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ABSTRACT

Various types of acceptance reliability sampling plans have been reported in the literature. Most of the existing studies deal with the lifetimes of items described by continuous distributions. However, discrete lifetimes are also frequently encountered in practice. For example, field failures are often collected and reported daily, weekly, and so forth. Items often operate in cycles and the experimenter observes the number of cycles successfully completed prior to failure. In this study, we develop a variables acceptance reliability sampling plan for items having discrete lifetimes. Furthermore, the reliability improvement in the population reliability characteristics after the testing procedure, as compared to those before the test, is discussed in detail. The reliability sampling acceptance test is applied for heterogeneous populations using the alternative discrete hazard rate for the corresponding proportional hazards model. Some numerical examples illustrate our findings.

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1. Introduction

Based on a random sample drawn from a lot provided by a supplier, a consumer can decide on whether to accept or to reject this lot of products. This is often done by means of the corresponding acceptance sampling plan. Thus, from products grouped into lots, a sample is picked at random and, on the basis of the information obtained from this sample, the entire lot is accepted or rejected. As lots of poor quality can be screened by sampling tests, the acceptance sampling plan has become an essential tool in statistical quality control. A general introduction to the theory and practice of acceptance sampling plans can be found in Stephens [1] and Montgomery [2]. Specifically, when the lifetime of a product is the main characteristic of its quality, the corresponding sampling plans are called life test reliability sampling plans.

Various types of acceptance reliability sampling plans have been suggested and studied in the literature. For instance, in life testing, a fixed number of items are often tested for some fixed period of time (Type I censoring) or until some fixed number of items on test

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fail (Type II censoring). In the initial stage of development, the reliability sampling plans for items described by the exponential lifetime distribution were mainly investigated. See Epstein [3], Epstein and Sobel [4,5], Blugren and Hewette [6], Fairbanks [7] for different acceptance reliability sampling plans in this case. Ferting and Mann [8] and Schneider [9] considered the Weibull distribution for the development of acceptance reliability sampling plans. Later on, more sophisticated reliability sampling plans have been developed. See, for instance, Seidel [10], Edgeman and Salzberg [11], Sohn and Jang [12], Pérez-González and Fernández [13], Kim and Yum [14], Lam and Choy [15], Tsai and Wu [16], Balakrishnan et al. [17], Aslam and Jun [18] and Aslam et al. [19–22]. Recently, Cha [23] has studied the effect of reliability sampling plan on the population of products. Aslam et al. [24] and Bhattacharya and Aslam [25] have studied the modified multiple dependent state sampling plan and the generalized multiple dependent state sampling plan, respectively. Multiple dependent state (MDS) sampling plan is applied for the reduction of inspection cost and makes a decision by utilizing the current sample and/or previous sample information. Aslam et al. [24] considered the designing of a modified version of the MDS sampling plan to achieve more reduction in the sample size with the desired protection. Bhattacharya and Aslam [25] considered a modified version of MDS which is found to be efficient in minimizing sample size as compared to the MDS plan. The sampling plan proposed in this paper is different from MDS sampling plan in the sense that it utilizes only the current sample information.

As far as we know, the acceptance reliability sampling plans reported in the literature have been developed *only for the continuous lifetime distributions* of the manufactured items. However, items with lifetimes described by the discrete distributions can be frequently encountered in practice as well. For example, field failures are often collected and reported daily, weekly, and so forth. Items often operate in cycles and the experimenter observes the number of cycles successfully completed prior to failure.

Therefore, the goal and contribution of this paper are in developing a new variables acceptance reliability sampling plan for items with *lifetimes described by discrete distributions.* Moreover, the paper can be considered as a starting point for further studies of the acceptance reliability sampling plans for discrete lifetime distributions.

It is also important from practical and theoretical points of view to discuss in detail the changes in reliability characteristics of a population before and after the testing procedure. For instance, the following question should be answered: will the population lifetime after the successful test be larger (and in what stochastic sense) than that before the test? It is worth noting that the obtained results on the developed acceptance reliability sampling plan are general in the sense that they are valid for any discrete distribution.

This paper is organized as follows. In Section 2, as preliminaries for our discussion, some basic concepts for stochastic order are introduced and related properties are briefly discussed. A general reliability sampling plan for discrete lifetime distributions is developed in Section 3, where a testing procedure is formulated and the rule for acceptance and rejection of the lots is defined. In this section, to represent the variability (heterogeneity) in the population reliability characteristics, the discrete proportional hazard model is employed. The parameters of the proposed sampling plan are determined taking into account two types of risks (i.e., the producer's risk and the consumer's risk). To find the parameters more conveniently, a searching procedure is developed. In Section 4, we show the utility of the proposed sampling plan by comparing the lifetime distributions of the populations before and after the testing procedure. It is shown that the lifetime of the population which has passed the life testing procedure is stochastically larger than that of the original population. In Section 5, to describe variability in population reliability characteristics, the discrete additive hazard model is considered and discussed. Finally, in Section 5, some concluding remarks are given.

2. Preliminaries

In this section, we present the necessary definitions and distribution properties for discrete lifetimes. Let T be the discrete lifetime, with support in $\{1, 2, \ldots\}$. The discrete-time failure rate is defined as

$$\lambda(k) = \Pr(T = k|T > k-1) = \frac{\Pr(T = k)}{\Pr(T > k-1)}$$

$$= \frac{f(k)}{S(k-1)} = \frac{S(k-1) - S(k)}{S(k-1)}, \quad k = 1, 2, \ldots \tag{1}$$

where $S(k) \equiv \Pr(T > k)$, $F(k) \equiv \Pr(T \leq k)$ and $f(k) \equiv \Pr(T = k)$ are the corresponding survival function, cumulative distribution function (cdf) and the probability mass function (pmf), respectively. Then the relationships among these measures are given by

$$S(k) = \prod_{i=1}^{k} (1 - \lambda(i)), F(k) = 1 - S(k) \tag{2}$$

and

$$f(k) = \lambda(k) \prod_{i=1}^{k-1} (1 - \lambda(i))$$

respectively.

Note that the discrete failure rate (1) is always bounded, i.e., $\lambda(k) \leq 1$, as it is defined as a conditional probability, which is the crucial distinction from the case of continuous distributions, where $\lambda(t)$ as such is not a probability. Also, we have a product form for the survival function in (2) instead of the famous exponential representation in the continuous case.

