Aloqaily et al. *Boundary Value Problems* (2024) 2024:47
https://doi.org/10.1186/s13661-024-01854-2

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A novel stability analysis of functional equation in neutrosophic normed spaces

Ahmad Aloqaily[1,2], P. Agilan[3], K. Julietraja[4], S. Annadurai[3] and Nabil Mlaiki[1*]

*Correspondence:
nmlaiki@psu.edu.sa;
nmlaiki2012@gmail.com
[1]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

The analysis of stability in functional equations (FEs) within neutrosophic normed spaces is a significant challenge due to the inherent uncertainties and complexities involved. This paper proposes a novel approach to address this challenge, offering a comprehensive framework for investigating stability properties in such contexts. Neutrosophic normed spaces are a generalization of traditional normed spaces that incorporate neutrosophic logic. By providing a systematic methodology for addressing stability concerns in neutrosophic normed spaces, our approach facilitates enhanced understanding and control of complex systems characterized by indeterminacy and uncertainty. The primary focus of this research is to propose a novel class of Euler-Lagrange additive FE and investigate its Ulam-Hyers stability in neutrosophic normed spaces. Direct and fixed point techniques are utilized to achieve the required results.

Mathematics Subject Classification: 39B52; 39B82

Keywords: Euler-Lagrange additive functional equations; Ulam-Hyers stability; Neutrosophic normed spaces; Fixed point theory

1 Introduction

Lotfi A. Zadeh, a mathematician and computer scientist, introduced the idea of fuzzy sets in his groundbreaking paper entitled "Fuzzy Sets," published in 1965 [1]. Zadeh's motivation was to address the limitations of classical set theory, which relies on crisp, well-defined boundaries for membership. The extension of fuzzy sets known as intuitionistic fuzzy sets (IFS) was initially proposed by Atanassov in 1983 [2, 3]. IFS provide a framework for dealing with uncertainty, vagueness, and hesitation more comprehensively than traditional fuzzy sets. A neutrosophic set is a mathematical concept introduced as an extension of classical set theory [4, 5]. Neutrosophic sets provide a way to handle indeterminacy, uncertainty, and incomplete information more flexibly.

Fuzzy normed spaces (FNS) are mathematical structures that extend the concept of normed spaces to include fuzzy numbers. Katsaras introduced the idea of FNS, a vector space equipped with a fuzzy norm, where the norm values are fuzzy numbers rather than real numbers [6]. The idea of intuitionistic fuzzy normed spaces (IFNS) was proposed in 2006 [7]. IFNS represent a fusion of concepts from fuzzy mathematics, intuitionistic fuzzy sets, and normed spaces, providing a versatile framework for handling uncertainty and imprecision in mathematical modeling and analysis. Neutrosophic normed linear spaces [8, 9] extend the ideas of neutrosophic sets to linear algebraic structures, offering a more expressive way to represent and handle uncertainty in vector spaces. Neutrosophic concepts have been applied across a wide range of mathematical fields such as groups, subgroups [10], vector spaces [11], homomorphisms of rings [12, 13], linear transformations [14], number theory [15], graph theory [16], measure theory, integral theory, probability theory [17], etc. Neutrosophic normed linear spaces find applications in various scientific and real-time applications like decision-making, control systems, optimization, image processing, pattern recognition, medical diagnosis, finance and risk management, information retrieval, and artificial intelligence [18–24].

Several mathematicians have derived the results of fixed point (F-P) theory using neutrosophic concepts. Ishtiaq et al. [25] proved F-P results in the framework of an orthogonal neutrosophic metric space. The common F-P results in the context of a neutrosophic metric space were established using contraction mapping [26]. Salama and Alblowi [27] studied the neutrosophic topological spaces, whereas Al-Omeri et al. [28] analyzed a neutrosophic cone metric space. Riaz et al. [29] discussed the F-P results for ξ-chainable neutrosophic and generalized neutrosophic cone metric spaces. Sharma et al. [30] studied the generalized summability using difference operators on neutrosophic normed spaces. One of the most prevalent stability theories was introduced by Ulam [31] and further developed by Hyers [32]. This theory is called the Ulam-Hyers stability and has applications in various branches of mathematics, including differential equations, functional analysis, and dynamic systems. Many researchers have worked on this theory and established the stability results in different normed spaces [33–40]. Recently, Agilan et al. introduced new kinds of FEs and established the Ulam-Hyers stability of the newly proposed equations in a variety of normed spaces [41–44]. Some related applications are also discussed in [45–48].

This article presents a novel category of Euler-Lagrange additive FE. The Ulam-Hyers stability of the newly introduced equation is analyzed in neutrosophic normed linear spaces using two techniques: direct and fixed point techniques. The stability analysis of the newly introduced equation holds significance due to the distinctive characteristics and potentially broad applications of neutrosophic normed spaces. It is noteworthy that, for the first time in the literature, the stability of an FE has been examined within neutrosophic normed spaces. The uniqueness of this endeavor underscores the importance of this research.

This study aims to achieve the following primary objectives:

(a) Extend and enhance the current body of work on neutrosophic normed linear spaces.

(b) Ascertain the uniqueness of the solution for the newly proposed Euler-Lagrange additive FE.

(c) Derive the Ulam-Hyers stability of the newly proposed equation in neutrosophic normed linear spaces using the F-P method.

2 Definitions on neutrosophic normed spaces

Definition 2.1 The Seven-tuple $(\mathbb{A}, \mathfrak{A}_a, \mathfrak{B}_a, \mathfrak{C}_a *, \diamond, \oslash)$ is said to be a neutrosophic normed space (for short, NNS) if $\mathbb{A}$ is a vector space, $*$ is a continuous κ-norm, $\diamond$ and $\oslash$ is a continuous $\kappa-$ conorm, and $\mathfrak{A}_a$, $\mathfrak{B}_a$, $\mathfrak{C}_a$ are fuzzy sets on $\mathbb{A} \times (0, \infty)$ satisfying the following conditions. For every $p, q \in \mathbb{A}$ and $s, \kappa > 0$,

(A1) $\mathfrak{A}_a(p,\kappa) + \mathfrak{B}_a(p,\kappa) + \mathfrak{C}_a(p,\kappa) \le 3$,
(A2) $0 \le \mathfrak{A}_a(p,\kappa) \le 1$, $0 \le \mathfrak{B}_a(p,\kappa) \le 1$, $0 \le \mathfrak{C}_a(p,\kappa) \le 1$,
(A3) $\mathfrak{A}_a(p,\kappa) > 0$,
(A4) $\mathfrak{A}_a(p,\kappa) = 1$, if and only if $p = 0$.
(A5) $\mathfrak{A}_a(\alpha p,\kappa) = \mathfrak{A}_a(p,\frac{\kappa}{|\alpha|})$ for each $\alpha \neq 0$,
(A6) $\mathfrak{A}_a(p,\kappa) * \mathfrak{A}_a(q,s) \le \mathfrak{A}_a(p+q,\kappa+s)$,
(A7) $\mathfrak{A}_a(p,\cdot) : (0,\infty) \to [0,1]$ is continuous,
(A8) $\lim_{\kappa\to\infty} \mathfrak{A}_a(p,\kappa) = 1$ and $\lim_{\kappa\to 0} \mathfrak{A}_a(p,\kappa) = 0$,
(A9) $\mathfrak{B}_a(p,\kappa) < 1$,
(A10) $\mathfrak{B}_a(p,\kappa) = 0$, if and only if $p = 0$.
(A11) $\mathfrak{B}_a(\alpha p,\kappa) = \mathfrak{B}_a(p,\frac{\kappa}{|\alpha|})$ for each $\alpha \neq 0$,
(A12) $\mathfrak{B}_a(p,\kappa) \diamond \mathfrak{B}_a(q,s) \ge \mathfrak{B}_a(p+q,\kappa+s)$,
(A13) $\mathfrak{B}_a(p,\cdot) : (0,\infty) \to [0,1]$ is continuous,
(A14) $\lim_{\kappa\to\infty} \mathfrak{B}_a(p,\kappa) = 0$ and $\lim_{\kappa\to 0} \mathfrak{B}_a(p,\kappa) = 1$
(A15) $\mathfrak{C}_a(p,\kappa) < 1$,
(A16) $\mathfrak{C}_a(p,\kappa) = 0$, if and only if $p = 0$.
(A17) $\mathfrak{C}_a(\alpha p,\kappa) = \mathfrak{C}_a(p,\frac{\kappa}{|\alpha|})$ for each $\alpha \neq 0$,
(A18) $\mathfrak{C}_a(p,\kappa) \oslash \mathfrak{C}_a(q,s) \ge \mathfrak{C}_a(p+q,\kappa+s)$,
(A19) $\mathfrak{C}_a(p,\cdot) : (0,\infty) \to [0,1]$ is continuous,
(A20) $\lim_{\kappa\to\infty} \mathfrak{C}_a(p,\kappa) = 0$ and $\lim_{\kappa\to 0} \mathfrak{C}_a(p,\kappa) = 1$.

3 Stability results: direct method

The newly proposed Euler-Lagrange additive FE is as follows:

$$
\begin{aligned}
&\mathfrak{T}_1\mathscr{A}_1(\mathcal{P}_1\mathscr{X}_a + \mathcal{Q}_1\mathscr{V}_a + \mathcal{R}_1\mathscr{W}_a) + \mathcal{P}_1\mathscr{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a - \mathscr{V}_a)\big) \\
&\quad + \mathcal{Q}_1\mathscr{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a - \mathscr{W}_a)\big) + \mathcal{R}_1\mathscr{A}_1\big(\mathcal{P}_1\mathcal{Q}_1(\mathscr{W}_a - \mathscr{X}_a)\big) \\
&= \mathfrak{T}_1\big(\mathcal{P}_1\mathscr{A}_1(\mathscr{X}_a) + \mathcal{Q}_1\mathscr{A}_1(\mathscr{V}_a) + \mathcal{R}_1\mathscr{A}_1(\mathscr{W}_a)\big),
\end{aligned}
\tag{3.1}
$$

where $\mathcal{P}_1,\mathcal{Q}_1,\mathcal{R}_1 \in \mathbb{R}$ with $\mathcal{P}_1$, $\mathcal{Q}_1$, $\mathcal{R}_1 \neq 0$ and $\mathfrak{T}_1 = \mathcal{P}_1 + \mathcal{Q}_1 + \mathcal{R}_1 \neq 0$ in neutrosophic normed space using direct and F-P methods.

