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An neutrosophic topological operator and its application in the building of new neutrosophic sets

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Abstract

In this article, we use the notion of neutrosophic local function to introduce a new neutrosophic operator in the context of a neutrosophic topological space equipped with a neutrosophic ideal. Also, we introduce and study some new classes of neutrosophic sets defined in terms of the neutrosophic local function and the new notion of neutrosophic operator given.

Keywords: Neutrosophic set; neutrosophic topological space; neutrosophic ideal; neutrosophic local function

1 introduction

The notion of neutrosophic set has gained much relevance in recent years due to its various applications, some of which have been reported by Alias and Mohamad,[3] AboElHamd et al.[1] and Essa et al.[4] The key point of neutrosophic set theory is that indeterminacy is explicitly quantified and the truth membership function, the indeterminacy membership function and the falsity membership functions are independent. This theory was proposed by Smarandache[11] and has been studied by many researchers. In particular, Karataş and Kuru[5] introduced new neutrosophic set operations and defined the concept of neutrosophic topological space with them. Following this line of research, Albowi and Salama[2] introduced the notion of neutrosophic ideal, which was later used by Salama and Smarandache[9] to introduce the concept of neutrosophic local function, investigate its properties and analyze the relations between different neutrosophic ideals and neutrosophic topologies. Recently, the study of neutrosophic topologies has had a breakthrough with the foundation of revolutionary topologies by Smarandache,[12] which is inspiring work by other researchers. The purpose of this paper is to continue with this line of research, but introducing a new notion of neutrosophic operator by using the neutrosophic local function, investigate the main properties of this new neutrosophic operator and with it we construct new classes of neutrosophic sets in a neutrosophic topological space endowed with a neutrosophic ideal. The results presented here can potentially be used to develop new strands in the framework of Smarandache's revolutionary topologies.

2 Preliminaries

Throughout this paper, X denoted a nonempty set called the *universe of discourse*.

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Definition 2.1. [11] A *neutrosophic set* N on X is an object of the form

$$N = \{\langle x, \mu_N(x), \sigma_N(x), \gamma_N(x)\rangle : x \in X\},$$

where $\mu_N, \sigma_N, \gamma_N$ are functions from X to $[0, 1]$ and $0 \leq \mu_N(x) + \sigma_N(x) + \gamma_N(x) \leq 3$.

We denote by $\mathcal{N}(X)$ the collection of all neutrosophic sets over X.

Definition 2.2. [5] For $N, M \in \mathcal{N}(X)$ we define the following:

1. (Inclusion) N is called a *neutrosophic subset* of M, denoted by $N \sqsubseteq M$, if $\mu_N(x) \leq \mu_M(x)$, $\sigma_N(x) \geq \sigma_M(x)$ and $\gamma_N(x) \geq \gamma_M(x)$ for all $x \in X$. Also, we can say that M is a neutrosophic super set of N.
2. (Equality) N is called *neutrosophic equal* to M, denoted by $N = M$, if $N \sqsubseteq M$ and $M \sqsubseteq N$.
3. (Universal set) N is called the *neutrosophic universal set*, denoted by $\widetilde{X}$, if $\mu_N(x) = 1, \sigma_N(x) = 0$ and $\gamma_N(x) = 0$ for all $x \in X$.
4. (Empty set) N is called the *neutrosophic empty set*, denoted by $\widetilde{\emptyset}$, if $\mu_N(x) = 0, \sigma_N(x) = 1$ and $\gamma_N(x) = 1$ for all $x \in X$.
5. (Intersection) The *neutrosophic intersection* of N and M, denoted by $N \sqcap M$, is defined as
$$N \sqcap M = \{(x, \mu_N(x) \wedge \mu_M(x), \sigma_N(x) \vee \sigma_M(x), \gamma_N(x) \vee \gamma_M(x)\rangle : x \in X\}.$$
6. (Union) The *neutrosophic union* of N and M, denoted by $N \sqcup M$, is defined as
$$N \sqcup M = \{\langle x, \mu_N(x) \vee \mu_M(x), \sigma_N(x) \wedge \sigma_M(x), \gamma_N(x) \wedge \gamma_M(x)\rangle : x \in X\}.$$
7. (Complement) The *neutrosophic complement* of N, denoted by N^c, is defined as
$$N^c = \{\langle x, \gamma_N(x), 1 - \sigma_N(x), \mu_N(x)\rangle : x \in X\}.$$

Proposition 2.3. [5] If $N, M, O \in \mathcal{N}(X)$, then we have the following properties:

1. $N \sqcap N = N$ and $N \sqcup N = N$.
2. $N \sqcap M = M \sqcap N$ and $N \sqcup M = M \sqcup N$.
3. $N \sqcap \tilde{\emptyset} = \tilde{\emptyset}$ and $N \sqcap \tilde{X} = N$.
4. $N \sqcup \tilde{\emptyset} = N$ and $N \sqcup \tilde{X} = \tilde{X}$.
5. $N \sqcap (M \sqcap O) = (N \sqcap M) \sqcap O$ and $N \sqcup (M \sqcup O) = (N \sqcup M) \sqcup O$.
6. $(N^c)^c = N$.

