} \vartheta_j (\mu_{ij}^+, \mu_{ij}^-)_{\psi_1} \cdot (\mu_{ij}^+, \mu_{ij}^-)_{\psi_2}}{\sum_j \vartheta_j}, \\ \|\psi_1\| &= \sqrt{\langle \psi_1, \psi_1 \rangle}. \end{aligned}$$

Proposition 2 *The similarity measure defined in Definition 13 provides the circumstances listed below:*

1. $0 \leq \text{Sim}_W(\psi_1, \psi_2) \leq 1$
2. $\text{Sim}_W(\psi_1, \psi_2) = 1 \Leftrightarrow \psi_1 = \psi_2$
3. $\text{Sim}_W(\psi_1, \psi_2) = \text{Sim}_W(\psi_2, \psi_1)$.

Example 3 Let us take ψ_1 and ψ_2 , BFS-matrices of (ψ_1, p) and (ψ_2, p) as given below:

$$\begin{aligned} \psi_1 &= \begin{pmatrix} (0.513, -0.279) & (0.312, -0.700) & (0.624, -0.861) \\ (0.342, -0.489) & (0.317, -0.817) & (0.754, -0.426) \\ (0.329, -0.544) & (0.604, -0.152) & (0.214, -0.191) \end{pmatrix}, \\ \psi_2 &= \begin{pmatrix} (0.641, -0.040) & (0.542, -0.842) & (0.384, -0.216) \\ (0.045, -0.343) & (0.941, -0.525) & (0.782, -0.013) \\ (0.253, -0.517) & (0.317, -0.544) & (0.262, -0.415) \end{pmatrix}. \end{aligned}$$

Then similarity measure between (ψ_1, p) and (ψ_2, p) will be determined as follows:

$$\begin{aligned} \langle \psi_1, \psi_2 \rangle &= 3.7960 \\ \|\psi_1\| &= 2.1909 \\ \|\psi_2\| &= 2.1123 \\ \therefore \text{Sim}(\psi_1, \psi_2) &= 0.8203. \end{aligned}$$

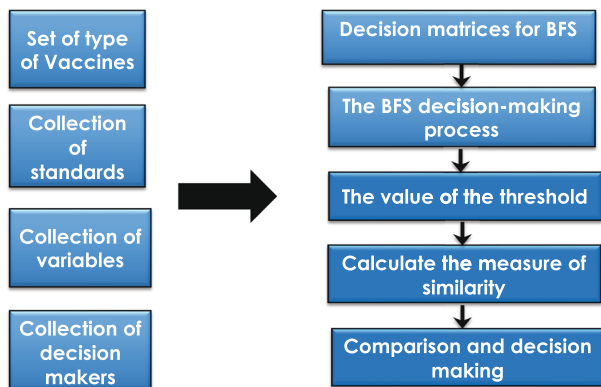


Fig. 1 Diagram of Algorithm 1

Additionally, consider $\vartheta_1 = 0.12$, $\vartheta_1 = 0.67$ and $\vartheta_1 = 0.54$ as weights of p_1 , p_2 and p_3 . After, the subsequent computations can be made:

$$\begin{aligned}\langle \psi_1, \psi_2 \rangle &= 1.9185 \\ \|\psi_1\| &= 1.5352 \\ \|\psi_2\| &= 1.5428 \\ \therefore \text{Sim}_W(\psi_1, \psi_2) &= 0.8100.\end{aligned}$$

Algorithm 1

- Step 1: Notice the set of type of vaccines $L = \{L_1, L_2, \dots, L_n\}$.
 Step 2: Notice the sets of requirement $M = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$ and parameters $E = \{e_1, e_2, \dots, e_m\}$ and a group of decision makers $D = \{d_1, d_2, \dots, d_k\}$.
 Step 3: Create a BFSS (ψ, E) model to compute the similarity measure.
 Step 4: Determine the BFS-decision matrix for each vaccine type by each decision maker.
 Step 5: Search the unanimous decision matrix and make a table with (ψ, E) and (ψ_i, E) .
 Step 6: Determine the degree of similarity between each BFSS (ψ_i, E) and the model BFSS (ψ, E) .
 Step 7: Determine the threshold measure $\xi \in]0, 1[$.
 Step 8: The vaccine with the highest similarity measure, $\geq \xi$, is the best option.
 Step 9: Make a final classification.