Assume that $(\mathcal{M},\mathfrak{A}'_a,\mathfrak{B}'_a,\mathfrak{C}'_a)$ is a neutrosophic normed linear space and $(\mathcal{M},\mathfrak{A}_a,\mathfrak{B}_a,\mathfrak{C}_a)$ is a neutrosophic Banach space. Let $\mathcal{L}$ be a linear space. Then,

$$
\begin{aligned}
\mathfrak{Z}(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a) &= \mathfrak{T}_1\mathscr{A}_1(\mathcal{P}_1\mathscr{X}_a + \mathcal{Q}_1\mathscr{V}_a + \mathcal{R}_1\mathscr{W}_a) + \mathcal{P}_1\mathscr{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a - \mathscr{V}_a)\big) \\
&\quad + \mathcal{Q}_1\mathscr{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a - \mathscr{W}_a)\big) + \mathcal{R}_1\mathscr{A}_1\big(\mathcal{P}_1\mathcal{Q}_1(\mathscr{W}_a - \mathscr{X}_a)\big) \\
&\quad - \mathfrak{T}_1\big(\mathcal{P}_1\mathscr{A}_1(\mathscr{X}_a) + \mathcal{Q}_1\mathscr{A}_1(\mathscr{V}_a) + \mathcal{R}_1\mathscr{A}_1(\mathscr{W}_a)\big),
\end{aligned}
$$

where $\mathfrak{T}_1 = \mathcal{P}_1 + \mathcal{Q}_1 + \mathcal{R}_1$, $\mathcal{P}_1,\mathcal{Q}_1,\mathcal{R}_1 \in \mathbb{R}$ and $\mathcal{P}_1$, $\mathcal{Q}_1$, $\mathcal{R}_1 \neq 0$ $\forall \mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a \in \mathcal{L}$.

Theorem 3.1 *Let $N : \mathcal{L} \times \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{M}$ be a mapping with the condition $0 < (\frac{\mathscr{X}_a}{\mathfrak{T}_1})^{\eta} < 1$, then*

$$
\left.
\begin{aligned}
&\mathfrak{A}'_a\big(N\big(\mathfrak{T}_1^{n\eta}\mathscr{X}_a,\mathfrak{T}_1^{n\eta}\mathscr{V}_a,\mathfrak{T}_1^{n\eta}\mathscr{W}_a\big),\upsilon\big) \ge \mathfrak{A}'_a\big(\mathscr{X}_a^{n\eta}N(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big), \\
&\mathfrak{B}'_a\big(N\big(\mathfrak{T}_1^{n\eta}\mathscr{X}_a,\mathfrak{T}_1^{n\eta}\mathscr{V}_a,\mathfrak{T}_1^{n\eta}\mathscr{W}_a\big),\upsilon\big) \le \mathfrak{B}'_a\big(\mathscr{X}_a^{n\eta}N(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big), \\
&\mathfrak{C}'_a\big(N\big(\mathfrak{T}_1^{n\eta}\mathscr{X}_a,\mathfrak{T}_1^{n\eta}\mathscr{V}_a,\mathfrak{T}_1^{n\eta}\mathscr{W}_a\big),\upsilon\big) \le \mathfrak{C}'_a\big(\mathscr{X}_a^{n\eta}N(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big)
\end{aligned}
\right\}
\tag{3.2}
$$

and

$$\left.\begin{aligned}
&\lim_{n\to\infty}\mathfrak{A}'_a\big(N\big(\mathfrak{T}_1^{\eta n}\mathscr{X}_a,\mathfrak{T}_1^{\eta n}\mathscr{V}_a,\mathfrak{T}_1^{\eta n}\mathscr{W}_a\big),a^{\eta n}\upsilon\big)=1,\\
&\lim_{n\to\infty}\mathfrak{B}'_a\big(N\big(\mathfrak{T}_1^{\eta n}\mathscr{X}_a,\mathfrak{T}_1^{\eta n}\mathscr{V}_a,\mathfrak{T}_1^{\eta n}\mathscr{W}_a\big),a^{\eta n}\upsilon\big)=0,\\
&\lim_{n\to\infty}\mathfrak{C}'_a\big(N\big(\mathfrak{T}_1^{\eta n}\mathscr{X}_a,\mathfrak{T}_1^{\eta n}\mathscr{V}_a,\mathfrak{T}_1^{\eta n}\mathscr{W}_a\big),a^{\eta n}\upsilon\big)=0.
\end{aligned}\right\}\tag{3.3}$$

Assume that a mapping $\mathscr{A}_1:\mathcal{L}\to\mathcal{M}$ satisfies the inequality

$$\left.\begin{aligned}
&\mathfrak{A}_a\big(\mathfrak{Z}(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(N(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big),\\
&\mathfrak{B}_a\big(\mathfrak{Z}(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(N(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big),\\
&\mathfrak{C}_a\big(\mathfrak{Z}(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(N(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big)
\end{aligned}\right\}\tag{3.4}$$

and $\exists$ *unique additive function* $\mathcal{A}_1:\mathcal{L}\longrightarrow\mathcal{M}$

$$\left.\begin{aligned}
&\mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\mathfrak{T}_1|\mathfrak{T}_1-\mathscr{X}_a|\upsilon\big),\\
&\mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\mathfrak{T}_1|\mathfrak{T}_1-\mathscr{X}_a|\upsilon\big),\\
&\mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\mathfrak{T}_1|\mathfrak{T}_1-\mathscr{X}_a|\upsilon\big)
\end{aligned}\right\}\tag{3.5}$$

with the conditions $\eta\in\{1,-1\}$, *where* $\mathfrak{T}_1=\mathcal{P}_1+\mathcal{Q}_1+\mathcal{R}_1$.

Proof Let us consider $(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a)$ by $(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a)$ in (3.4), we reach

$$\left.\begin{aligned}
&\mathfrak{A}_a\big(\mathfrak{T}_1\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)-\mathfrak{T}_1^2\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big),\\
&\mathfrak{B}_a\big(\mathfrak{T}_1\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)-\mathfrak{T}_1^2\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big),\\
&\mathfrak{C}_a\big(\mathfrak{T}_1\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)-\mathfrak{T}_1^2\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big).
\end{aligned}\right\}\tag{3.6}$$

By applying the conditions of neutrosophic normed space, we arrive

$$\left.\begin{aligned}
&\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)}{\mathfrak{T}_1}-\mathscr{A}_1(\mathscr{X}_a),\frac{\upsilon}{\mathfrak{T}_1^2}\right)\geq\mathfrak{A}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big),\\
&\mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)}{\mathfrak{T}_1}-\mathscr{A}_1(\mathscr{X}_a),\frac{\upsilon}{\mathfrak{T}_1^2}\right)\leq\mathfrak{B}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big),\\
&\mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)}{\mathfrak{T}_1}-\mathscr{A}_1(\mathscr{X}_a),\frac{\upsilon}{\mathfrak{T}_1^2}\right)\leq\mathfrak{C}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big).
\end{aligned}\right\}\tag{3.7}$$

Replacing $\mathscr{X}_a$ by $\mathfrak{T}_1^n\mathscr{X}_a$ in (3.7), we get

$$\left.\begin{aligned}
&\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+1}\mathscr{X}_a)}{\mathfrak{T}_1}-\mathscr{A}_1\big(\mathfrak{T}_1^n\mathscr{X}_a\big),\frac{\upsilon}{\mathfrak{T}_1^2}\right)\geq\mathfrak{A}'_a\big(N\big(\mathfrak{T}_1^n\mathscr{X}_a,\mathfrak{T}_1^n\mathscr{X}_a,\mathfrak{T}_1^n\mathscr{X}_a\big),\upsilon\big),\\
&\mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+1}\mathscr{X}_a)}{\mathfrak{T}_1}-\mathscr{A}_1\big(\mathfrak{T}_1^n\mathscr{X}_a\big),\frac{\upsilon}{\mathfrak{T}_1^2}\right)\leq\mathfrak{B}'_a\big(N\big(\mathfrak{T}_1^n\mathscr{X}_a,\mathfrak{T}_1^n\mathscr{X}_a,\mathfrak{T}_1^n\mathscr{X}_a\big),\upsilon\big),\\
&\mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+1}\mathscr{X}_a)}{\mathfrak{T}_1}-\mathscr{A}_1\big(\mathfrak{T}_1^n\mathscr{X}_a\big),\frac{\upsilon}{\mathfrak{T}_1^2}\right)\leq\mathfrak{C}'_a\big(N\big(\mathfrak{T}_1^n p,\mathfrak{T}_1^n\mathscr{X}_a,\mathfrak{T}_1^n\mathscr{X}_a\big),\upsilon\big).
\end{aligned}\right\}\tag{3.8}$$

Using (3.8) and conditions of neutrosophic normed space, we arrive

$$\left.\begin{aligned}
&\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+1}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+1)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{n}\mathscr{X}_a)}{\mathfrak{T}_1^{n}},\frac{\upsilon}{\mathfrak{T}_1^{n+2}}\right)\geq\mathfrak{A}'_a\left(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\frac{\upsilon}{\mathscr{X}_a^{n}}\right),\\
&\mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+1}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+1)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{n}\mathscr{X}_a)}{\mathfrak{T}_1^{n}},\frac{\upsilon}{\mathfrak{T}_1^{n+2}}\right)\leq\mathfrak{B}'_a\left(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\frac{\upsilon}{\mathscr{X}_a^{n}}\right),\\
&\mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+1}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+1)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{n}\mathscr{X}_a)}{\mathfrak{T}_1^{n}},\frac{\upsilon}{\mathfrak{T}_1^{n+2}}\right)\leq\mathfrak{C}'_a\left(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\frac{\upsilon}{\mathscr{X}_a^{n}}\right).
\end{aligned}\right\}\tag{3.9}$$

Let υ by $\mathscr{X}_a^{n}\upsilon$ in (3.9), then

$$\left.\begin{aligned}
&\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+1}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+1)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{n}\mathscr{X}_a)}{\mathfrak{T}_1^{n}},\frac{\upsilon\cdot\mathscr{X}_a^{n}}{\mathfrak{T}_1^{n+2}}\right)\geq\mathfrak{A}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big),\\
&\mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+1}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+1)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{n}\mathscr{X}_a)}{\mathfrak{T}_1^{n}},\frac{\upsilon\cdot\mathscr{X}_a^{n}}{\mathfrak{T}_1^{n+2}}\right)\leq\mathfrak{B}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big),\\
&\mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+1}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+1)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{n}\mathscr{X}_a)}{\mathfrak{T}_1^{n}},\frac{\upsilon\cdot\mathscr{X}_a^{n}}{\mathfrak{T}_1^{n+2}}\right)\leq\mathfrak{C}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big).
\end{aligned}\right\}\tag{3.10}$$