Proposition 2.4. [7] Let $N, M \in \mathcal{N}(X)$. Then, $N \sqsubseteq M$ if and only if $M^c \sqsubseteq N^c$.

The union and intersection operations given in Definition 2.2 can be extended as follows.

Definition 2.5. [8] For $\{N_j : j \in J\} \subseteq \mathcal{N}(X)$ we define the following operations:

1. (Arbitrary intersection) The *arbitrary neutrosophic intersection* of the collecction $\{N_j : j \in J\}$, denoted by ${}_{j\in J}N_j$, is defined as
$${}_{j\in J}N_j = \left\{\left\langle x, \inf_{j\in J} \mu_{N_j}(x), \sup_{j\in J} \sigma_{N_j}(x), \sup_{j\in J} \gamma_{N_j}(x)\right\rangle : x \in X\right\}.$$

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2. (Arbitrary union) The *arbitrary neutrosophic union* of the collecction $\{N_j : j \in J\}$, denoted by $\bigsqcup\limits_{j\in J} N_j$, is defined as
$$\bigsqcup_{j\in J} N_j = \left\{ \left\langle x, \sup_{j\in J} \mu_{N_j}(x), \inf_{j\in J} \sigma_{N_j}(x), \inf_{j\in J} \gamma_{N_j}(x) \right\rangle : x \in X \right\}.$$

Proposition 2.6. [5] If $\{N_j : j \in J\} \subseteq \mathcal{N}(X)$ and $M \in \mathcal{N}(X)$, then we have the following properties:

1. $M \sqcap \left(\bigsqcup\limits_{j\in J} N_j \right) = \bigsqcup\limits_{j\in J} (M \sqcap N_j)$.
2. $M \sqcup ({}_{j\in J} N_j) =_{j\in J} (M \sqcup N_j)$.
3. $({}_{j\in J} N_j)^c = \bigsqcup\limits_{j\in J} N_j^c$.
4. $\left(\bigsqcup\limits_{j\in J} N_j \right)^c =_{j\in J} N_j^c$.

Definition 2.7. [5] A *neutrosophic topology* on a set X is a collection $\tau \subseteq \mathcal{N}(X)$ which satisfies the following conditions:

1. $\widetilde{\emptyset}$ and $\widetilde{X}$ are in τ.
2. The intersection of two neutrosophic sets belonging to τ is in τ.
3. The union of any collection of neutrosophic sets belonging to τ is in τ.

A set X for which a neutrosophic topology τ has been defined is called a *neutrosophic topological space* and is denoted as a pair (X, τ). If $N \in \tau$, then N is called a *neutrosophic open set* and if $N^c \in \tau$, then N is called a *neutrosophic closed set*. We denote by τ^c the collection of all neutrosophic closed sets in the neutrosophic topological space (X, τ).

Definition 2.8. [5] Let (X, τ) be a neutrosophic topological space and $N \in \mathcal{N}(X)$. The *neutrosophic closure* of N, denoted by $Cl(N)$, is defined as
$$Cl(N) = \{F \in \mathcal{N}(X) : N \sqsubseteq F \text{ and } F \in \tau^c\};$$
while the *neutrosophic interior* of N, denoted by $Int(N)$, is defined as
$$Int(N) = \bigsqcup \{U \in \mathcal{N}(X) : U \sqsubseteq N \text{ and } U \in \tau\}.$$

Proposition 2.9. [5] Let (X, τ) be a neutrosophic topological space and $N, M \in \mathcal{N}(X)$. Then, the following conditions hold:

1. $N \sqsubseteq Cl(N)$ and $Int(N) \sqsubseteq N$.
2. If $N \sqsubseteq M$, then $Cl(N) \sqsubseteq Cl(M)$ and $Int(N) \sqsubseteq Int(M)$.
3. $N \in \tau^c$ if and only if $N = Cl(N)$.
4. $N \in \tau$ if and only if $N = Int(N)$.

Now, we present the concept of neutrosophic point given by Ray and Dey,[6] and some of the properties associated with this concept are described.

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Definition 2.10. [6] A neutrosophic set $M = \{\langle x, \mu_M(x), \sigma_M(x), \gamma_M(x)\rangle : x \in X\}$ is called a *neutrosophic point* if for any element $y \in X$, $\mu_M(y) = a$, $\sigma_M(y) = b$, $\gamma_M(y) = c$ for $y = x$ and $\mu_M(y) = 0$, $\sigma_M(y) = 1$, $\gamma_M(y) = 1$ for $y \neq x$, where $a \in (0, 1]$ and $b, c \in [0, 1)$. In this case, the neutrosophic point M is denoted by $M^x_{a,b,c}$ or simply by $x_{a,b,c}$. Also, x is called the *support* of the neutrosophic point $x_{a,b,c}$. The neutrosophic point $x_{1,0,0}$ is called a *neutrosophic crisp point*.

Definition 2.11. [6] Let $N \in \mathcal{N}(X)$. A neutrosophic point $x_{a,b,c}$ is said to belong to N, denoted by $x_{a,b,c} \in N$, if $\mu_N(x) \geq a$, $\sigma_N(x) \leq b$ and $\gamma_N(x) \leq c$.