For our further discussion on modelling variability in the population reliability characteristics and for relevant stochastic comparisons, we need to introduce some basic concepts for stochastic orders. For more details, the readers could refer to Shaked and Shanthikumar [26] and Cha and Finkelstein [27].

Definition 2.1: Let Z_1 and Z_2 be two nonnegative continuous or discrete random variables with respective failure rate functions $\lambda_1(t)$ and $\lambda_2(t)$, such that

$$\lambda_1(t) \geq \lambda_2(t), \text{ for all } t \geq 0$$

Then Z_1 is said to be smaller than Z_2 in the sense of failure rate order, which is denoted by $Z_1 \leq_{fr} Z_2$.

Definition 2.2: Let Z_1 and Z_2 be two nonnegative continuous or discrete random variables with respective survival functions $S_1(t)$ and $S_2(t)$, such that

$$S_1(t) \leq S_2(t), \text{ for all } t \geq 0$$

Then Z_1 is said to be smaller than Z_2 in the sense of usual stochastic order, which is denoted by $Z_1 \leq_{st} Z_2$.

Definition 2.3: Let Z_1 and Z_2 be two nonnegative continuous (discrete) random variables with respective probability density functions (probability mass functions) $f_1(t)$ and $f_2(t)$, such that

$$\frac{f_1(t)}{f_2(t)} \text{ is decreasing for all } t \geq 0$$

Then Z_1 is said to be smaller than Z_2 in the sense of likelihood ratio order, which is denoted by $Z_1 \leq_{lr} Z_2$.

Due to Shaked and Shanthikumar [26], it holds that

$$Z_1 \leq_{lr} Z_2 \Rightarrow Z_1 \leq_{fr} Z_2 \Rightarrow Z_1 \leq_{st} Z_2 \tag{3}$$

Furthermore, for stochastic comparison of populations to be discussed in the next section, we need the following lemma. The proof can also be found in Shaked and Shanthikumar [26].

Lemma 2.1:

(i) *Let Z_1 and Z_2 be two nonnegative random variables satisfying $Z_1 \leq_{st} Z_2$. Then $E[Z_1] \leq E[Z_2]$.*
(ii) *If $Z_1 \leq_{st} Z_2$ and $g(\cdot)$ is any increasing [decreasing] function, then $g(Z_1) \leq_{st} [\geq_{st}] g(Z_2)$.*

Now we are ready to describe the population distribution. Note that the quality level of the produced items is variable due to many different reasons such as the quality of resources and components used in the production process, human errors, uncontrollable significant quality factors, and so forth. Thus, when the lifetime of a product is its main quality characteristic, defective resources and components may result in shorter lifetime of the corresponding population of the produced item, and vice versa. Thus, the described heterogeneity in manufactured items is the intrinsic property of real-life populations of products.

For a continuous lifetime with the failure rate $\lambda(t)$, such variability is often modelled by the following well-known proportional hazard model (Leemis [28], Ebeling [29], Badıa et al. [30], Kumar and Klefsjö [31]) or the relative risk model (Kalbfleisch and Prentice [32])

$$\lambda(t) = \theta \lambda_0(t), t \geq 0 \tag{4}$$

where $\theta > 0$ and $\lambda_0(t)$ is the baseline failure rate function.

However, for a discrete random variable, the discrete failure rate is always bounded by 1 and the proportional hazard model in a continuous setting cannot be directly applied to this case. This, along with other deficiencies, can be regarded as a critical restriction, especially from a modelling point of view. Due to this reason, the alternative discrete failure rate function has been suggested in the literature (e.g., Roy and Gupta, [33]) in the following way

$$r(k) = -\ln \frac{S(k)}{S(k-1)}, k = 1, 2, \ldots$$

As

$$r(k) = -\ln(1 - \lambda(k)) \text{ or } \lambda(k) = 1 - \exp(-r(k)) \tag{5}$$

the alternative failure rate function can be considered as a suitable transformation of $\lambda(k)$. Taking into account (5), define the baseline alternative failure rate as

$$r_0(k) = -\ln[1 - \lambda_0(k)]$$

where $\lambda_0(k)$ is the baseline discrete failure rate function of the produced items. Since the discrete failure rate $r(k)$ can take values in $(0, \infty)$, similar to the continuous setting in (4), we can now employ the proportional hazard model for the discrete lifetime as follows:

$$r(k) = \theta r_0(k) \tag{6}$$

Then, from (5) and (6),

$$\lambda(k) = 1 - \exp[-r(k)] = 1 - \exp[-\theta r_0(k)] = 1 - (\exp[-r_0(k)])^\theta = 1 - (1 - \lambda_0(k))^\theta \tag{7}$$

and throughout this paper, we assume the model (7) for describing variability of reliability characteristics of the manufactured items. Throughout this paper, it is also assumed that the items in the same lot are manufactured under a sufficiently stable manufacturing environment so that they share common failure time distribution following (7), which is the usual assumption employed in variables sampling plans (see, e.g., Fertig and Mann [8]).

Denote by T_θ the lifetime of an item which corresponds to the failure rate function in (7). Then $\theta_1 \leq \theta_2$ implies that $T_{\theta_1} \geq_{fr} T_{\theta_2}$ and, accordingly, due to (3), $T_{\theta_1} \geq_{st} T_{\theta_2}$. Thus, larger θ corresponds to a shorter lifetime, and vice versa.

3. Sampling plan for discrete lifetimes

In this paper, the quality of a lot will be defined in terms of the mean time to failure of the item drawn from the lot. Observe that the mean of T_θ can be written as

$$E[T_\theta] = \sum_{k=0}^{\infty} \prod_{i=1}^{k} (1 - \lambda_0(i))^\theta$$

where $\prod_{i=1}^{0} (\cdot) \equiv 1$, and, then, under the model (7), $\theta_1 \leq \theta_2$ implies

$$E[T_{\theta_1}] = \sum_{k=0}^{\infty} \prod_{i=1}^{k} (1 - \lambda_0(i))^{\theta_1} \geq \sum_{k=0}^{\infty} \prod_{i=1}^{k} (1 - \lambda_0(i))^{\theta_2} = E[T_{\theta_2}]$$

Thus, as θ increases, the mean time to failure of an item monotonically decreases.