It is easy to see that

$$\frac{\mathscr{A}_1(\mathfrak{T}_1^{n}\mathscr{X}_a)}{\mathfrak{T}_1^{n}}-\mathscr{A}_1(\mathscr{X}_a)=\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}+1}\mathscr{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}}\mathscr{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}}.\tag{3.11}$$

From (3.10) and (3.11), we reach

$$\left.\begin{aligned}
&\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n}\mathscr{X}_a)}{\mathfrak{T}_1^{n}}-\mathscr{A}_1(\mathscr{X}_a),\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right)\\
&\quad=\mathfrak{A}_a\left(\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}+1}\mathscr{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}}\mathscr{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}},\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right),\\
&\mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n}\mathscr{X}_a)}{\mathfrak{T}_1^{n}}-\mathscr{A}_1(\mathscr{X}_a),\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right)\\
&\quad=\mathfrak{B}_a\left(\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}+1}\mathscr{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}}\mathscr{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}},\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right),\\
&\mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n}\mathscr{X}_a)}{\mathfrak{T}_1^{n}}-\mathscr{A}_1(\mathscr{X}_a),\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right)\\
&\quad=\mathfrak{C}_a\left(\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}+1}\mathscr{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}}\mathscr{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}},\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right).
\end{aligned}\right\}\tag{3.12}$$

From (3.12), we arrive

$$\left.\begin{aligned}
&\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^n\mathscr{X}_a)}{\mathfrak{T}_1^n}-\mathscr{A}_1(\mathscr{X}_a),\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right)\\
&\quad\geq\prod_{\mathcal{J}=0}^{n-1}\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}+1}\mathscr{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}}\mathscr{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}},\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon r}{\mathfrak{T}_1^{\mathcal{J}+2}}\right),\\
&\mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^n\mathscr{X}_a)}{\mathfrak{T}_1^n}-\mathscr{A}_1(\mathscr{X}_a),\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right)\\
&\quad\leq\coprod_{\mathcal{J}=0}^{n-1}\mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}+1}\mathscr{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}}\mathscr{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}},\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right),\\
&\mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^n\mathscr{X}_a)}{\mathfrak{T}_1^n}-\mathscr{A}_1(\mathscr{X}_a),\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right)\\
&\quad\leq\coprod_{\mathcal{J}=0}^{n-1}\mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}+1}\mathscr{X}_a)}{\mathfrak{T}_1^{(\mathcal{J}+1)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{\mathcal{J}}\mathscr{X}_a)}{\mathfrak{T}_1^{\mathcal{J}}},\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right),
\end{aligned}\right\} \tag{3.13}$$

where

$$\prod_{\mathcal{J}=0}^{n-1}Q_j=Q_1*Q_2*\cdots*Q_n\quad\text{and}\quad\coprod_{\mathcal{J}=0}^{n-1}R_j=R_1\diamond R_2\diamond\cdots\diamond R_n\quad\text{and}$$
$$\coprod_{\mathcal{J}=0}^{n-1}S_j=S_1\oslash S_2\oslash\cdots\oslash S_n.$$

Using the above conditions, we have

$$\left.\begin{aligned}
&\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^n\mathscr{X}_a)}{\mathfrak{T}_1^n}-\mathscr{A}_1(\mathscr{X}_a),\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right)\\
&\quad\geq\prod_{\mathcal{J}=0}^{n-1}\mathfrak{A}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big)=\mathfrak{A}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big),\\
&\mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^n\mathscr{X}_a)}{\mathfrak{T}_1^n}-\mathscr{A}_1(\mathscr{X}_a),\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right)\\
&\quad\leq\coprod_{\mathcal{J}=0}^{n-1}\mathfrak{B}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big)=\mathfrak{B}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big),\\
&\mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^n\mathscr{X}_a)}{\mathfrak{T}_1^n}-\mathscr{A}_1(\mathscr{X}_a),\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{\mathcal{J}+2}}\right)\\
&\quad\leq\coprod_{\mathcal{J}=0}^{n-1}\mathfrak{C}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big)=\mathfrak{C}'_a\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big).
\end{aligned}\right\} \tag{3.14}$$

Considering $\mathscr{X}_a$ by $\mathfrak{T}_1^m \mathscr{X}_a$ in (3.14) and dividing by $\mathfrak{T}_1^m$, we get

$$\left.\begin{aligned}
&\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+m}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+m)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{m}\mathscr{X}_a)}{\mathfrak{T}_1^{m}},\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{(\mathcal{J}+m+2)}}\right)\\
&\quad\geq \mathfrak{A}_a'\big(N\big(\mathfrak{T}_1^m\mathscr{X}_a,\mathfrak{T}_1^m\mathscr{X}_a,\mathfrak{T}_1^m\mathscr{X}_a\big),\upsilon\big)\\
&\quad= \mathfrak{A}_a'\left(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\frac{\upsilon}{\mathscr{X}_a^m}\right),\\
&\mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+m}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+m)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{m}\mathscr{X}_a)}{\mathfrak{T}_1^{m}},\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{(\mathcal{J}+m+2)}}\right)\\
&\quad\leq \mathfrak{B}_a'\big(N\big(\mathfrak{T}_1^m\mathscr{X}_a,\mathfrak{T}_1^m\mathscr{X}_a,\mathfrak{T}_1^m\mathscr{X}_a\big),\upsilon\big)\\
&\quad= \mathfrak{B}_a'\left(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\frac{\upsilon}{\mathscr{X}_a^m}\right),\\
&\mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+m}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+m)}}-\frac{\mathscr{C}(\mathfrak{T}_1^{m}\mathscr{X}_a)}{\mathfrak{T}_1^{m}},\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}\upsilon}{\mathfrak{T}_1^{(\mathcal{J}+m+2)}}\right)\\
&\quad\leq \mathfrak{C}_a'\big(N\big(\mathfrak{T}_1^m\mathscr{X}_a,\mathfrak{T}_1^m\mathscr{X}_a,\mathfrak{T}_1^m\mathscr{X}_a\big),\upsilon\big)\\
&\quad= \mathfrak{B}_a'\left(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\frac{\upsilon}{\mathscr{X}_a^m}\right).
\end{aligned}\right\}$$

Replacing υ by $\mathscr{X}_a^m\upsilon$ in the above inequality, we have

$$\left.\begin{aligned}
&\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+m}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+m)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{m}\mathscr{X}_a)}{\mathfrak{T}_1^{m}},\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}+m}\upsilon}{\mathfrak{T}_1^{(\mathcal{J}+m+2)}}\right)\geq \mathfrak{A}_a'\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big),\\
&\mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+m}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+m)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{m}\mathscr{X}_a)}{\mathfrak{T}_1^{m}},\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}+m}\upsilon}{\mathfrak{T}_1^{(\mathcal{J}+m+2)}}\right)\leq \mathfrak{B}_a'\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big),\\
&\mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+m}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+m)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{m}\mathscr{X}_a)}{\mathfrak{T}_1^{m}},\sum_{\mathcal{J}=0}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}+m}\upsilon}{\mathfrak{T}_1^{(\mathcal{J}+m+2)}}\right)\leq \mathfrak{C}_a'\big(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\upsilon\big),
\end{aligned}\right\}\quad(3.15)$$

$$\left.\begin{aligned}
&\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+m}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+m)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{m}\mathscr{X}_a)}{\mathfrak{T}_1^{m}},\upsilon\right)\geq \mathfrak{A}_a'\left(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\frac{\upsilon}{\sum_{\mathcal{J}=m}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}}{\mathfrak{T}_1^{\mathcal{J}+2}}}\right),\\
&\mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+m}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+m)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{m}\mathscr{X}_a)}{\mathfrak{T}_1^{m}},\upsilon\right)\leq \mathfrak{B}_a'\left(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\frac{\upsilon}{\sum_{\mathcal{J}=m}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}}{\mathfrak{T}_1^{\mathcal{J}+2}}}\right),\\
&\mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^{n+m}\mathscr{X}_a)}{\mathfrak{T}_1^{(n+m)}}-\frac{\mathscr{A}_1(\mathfrak{T}_1^{m}\mathscr{X}_a)}{\mathfrak{T}_1^{m}},\upsilon\right)\leq \mathfrak{C}_a'\left(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\frac{\upsilon}{\sum_{\mathcal{J}=m}^{n-1}\frac{\mathscr{X}_a^{\mathcal{J}}}{\mathfrak{T}_1^{\mathcal{J}+2}}}\right).
\end{aligned}\right\}\quad(3.16)$$

Since $0<\mathscr{X}_a<1$ and $\sum_{\mathcal{J}=0}^{n}(\frac{\mathscr{X}_a}{\mathfrak{T}_1})^{\mathcal{J}}<\infty$. The sequence $\{\frac{\mathscr{A}_1(\mathfrak{T}_1^n\mathscr{X}_a)}{\mathfrak{T}_1^n}\}$ is Cauchy in $(Y,\mathfrak{A}_a,\mathfrak{B}_a,\mathfrak{C}_a)$. Since $(Y,\mathfrak{A}_a,\mathfrak{B}_a,\mathfrak{C}_a)$ is a complete NSN-space, this sequence converges to some point $\mathcal{A}_1(\mathscr{X}_a)\in Y$. Defining $\mathcal{A}_1:\mathcal{L}\longrightarrow\mathcal{M}$ by

$$\lim_{n\to\infty}\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^n\mathscr{X}_a)}{\mathfrak{T}_1^n}-\mathcal{A}_1(\mathscr{X}_a),\upsilon\right)=1,$$

$$\lim_{n\to\infty} \mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^n \mathscr{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathscr{X}_a), \upsilon\right) = 0,$$
$$\lim_{n\to\infty} \mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^n \mathscr{X}_a)}{\mathfrak{T}_1^n} - \mathcal{A}_1(\mathscr{X}_a), \upsilon\right) = 0.$$

Finally

$$\frac{\mathscr{A}_1(\mathfrak{T}_1^n \mathscr{X}_a)}{\mathfrak{T}_1^n} \xrightarrow{\text{NSN}} \mathcal{A}_1(\mathscr{X}_a), \quad \text{as } n \to \infty.$$