Proposition 2.12. [6] Let $\{N_j : j \in J\} \subseteq \mathcal{N}(X)$ and let $x_{a,b,c}$ be a neutrosophic point on X. The following properties hold:

1. $x_{a,b,c} \in {}_{j\in J}N_j$ if and only if $x_{a,b,c} \in N_j$ for each $j \in J$.
2. $x_{a,b,c} \in N_j$ for some $j \in J$ implies that $x_{a,b,c} \in \bigsqcup\limits_{j\in J} N_j$.
3. $x_{a,b,c} \in \bigsqcup\limits_{j\in J} N_j$ implies that there exists a neutrosophic set N such that $x_{a,b,c} \in N \sqsubseteq \bigsqcup\limits_{j\in J} N_j$.

Remark 2.13. It is important to note that $\widetilde{\emptyset}$ is not the only neutrosophic set that does not have points belonging to it. For example, if $X = \{x, y\}$, then $N = \{\langle x, 0, 0.5, 1\rangle, \langle y, 0, 0.4, 1\rangle\}$ is a neutrosophic set over X for which there are no neutrosophic points belonging to it.

Let $\mathcal{N}_p(X) = \{N \in \mathcal{N}(X) :$ there exists a neutrosophic point $x_{a,b,c} \in N\}$ and let $\mathcal{N}'(X) = \{\widetilde{\emptyset}\} \cup \mathcal{N}_p(X)$. In the remainder of this paper, we will use the definitions and results described previously, restricted to the collection $\mathcal{N}'(X)$.

Definition 2.14. [10] Let (X, τ) be a neutrosophic topological space and $N \in \mathcal{N}'(X)$. The *neutrosophic point-closure* of N, denoted by $Cl_p(N)$, is defined as

$$Cl_p(N) = \bigsqcup\{x_{a,b,c} \in \mathcal{N}'(X) : U \sqcap N \neq \widetilde{\emptyset} \text{ for every } U \in \tau(x_{a,b,c})\},$$

where $\tau(x_{a,b,c}) = \{U \in \tau : x_{a,b,c} \in U\}$.

Remark 2.15. In general, in a neutrosophic topological space (X, τ) it is not true that $Cl(N) = Cl_p(N)$ for each $N \in \mathcal{N}'(X)$. Moreover, none of the neutrosophic inclusions $Cl(N) \sqsubseteq Cl_p(N)$ and $Cl_p(N) \sqsubseteq Cl(N)$ is true in general, as we can see in.[10]

According to,[10] the colection $\tau_p = \{N \in \mathcal{N}'(X) : Cl_p(N^c) = N^c\}$ is a neutrosophic topology on X and Cl_p is the neutrosophic closure in the neutrosophic topological space (X, τ_p). We say that a neutrosophic set N is neutrosophic τ_p-open, if $N \in \tau_p$. The complement of a neutrosophic τ_p-open set we will call it a neutrosophic τ_p-closed set. We denote by Int_p the neutrosophic interior in the neutrosophic topological space (X, τ_p). Let us note that M is τ_p-open neutrosophic if and only if $Int_p(M) = M$; while M is τ_p-closed neutrosophic if and only if $Cl_p(M) = M$.

3 Neutrosophic ideals and new properties of the neutrosophic local function

Definition 3.1. [2] A *neutrosophic ideal* on a set X is a nonempty collection $\mathcal{L} \subseteq \mathcal{N}'(X)$, which satisfies the following conditions:

1. $N \in \mathcal{L}$ and $M \sqsubseteq N$ imply that $M \in \mathcal{L}$. (Hereditary property)
2. $N, M \in \mathcal{L}$ imply that $N \sqcup M \in \mathcal{L}$. (Finite additivity property)

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Proposition 3.2. Let $\{\mathcal{L}_j : j \in J\}$ be any nonempty collection of neutrosophic ideals on a set X. Then, $\bigcap_{j\in J} \mathcal{L}_j$ is a neutrosophic ideal on X.

Proof. We verify that $\mathcal{L} = \bigcap_{j\in J} \mathcal{L}_j$ satisfies the two conditions of a neutrosophic ideal on X.

(1) If $N \in \mathcal{L}$ and $M \sqsubseteq N$, then $N \in \mathcal{L}_j$ for each $j \in J$ and so, by the hereditary property of each $\mathcal{L}_j$, we have $M \in \mathcal{L}_j$ for each $j \in J$. Hence, $M \in \bigcap_{j\in J} \mathcal{L}_j = \mathcal{L}$.

(2) If $N, M \in \mathcal{L}$, then $N, M \in \mathcal{L}_j$ for each $j \in J$. By the additive property of each $\mathcal{L}_j$, $N \sqcup M \in \mathcal{L}_j$ for each $j \in J$. Thus, $N \sqcup M \in \bigcap_{j\in J} \mathcal{L}_j = \mathcal{L}$. □

Proposition 3.3. Let $\mathcal{L}$ and $\mathcal{L}'$ be two neutrosophic ideals on a set X. Then, the collection $\mathcal{L} \vee \mathcal{L}' = \{L \sqcup L' : L \in \mathcal{L} \text{ and } L' \in \mathcal{L}'\}$ is a neutrosophic ideal on X.