We will now develop a reliability sampling plan which assures that the mean time to failure of the item should be greater than a certain level. As in the reliability sampling plans for continuous lifetime, we assume that the consumer requires that the lot acceptance probability should be smaller than the specified consumer's risk β at a lower quality level m_2, whereas the producer requires that the lot rejection probability should be smaller than the specified producer's risk α at a higher quality level m_1, $m_1 > m_2$, where m_1 and m_2 represent the mean time to failure of the item at the two quality levels. However, as $E[T_\theta]$ is monotonic in θ, we can find θ_1 and θ_2 ($\theta_1 < \theta_2$) which satisfy $E[T_{\theta_i}] = m_i, i = 1, 2$, respectively. Then, in terms of the value of θ, the lower quality level is defined by $\theta = \theta_2$ and the higher level is defined by $\theta = \theta_1$. Therefore, throughout this paper, for convenience, the quality levels will be represented by using the value of θ, instead of the value of the mean time to failure.

We will formally define now the proposed sampling plan for discrete lifetime models. From the lot to be tested, n items are drawn at random and they are put into the life test for the time interval $(0, k_0]$, where $k_0 \geq 1$ is a specified discrete testing time (a positive integer). Define N as the number of items which have failed in $(0, k_0]$ and c (integer) as the fixed threshold number. If $N > c$, then the lot is rejected; otherwise the lot is accepted. Thus (n, c) are two parameters of the proposed sampling plan. As usual, the lot size is assumed to be large enough, allowing the use of a binomial distribution in obtaining the lot acceptance probability (see, e.g., Stephens [1]). Therefore, under the model in (7), the acceptance probability of the lot as the function of θ is given by

$$L(\theta) = \sum_{i=0}^{c} \binom{n}{i} p(\theta)^i (1-p(\theta))^{n-i}$$

where $p(\theta)$ is defined by

$$p(\theta) = 1 - \prod_{j=1}^{k_0} (1 - \lambda_0(j))^\theta$$

The parameters (n, c) will be determined so that the consumer's risk and the producer's risk are balanced as follows:

$$L(\theta_2) = \beta \text{ and } L(\theta_1) = 1 - \alpha \tag{8}$$

Note that both n and c are integers and there cannot be exact values of n and c satisfying the two equations in (8). In this case, we will find the integers n and c which achieve the nearest acceptance probabilities $L(\theta_2) \approx \beta$ and $L(\theta_1) \approx 1 - \alpha$.

In determining two parameters n and c, an efficient searching procedure will be provided. For this, we need to state the following proposition.

Proposition 3.1: *The acceptance probability*

$$L(\theta) = \sum_{i=0}^{c} \binom{n}{i} p(\theta)^i (1-p(\theta))^{n-i}$$

is decreasing in n for any fixed $\theta > 0$ *and c.*

The proof of Proposition 3.1 is rather straightforward. By taking the ratio of the probability mass functions, it can be easily shown that $X_{n,p(\theta)} \leq_{lr} X_{n+1,p(\theta)}$, where $X_{n,p}$ is the binomial random variable with the number of trials n and the success probability p. Then, due to (3), we have $X_{n,p(\theta)} \leq_{st} X_{n+1,p(\theta)}$. As $L(\theta)$ corresponds to the cdf of a binomial random variable, we arrive at the formulated result.

Using Proposition 3.1, parameters (n, c) satisfying both equations in (8) can be determined. For a fixed $n \geq 1$, define $c(n)$ as the nonnegative integer of c satisfying

$$L(\theta_1) = \sum_{i=0}^{c} \binom{n}{i} p(\theta_1)^i (1 - p(\theta_1))^{n-i} = 1 - \alpha$$

Then, by the definition of $c(n)$, we have

$$\sum_{i=0}^{c(n)} \binom{n}{i} p(\theta_1)^i (1 - p(\theta_1))^{n-i} = 1 - \alpha$$

Due to Proposition 3.1, for this 'fixed $c(n)$', $\sum_{i=0}^{c(n)} \binom{n}{i} p(\theta_1)^i (1 - p(\theta_1))^{n-i}$ is decreasing in n and

$$\sum_{i=0}^{c(n)} \binom{n+1}{i} p(\theta_1)^i (1 - p(\theta_1))^{n+1-i} \leq \sum_{i=0}^{c(n)} \binom{n}{i} p(\theta_1)^i (1 - p(\theta_1))^{n-i} = 1 - \alpha$$

which implies that, for the integer $c(n + 1)$ such that $c(n + 1) \geq c(n)$,

$$\sum_{i=0}^{c(n+1)} \binom{n+1}{i} p(\theta_1)^i (1 - p(\theta_1))^{n+1-i} = 1 - \alpha$$

Therefore, $c(n) \leq c(n + 1)$, $n = 1, 2, \ldots$, which provides a lower bound in the following sequential procedure for searching parameters (n, c).

Proposition 3.2 (Sequential Procedure): *The parameters (n, c) satisfying both equations in (8) can be obtained by the following procedure:*

(Step 1)

Fix $n = 1$ and find the integer $c(1)$, $0 \leq c(1) \leq 1$, such that

$$\sum_{i=0}^{c(1)} \binom{n}{i} p(\theta_1)^i (1 - p(\theta_1))^{n-i} \approx 1 - \alpha$$

If $\sum_{n=0}^{c(1)} \binom{n}{i} p(\theta_2)^i (1 - p(\theta_2))^{n-i} \approx \beta$, then choose $(1, c(1))$ as the desired parameters; otherwise go to Step 2.

(Step 2)

Fix $n = 2$ and find the integer $c(2)$, where $c(1) \le c(2) \le 2$, such that

$$\sum_{n=0}^{c(2)} \binom{n}{i} p(\theta_1)^i (1 - p(\theta_1))^{n-i} \approx 1 - \alpha$$

If $\sum_{n=0}^{c(2)} \binom{n}{i} p(\theta_2)^i (1 - p(\theta_2))^{n-i} \approx \beta$, then choose $(2, c(2))$ as the desired parameters; otherwise go to Step 3.