Consider $m = 0$ in (3.16), then

$$\left.\begin{aligned}
&\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^n \mathscr{X}_a)}{\mathfrak{T}_1^n} - \mathscr{A}_1(\mathscr{X}_a), \upsilon\right) \geq \mathfrak{A}'_a\left(N(\mathscr{X}_a, \mathscr{X}_a, \mathscr{X}_a), \frac{\upsilon}{\sum_{\mathcal{J}=0}^{n-1} \frac{\mathscr{X}_a^{\mathcal{J}}}{\mathfrak{T}_1^{\mathcal{J}+2}}}\right),\\
&\mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^n \mathscr{X}_a)}{\mathfrak{T}_1^n} - \mathscr{A}_1(\mathscr{X}_a), \upsilon\right) \leq \mathfrak{B}'_a\left(N(\mathscr{X}_a, \mathscr{X}_a, \mathscr{X}_a), \frac{\upsilon}{\sum_{\mathcal{J}=0}^{n-1} \frac{\mathscr{X}_a^{\mathcal{J}}}{\mathfrak{T}_1^{\mathcal{J}+2}}}\right),\\
&\mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1^n \mathscr{X}_a)}{\mathfrak{T}_1^n} - \mathscr{A}_1(\mathscr{X}_a), \upsilon\right) \leq \mathfrak{C}'_a\left(N(\mathscr{X}_a, \mathscr{X}_a, \mathscr{X}_a), \frac{\upsilon}{\sum_{\mathcal{J}=0}^{n-1} \frac{\mathscr{X}_a^{\mathcal{J}}}{\mathfrak{T}_1^{\mathcal{J}+2}}}\right).
\end{aligned}\right\} \tag{3.17}$$

As $n \longrightarrow \infty$ in (3.17) and

$$\left.\begin{aligned}
&\mathfrak{A}_a\big(\mathcal{A}_1(\mathscr{X}_a) - \mathscr{A}_1(\mathscr{X}_a), t\big) \geq \mathfrak{A}'_a\big(N(\mathscr{X}_a, \mathscr{X}_a, \mathscr{X}_a), \mathfrak{T}_1\upsilon(\mathfrak{T}_1 - \mathscr{X}_a)\big),\\
&\mathfrak{B}_a\big(\mathcal{A}_1(\mathscr{X}_a) - \mathscr{A}_1(\mathscr{X}_a), t\big) \leq \mathfrak{B}'_a\big(N(\mathscr{X}_a, \mathscr{X}_a, \mathscr{X}_a), \mathfrak{T}_1\upsilon(\mathfrak{T}_1 - \mathscr{X}_a)\big),\\
&\mathfrak{C}_a\big(\mathcal{A}_1(\mathscr{X}_a) - \mathscr{A}_1(\mathscr{X}_a), t\big) \leq \mathfrak{C}'_a\big(N(\mathscr{X}_a, \mathscr{X}_a, \mathscr{X}_a), \mathfrak{T}_1\upsilon(\mathfrak{T}_1 - \mathscr{X}_a)\big).
\end{aligned}\right\} \tag{3.18}$$

Finally, $\mathcal{A}_1$ satisfies (3.1), taking $(\mathscr{X}_a, \mathscr{V}_a, \mathscr{W}_a)$ by $(\mathfrak{T}_1^n\mathscr{X}_a, \mathfrak{T}_1^n\mathscr{V}_a, \mathfrak{T}_1^n\mathscr{W}_a)$ in (3.4).

$$\left.\begin{aligned}
&\mathfrak{A}_a\left(\frac{1}{\mathfrak{T}_1^n}\mathfrak{Z}\big(\mathfrak{T}_1^n\mathscr{X}_a, \mathfrak{T}_1^n\mathscr{V}_a, \mathfrak{T}_1^n\mathscr{W}_a\big), \upsilon\right) \geq \mathfrak{A}'_a\big(N\big(\mathfrak{T}_1^n\mathscr{X}_a, \mathfrak{T}_1^n\mathscr{V}_a, \mathfrak{T}_1^n\mathscr{W}_a\big), \mathfrak{T}_1^n\upsilon\big),\\
&\mathfrak{B}_a\left(\frac{1}{\mathfrak{T}_1^n}\mathfrak{Z}\big(\mathfrak{T}_1^n\mathscr{X}_a, \mathfrak{T}_1^n\mathscr{V}_a, \mathfrak{T}_1^n\mathscr{W}_a\big), \upsilon\right) \leq \mathfrak{B}'_a\big(N\big(\mathfrak{T}_1^n\mathscr{X}_a, \mathfrak{T}_1^n\mathscr{V}_a, \mathfrak{T}_1^n\mathscr{W}_a\big), \mathfrak{T}_1^n\upsilon\big),\\
&\mathfrak{C}_a\left(\frac{1}{\mathfrak{T}_1^n}\mathfrak{Z}\big(\mathfrak{T}_1^n\mathscr{X}_a, \mathfrak{T}_1^n\mathscr{V}_a, \mathfrak{T}_1^n\mathscr{W}_a\big), \upsilon\right) \leq \mathfrak{C}'_a\big(N\big(\mathfrak{T}_1^n\mathscr{X}_a, \mathfrak{T}_1^n\mathscr{V}_a, \mathfrak{T}_1^n\mathscr{W}_a\big), \mathfrak{T}_1^n\upsilon\big).
\end{aligned}\right\} \tag{3.19}$$

Here,

$$\begin{aligned}
&\mathfrak{A}_a\big(\mathcal{P}_1\mathcal{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a - \mathscr{V}_a)\big) + \mathcal{Q}_1\mathcal{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a - \mathscr{W}_a)\big)\\
&\qquad + \mathcal{R}_1\mathcal{A}_1\big(\mathcal{P}_1\mathcal{Q}_1(\mathscr{W}_a - \mathscr{X}_a)\big) + \mathfrak{T}_1\mathcal{A}_1(\mathcal{P}_1\mathscr{X}_a + \mathcal{Q}_1\mathscr{V}_a + \mathcal{R}_1\mathscr{W}_a)\\
&\qquad - \mathfrak{T}_1\big(\mathcal{P}_1\mathcal{A}_1(\mathscr{X}_a) + \mathcal{Q}_1\mathcal{A}_1(\mathscr{V}_a) + \mathcal{R}_1\mathcal{A}_1(\mathscr{W}_a)\big)\big)\\
&\geq \mathfrak{A}_a\left(\mathcal{P}_1\mathcal{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a - \mathscr{V}_a)\big) - \frac{\mathcal{P}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a - \mathscr{V}_a)\big), \frac{\upsilon}{6}\right)\\
&\qquad * \mathfrak{A}_a\left(\mathcal{Q}_1\mathcal{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a - \mathscr{W}_a)\big) - \frac{\mathcal{Q}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a - \mathscr{W}_a)\big), \frac{\upsilon}{6}\right)
\end{aligned}$$

$$
\begin{aligned}
&* \mathfrak{A}_a\left(\mathfrak{T}_1\mathcal{A}_1(\mathcal{P}_1\mathscr{X}_a+\mathcal{Q}_1\mathscr{V}_a+\mathcal{R}_1\mathscr{W}_a)+\frac{\mathfrak{T}_1}{\mathfrak{T}_1^n}\mathscr{A}_1(\mathcal{P}_1\mathscr{X}_a+\mathcal{Q}_1\mathscr{V}_a+\mathcal{R}_1\mathscr{W}_a),\frac{\upsilon}{6}\right)\\
&* \mathfrak{A}_a\left(-\mathfrak{T}_1\big(\mathcal{P}_1\mathcal{A}_1(\mathscr{X}_a)+\mathcal{Q}_1\mathcal{A}_1(\mathscr{V}_a)+\mathcal{R}_1\mathcal{A}_1(\mathscr{W}_a)\big)\right.\\
&\left.+\frac{\mathfrak{T}_1}{\mathfrak{T}_1^n}\big(\mathcal{P}_1\mathscr{A}_1(\mathscr{X}_a)+\mathcal{Q}_1\mathscr{A}_1(\mathscr{V}_a)+\mathcal{R}_1\mathscr{A}_1(\mathscr{W}_a)\big),\frac{\upsilon}{6}\right)\\
&* \mathfrak{A}_a\left(\frac{\mathcal{P}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a-\mathscr{V}_a)\big)+\frac{\mathcal{Q}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a-\mathscr{W}_a)\big)\right.\\
&+\frac{\mathcal{R}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{P}_1\mathcal{Q}_1(\mathscr{W}_a-\mathscr{X}_a)\big)+\frac{\mathfrak{T}_1}{\mathfrak{T}_1^n}\mathscr{A}_1(\mathcal{P}_1\mathscr{X}_a+\mathcal{Q}_1\mathscr{V}_a+\mathcal{R}_1\mathscr{W}_a)\\
&\left.-\frac{\mathfrak{T}_1}{\mathfrak{T}_1^n}\big(\mathcal{P}_1\mathscr{A}_1(\mathscr{X}_a)+\mathcal{Q}_1\mathscr{A}_1(\mathscr{V}_a)+\mathcal{R}_1\mathscr{A}_1(\mathscr{W}_a)\big),\frac{\upsilon}{6}\right)
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
&\mathfrak{B}_a\big(\mathcal{P}_1\mathcal{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a-\mathscr{V}_a)\big)+\mathcal{Q}_1\mathcal{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a-\mathscr{W}_a)\big)\\
&\quad+\mathcal{R}_1\mathcal{A}_1\big(\mathcal{P}_1\mathcal{Q}_1(\mathscr{W}_a-\mathscr{X}_a)\big)+\mathfrak{T}_1\mathcal{A}_1(\mathcal{P}_1\mathscr{X}_a+\mathcal{Q}_1\mathscr{V}_a+\mathcal{R}_1\mathscr{W}_a)\\
&\quad-\mathfrak{T}_1\big(\mathcal{P}_1\mathcal{A}_1(\mathscr{X}_a)+\mathcal{Q}_1\mathcal{A}_1(\mathscr{V}_a)+\mathcal{R}_1\mathcal{A}_1(\mathscr{W}_a)\big)\big)\\
&\geq \mathfrak{B}_a\left(\mathcal{P}_1\mathcal{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a-\mathscr{V}_a)\big)-\frac{\mathcal{P}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a-\mathscr{V}_a)\big),\frac{\upsilon}{6}\right)\\
&\quad\diamond \mathfrak{B}_a\left(\mathcal{Q}_1\mathcal{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a-\mathscr{W}_a)\big)-\frac{\mathcal{Q}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a-\mathscr{W}_a)\big),\frac{\upsilon}{6}\right)\\
&\quad\diamond \mathfrak{B}_a\left(\mathfrak{T}_1\mathcal{A}_1(\mathcal{P}_1\mathscr{X}_a+\mathcal{Q}_1\mathscr{V}_a+\mathcal{R}_1\mathscr{W}_a)+\frac{\mathfrak{T}_1}{\mathfrak{T}_1^n}\mathscr{A}_1(\mathcal{P}_1\mathscr{X}_a+\mathcal{Q}_1\mathscr{V}_a+\mathcal{R}_1\mathscr{W}_a),\frac{\upsilon}{6}\right)\\
&\quad\diamond \mathfrak{B}_a\left(-\mathfrak{T}_1\big(\mathcal{P}_1\mathcal{A}_1(\mathscr{X}_a)+\mathcal{Q}_1\mathcal{A}_1(\mathscr{V}_a)+\mathcal{R}_1\mathcal{A}_1(\mathscr{W}_a)\big)\right.\\
&\quad\left.+\frac{\mathfrak{T}_1}{\mathfrak{T}_1^n}\big(\mathcal{P}_1\mathscr{A}_1(\mathscr{X}_a)+\mathcal{Q}_1\mathscr{A}_1(\mathscr{V}_a)+\mathcal{R}_1\mathscr{A}_1(\mathscr{W}_a)\big),\frac{\upsilon}{6}\right)\\
&\quad\diamond \mathfrak{B}_a\left(\frac{\mathcal{P}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a-\mathscr{V}_a)\big)+\frac{\mathcal{Q}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a-\mathscr{W}_a)\big)\right.\\
&\quad+\frac{\mathcal{R}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{P}_1\mathcal{Q}_1(\mathscr{W}_a-\mathscr{X}_a)\big)+\frac{\mathfrak{T}_1}{\mathfrak{T}_1^n}\mathscr{A}_1(\mathcal{P}_1\mathscr{X}_a+\mathcal{Q}_1\mathscr{V}_a+\mathcal{R}_1\mathscr{W}_a)\\
&\quad\left.-\frac{\mathfrak{T}_1}{\mathfrak{T}_1^n}\big(\mathcal{P}_1\mathscr{A}_1(\mathscr{X}_a)+\mathcal{Q}_1\mathscr{A}_1(\mathscr{V}_a)+\mathcal{R}_1\mathscr{A}_1(\mathscr{W}_a)\big),\frac{\upsilon}{6}\right)
\end{aligned}
\tag{3.21}
$$