Proof. We verify that $\mathcal{L} \vee \mathcal{L}'$ satisfies the two conditions of a neutrosophic ideal on X.

(1) Suppose that $N \in \mathcal{L} \vee \mathcal{L}'$ and $M \sqsubseteq N$. Then, there exist $N_1 \in \mathcal{L}$ and $N_2 \in \mathcal{L}'$ such that $N = N_1 \sqcup N_2$. Thus, $M = M \sqcap N = M \sqcap (N_1 \sqcup N_2) = (M \sqcap N_1) \sqcup (M \sqcap N_2)$ and by the hereditary property of each neutrosophic ideal, we have $M \sqcap N_1 \in \mathcal{L}$ and $M \sqcap N_2 \in \mathcal{L}'$. Therefore, $M \in \mathcal{L} \vee \mathcal{L}'$.

(2) Let $N, M \in \mathcal{L} \vee \mathcal{L}'$. Then, there exist $N_1, M_1 \in \mathcal{L}$ and $N_2, M_2 \in \mathcal{L}'$ such that $N = N_1 \sqcup N_2$ and $M = M_1 \sqcup M_2$. By the additive property of each neutrosophic ideal, $N_1 \sqcup M_1 \in \mathcal{L}$ and $N_2 \sqcup M_2 \in \mathcal{L}'$. Since $N \sqcup M = (N_1 \sqcup N_2) \sqcup (M_1 \sqcup M_2) = (N_1 \sqcup M_1) \sqcup (N_2 \sqcup M_2)$, we conclude that $N \sqcup M \in \mathcal{L} \vee \mathcal{L}'$. □

Given a neutrosophic topological space (X, τ), a neutrosophic ideal $\mathcal{L}$ on X and $N \in \mathcal{N}'(X)$, the *neutrosophic local function*[9] of N, denoted by $N^\star(\mathcal{L}, \tau)$, is defined as

$$N^\star(\mathcal{L}, \tau) = \bigsqcup\{x_{a,b,c} \in \mathcal{N}'(X) : U \sqcap N \notin \mathcal{L} \text{ for every } U \in \tau(x_{a,b,c})\}.$$

We will denote $N^\star(\mathcal{L}, \tau)$ by $N^\star$ or $N^\star(\mathcal{L})$. Clearly, if (X, τ) is a neutrosophic topological space and $\mathcal{L}$ is a neutrosophic ideal on X, then $\widetilde{\emptyset}^\star = \widetilde{\emptyset}$, because for every neutrosophic point $x_{a,b,c} \in \mathcal{N}'(X)$ and every $U \in \tau(x_{a,b,c})$, $\widetilde{\emptyset} \sqcap U = \widetilde{\emptyset} \in \mathcal{L}$.

Lemma 3.4. [9,10] Let (X, τ) be a neutrosophic topological space with two neutrosophic ideals $\mathcal{L}$, $\mathcal{L}'$ on X. If $N, M \in \mathcal{N}'(X)$, then the following properties hold:

1. If $N \sqsubseteq M$, then $N^\star \sqsubseteq M^\star$.
2. If $\mathcal{L} \subseteq \mathcal{L}'$, then $N^\star(\mathcal{L}') \sqsubseteq N^\star(\mathcal{L})$.
3. $N^\star = Cl_p(N^\star) \sqsubseteq Cl_p(N)$ ($N^\star$ is a neutrosophic τ_p-closed set).
4. $(N \sqcup M)^\star = N^\star \sqcup M^\star$.
5. $(N^\star)^\star \sqsubseteq N^\star$.
6. $(N \sqcap M)^\star \sqsubseteq N^\star \sqcap M^\star$.
7. If $M \in \mathcal{L}$, then $(N \sqcup M)^\star = N^\star$.

Proposition 3.5. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For each $N \in \mathcal{L}$, we have $N^\star = \widetilde{\emptyset}$.

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Proof. Let $N \in \mathcal{L}$ and suppose that $N^\star \neq \widetilde{\emptyset}$. Then, there exists a neutrosophic point $x_{a,b,c} \in N^\star$ and so, $U \sqcap N \notin \mathcal{L}$ for each $U \in \tau(x_{a,b,c})$. On the other side, as $N \in \mathcal{L}$ and $U \sqcap N \sqsubseteq N$, by the hereditary property of $\mathcal{L}$, we have $U \sqcap N \in \mathcal{L}$, which leads to a contradiction. Therefore, $N^\star = \widetilde{\emptyset}$. □

Theorem 3.6. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. The following properties are equivalent:

1. $\tau \cap \mathcal{L} = \{\widetilde{\emptyset}\}$.
2. If $L \in \mathcal{L}$, then $Int(L) = \widetilde{\emptyset}$.
3. $O \sqsubseteq O^\star$ for each $O \in \tau$.
4. $\widetilde{X} = \widetilde{X}^\star$.