(Step 3)

Fix $n = 3$ and find the integer $c(3)$, where $c(2) \le c(3) \le 3$, such that

$$\sum_{n=0}^{c(3)} \binom{n}{i} p(\theta_1)^i (1 - p(\theta_1))^{n-i} \approx 1 - \alpha$$

If $\sum_{n=0}^{c(3)} \binom{n}{i} p(\theta_2)^i (1 - p(\theta_2))^{n-i} \approx \beta$, then choose $(3, c(3))$ as the desired parameters; otherwise go to the next Step 4 with $n = 4, \dots$, and so on.

Table 1 illustrates the proposed sequential procedure for obtaining parameters (n, c) for different values of θ_1 and θ_2. In this table, the baseline discrete failure rate function is set as $\lambda_0(i) = 0.1$, $i = 1, 2, \dots$. The two risks are $\alpha = 0.05$, $\beta = 0.01$, and the testing time is $k_0 = 5$.

As the exact values for n and c matching the requirement for α (producer's risk) and β (consumer's risk) cannot be obtained, in applying the sequential procedure in Proposition 3.2, we have to determine the precision thresholds in advance. That is, for pre-determined precision thresholds $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, if $|L(\theta_1) - (1 - \alpha)| < \varepsilon_1$ and $|L(\theta_2) - \beta| < \varepsilon_2$ for some n and c, then it is set that these n and c achieve the nearest acceptance probabilities $L(\theta_2) \approx \beta$ and $L(\theta_1) \approx 1 - \alpha$. For example, in Table 1, when $\theta_1 = 2$ and $\theta_2 = 4$, the precision thresholds $\varepsilon_1 = 0.003$ and $\varepsilon_2 = 0.001$ were applied. For the combination $(n = 48, c = 36)$, $L(\theta_1) = 0.947$ and $L(\theta_2) = 0.0107$. Thus, this combination $(n = 48, c = 36)$ achieves the nearest acceptance probabilities $L(\theta_2) \approx \beta$ and $L(\theta_1) \approx 1 - \alpha$.

As can be seen from Table 1, if there exists a substantial difference between θ_1 and θ_2, then the number of testing items n and the threshold number for failures c are small, and vice versa. This result can be explained intuitively as follows. If the difference in the quality levels is larger, then it would be possible to identify lots of good and bad quality levels with a smaller number of tested items.

Table 1. The set of parameters (n, c) for $\alpha = 0.05$, $\beta = 0.01$.

θ_1	θ_2	(n, c)	θ_1	θ_2	(n, c)
1	2	$(n = 59, c = 30)$	2	4	$(n = 48, c = 36)$
	2.5	$(n = 35, c = 19)$		4.5	$(n = 41, c = 32)$
	3	$(n = 23, c = 13)$		5	$(n = 28, c = 22)$
	3.5	$(n = 17, c = 10)$		5.5	$(n = 25, c = 20)$
	4	$(n = 12, c = 7)$		6	$(n = 21, c = 17)$

4. Reliability improvement after testing

In this section, the reliability improvement in the population after the testing procedure is discussed in detail. First, the population distribution before the reliability testing procedure is described. As discussed earlier, reliability indices of the population are randomly variable due to many different reasons. This variability was described by the variable θ and it is natural to assume that it is random (to be denoted for convenience as Θ). Let $\pi(\theta)$ be its probability density function (pdf). Denote by T the lifetime of an item randomly selected from the population before the testing procedure. Its survival function, the probability mass function and the failure rate function are denoted by $S_m(k), f_m(k)$ and $\lambda_m(k)$, respectively, where 'm' stands for 'mixture', as these quantities will be defined as the corresponding mixtures. Furthermore, in the following, the conditional pdf of Θ given $T > k$, $(\Theta|T > k)$, is denoted by $\pi(\theta|k)$. Thus, the initial pdf $\pi(\theta)$ is updated by information that an item had survived in $[0, k)$.

Then, under the model in (7), the mixture survival function of T is given by

$$S_m(k) = \int_0^\infty \left[\prod_{i=1}^{k} (1 - \lambda_0(i))^\theta\right] \pi(\theta) d\theta \tag{9}$$

and the mixture probability mass function is given by

$$f_m(k) = \int_0^\infty (1 - (1 - \lambda_0(k))^\theta) \left[\prod_{i=1}^{k-1} (1 - \lambda_0(i))^\theta\right] \pi(\theta) d\theta \tag{10}$$

whereas the corresponding failure rate function is

$$\lambda_m(k) = \frac{S_m(k-1) - S_m(k)}{S_m(k-1)} = \frac{\int_0^\infty \{1 - (1 - \lambda_0(k))^\theta\} \left[\prod_{i=1}^{k-1} (1 - \lambda_0(i))^\theta\right] \pi(\theta) d\theta}{\int_0^\infty \left[\prod_{i=1}^{k-1} (1 - \lambda_0(i))^\theta\right] \pi(\theta) d\theta}$$

where $\prod_{i=1}^{k-1} [\cdot] \equiv 1$ when $k = 1$. Note that, the mixture failure rate can be represented as the following conditional expectation

$$\begin{aligned} \lambda_m(k) &= \frac{\int_0^\infty \{1 - (1 - \lambda_0(k))^\theta\} \left[\prod_{i=1}^{k-1} (1 - \lambda_0(i))^\theta\right] \pi(\theta) d\theta}{\int_0^\infty \left[\prod_{i=1}^{k-1} (1 - \lambda_0(i))^\rho\right] \pi(\rho) d\rho} \\ &= E_{(\Theta|T>k-1)}[\{1 - (1 - \lambda_0(k))^\Theta\}] \end{aligned} \tag{11}$$

where $E_{(Z|T>k-1)}[\,\cdot\,]$ represents the expectation with respect to the conditional pdf of $(Z|T > k-1)$:

$$\pi(\theta|k-1) \equiv \frac{\left[\prod_{i=1}^{k-1} (1-\lambda_0(i))^{\theta}\right]\pi(\theta)}{\int_0^{\infty}\left[\prod_{i=1}^{k-1} (1-\lambda_0(i))^{\rho}\right]\pi(\rho)d\rho}$$

For more general discussions on the mixture distributions for discrete lifetimes, see Cha and Finkelstein [34].