and

$$
\begin{aligned}
&\mathfrak{C}_a\big(\mathcal{P}_1\mathcal{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a-\mathscr{V}_a)\big)+\mathcal{Q}_1\mathcal{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a-\mathscr{W}_a)\big)\\
&\quad+\mathcal{R}_1\mathcal{A}_1\big(\mathcal{P}_1\mathcal{Q}_1(\mathscr{W}_a-\mathscr{X}_a)\big)+\mathfrak{T}_1\mathcal{A}_1(\mathcal{P}_1\mathscr{X}_a+\mathcal{Q}_1\mathscr{V}_a+\mathcal{R}_1\mathscr{W}_a)\\
&\quad-\mathfrak{T}_1\big(\mathcal{P}_1\mathcal{A}_1(\mathscr{X}_a)+\mathcal{Q}_1\mathcal{A}_1(\mathscr{V}_a)+\mathcal{R}_1\mathcal{A}_1(\mathscr{W}_a)\big)\big)\\
&\geq \mathfrak{C}_a\left(\mathcal{P}_1\mathcal{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a-\mathscr{V}_a)\big)-\frac{\mathcal{P}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a-\mathscr{V}_a)\big),\frac{\upsilon}{6}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\oslash \mathfrak{C}_a\left(\mathcal{Q}_1\mathcal{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a-\mathscr{W}_a)\big)-\frac{\mathcal{Q}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a-\mathscr{W}_a)\big),\frac{\upsilon}{6}\right)\\
&\oslash \mathfrak{C}_a\left(\mathfrak{T}_1\mathcal{A}_1(\mathcal{P}_1\mathscr{X}_a+\mathcal{Q}_1\mathscr{V}_a+\mathcal{R}_1\mathscr{W}_a)+\frac{\mathfrak{T}_1}{\mathfrak{T}_1^n}\mathscr{A}_1(\mathcal{P}_1\mathscr{X}_a+\mathcal{Q}_1\mathscr{V}_a+\mathcal{R}_1\mathscr{W}_a),\frac{\upsilon}{6}\right)\\
&\oslash \mathfrak{C}_a\bigg(-\mathfrak{T}_1\big(\mathcal{P}_1\mathcal{A}_1(\mathscr{X}_a)+\mathcal{Q}_1\mathcal{A}_1(\mathscr{V}_a)+\mathcal{R}_1\mathcal{A}_1(\mathscr{W}_a)\big)\\
&+\frac{\mathfrak{T}_1}{\mathfrak{T}_1^n}\big(\mathcal{P}_1\mathscr{A}_1(\mathscr{X}_a)+\mathcal{Q}_1\mathscr{A}_1(\mathscr{V}_a)+\mathcal{R}_1\mathscr{A}_1(\mathscr{W}_a)\big),\frac{\upsilon}{6}\bigg)\\
&\oslash \mathfrak{C}_a\bigg(\frac{\mathcal{P}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{Q}_1\mathcal{R}_1(\mathscr{X}_a-\mathscr{V}_a)\big)+\frac{\mathcal{Q}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{P}_1\mathcal{R}_1(\mathscr{V}_a-\mathscr{W}_a)\big)\\
&+\frac{\mathcal{R}_1}{\mathfrak{T}_1^n}\mathscr{A}_1\big(\mathcal{P}_1\mathcal{Q}_1(\mathscr{W}_a-\mathscr{X}_a)\big)+\frac{\mathfrak{T}_1}{\mathfrak{T}_1^n}\mathscr{A}_1(\mathcal{P}_1\mathscr{X}_a+\mathcal{Q}_1\mathscr{V}_a+\mathcal{R}_1\mathscr{W}_a)\\
&-\frac{\mathfrak{T}_1}{\mathfrak{T}_1^n}\big(\mathcal{P}_1\mathscr{A}_1(\mathscr{X}_a)+\mathcal{Q}_1\mathscr{A}_1(\mathscr{V}_a)+\mathcal{R}_1\mathscr{A}_1(\mathscr{W}_a)\big),\frac{\upsilon}{6}\bigg).
\end{aligned}
\tag{3.22}
$$

Also,

$$
\left.
\begin{aligned}
&\lim_{n\to\infty}\mathfrak{A}_a\left(\frac{1}{\mathfrak{T}_1^n}\mathfrak{Z}\big(\mathfrak{T}_1^n\mathscr{X}_a,\mathfrak{T}_1^n\mathscr{V}_a,\mathfrak{T}_1^n\mathscr{W}_a\big),\frac{\upsilon}{6}\right)=1,\\
&\lim_{n\to\infty}\mathfrak{B}_a\left(\frac{1}{\mathfrak{T}_1^n}\mathfrak{Z}\big(\mathfrak{T}_1^n\mathscr{X}_a,\mathfrak{T}_1^n\mathscr{V}_a,\mathfrak{T}_1^n\mathscr{W}_a\big),\frac{\upsilon}{6}\right)=0,\\
&\lim_{n\to\infty}\mathfrak{C}_a\left(\frac{1}{\mathfrak{T}_1^n}\mathfrak{Z}\big(\mathfrak{T}_1^n\mathscr{X}_a,\mathfrak{T}_1^n\mathscr{V}_a,\mathfrak{T}_1^n\mathscr{W}_a\big),\frac{\upsilon}{6}\right)=0.
\end{aligned}
\right\}
\tag{3.23}
$$

To prove uniqueness

$$
\begin{aligned}
&\mathfrak{A}_a\big(\mathcal{A}_1(\mathscr{X}_a)-\mathcal{A}_1'(\mathscr{X}_a),\upsilon\big)\\
&\quad\geq \mathfrak{A}_a\left(\mathcal{A}_1\big(\mathfrak{T}_1^n\mathscr{X}_a\big)-\mathscr{A}_1\big(\mathfrak{T}_1^n\mathscr{X}_a\big),\frac{\upsilon.\mathfrak{T}_1^n}{2}\right)*\mathfrak{A}_a\left(\mathscr{A}_1\big(\mathfrak{T}_1^n\mathscr{X}_a\big)-\mathcal{A}_1'\big(\mathfrak{T}_1^n\mathscr{X}_a\big),\frac{\upsilon.\mathfrak{T}_1^n}{2}\right)\\
&\quad\geq \mathfrak{A}_a'\left(N\big(\mathfrak{T}_1^n\mathscr{X}_a,\mathfrak{T}_1^n\mathscr{X}_a,\mathfrak{T}_1^n\mathscr{X}_a\big),\frac{\upsilon\mathfrak{T}_1^{n+1}}{2}|\mathfrak{T}_1-\mathscr{X}_a|\right)\\
&\quad\geq \mathfrak{A}_a'\left(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\frac{\upsilon\mathfrak{T}_1^{n+1}|\mathfrak{T}_1-\mathscr{X}_a|}{2\cdot\mathscr{X}_a^n}\right),\\
&\mathfrak{B}_a\big(\mathcal{A}_1(\mathscr{X}_a)-\mathcal{A}_1'(\mathscr{X}_a),\upsilon\big)\\
&\quad\leq \mathfrak{B}_a\left(\mathcal{A}_1\big(\mathfrak{T}_1^n\mathscr{X}_a\big)-\mathscr{A}_1\big(\mathfrak{T}_1^n\mathscr{X}_a\big),\frac{\upsilon.\mathfrak{T}_1^n}{2}\right)\diamond\mathfrak{B}_a\left(\mathscr{A}_1\big(\mathfrak{T}_1^n\mathscr{X}_a\big)-\mathcal{A}_1'\big(\mathfrak{T}_1^n\mathscr{X}_a\big),\frac{\upsilon.\mathfrak{T}_1^n}{2}\right)\\
&\quad\leq \mathfrak{B}_a'\left(N\big(\mathfrak{T}_1^n\mathscr{X}_a,\mathfrak{T}_1^n\mathscr{X}_a,\mathfrak{T}_1^n\mathscr{X}_a\big),\frac{t\mathfrak{T}_1^{n+1}}{2}|\mathfrak{T}_1-\mathscr{X}_a|\right)\\
&\quad\leq \mathfrak{B}_a'\left(N(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\frac{\upsilon\mathfrak{T}_1^{n+1}|\mathfrak{T}_1-\mathscr{X}_a|}{2\cdot\mathscr{X}_a^n}\right),\\
&\mathfrak{C}_a\big(\mathcal{A}_1(\mathscr{X}_a)-\mathcal{A}_1'(\mathscr{X}_a),\upsilon\big)\\
&\quad\leq \mathfrak{C}_a\left(\mathcal{A}_1\big(\mathfrak{T}_1^n\mathscr{X}_a\big)-\mathscr{A}_1\big(\mathfrak{T}_1^n\mathscr{X}_a\big),\frac{\upsilon.\mathfrak{T}_1^n}{2}\right)\diamond\mathfrak{C}_a\left(\mathscr{A}_1\big(\mathfrak{T}_1^n\mathscr{X}_a\big)-\mathcal{A}_1'\big(\mathfrak{T}_1^n\mathscr{X}_a\big),\frac{\upsilon.\mathfrak{T}_1^n}{2}\right)
\end{aligned}
$$