Proof. $(1) \Rightarrow (2)$ Let $L \in \mathcal{L}$ and suppose that $Int(L) \neq \widetilde{\emptyset}$. Then, there exists a neutrosophic point $x_{a,b,c} \in Int(L)$. Thus, $Int(L) \in \tau(x_{a,b,c})$ and $Int(L) \sqsubseteq L$. By the hereditary property of $\mathcal{L}$, it follows that $Int(L) \in \mathcal{L}$ and so, $\widetilde{\emptyset} \neq Int(L) \in \tau \cap \mathcal{L}$, contradicting the fact that $\tau \cap \mathcal{L} = \{\widetilde{\emptyset}\}$.

$(2) \Rightarrow (3)$ Let $x_{a,b,c} \in O$ and assume that $x_{a,b,c} \notin O^\star$. Then, there exists $V \in \tau(x_{a,b,c})$ such that $O \sqcap V \in \mathcal{L}$. Since $O \in \tau$, it follows that $O \sqcap V \in \tau(x_{a,b,c})$ and so, $Int(O \sqcap V) = O \sqcap V \in \tau(x_{a,b,c})$, which implies that $Int(O \sqcap V) \neq \widetilde{\emptyset}$ and $O \sqcap V \in \mathcal{L}$. This contradicts the fact that $Int(L) = \widetilde{\emptyset}$ for each $L \in \mathcal{L}$.

$(3) \Rightarrow (4)$ It follows from the fact that $\widetilde{X}$ is a neutrosophic open set.

$(4) \Rightarrow (1)$ If $\widetilde{X} = \widetilde{X}^\star$ then $\widetilde{X} = \bigsqcup\{x_{a,b,c} \in \mathcal{N}'(X) : O \sqcap \widetilde{X} \notin \mathcal{L} \text{ for each } O \in \tau(x_{a,b,c})\}$, and this implies that $\tau \cap \mathcal{L} = \{\widetilde{\emptyset}\}$. □

Definition 3.7. Let (X, τ) be a neutrosophic topological space. A neutrosophic ideal $\mathcal{L}$ on X is called τ-*boundary*, if one of the equivalent properties of Theorem 3.6 is satisfied.

4 Neutrosophic Ψ-operator and related neutrosophic sets

Definition 4.1. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For each $N \in \mathcal{N}'(X)$, we define the *neutrosophic complement local function* of N as $\Psi(N) = ((N^c)^\star)^c$.

In Table 1 we summarize the main equalities related to the neutrosophic operator Ψ, which are obtained by applying the neutrosophic complement operation or the local neutrosophic function from equation (1).

(1) $\Psi(N) = ((N^c)^\star)^c$	(2) $[\Psi(N)]^c = (N^c)^\star$
(3) $[\Psi(N)]^\star = (((N^c)^\star)^c)^\star$	(4) $\Psi(N^c) = (N^\star)^c$
(5) $[\Psi(N^c)]^c = N^\star$	(6) $[\Psi(N^c)]^\star = ((N^\star)^c)^\star$
(7) $\Psi(N^\star) = (((N^\star)^c)^\star)^c$	(8) $[\Psi(N^\star)]^c = ((N^\star)^c)^\star$

Table 1: Equalities related to the neutrosophic operator Ψ.

Remark 4.2. From the equalities (6) and (8) of Table 1, we can deduce that $[\Psi(N^c)]^\star = [\Psi(N^\star)]^c$.

In the following proposition we present interesting properties related to the neutrosophic operator Ψ.

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Proposition 4.3. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. Then, we have the following properties:

1. If $N, M \in \mathcal{N}'(X)$ and $N \sqsubseteq M$, then $\Psi(N) \sqsubseteq \Psi(M)$. ($\Psi$ is monotone)
2. $\Psi(N \sqcap M) = \Psi(N) \sqcap \Psi(M)$ for each $N, M \in \mathcal{N}'(X)$.
3. $\Psi(N) \sqsubseteq \Psi(\Psi(N))$ for each $N \in \mathcal{N}'(X)$.
4. $\Psi(\widetilde{X}) = \widetilde{X}$.
5. $O \sqsubseteq \Psi(O)$ for each $O \in \tau_p$. (Ψ is expansive on τ_p)
6. $Int_p(N) \sqsubseteq \Psi(N)$ for each $N \in \mathcal{N}'(X)$.

Proof. (1) Let $N, M \in \mathcal{N}'(X)$ such that $N \sqsubseteq M$. Then, $M^c \sqsubseteq N^c$ and by part (1) of Lemma 3.4, $(M^c)^\star \sqsubseteq (N^c)^\star$. Therefore, $\Psi(N) = ((N^c)^\star)^c \sqsubseteq ((M^c)^\star)^c = \Psi(M)$.
(2) If $N, M \in \mathcal{N}'(X)$, then

$$\begin{aligned}\Psi(N \sqcap M) &= (((N \sqcap M)^c)^\star)^c = ((N^c \sqcup M^c)^\star)^c \\ &= ((N^c)^\star \sqcup (M^c)^\star)^c = ((N^c)^\star)^c \sqcap ((M^c)^\star)^c \\ &= \Psi(N) \sqcap \Psi(M).\end{aligned}$$