The lifetime distribution of items in the population which has passed the reliability testing procedure is different from that before the testing. We will now derive it. Denote by T_s the lifetime of the item randomly selected from this population. Its survival function, the probability mass function and the failure rate function are denoted by $S_{ms}(k)$, $f_{ms}(k)$ and $\lambda_{ms}(k)$, respectively. Denote θ

Furthermore, in the following, the pdf of $\Theta_s \equiv (\Theta|N \le c)$ in the population which has passed the testing procedure and, obviously, differs from initial $\pi(\theta)$, is denoted by $\pi_s(\theta)$and the conditional pdf of Θ_s given $T_s > k$, $(\Theta_s|T_s > k)$, is denoted by $\pi_s(\theta|k)$.

Theorem 4.1: *The survival function, the probability density function and the failure rate function of T_s are given by*

$$S_{ms}(k) = \int_0^{\infty}\left[\prod_{i=1}^{k} (1-\lambda_0(k))^{\theta}\right]\pi_s(\theta)d\theta$$

$$f_{ms}(k) = \int_0^{\infty} (1-(1-\lambda_0(k))^{\theta})\left[\prod_{i=1}^{k-1} (1-\lambda_0(k))^{\theta}\right]\pi_s(\theta)d\theta$$

$$\lambda_{ms}(k) = \int_0^{\infty} \{1-(1-\lambda_0(k))^{\theta}\}\pi_s(\theta|k-1)d\theta \tag{12}$$

respectively, where

$$\pi_s(\theta) = \frac{\sum_{i=0}^{c}\binom{n}{i} p(\theta)^i(1-p(\theta))^{n-i}\pi(\theta)}{\int_0^{\infty}\sum_{i=0}^{c}\binom{n}{i} p(\rho)^i(1-p(\rho))^{n-i}\pi(\rho)d\rho}$$

and

$$\pi_s(\theta|k-1) = \frac{\left[\prod_{i=1}^{k-1} (1-\lambda_0(i))^{\theta}\right]\pi_s(\theta)}{\int_0^{\infty}\left[\prod_{i=1}^{k-1} (1-\lambda_0(i))^{\rho}\right]\pi_s(\rho)d\rho}$$

Proof: In the original population (i.e., the population before the testing procedure), the 'reliability level' Θ of the population is described by $\pi(\theta)$. However, after the reliability testing, for the items in the population which has passed the reliability testing procedure, we have additional information that, for n items randomly selected from the lot, $N \leq c$. Thus, for accepted lots, the distribution of Θ should be updated to the conditional distribution of $(\Theta|N \leq c)$. As the conditional distribution of $(N \leq c|\Theta = \theta)$ is given by

$$P(N \leq c|\Theta = \theta) = \sum_{i=0}^{c} \binom{n}{i} p(\theta)^i (1 - p(\theta))^{n-i}$$

the conditional distribution of $(\Theta|N \leq c)$ is obtained as

$$\pi_s(\theta) = \frac{\sum_{i=0}^{c} \binom{n}{i} p(\theta)^i (1 - p(\theta))^{n-i} \pi(\theta)}{\int_0^{\infty} \sum_{i=0}^{c} \binom{n}{i} p(\rho)^i (1 - p(\rho))^{n-i} \pi(\rho) d\rho}$$

Then by replacing $\pi(\theta)$ in (9), (10) and (11) with $\pi_s(\theta)$, respectively, we have the desired result. ■

We will now analyze the reliability improvement in the population after the testing procedure by comparing the population distributions before and after the reliability test. The following theorem states this improvement in the form of the corresponding failure rate ordering.

Theorem 4.2: *If the proposed reliability sampling plan is applied, then*

(i) $\Theta \geq_{lr} (\Theta|N \leq c)$, *for any given* $\pi(\theta)$;
(ii) $\lambda_m(k) \geq \lambda_{ms}(k)$, *for all* $k \geq 1$,

for any given $\pi(\theta)$, *that is,* $T \leq_{fr} T_s$, *for any* $\pi(\theta)$.

Proof: Consider the following ratio of the corresponding pdfs

$$\frac{\pi_s(\theta)}{\pi(\theta)} \equiv \frac{\sum_{i=0}^{c} \binom{n}{i} p(\theta)^i (1 - p(\theta))^{n-i}}{\int_0^{\infty} \sum_{i=0}^{c} \binom{n}{i} p(\rho)^i (1 - p(\rho))^{n-i} \pi(\rho) d\rho}$$

where $\sum_{i=0}^{c} \binom{n}{i} p(\theta)^i (1 - p(\theta))^{n-i}$ is the cdf of a binomial random variable $X_{n,p(\theta)}$. Observe that

$$p(\theta) = 1 - \prod_{j=1}^{k_0} (1 - \lambda_0(j))^{\theta}$$

and, for $\rho_1 < \rho_2$,

$$\frac{\binom{n}{i} p(\rho_1)^i (1-p(\rho_1))^{n-i}}{\binom{n}{i} p(\rho_2)^i (1-p(\rho_2))^{n-i}} = \left(\frac{1-p(\rho_1)}{1-p(\rho_2)}\right)^n \left(\frac{p(\rho_1)}{p(\rho_2)} \frac{1-p(\rho_2)}{1-p(\rho_1)}\right)^i$$

is decreasing in i as $p(\rho_1) < p(\rho_2)$. This means that $X_{n,p(\rho_1)} \le_{lr} X_{n,p(\rho_2)}$, also implying that

$$X_{n,p(\rho_1)} \le_{st} X_{n,p(\rho_2)}$$

and thus

$$\sum_{i=0}^{c} \binom{n}{i} p(\rho_1)^i (1-p(\rho_1))^{n-i} \ge \sum_{i=0}^{c} \binom{n}{i} p(\rho_2)^i (1-p(\rho_2))^{n-i}$$

Then, for $\rho_1 < \rho_2$,

$$\frac{\pi_s(\rho_1)}{\pi(\rho_1)} = \frac{\sum_{i=0}^{c} \binom{n}{i} p(\rho_1)^i (1-p(\rho_1))^{n-i}}{\int_0^\infty \sum_{i=0}^{c} \binom{n}{i} p(\rho)^i (1-p(\rho))^{n-i} \pi(\rho) d\rho}$$
$$\ge \frac{\sum_{i=0}^{c} \binom{n}{i} p(\rho_2)^i (1-p(\rho_2))^{n-i}}{\int_0^\infty \sum_{i=0}^{c} \binom{n}{i} p(\rho)^i (1-p(\rho))^{n-i} \pi(\rho) d\rho} = \frac{\pi_s(\rho_2)}{\pi(\rho_2)}$$

which implies that $\pi_s(\theta)/\pi(\theta)$ is decreasing in θ. Therefore, $\Theta \ge_{lr} (\Theta | N \le c)$, which competes for the proof of (i).