$$\leq \mathfrak{C}'_a\left(N\big(\mathfrak{T}_1^n\mathscr{X}_a, \mathfrak{T}_1^n\mathscr{X}_a, \mathfrak{T}_1^n\mathscr{X}_a\big), \frac{\upsilon\mathfrak{T}_1^{n+1}}{2}|\mathfrak{T}_1 - \mathscr{X}_a|\right)$$
$$\leq \mathfrak{C}'_a\left(N(\mathscr{X}_a, \mathscr{X}_a, \mathscr{X}_a), \frac{\upsilon\mathfrak{T}_1^{n+1}|\mathfrak{T}_1 - \mathscr{X}_a|}{2\cdot\mathscr{X}_a^n}\right).$$

Since $\lim_{n\to\infty}\frac{\upsilon\mathfrak{T}_1^{n+1}|\mathfrak{T}_1-\mathscr{X}_a|}{2\mathscr{X}_a^n} = \infty$,

$$\left.\begin{aligned}
&\lim_{n\to\infty}\mathfrak{A}'_a\left(N(\mathscr{X}_a, \mathscr{X}_a, \mathscr{X}_a), \frac{\upsilon\mathfrak{T}_1^{n+1}|\mathfrak{T}_1 - \mathscr{X}_a|}{2\cdot\mathscr{X}_a^n}\right) = 1,\\
&\lim_{n\to\infty}\mathfrak{B}'_a\left(N(\mathscr{X}_a, \mathscr{X}_a, \mathscr{X}_a), \frac{\upsilon\mathfrak{T}_1^{n+1}|\mathfrak{T}_1 - \mathscr{X}_a|}{2\cdot\mathscr{X}_a^n}\right) = 0,\\
&\lim_{n\to\infty}\mathfrak{C}'_a\left(N(\mathscr{X}_a, \mathscr{X}_a, \mathscr{X}_a), \frac{\upsilon\mathfrak{T}_1^{n+1}|\mathfrak{T}_1 - \mathscr{X}_a|}{2\cdot\mathscr{X}_a^n}\right) = 0.
\end{aligned}\right\}$$

Hence,

$$\left.\begin{aligned}
&\mathfrak{A}_a\big(\mathcal{A}_1(\mathscr{X}_a) - \mathcal{A}'_1(\mathscr{X}_a), \upsilon\big) = 1,\\
&\mathfrak{B}_a\big(\mathcal{A}_1(\mathscr{X}_a) - \mathcal{A}'_1(\mathscr{X}_a), \upsilon\big) = 0,\\
&\mathfrak{B}_a\big(\mathcal{A}_1(\mathscr{X}_a) - \mathcal{A}'_1(\mathscr{X}_a), \upsilon\big) = 0.
\end{aligned}\right\}$$

Thus, $\mathcal{A}_1(\mathscr{X}_a) = \mathcal{A}'_1(\mathscr{X}_a)$. Hence, $\mathcal{A}_1(\mathscr{X}_a)$ is unique.

Method 2: Assume that $\eta = -1$. Substituting p by $\frac{\mathscr{X}_a}{\mathfrak{T}_1}$ in (3.6) gives

$$\left.\begin{aligned}
&\mathfrak{A}_a\left(\mathfrak{T}_1\mathscr{A}_1(\mathscr{X}_a) - \mathfrak{T}_1^2\omega\left(\frac{\mathscr{X}_a}{\mathfrak{T}_1}\right), \upsilon\right) \geq \mathfrak{A}'_a\left(N\left(\frac{\mathscr{X}_a}{2}, \frac{\mathscr{X}_a}{2}, \frac{\mathscr{X}_a}{2}\right), \upsilon\right),\\
&\mathfrak{B}_a\left(\mathfrak{T}_1\mathscr{A}_1(\mathscr{X}_a) - \mathfrak{T}_1^2\omega\left(\frac{\mathscr{X}_a}{\mathfrak{T}_1}\right), \upsilon\right) \leq \mathfrak{B}'_a\left(N\left(\frac{\mathscr{X}_a}{2}, \frac{\mathscr{X}_a}{2}, \frac{\mathscr{X}_a}{2}\right), \upsilon\right),\\
&\mathfrak{C}_a\left(\mathfrak{T}_1\mathscr{A}_1(\mathscr{X}_a) - \mathfrak{T}_1^2\omega\left(\frac{\mathscr{X}_a}{\mathfrak{T}_1}\right), \upsilon\right) \leq \mathfrak{C}'_a\left(N\left(\frac{\mathscr{X}_a}{2}, \frac{\mathscr{X}_a}{2}, \frac{\mathscr{X}_a}{2}\right), \upsilon\right).
\end{aligned}\right\} \tag{3.24}$$

□

Corollary 3.2 *Let $\mathscr{A}_1$ be an approximately additive mapping from $(Z, \mathfrak{A}'_a, \mathfrak{B}'_a, \mathfrak{C}'_a)$ in a neutrosophic normed space and $(\mathcal{M}, \mathfrak{A}_a, \mathfrak{B}_a, \mathfrak{C}_a)$ be a neutrosophic Banach space that satisfies the inequality*

$$\mathfrak{A}_a\big(\mathfrak{Z}(\mathscr{X}_a, \mathscr{V}_a, \mathscr{W}_a), \upsilon\big)$$
$$\geq \begin{cases}
\mathfrak{A}'_a(\mathcal{S}, \upsilon),\\
\mathfrak{A}'_a\big(\mathcal{S}\big(\|\mathscr{X}_a\|^{\mathfrak{E}} + \|\mathscr{V}_a\|^{\mathfrak{F}} + \|\mathscr{W}_a\|^{\mathfrak{G}}\big), \upsilon\big), & \mathfrak{E}, \mathfrak{F}, \mathfrak{G} \neq 1,\\
\mathfrak{A}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{F}}\|\mathscr{W}_a\|^{\mathfrak{G}}, \upsilon\big), & \mathfrak{E} + \mathfrak{F} + \mathfrak{G} \neq 1,\\
\mathfrak{A}'_a\big(\mathcal{S}\big\{\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{F}}\|\mathscr{W}_a\|^{\mathfrak{G}} & \\
\quad + \big(\|\mathscr{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} + \|\mathscr{V}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} + \|\mathscr{W}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}\big)\big\}, \upsilon\big), & \mathfrak{E} + \mathfrak{F} + \mathfrak{G} \neq 1,
\end{cases} \tag{3.25}$$

$$\mathfrak{B}_a\big(\mathfrak{Z}(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big) \tag{3.26}$$

$$\leq \begin{cases} \mathfrak{B}'_a(\mathcal{S},\upsilon), \\ \mathfrak{B}'_a\big(\mathcal{S}\big(\|\mathscr{X}_a\|^{\mathfrak{E}}+\|\mathscr{V}_a\|^{\mathfrak{F}}+\|\mathscr{W}_a\|^{\mathfrak{G}}\big),\upsilon\big), \quad \mathfrak{E},\mathfrak{F},\mathfrak{G}\neq 1, \\ \mathfrak{B}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{X}_a\|^{\mathfrak{F}}\|\mathscr{V}_a\|^{\mathfrak{G}},\upsilon\big), \quad \mathfrak{E}+\mathfrak{F}+\mathfrak{G}\neq 1, \\ \mathfrak{B}'_a\big(\mathcal{S}\big\{\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{F}}\|\mathscr{W}_a\|^{\mathfrak{G}} \\ \quad +\big(\|\mathscr{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}+\|\mathscr{V}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}+\|\mathscr{W}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}\big)\big\},\upsilon\big), \quad \mathfrak{E}+\mathfrak{F}+\mathfrak{G}\neq 1, \end{cases} \tag{3.27}$$

$$\mathfrak{C}_a\big(\mathfrak{Z}(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big) \tag{3.28}$$

$$\leq \begin{cases} \mathfrak{C}'_a(\mathcal{S},\upsilon), \\ \mathfrak{C}'_a\big(\mathcal{S}\big(\|\mathscr{X}_a\|^{\mathfrak{E}}+\|\mathscr{V}_a\|^{\mathfrak{F}}+\|\mathscr{W}_a\|^{\mathfrak{G}}\big),\upsilon\big), \quad \mathfrak{E},\mathfrak{F},\mathfrak{G}\neq 1, \\ \mathfrak{C}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{X}_a\|^{\mathfrak{F}}\|\mathscr{V}_a\|^{\mathfrak{G}},\upsilon\big), \quad \mathfrak{E}+\mathfrak{F}+\mathfrak{G}\neq 1, \\ \mathfrak{C}'_a\big(\mathcal{S}\big\{\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{F}}\|\mathscr{W}_a\|^{\mathfrak{G}} \\ \quad +\big(\|\mathscr{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}+\|\mathscr{V}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}+\|\mathscr{W}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}\big)\big\},\upsilon\big), \quad \mathfrak{E}+\mathfrak{F}+\mathfrak{G}\neq 1 \end{cases} \tag{3.29}$$