(3) Let $N \in \mathcal{N}'(X)$. By part (5) of Lemma 3.4, we have $((N^c)^\star)^\star \sqsubseteq (N^c)^\star$, which implies that $\Psi(N) = ((N^c)^\star)^c \sqsubseteq (((N^c)^\star)^\star)^c$. Now, applying Definition 4.1 to the neutrosophic set $\Psi(N)$, we obtain that $\Psi(\Psi(N)) = (([\Psi(N)]^c)^\star)^c$ and by equation (2) of Table 1, it follows that $\Psi(\Psi(N)) = (((N^c)^\star)^\star)^c$. Therefefore, we conclude that $\Psi(N) \sqsubseteq ((N^c)^\star)^\star)^c = \Psi(\Psi(N))$.
(4) By definition we have $\Psi(\widetilde{X}) = ((\widetilde{X}^c)^\star)^c = (\widetilde{\emptyset}^\star)^c = \widetilde{\emptyset}^c = \widetilde{X}$.
(5) If $O \in \tau_p$, then O^c is a neutrosophic τ_p-closed set and so $Cl_p(O^c) = O^c$. By equation (2) of Table 1 and part (3) of Lemma 3.4, we obtain that $[\Psi(O)]^c = (O^c)^\star \sqsubseteq Cl_p(O^c) = O^c$ and hence $O \sqsubseteq \Psi(O)$ for each $O \in \tau_p$.
(6) Since $Int_p(N) \in \tau_p$, by part (5), we have $Int_p(N) \sqsubseteq \Psi(Int_p(N))$ and as $Int_p(N) \sqsubseteq N$, by part (1), we conclude that $Int_p(N) \sqsubseteq \Psi(Int_p(N)) \sqsubseteq \Psi(N)$. ☐

Definition 4.4. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. A subset $N \in \mathcal{N}'(X)$ is said to be:

1. *neutrosophic $\star$-perfect*, if $N = N^\star$
2. *neutrosophic $\star$-dense*, if $N^\star = \widetilde{X}$.
3. *neutrosophic $\star$-condensed*, if $[\Psi(N)]^\star = N^\star$.
4. *neutrosophic Ψ-condensed*, if $\Psi(N^\star) = \Psi(N)$.
5. *neutrosophic $\Psi^\star$-condensed*, if it is neutrosophic $\star$-condensed and neutrosophic Ψ-condensed.
6. *neutrosophic non $\Psi^\star$-condensed*, if $\Psi(N^\star) = \widetilde{\emptyset}$.
7. *neutrosophic $\star$-congruent*, if $[\Psi(N)]^\star = N$.
8. *neutrosophic Ψ-congruent*, if $\Psi(N^\star) = N$.
9. *neutrosophic $\Psi^\star$-congruent*, if it is neutrosophic $\star$-congruent and neutrosophic Ψ-congruent.

Proposition 4.5. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $N \in \mathcal{N}'(X)$, then we have the following properties:

1. If N is neutrosophic $\star$-perfect, then it is neutrosophic Ψ-condensed.
2. N is neutrosophic Ψ-condensed if and only if N^c is neutrosophic $\star$-condensed.

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3. N is neutrosophic $\Psi^{\star}$-condensed if and only if N^c is neutrosophic $\Psi^{\star}$-condensed.

4. N is neutrosophic Ψ-congruent if and only if N^c is neutrosophic $\star$-congruent.

5. N is neutrosophic $\Psi^{\star}$-congruent if and only if N^c is neutrosophic $\Psi^{\star}$-congruent.

6. If N is neutrosophic Ψ-condensed and neutrosophic non $\Psi^{\star}$-condensed, then N^c is neutrosophic $\star$-dense.

7. If N is neutrosophic $\star$-condensed and N^c is neutrosophic non $\Psi^{\star}$-condensed, then N is neutrosophic $\star$-dense.

8. If N is neutrosophic non $\Psi^{\star}$-condensed and neutrosophic $\star$-perfect, then N^c is neutrosophic $\star$-dense.

Proof. (1) From Definition 4.4, we have:

$$\begin{aligned} N \text{ is neutrosophic } \star\text{-perfect} &\iff N = N^{\star} \\ &\implies \Psi(N) = \Psi(N^{\star}) \\ &\implies N \text{ is neutrosophic } \Psi\text{-condensed.} \end{aligned}$$

(2) By Remark 4.2 and equation (2) of Table 1, we get that

$$\begin{aligned} N \text{ is neutrosophic } \Psi\text{-condensed} &\iff \Psi(N^{\star}) = \Psi(N) \\ &\iff [\Psi(N^{\star})]^c = [\Psi(N)]^c \\ &\iff [\Psi(N^c)]^{\star} = (N^c)^{\star} \\ &\iff N^c \text{ is neutrosophic } \star\text{-condensed.} \end{aligned}$$

(3) The proof follows from (2).
(4) By Remark 4.2, we obtain that

$$\begin{aligned} N \text{ is neutrosophic } \Psi\text{-congruent} &\iff \Psi(N^{\star}) = N \\ &\iff [\Psi(N^{\star})]^c = N^c \\ &\iff [\Psi(N^c)]^{\star} = N^c \\ &\iff N^c \text{ is neutrosophic } \star\text{-congruent.} \end{aligned}$$