We will now prove (ii). Observe that

$$\lambda_m(k) = \int_0^\infty \{1-(1-\lambda_0(k))^\theta\} \pi_s(\theta|k-1) d\theta = E_{(\Theta|T>k-1)}[\{1-(1-\lambda_0(k))^\Theta\}]$$

and $\lambda_{ms}(k)$ in (12) can be expressed as

$$\lambda_{ms}(k) = E_{(\Theta_s|T_s>k-1)}[\{1-(1-\lambda_0(k))^{\Theta_s}\}]$$

where $E_{(\Theta_s|T_s>k-1)}[\cdot]$ denotes the expectation with respect to the conditional pdf of $(\Theta_s|T_s > k-1)$, $\pi_s(\theta|k-1)$. Consider the following ratio of the corresponding pdfs:

$$\frac{\pi_s(\theta|k-1)}{\pi(\theta|k-1)}$$

$$= \frac{\int_0^\infty \left[\prod_{i=1}^{k-1} (1-\lambda_0(k))^\rho\right] \pi(\rho)d\rho}{\int_0^\infty \left[\prod_{i=1}^{k-1} (1-\lambda_0(k))^\rho\right] \sum_{i=0}^{c} \binom{n}{i} p(\rho)^i(1-p(\rho))^{n-i}\pi(\rho)d\rho}$$

$$\cdot \left(\sum_{i=0}^{c} \binom{n}{i} p(\theta)^i(1-p(\theta))^{n-i}\right)$$

which is decreasing in θ. This implies that

$$(\Theta_s|T_s > k-1) \leq_{lr} (\Theta|T > k-1), \text{ for all } k \geq 1$$

and, due to Lemma 2.1, we have

$$\begin{aligned}\lambda_m(k) &= E_{(\Theta|T>k-1)}[\{1-(1-\lambda_0(k))^{\Theta}\}] \geq E_{(\Theta_s|T_s>k-1)}[\{1-(1-\lambda_0(k))^{\Theta_s}\}] \\ &= \lambda_{ms}(k), \text{ for all } k \geq 1\end{aligned}$$

This completes the proof of the theorem. ■

We now present an example which illustrates the theoretical findings of Theorem 4.2.

Example 4.1: Suppose that the baseline failure rate is given by $\lambda_0(i) = 0.1, i = 1, 2, \ldots$. Let $\theta_1 = 1, \theta_2 = 3$. The two risks are $\alpha = 0.05$, $\beta = 0.01$, and the testing time is $k_0 = 5$. Furthermore, to see how the shape of the distribution of Θ changes after the acceptance test, we employ the uniform distribution for Θ: $\pi(\theta) = 0.2$, if $0.5 \leq \theta \leq 5.5$; $\pi(\theta) = 0$, otherwise. Then, referring to Table 1 in Section 2, the desired sampling plan is characterized by two parameters $(n = 23, c = 13)$. Under the above setting, the failure rate functions $\lambda_m(k)$ and $\lambda_{ms}(k)$ are given in Figure 1.

We see that the population failure rate after the test is clearly smaller than that before it, which describes the corresponding reliability improvement.

Define $\Pi(\theta) \equiv \int_0^\theta \pi(\rho)d\rho$ and $\Pi_s(\theta) \equiv \int_0^\theta \pi_s(\rho)d\rho$ as the corresponding cdf's. The cdf's and the pdf's of Θ and $(\Theta|N \leq c)$ are given in Figures 2 and 3.

Using numerical examples, we will now discuss the impact of the testing time k_0 on the corresponding reliability sampling plan. For this, under the setting of Example 4.1, five different testing times are chosen: $k_0 = 3, 4, 5, 6, 7$. For each case, the parameters (n, c) satisfying both equations in (8) are as follows:

$$(k_0 = 3; n = 31, c = 12), (k_0 = 4; n = 25, c = 12), (k_0 = 5; n = 23, c = 13)$$

$$(k_0 = 6; n = 22, c = 14), (k_0 = 7; n = 19, c = 13)$$

Our numerical experiments showed that the corresponding failure rate functions $\lambda_{ms}(k)$ for different values of testing time k_0 are practically undistinguishable. Moreover, the acceptance probability functions $L(\theta)$ for different values of testing time k_0 are very close (Figure 4).

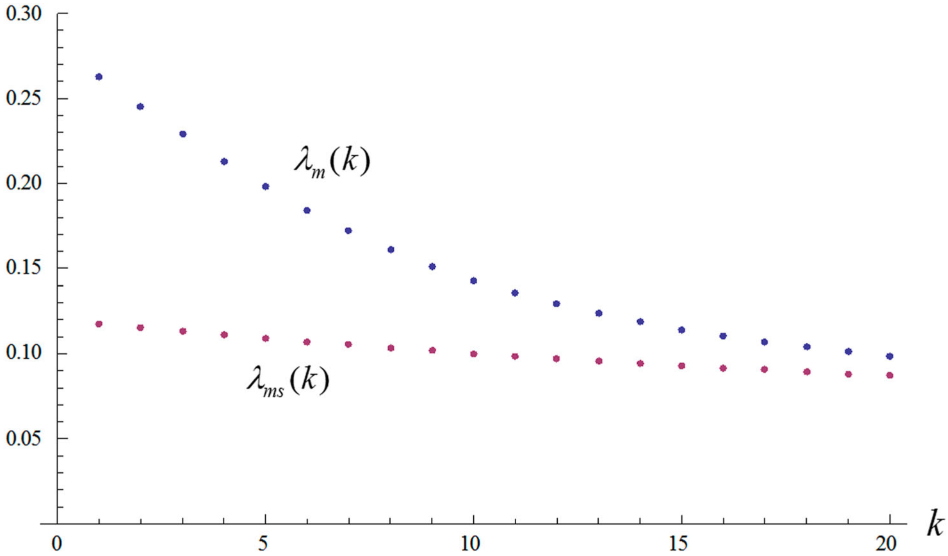

Figure 1. Failure Rate Functions $\lambda_m(k)$ and $\lambda_{ms}(k)$.