The materials required of Algorithm 1 are illustrated in Figs. 1 and 2.

5.1 Numerical example

A government intends to purchase vaccines from corporations that have the vaccine due to the COVID-19 outbreak. The government selected two decision makers to determine the best type of vaccines from the five types of vaccines suitable for its conditions along with particularities $C = \{\alpha_1, \alpha_2, \alpha_3\}$. Here,

- α_1 = Sustainability
- α_2 = Efficiency
- α_3 = Cost

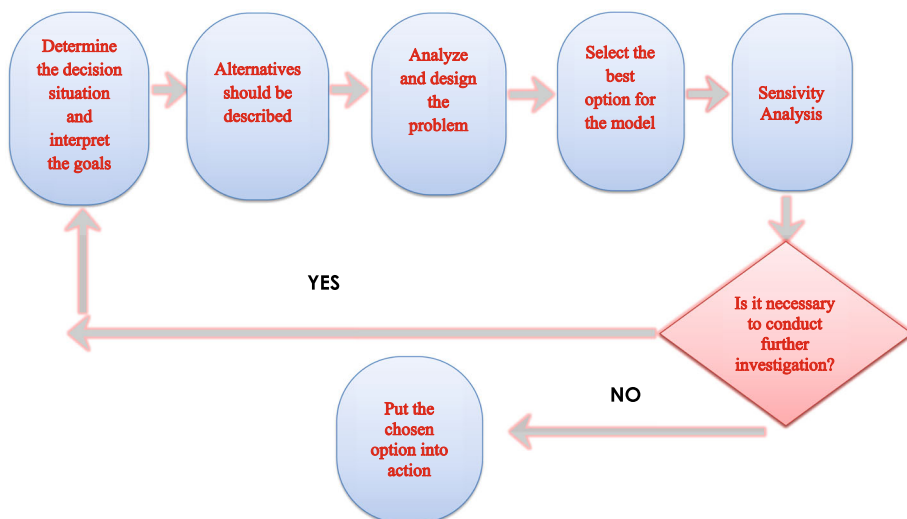


Fig. 2 An example of decision making algorithm

Take into account the collection of variables

$$E = \{e_1 = \text{unimpressed}, e_2 = \text{impressed}, e_3 = \text{extremely impressed}\},$$

which are linguistic identifiers that define the degree of satisfaction with decision-making when evaluating a vaccine. Assume that (ψ, E) is the regular BFSS, which is considered as a model BFSS by the decision maker. A positive grade is given if the effectiveness of the type of vaccine is more appropriate than the country's expectation standard. A negative grade is also granted if the suitability of the type of vaccine is less effective based on certain parameters. The government determines two decision makers from the management committee to assign the best form of vaccine to administer to the population. Conclude that (ψ_i, E) is the BFSS model of the vaccine kind, with $L_i, i = 1, 2, 3, 4, 5$ being assigned by each decision manager. Then, by considering the maximum of the positive values and the minimum of the negative values, we arrive at a consensus BFS decision. To determine the SM between these BFSSs, we will compare the vaccines' characteristics with the analysis in Table 3. BFS decision matrices are shown in Tables 1 and 2. Assume that the threshold value is 0.8. When we class with this value to the determined value of SM in Table 3, it would seem that the BFSS model (ψ, E) is very similar to (ψ_1, E) . Also, it means that, it is unlike any of the other sets. We acknowledge that the form of vaccine L_1 is the best option for satisfying certain requirements. Since L_1 is the most similar to model, the government should select it as their country's vaccine provider.

In Table 4, the SMs between BFS-model and vaccine form are listed as below:

To compute the weighted SMs, we use the weights $w_1 = 0.119$, $w_2 = 0.452$, and $w_3 = 0.817$, which correspond to e_1, e_2 , and e_3 , respectively. By this way we calculate the weighted SMs. Table 5 becomes

$$L_1 > L_4 > L_5 > L_2 > L_3$$

when ranking vaccine types based on weighted similarity measure.

Thus, as shown in the Tables 4, 5, it can be easily said that L_1 is the most suitable and preferable vaccine option.