(5) The proof follows from (4).
(6) Assume that N neutrosophic Ψ-condensed and N is neutrosophic non $\Psi^{\star}$-condensed. Then, $\Psi(N^{\star}) = \Psi(N)$ and $\Psi(N^{\star}) = \widetilde{\emptyset}$, which implies that $\Psi(N) = \widetilde{\emptyset}$. Thus, $[\Psi(N)]^c = \widetilde{X}$ and by equation (2) of Table 1, it follows that $(N^c)^{\star} = \widetilde{X}$. Therefore, N^c is neutrosophic $\star$-dense.
(7) The proof follows from (2) and (6).
(8) Assume that N is neutrosophic non $\Psi^{\star}$-condensed and neutrosophic $\star$-perfect. Then, $\Psi(N^{\star}) = \widetilde{\emptyset}$ and $N^{\star} = N$, which implies that $\Psi(N) = \Psi(N^{\star}) = \widetilde{\emptyset}$. By equation (2) of Table 1, it follows that $(N^c)^{\star} = [\Psi(N)]^c = \widetilde{X}$ and so, N^c is neutrosophic $\star$-dense. □

Theorem 4.6. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $N \in \mathcal{N}'(X)$ and N^c is a neutrosophic $\star$-perfect set, then the following properties are equivalent:

1. N is neutrosophic Ψ-congruent

2. N is neutrosophic Ψ-condensed.

Proof. (1) $\implies$ (2) Suppose that N is neutrosophic Ψ-congruent. Then, $\Psi(N^{\star}) = N$. Since N^c is neutrosophic $\star$-perfect, $(N^c)^{\star} = N^c$, which implies that $\Psi(N^{\star}) = N = ((N^c)^{\star})^c = \Psi(N)$, which shows that N is neutrosophic Ψ-condensed.
(2) $\implies$ (1) Assume that N is neutrosophic Ψ-condensed. Then, $\Psi(N^{\star}) = \Psi(N)$. Since N^c is neutrosophic $\star$-perfect, $(N^c)^{\star} = N^c$ and by equation (2) of Table 1, it follows that $[\Psi(N)]^c = N^c$, which implies that $\Psi(N) = N$. Therefore, $\Psi(N^{\star}) = \Psi(N) = N$ and so, N is neutrosophic Ψ-congruent □

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Corollary 4.7. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $N \in \mathcal{N}'(X)$ is neutrosophic $\star$-perfect, then the following properties are equivalent:

1. N is neutrosophic $\star$-congruent
2. N is neutrosophic $\star$-condensed.

Proof. It is deduced from Theorem 4.6 by using parts (2) and (4) of Proposition 4.5. □

Proposition 4.8. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal τ-boundary on X. If $N \in \mathcal{L}$, then N is neutrosophic non $\Psi^\star$-condensed.

Proof. If $N \in \mathcal{L}$, then by Proposition 3.5, we have $N^\star = \widetilde{\emptyset}$ and by using equation (2) of Table 1, $\Psi(N^\star) = \Psi(\widetilde{\emptyset}) = \Psi(\widetilde{X}^c) = (\widetilde{X}^\star)^c = \widetilde{X}^c = \widetilde{\emptyset}$. Therefore, N is neutrosophic non $\Psi^\star$-condensed. □

Proposition 4.9. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For $N \in \mathcal{N}'(X)$, we have the following properties:

1. If N is neutrosophic non $\Psi^\star$-condensed and $M \sqsubseteq N$, then M is neutrosophic non $\Psi^\star$-condensed.
2. If N is neutrosophic non $\Psi^\star$-condensed and $M \in \mathcal{N}'(X)$, then $N \sqcap M$ is neutrosophic non $\Psi^\star$-condensed.
3. If N is neutrosophic non $\Psi^\star$-condensed and $L \in \mathcal{L}$, then $N \sqcup L$ is neutrosophic non $\Psi^\star$-condensed.
4. If N is neutrosophic non $\Psi^\star$-condensed, then $N^\star$ is neutrosophic non $\Psi^\star$-condensed.
5. If N is neutrosophic non $\Psi^\star$-condensed, then for each $x_{a,b,c} \in \mathcal{N}'(X)$ and each $U \in \tau(x_{a,b,c})$, $\Phi(N^c) \sqcap U \neq \widetilde{\emptyset}$.
6. If $\mathcal{J}$ is a neutrosophic ideal on X such that $\mathcal{J} \subseteq \mathcal{L}$ and N is neutrosophic non $\Psi^\star$-condensed, with respect to $\mathcal{J}$, then N is neutrosophic non $\Psi^\star$-condensed with respect to $\mathcal{L}$.