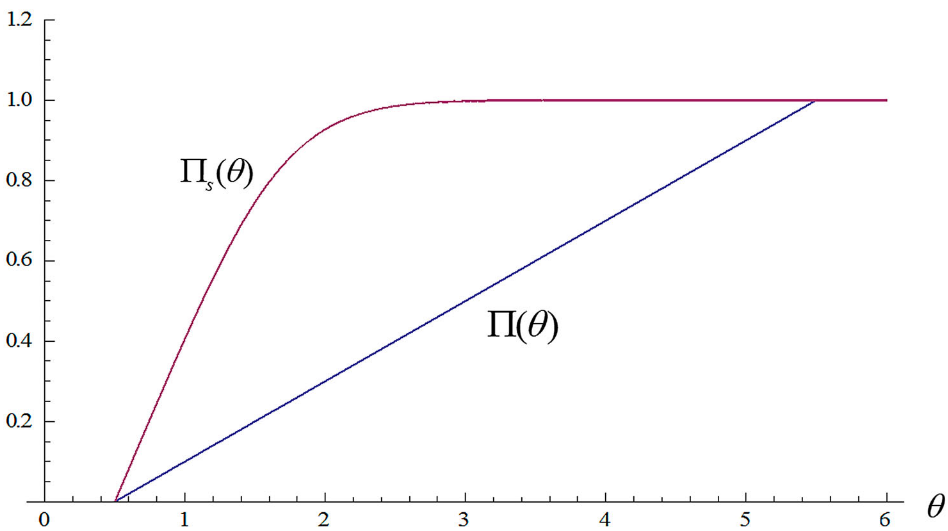

Figure 2. The Cdf's $\Pi(\theta)$ and $\Pi_s(\theta)$.

Therefore, we find that the testing time does not have a noticeable impact on the reliability characteristics of the accepted lots. However, as the testing time k_0 decreases, the required sample size n and, therefore, related costs increase rapidly. Hence, when considering this cost aspect, a balance between the sample size n and the testing time k_0 should be taken into account. The corresponding optimal problem can constitute a topic for future research.

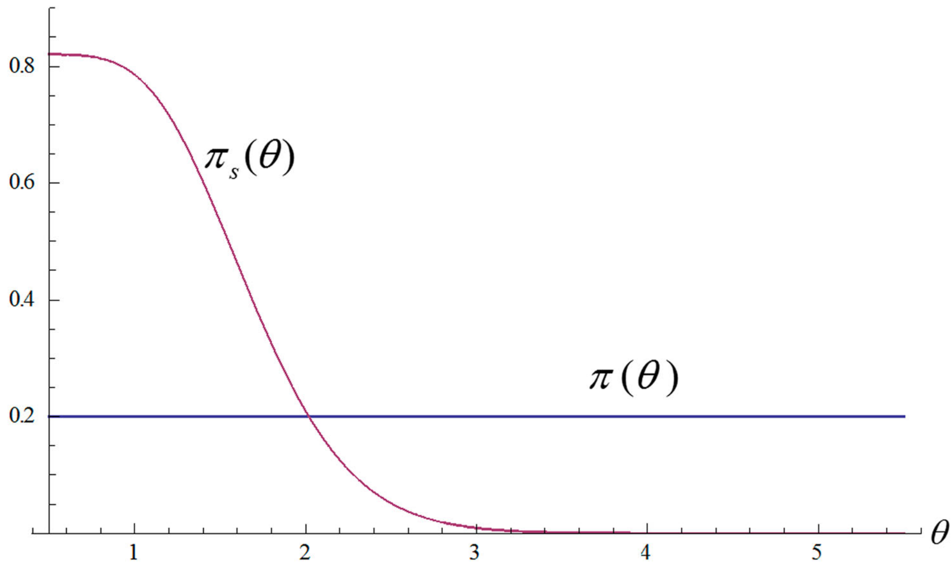

Figure 3. The pdf's $\pi(\theta)$ and $\pi_s(\theta)$.

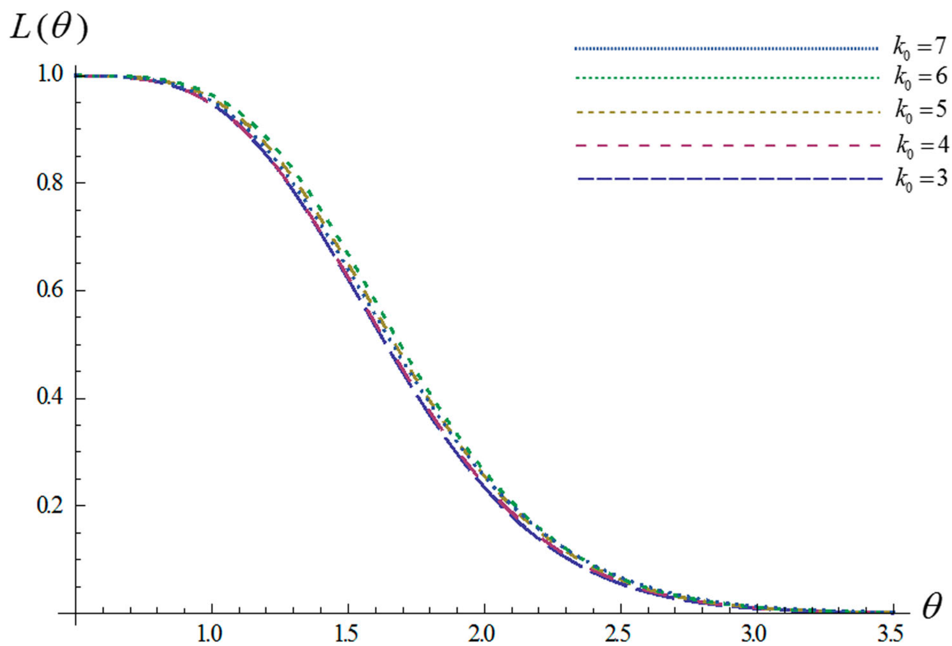

Figure 4. The acceptance probability $L(\theta)$ for different values of testing time k_0.