Table 1 BFS decision matrix 1

	(e_1, α_1)	(e_1, α_2)	(e_1, α_3)	(e_2, α_1)	(e_2, α_2)	(e_2, α_3)	(e_3, α_1)	(e_3, α_2)	(e_3, α_3)
(ψ_1, E)	$(0.251,$ $-0.798)$	$(0.375,$ $-0.692)$	$(0.491,$ $-0.693)$	$(0.109,$ $-0.910)$	$(0.262,$ $-0.823)$	$(0.501,$ $-0.652)$	$(0.114,$ $-0.513)$	$(0.730,$ $-0.215)$	$(0.177,$ $-0.807)$
(ψ_2, E)	$(0.322,$ $-0.358)$	$(0.536,$ $-0.604)$	$(0.809,$ $-0.540)$	$(0.650,$ $-0.880)$	$(0.251,$ $-0.652)$	$(0.504,$ $-0.790)$	$(0.163,$ $-0.542)$	$(0.317,$ $-0.289)$	$(0.627,$ $-0.943)$
(ψ_3, E)	$(0.556,$ $-0.215)$	$(0.941,$ $-0.172)$	$(0.317,$ $-0.429)$	$(0.532,$ $-0.609)$	$(0.890,$ $-0.462)$	$(0.122,$ $-0.945)$	$(0.214,$ $-0.604)$	$(0.119,$ $-0.648)$	$(0.940,$ $-0.786)$
(ψ_4, E)	$(0.329,$ $-0.417)$	$(0.212,$ $-0.413)$	$(0.785,$ $-0.321)$	$(0.117,$ $-0.850)$	$(0.150,$ $-0.513)$	$(0.175,$ $-0.782)$	$(0.450,$ $-0.116)$	$(0.517,$ $-0.902)$	$(0.113,$ $-0.551)$
(ψ_5, E)	$(0.641,$ $-0.040)$	$(0.891,$ $-0.349)$	$(0.116,$ $-0.415)$	$(0.045,$ $-0.343)$	$(0.941,$ $-0.525)$	$(0.782,$ $-0.013)$	$(0.351,$ $-0.789)$	$(0.814,$ $-0.351)$	$(0.553,$ $-0.652)$

Table 2 BFS decision matrix 2

	(e_1, α_1)	(e_1, α_2)	(e_1, α_3)	(e_2, α_1)	(e_2, α_2)	(e_2, α_3)	(e_3, α_1)	(e_3, α_2)	(e_3, α_3)
(ψ_1, E)	(0.344, -0.421)	(0.634, -0.177)	(0.821, -0.912)	(0.536, -0.945)	(0.251, -0.103)	(0.652, -0.945)	(0.153, -0.567)	(0.998, -0.324)	(0.665, -0.814)
(ψ_2, E)	(0.820, -0.359)	(0.454, -0.013)	(0.284, -0.724)	(0.329, -0.544)	(0.804, -0.281)	(0.100, -0.561)	(0.172, -0.803)	(0.311, -0.382)	(0.542, -0.719)
(ψ_3, E)	(0.212, -0.512)	(0.627, -0.409)	(0.943, -0.786)	(0.431, -0.512)	(0.132, -0.781)	(0.160, -0.562)	(0.521, -0.716)	(0.261, -0.490)	(0.163, -0.054)
(ψ_4, E)	(0.263, -0.152)	(0.730, -0.533)	(0.642, -0.356)	(0.991, -0.357)	(0.820, -0.414)	(0.823, -0.604)	(0.410, -0.370)	(0.880, -0.662)	(0.942, -0.480)
(ψ_5, E)	(0.423, -0.517)	(0.542, -0.842)	(0.384, -0.216)	(0.258, -0.343)	(0.570, -0.214)	(0.215, -0.066)	(0.253, -0.517)	(0.317, -0.544)	(0.262, -0.415)