Proof. (1) Suppose that N is neutrosophic non $\Psi^\star$-condensed and $M \sqsubseteq N$. Then $\Psi(N^\star) = \widetilde{\emptyset}$ and $M^\star \sqsubseteq N^\star$. Thus, $\Psi(M^\star) \sqsubseteq \Psi(N^\star) = \widetilde{\emptyset}$, which means that $\Psi(M^\star) = \widetilde{\emptyset}$ and hence, M is neutrosophic non $\Psi^\star$-condensed.
(2) Since $N \sqcap M \sqsubseteq N$ for each $M \in \mathcal{N}'(X)$, the result follows from part (1).
(3) Assume that N is neutrosophic non $\Psi^\star$-condensed and $L \in \mathcal{L}$. Then $\Psi(N^\star) = \widetilde{\emptyset}$ and $L^\star = \widetilde{\emptyset}$, which implies that $(N \sqcup L)^\star = N^\star \sqcup L^\star = N^\star$ and $\Psi((N \sqcup L)^\star) = \Psi(N^\star) = \widetilde{\emptyset}$. Therefore, $N \sqcup L$ is neutrosophic non $\Psi^\star$-condensed.
(4) Suppose that N is neutrosophic non $\Psi^\star$-condensed. Then $\Psi(N^\star) = \widetilde{\emptyset}$ and $(N^\star)^\star \sqsubseteq N^\star$. Hence $\Psi((N^\star)^\star) \sqsubseteq \Psi(N^\star) = \widetilde{\emptyset}$ and so $N^\star$ is neutrosophic non $\Psi^\star$-condensed.
(5) Assume that N is neutrosophic non $\Psi^\star$-condensed, i.e. $\Psi(N^\star) = \widetilde{\emptyset}$. Then $[\Psi(N^\star)]^c = \widetilde{X}$ and so, by Remark 4.2, $[\Psi(N^c)]^\star = \widetilde{X}$. Therefore, for each $x_{a,b,c} \in \mathcal{N}'(X)$ and each $U \in \tau(x_{a,b,c})$, $U \sqcap \Psi(N^c) \notin \mathcal{L}$, which implies that $U \sqcap \Psi(N^c) \neq \emptyset$, for each $x_{a,b,c} \in \mathcal{N}'(X)$ and each $U \in \tau(x_{a,b,c})$.
(6) Let $\mathcal{J}$ be a neutrosophic ideal on X such that $\mathcal{J} \sqsubseteq \mathcal{L}$ and N be a neutrosophic non $\Psi^\star$-condensed set with respect to $\mathcal{J}$. Then $\Psi(N^\star(\mathcal{J})) = \widetilde{\emptyset}$ and by part (2) of Theorem 3.4, we have $N^\star(\mathcal{L}) \sqsubseteq N^\star(\mathcal{J})$, which implies that $\Psi(N^\star(\mathcal{L})) \sqsubseteq \Psi(N^\star(\mathcal{J})) = \widetilde{\emptyset}$. Therefore, $\Psi(N^\star(\mathcal{L})) = \widetilde{\emptyset}$ and so, N is neutrosophic non $\Psi^\star$-condensed with respect to $\mathcal{L}$. □

Proposition 4.10. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For $N \in \mathcal{N}'(X)$, we have the following properties:

1. N is neutrosophic non $\Psi^\star$-condensed if and only if $(N^\star)^c$ is neutrosophic $\star$-dense.
2. N is neutrosophic non $\Psi^\star$-condensed if and only if $\Psi(N^c)$ is neutrosophic $\star$-dense.

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3. N^c is neutrosophic non $\Psi^\star$-condensed if and only if $\Psi(N)$ is neutrosophic $\star$-dense.

Proof. The proofs of (1) and (2) are obtained from Definition 4.4 and equation (8) of Table 1 as follows:

$$\begin{aligned} N \text{ is neutrosophic non } \Psi^\star\text{-condensed} &\iff \Psi(N^\star) = \widetilde{\emptyset} \\ &\implies [\Psi(N^\star)]^c = \widetilde{X} \\ &\iff ((N^\star)^c)^\star = \widetilde{X} \\ &\iff (N^\star)^c \text{ is neutrosophic } \star\text{-dense} \\ &\iff \Psi(N^c) \text{ is neutrosophic } \star\text{-dense.} \end{aligned}$$

(3) The proof follows from (2) by changing N to N^c. □

Definition 4.11. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. For each $N \in \mathcal{N}'(X)$, the *neutrosophic $\star$-frontier* of N, denoted by $Fr^\star(N)$, is defined as $Fr^\star(N) = N^\star \sqcap (N^c)^\star$.

Proposition 4.12. Let (X, τ) be a neutrosophic topological space and $\mathcal{L}$ be a neutrosophic ideal on X. If $N \in \mathcal{N}'(X)$ is neutrosophic $\star$-dense and $\Psi((Fr^\star(N)) = \widetilde{\emptyset}$, then N^c is neutrosophic non $\Psi^\star$-condensed.

Proof. Assume that $N \in \mathcal{N}'(X)$ is neutrosophic $\star$-dense and $\Psi((Fr^\star(N)) = \widetilde{\emptyset}$. Then $N^\star = \widetilde{X}$ and $\Psi(N^\star \sqcap (N^c)^\star) = \widetilde{\emptyset}$. Thus, by parts (2) and (4) of Proposition 4.3, we have $\Psi(N^\star) \sqcap \Psi((N^c)^\star) = \widetilde{\emptyset}$ and $\Psi(N^\star) = \Psi(\widetilde{X}) = \widetilde{X}$, respectively. Therefore, $\Psi((N^c)^\star) = \widetilde{X} \sqcap \Psi((N^c)^\star) = \widetilde{\emptyset}$ and so, N^c is neutrosophic non $\Psi^\star$-condensed. □

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