such that

$$\mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)$$

$$\geq \begin{cases} \mathfrak{A}'_a\big(\mathcal{S},\mathfrak{T}_1 t\upsilon|\mathfrak{T}_1-1|\big), \\ \mathfrak{A}'_a\big(\big[\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}|\mathfrak{T}_1|^{\mathfrak{E}}+\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{F}}|\mathfrak{T}_1|^{\mathfrak{F}}+\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{G}}|\mathfrak{T}_1|^{\mathfrak{G}}\big], \\ \quad \mathfrak{T}_1\upsilon\big[\big|\mathfrak{T}_1-\mathfrak{T}_1^{\mathfrak{E}}\big|+\big|\mathfrak{T}_1-\mathfrak{T}_1^{\mathfrak{F}}\big|+\big|\mathfrak{T}_1-\mathfrak{T}_1^{\mathfrak{G}}\big|\big]\big), \\ \mathfrak{A}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}|\mathfrak{T}_1|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}},\mathfrak{T}_1\upsilon\big|\mathfrak{T}_1-\mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}\big|\big), \\ \mathfrak{A}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}|\mathfrak{T}_1|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} \\ \quad +\big[\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}|\mathfrak{T}_1|^{\mathfrak{E}}+\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{F}}|\mathfrak{T}_1|^{\mathfrak{F}}+\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{G}}|\mathfrak{T}_1|^{\mathfrak{G}}\big], \\ \quad \mathfrak{T}_1\upsilon\big|\mathfrak{T}_1-\mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}\big|\big), \end{cases}$$

$$\mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)$$

$$\leq \begin{cases} \mathfrak{B}'_a\big(\mathcal{S},\mathfrak{T}_1\upsilon|\mathfrak{T}_1-1|\big), \\ \mathfrak{B}'_a\big(\big[\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}|\mathfrak{T}_1|^{\mathfrak{E}}+\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{F}}|\mathfrak{T}_1|^{\mathfrak{F}}+\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{G}}|\mathfrak{T}_1|^{\mathfrak{G}}\big], \\ \quad \mathfrak{T}_1\upsilon\big[\big|\mathfrak{T}_1-\mathfrak{T}_1^{\mathfrak{E}}\big|+\big|\mathfrak{T}_1-\mathfrak{T}_1^{\mathfrak{F}}\big|+\big|\mathfrak{T}_1-\mathfrak{T}_1^{\mathfrak{G}}\big|\big]\big), \\ \mathfrak{B}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}|\mathfrak{T}_1|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}},\mathfrak{T}_1\upsilon\big|\mathfrak{T}_1-\mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}\big|\big), \\ \mathfrak{B}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}|\mathfrak{T}_1|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} \\ \quad +\big[\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}|\mathfrak{T}_1|^{\mathfrak{E}}+\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{F}}|\mathfrak{T}_1|^{\mathfrak{F}}+\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{G}}|\mathfrak{T}_1|^{\mathfrak{G}}\big], \\ \quad \mathfrak{T}_1\upsilon\big|\mathfrak{T}_1-\mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}\big|\big), \end{cases} \tag{3.30}$$

$$\mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a) - \mathcal{A}_1(\mathscr{X}_a), \upsilon\big)$$
$$\leq \begin{cases} \mathfrak{C}'_a\big(\mathcal{S}, \mathfrak{T}_1\upsilon|\mathfrak{T}_1 - 1|\big), \\ \mathfrak{C}'_a\big(\big[\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}|\mathfrak{T}_1|^{\mathfrak{E}} + \mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{F}}|\mathfrak{T}_1|^{\mathfrak{F}} + \mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{G}}|\mathfrak{T}_1|^{\mathfrak{G}}\big], \\ \quad \mathfrak{T}_1\upsilon\big[\big|\mathfrak{T}_1 - \mathfrak{T}_1^{\mathfrak{E}}\big| + \big|\mathfrak{T}_1 - \mathfrak{T}_1^{\mathfrak{F}}\big| + \big|\mathfrak{T}_1 - \mathfrak{T}_1^{\mathfrak{G}}\big|\big]\big), \\ \mathfrak{C}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}|\mathfrak{T}_1|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}, \mathfrak{T}_1\upsilon\big|\mathfrak{T}_1 - \mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}\big|\big), \\ \mathfrak{C}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}|\mathfrak{T}_1|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} \\ \quad + \big[\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}|\mathfrak{T}_1|^{\mathfrak{E}} + \mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{F}}|\mathfrak{T}_1|^{\mathfrak{F}} + \mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{G}}|\mathfrak{T}_1|^{\mathfrak{G}}\big], \\ \quad \mathfrak{T}_1\upsilon\big|\mathfrak{T}_1 - \mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}\big|\big). \end{cases}$$

Proof Let

$$N(\mathscr{X}_a, \mathscr{V}_a, \mathscr{W}_a) = \begin{cases} \mathcal{S}, \\ \mathcal{S}\big(\|\mathscr{X}_a\|^{\mathfrak{E}} + \|\mathscr{V}_a\|^{\mathfrak{F}} + \|\mathscr{W}_a\|^{\mathfrak{G}}\big), \\ \mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}\|y\|^{\mathfrak{F}}\|z\|^{\mathfrak{G}}, \\ \mathcal{S}\big\{\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{F}}\|\mathscr{W}_a\|^{\mathfrak{G}} \\ \quad + \big(\|\mathscr{X}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} + \|\mathscr{V}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}} + \|\mathscr{W}_a\|^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}\big)\big\}, \end{cases}$$

and

$$\mathscr{X}_a = \begin{cases} \mathfrak{T}_1^0, \\ \mathfrak{T}_1^A + \mathfrak{T}_1^B + \mathfrak{T}_1^C, \\ \mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}, \\ \mathfrak{T}_1^{\mathfrak{E}+\mathfrak{F}+\mathfrak{G}}. \end{cases}$$

□

4 Stability results: fixed point method [49]

Theorem 4.1 *Let* $N : \mathcal{L} \times \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{M}$ *such that*

$$\left.\begin{aligned} &\lim_{n\to\infty} \mathfrak{A}'_a\big(K\big(\mathfrak{D}_i^n\mathscr{X}_a, \mathfrak{D}_i^n\mathscr{V}_a, \mathfrak{D}_i^n\mathscr{W}_a\big), \mathfrak{D}^n\upsilon\big) = 1, \\ &\lim_{n\to\infty} \mathfrak{B}'_a\big(K\big(\mathfrak{D}_i^n\mathscr{X}_a, \mathfrak{D}_i^n\mathscr{V}_a, \mathfrak{D}_i^n\mathscr{W}_a\big), \mathfrak{D}^n\upsilon\big) = 0, \\ &\lim_{n\to\infty} \mathfrak{C}'_a\big(K\big(\mathfrak{D}_i^n\mathscr{X}_a, \mathfrak{D}_i^n\mathscr{V}_a, \mathfrak{D}_i^n\mathscr{W}_a\big), \mathfrak{D}^n\upsilon\big) = 0, \end{aligned}\right\} \tag{4.1}$$

where

$$\mathfrak{D}_i = \begin{cases} \mathfrak{T}_1 & \text{if } i = 0, \\ \frac{1}{\mathfrak{T}_1} & \text{if } i = 1, \end{cases} \tag{4.2}$$

then

$$\left.\begin{aligned}&\mathfrak{A}_a\big(\mathfrak{Z}(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(K(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big),\\&\mathfrak{B}_a\big(\mathfrak{Z}(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(K(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big),\\&\mathfrak{C}_a\big(\mathfrak{Z}(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(K(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big).\end{aligned}\right\}\tag{4.3}$$

If $L = L(i)$ *and*

$$\mathscr{A}_1(\mathscr{X}_a)=\frac{1}{\mathfrak{T}_1}K\left(\frac{\mathscr{X}_a}{\mathfrak{T}_1},\frac{\mathscr{X}_a}{\mathfrak{T}_1},\frac{\mathscr{X}_a}{\mathfrak{T}_1}\right),\tag{4.4}$$

and

$$\left.\begin{aligned}&\mathfrak{A}'_a\left(L\frac{\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a)}{\mathfrak{D}_i},\upsilon\right)=\mathfrak{A}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big),\\&\mathfrak{B}'_a\left(L\frac{\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a)}{\mathfrak{D}_i},\upsilon\right)=\mathfrak{B}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big),\\&\mathfrak{C}'_a\left(L\frac{\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a)}{\mathfrak{D}_i},\upsilon\right)=\mathfrak{C}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\end{aligned}\right\}\tag{4.5}$$

also

$$\left.\begin{aligned}&\mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\left(\mathscr{A}_1(\mathscr{X}_a),\frac{L^{1-i}}{1-L}\upsilon\right),\\&\mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\left(\mathscr{A}_1(\mathscr{X}_a),\frac{L^{1-i}}{1-L}\upsilon\right),\\&\mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\left(\mathscr{A}_1(\mathscr{X}_a),\frac{L^{1-i}}{1-L}\upsilon\right).\end{aligned}\right\}\tag{4.6}$$

Proof Let

$$\begin{aligned}&\mathcal{S}=\big\{h_1\mid h_1:\mathscr{X}_a\longrightarrow Y,\mathscr{A}_1(0)=0\big\},\\&d(h_1,f_1)=\inf\left\{L\in(0,\infty):\left\{\begin{aligned}&\mathfrak{A}_a(\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon)\geq\mathfrak{A}'_a(\mathscr{A}_1(\mathscr{X}_a),L\upsilon),\upsilon>0,\\&\mathfrak{B}_a(\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon)\leq\mathfrak{B}'_a(\mathscr{A}_1(\mathscr{X}_a),L\upsilon),\upsilon>0,\\&\mathfrak{C}_a(\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon)\leq\mathfrak{B}'_a(\mathscr{A}_1(\mathscr{X}_a),L\upsilon),\upsilon>0\end{aligned}\right\}\right\}.\end{aligned}\tag{4.7}$$