5. Additive hazard model for discrete lifetime

Until now, in describing the variability in the population reliability, the proportional hazard model in (6) and (7) has been assumed. In addition to the proportional hazard model, in modelling continuous distribution, the additive hazard model is also frequently employed.

In the continuous case, the failure rate function in the additive hazard model is defined as

$$\lambda(t) = \lambda_0(t) + \theta \tag{13}$$

where $\lambda_0(t)$ is the baseline failure rate. Similar to the case of the proportional hazard model, the additive hazard model in a continuous setting cannot be directly applied as the model in (13) is not bounded. Using the alternative failure rate $r(k)$, similar to the continuous setting in (6), we can now formulate the additive hazard model for the discrete lifetime as follows:

$$r(k) = r_0(k) + \theta \tag{14}$$

where $r_0(k) = -\ln[1 - \lambda_0(k)]$ and $\lambda_0(k)$ is the baseline discrete failure rate function. Under the model in (14),

$$\lambda(k) = 1 - \exp\{-r(k)\} = 1 - \exp\{-r_0(k) - \theta\} = 1 - \exp\{-\theta\}(1 - \lambda_0(k))$$

Thus,

$$S_m(k) = \int_0^\infty \left[\exp\{-k\theta\} \prod_{i=1}^{k} (1 - \lambda_0(i)) \right] \pi(\theta) d\theta$$

and the corresponding mixture probability mass function is given by

$$f_m(k) = \int_0^\infty [1 - \exp\{-\theta\}(1 - \lambda_0(k))] \left[\exp\{-(k-1)\theta\} \prod_{i=1}^{k-1} (1 - \lambda_0(i)) \right] \pi(\theta) d\theta$$

The corresponding failure rate function is

$$\lambda_m(k) = \frac{S_m(k-1) - S_m(k)}{S_m(k-1)}$$
$$= \frac{\int_0^\infty [1 - \exp\{-\theta\}(1 - \lambda_0(k))] \left[\exp\{-(k-1)\theta\} \prod_{i=1}^{k-1} (1 - \lambda_0(i)) \right] d\theta}{\int_0^\infty \exp\{-(k-1)\theta\} \prod_{i=1}^{k-1} (1 - \lambda_0(i)) \pi(\theta) d\theta}$$

Furthermore, the survival function, the probability density function and the failure rate function of T_s are

$$S_{ms}(k) = \int_0^\infty \left[\exp\{-k\theta\} \prod_{i=1}^{k} (1 - \lambda_0(i)) \right] \pi_s(\theta) d\theta$$

$$f_{ms}(k) = \int_0^\infty [1 - \exp\{-\theta\}(1 - \lambda_0(k))] \left[\exp\{-(k-1)\theta\} \prod_{i=1}^{k-1} (1 - \lambda_0(i)) \right] \pi_s(\theta) d\theta$$

$$\lambda_{ms}(k) = \int_0^\infty [1 - \exp\{-\theta\}(1 - \lambda_0(k))]\pi_s(\theta|k-1)d\theta$$

respectively, where

$$\pi_s(\theta) = \frac{\sum_{i=0}^{c} \binom{n}{i} p(\theta)^i (1 - p(\theta))^{n-i} \pi(\theta)}{\int_0^\infty \sum_{i=0}^{c} \binom{n}{i} p(\rho)^i (1 - p(\rho))^{n-i} \pi(\rho) d\rho}$$

and

$$\pi_s(\theta|k-1) = \frac{\exp\{-(k-1)\theta\} \prod_{i=1}^{k-1} (1 - \lambda_0(i))\pi_s(\theta)}{\int_0^\infty \exp\{-(k-1)\rho\} \prod_{i=1}^{k-1} (1 - \lambda_0(i))\pi_s(\rho) d\rho}$$

with $p(\theta) = 1 - \left[\exp\{-k\theta\} \prod_{j=1}^{k_0} (1 - \lambda_0(j))\right]$. Then, following similar procedures as those described in Section 4, we can obtain the same results on the ordering of lifetimes before and after the sampling test.

6. Concluding remarks

Until now, various types of acceptance reliability sampling plans have been developed and studied only for continuous lifetime models. However, in practice, discrete lifetimes are frequently encountered and the main objective of this paper is to develop a variables acceptance reliability sampling plan for items having discrete lifetimes.

A general reliability sampling plan for discrete lifetime distributions has been developed and its impact on reliability characteristics of items has been discussed by comparing the lifetime distributions of the populations before and after the testing procedure. It has been shown that the lifetime of the population which has passed the life testing procedure is stochastically larger than that of the original population in the sense of the failure rate ordering.

The limitation of the study is that the exact values for n and c matching the requirement for α (producer risk) and β (consumer risk) cannot be obtained and thus the producer's and consumer's risks are not exactly achieved. However, this is inevitable problem that cannot be improved in this field.

As mentioned earlier, in practice, field failures are often collected and reported daily, weekly, and so forth. Items often operate in cycles and the experimenter observes the number of cycles successfully completed prior to failure. In this case, lifetimes of the items are the discrete type and the conventional reliability acceptance testing plans for continuous lifetime distributions cannot be applied. The proposed study can be applied to such cases.

Our results are general, as they do not assume any particular distribution, and therefore, can be applied to any discrete distribution. We have considered two popular models (proportional and additive hazards models) for describing populations' variability (heterogeneity). Similar results can be obtained for the corresponding accelerated life model.

To the best of our knowledge, this is the first work to discuss a variables acceptance reliability sampling plan for discrete lifetime model. In the future, similar to the continuous model studied in Lam and Choy [15], Tsai and Wu [16], Balakrishnan et al. [17], Aslam and Jun [18] and Aslam et al. [19–22], more advanced types of sampling plans could be developed.

Neutrosophic statistics, introduced by Smarandache [35], is the extension of classical statistics and is applied when the data is coming from a complex process or from an uncertain environment. Recently, neutrosophic statistics theory has been applied to the inspection, inference, and process control (see, e.g., Aslam [36–38], Chen et al. [39]). The current study can be extended using neutrosophic statistics in the future research.

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