Table 3 Unanimous BFS decision matrix

	(e_1, α_1)	(e_1, α_2)	(e_1, α_3)	(e_2, α_1)	(e_2, α_2)	(e_2, α_3)	(e_3, α_1)	(e_3, α_2)	(e_3, α_3)
(ψ, E)	$(0.513,$ $-0.279)$	$(0.312,$ $-0.700)$	$(0.624,$ $-0.861)$	$(0.342,$ $-0.489)$	$(0.317,$ $-0.817)$	$(0.754,$ $-0.426)$	$(0.329,$ $-0.544)$	$(0.604,$ $-0.152)$	$(0.214,$ $-0.191)$
(ψ_1, E)	$(0.344,$ $-0.421)$	$(0.375,$ $-0.692)$	$(0.491,$ $-0.693)$	$(0.109,$ $-0.910)$	$(0.251,$ $-0.103)$	$(0.501,$ $-0.652)$	$(0.114,$ $-0.513)$	$(0.730,$ $-0.215)$	$(0.177,$ $-0.807)$
(ψ_2, E)	$(0.820,$ $-0.359)$	$(0.454,$ $-0.013)$	$(0.284,$ $-0.724)$	$(0.650,$ $-0.880)$	$(0.804,$ $-0.281)$	$(0.100,$ $-0.561)$	$(0.172,$ $-0.803)$	$(0.317,$ $-0.289)$	$(0.627,$ $-0.943)$
(ψ_3, E)	$(0.556,$ $-0.215)$	$(0.941,$ $-0.172)$	$(0.317,$ $-0.429)$	$(0.431,$ $-0.512)$	$(0.132,$ $-0.781)$	$(0.112,$ $-0.945)$	$(0.521,$ $-0.716)$	$(0.261,$ $-0.490)$	$(0.940,$ $-0.786)$
(ψ_4, E)	$(0.263,$ $-0.152)$	$(0.212,$ $-0.413)$	$(0.785,$ $-0.321)$	$(0.991,$ $-0.357)$	$(0.820,$ $-0.414)$	$(0.823,$ $-0.604)$	$(0.450,$ $-0.116)$	$(0.880,$ $-0.662)$	$(0.942,$ $-0.480)$
(ψ_5, E)	$(0.641,$ $-0.040)$	$(0.542,$ $-0.842)$	$(0.384,$ $-0.216)$	$(0.045,$ $-0.343)$	$(0.941,$ $-0.525)$	$(0.782,$ $-0.013)$	$(0.253,$ $-0.517)$	$(0.317,$ $-0.544)$	$(0.262,$ $-0.415)$

Table 4 SM between the BFSS method and the vaccine type

Type of Vaccines	$\text{Sim}(\psi, \psi_i)$
L_1	0.8523
L_2	0.7318
L_3	0.7430
L_4	0.7869
L_5	0.8203

Table 5 Weighted SM between the BFSS method and the vaccine type

Type of Vaccines	$\text{Sim}_W(\psi, \psi_i)$
L_1	0.8527
L_2	0.6813
L_3	0.6660
L_4	0.8033
L_5	0.7989

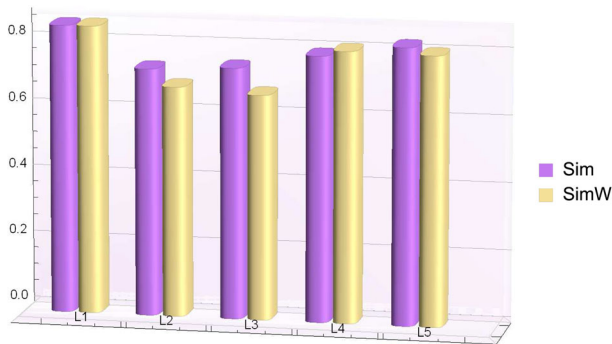
**Fig. 3** Diagram of Sim and Sim_W of type of vaccines provider

Table 4 becomes

$$L_1 > L_5 > L_4 > L_3 > L_2$$

when ranking vaccine types based on similarity measure. Table 5 shows the weighted SM between the BFSS model and the vaccine type as given in the follows:

The ranking of these two measures for each type of vaccines provider could be seen in Fig. 3.

6 Conclusion

The aim of this research is to define bipolar metric spaces and investigate some of their properties. The characteristic features of BFMSs have been developed, including open set, T_2 -space, compactness, completeness and nowhere dense. Following that, Baire Category and Uniform Convergence Theorems for bipolar metric spaces are presented. Afterwards, an application on determining the appropriate type of vaccine in the treatment process of

COVID-19 is given using similarity measure between bipolar fuzzy metric spaces. For vaccine selection and supply chain management, an advanced multi-attribute decision-making (MADM) algorithm is being developed using similarity measure between bipolar fuzzy metric spaces. The validity of the proposed MADM method for selecting the best proper form of vaccine is also established.

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