Hence, by (4.7)

$$\inf\left\{L\in(0,\infty):\begin{cases}\begin{cases}\mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\big\},\\ \mathfrak{A}_a\left(\frac{1}{\mathfrak{D}_i}\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a)-\frac{1}{\mathfrak{D}_i}\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a),\upsilon\right)\geq\mathfrak{A}'_a\big(\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a),\mathfrak{D}_i\upsilon\big)\big\},\\ \mathfrak{A}_a\left(\frac{1}{\mathfrak{D}_i}\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a)-\frac{1}{\mathfrak{D}_i}\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a),\upsilon\right)\geq\mathfrak{A}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big)\},\\ \mathfrak{A}_a\big(J\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big)\big\},\end{cases}\\ \begin{cases}\mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\big\},\\ \mathfrak{B}_a\left(\frac{1}{\mathfrak{D}_i}\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a)-\frac{1}{\mathfrak{D}_i}\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a),\upsilon\right)\leq\mathfrak{B}'_a\big(\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a),\mathfrak{D}_i\upsilon\big)\big\},\\ \mathfrak{B}_a\left(\frac{1}{\mathfrak{D}_i}\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a)-\frac{1}{\mathfrak{D}_i}\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a),\upsilon\right)\leq\mathfrak{B}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big)\big\},\\ \mathfrak{B}_a\big(J\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big)\big\},\end{cases}\\ \begin{cases}\mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\big\},\\ \mathfrak{C}_a\left(\frac{1}{\mathfrak{D}_i}\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a)-\frac{1}{\mathfrak{D}_i}\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a),\upsilon\right)\leq\mathfrak{C}'_a\big(\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a),\mathfrak{D}_i\upsilon\big)\big\},\\ \mathfrak{C}_a\left(\frac{1}{\mathfrak{D}_i}\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a)-\frac{1}{\mathfrak{D}_i}\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a),\upsilon\right)\leq\mathfrak{C}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big)\big\},\\ \mathfrak{C}_a\big(J\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big)\big\}\end{cases}\end{cases}\right\}$$

after that

$$\inf\left\{1\in(0,\infty):\begin{cases}\mathfrak{A}_a\big(\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)-\mathfrak{T}_1\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(K(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\mathfrak{T}_1\upsilon\big),\\ \mathfrak{B}_a\big(\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)-\mathfrak{T}_1\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(K(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\mathfrak{T}_1\upsilon\big),\\ \mathfrak{C}_a\big(\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)-\mathfrak{T}_1\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(K(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\mathfrak{T}_1\upsilon\big)\end{cases}\right\}.\tag{4.8}$$

Assuming $i=0$,

$$\inf\left\{L^{1-0}\in(0,\infty):\begin{cases}\mathfrak{A}_a\big(\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)-\mathfrak{T}_1\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(K(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\mathfrak{T}_1\upsilon\big),\\ \mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)}{\mathfrak{T}_1}-\mathscr{A}_1(\mathscr{X}_a),\upsilon\right)\geq\mathfrak{A}'_a\big(K(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\mathfrak{T}_1^2\upsilon\big),\\ \mathfrak{A}_a\big(J\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big),\\ \mathfrak{A}_a\big(J\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big),\\ \mathfrak{A}_a\big(J\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big)\end{cases}\right\},\tag{4.9}$$

$$\inf\left\{L^{1-0}\in(0,\infty):\left\{\begin{array}{l}\mathfrak{B}_a\big(\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)-\mathfrak{T}_1\mathscr{A}_1(\mathscr{X}_a),t\big)\leq\mathfrak{B}'_a\big(K(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\mathfrak{T}_1\upsilon\big),\\ \mathfrak{B}_a\left(\dfrac{\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)}{\mathfrak{T}_1}-\mathscr{A}_1(\mathscr{X}_a),\upsilon\right)\leq\mathfrak{B}'_a\big(K(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\mathfrak{T}_1^2\upsilon\big),\\ \mathfrak{B}_a\big(J\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big),\\ \mathfrak{B}_a\big(J\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big),\\ \mathfrak{B}_a\big(J\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big)\end{array}\right\}\right\},\tag{4.10}$$

$$\inf\left\{L^{1-0}\in(0,\infty):\left\{\begin{array}{l}\mathfrak{C}_a\big(\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)-\mathfrak{T}_1\mathscr{A}_1(\mathscr{X}_a),t\big)\leq\mathfrak{B}'_a\big(K(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\mathfrak{T}_1\upsilon\big),\\ \mathfrak{C}_a\left(\dfrac{\mathscr{A}_1(\mathfrak{T}_1\mathscr{X}_a)}{\mathfrak{T}_1}-\mathscr{A}_1(\mathscr{X}_a),\upsilon\right)\leq\mathfrak{C}'_a\big(K(\mathscr{X}_a,\mathscr{X}_a,\mathscr{X}_a),\mathfrak{T}_1^2\upsilon\big),\\ \mathfrak{C}_a\big(J\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big),\\ \mathfrak{C}_a\big(J\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big),\\ \mathfrak{C}_a\big(J\mathscr{A}_1(\mathscr{X}_a)-\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L\upsilon\big)\end{array}\right\}\right\}.\tag{4.11}$$

If $i=1$, and

$$\begin{array}{l}\inf\big\{L^{1-1}\in(0,\infty):\\ \left.\begin{array}{l}\left\{\begin{array}{l}\mathfrak{A}_a\left(\mathscr{A}_1(\mathscr{X}_a)-\mathfrak{T}_1\omega\left(\dfrac{\mathscr{X}_a}{\mathfrak{T}_1}\right),\upsilon\right)\geq\mathfrak{A}'_a\left(K\left(\dfrac{\mathscr{X}_a}{\mathfrak{T}_1},\dfrac{\mathscr{X}_a}{\mathfrak{T}_1}\right),\mathfrak{T}_1\upsilon\right),\\ \mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big),\\ \mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big),\\ \mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big),\end{array}\right.\\ \left\{\begin{array}{l}\mathfrak{B}_a\left(\mathscr{A}_1(\mathscr{X}_a)-\mathfrak{T}_1\omega\left(\dfrac{\mathscr{X}_a}{\mathfrak{T}_1},\dfrac{\mathscr{X}_a}{\mathfrak{T}_1}\right),t\right)\leq\mathfrak{B}'_a\left(K\left(\dfrac{\mathscr{X}_a}{\mathfrak{T}_1}\right),\mathfrak{T}_1\upsilon\right),\\ \mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big),\\ \mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big),\\ \mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big),\end{array}\right.\\ \left\{\begin{array}{l}\mathfrak{C}_a\left(\mathscr{A}_1(\mathscr{X}_a)-\mathfrak{T}_1\omega\left(\dfrac{\mathscr{X}_a}{\mathfrak{T}_1},\dfrac{\mathscr{X}_a}{\mathfrak{T}_1}\right),t\right)\leq\mathfrak{C}'_a\left(K\left(\dfrac{\mathscr{X}_a}{\mathfrak{T}_1}\right),\mathfrak{T}_1\upsilon\right),\\ \mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big),\\ \mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big),\\ \mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\end{array}\right.\end{array}\right\}\end{array}\tag{4.12}$$

and

$$\inf\left\{L^{1-i}\in(0,\infty):\begin{Bmatrix}\mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L^{1-i}\upsilon\big),\\ \mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L^{1-i}\upsilon\big),\\ \mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a)-J\mathscr{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L^{1-i}\upsilon\big)\end{Bmatrix}\right\}. \tag{4.13}$$

Hence property [49] Condition : 1 holds.

According to the Condition : 2, [49] there is a alternative fixed point $\mathcal{A}_1$ of J in $\mathcal{S}$ then

$$\lim_{n\to\infty}\mathfrak{A}_a\left(\frac{\mathscr{A}_1(\mathfrak{D}_i^n\mathscr{X}_a)}{\mathfrak{D}_i^n}-\mathcal{A}_1(\mathscr{X}_a),\upsilon\right)=1,$$
$$\lim_{n\to\infty}\mathfrak{B}_a\left(\frac{\mathscr{A}_1(\mathfrak{D}_i^n\mathscr{X}_a)}{\mathfrak{D}_i^n}-\mathcal{A}_1(\mathscr{X}_a),\upsilon\right)=0,$$
$$\lim_{n\to\infty}\mathfrak{C}_a\left(\frac{\mathscr{A}_1(\mathfrak{D}_i^n\mathscr{X}_a)}{\mathfrak{D}_i^n}-\mathcal{A}_1(\mathscr{X}_a),\upsilon\right)=0.$$

For $\mathscr{A}_1$ is additive, taking $(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a)$ by $(\mathfrak{D}_i^n\mathscr{X}_a,\mathfrak{D}_i^n\mathscr{V}_a,\mathfrak{D}_i^n\mathscr{W}_a)$.

Fixed point Condition : 3 of [49], $\mathcal{A}_1$ is the alternative fixed point of J in the set $\Delta=\{\mathcal{A}_1\in\mathcal{S}:d(\mathscr{A}_1,A)<\infty\}$, and $\mathcal{A}_1$ is a unique mapping then

$$\left.\begin{aligned}&\mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L^{1-i}\upsilon\big),\quad\mathscr{X}_a\in\mathcal{L},\\&\mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L^{1-i}\upsilon\big),\quad\mathscr{X}_a\in\mathcal{L},\\&\mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\big(\mathscr{A}_1(\mathscr{X}_a),L^{1-i}\upsilon\big),\quad\mathscr{X}_a\in\mathcal{L}.\end{aligned}\right\}$$

By applying [49] Condition : 4

$$\left.\begin{aligned}&\mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)\geq\mathfrak{A}'_a\left(\mathscr{A}_1(\mathscr{X}_a),\frac{L^{1-i}}{1-L}\upsilon\right),\\&\mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{B}'_a\left(\mathscr{A}_1(\mathscr{X}_a),\frac{L^{1-i}}{1-L}\upsilon\right),\\&\mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)\leq\mathfrak{C}'_a\left(\mathscr{A}_1(\mathscr{X}_a),\frac{L^{1-i}}{1-L}\upsilon\right).\end{aligned}\right\}$$

Hence proved. □

Corollary 4.2 *Let $\mathscr{A}_1$ be an approximately additive mapping from $(Z,\mathfrak{A}'_a,\mathfrak{B}'_a,\mathfrak{C}'_a)$ in a neutrosophic normed space and $(\mathcal{M},\mathfrak{A}_a,\mathfrak{B}_a,\mathfrak{C}_a)$ be a neutrosophic Banach space satisfying the inequality*

$$\begin{aligned}&\mathfrak{A}_a\big(\mathfrak{Z}(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big)\\&\quad\geq\begin{cases}\mathfrak{A}'_a(\mathcal{S},\upsilon),\\ \mathfrak{A}'_a\big(\mathcal{S}\big(\|\mathscr{X}_a\|^{\mathfrak{E}}+\|\mathscr{V}_a\|^{\mathfrak{E}}+\|\mathscr{W}_a\|^{\mathfrak{E}}\big),\upsilon\big),\quad\mathfrak{E}\neq1,\\ \mathfrak{A}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{E}}\|\mathscr{W}_a\|^{\mathfrak{E}},\upsilon\big),\quad3\mathfrak{E}\neq1,\\ \mathfrak{A}'_a\big(\mathcal{S}\big\{\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{E}}\|\mathscr{W}_a\|^{\mathfrak{E}}+\big(\|\mathscr{X}_a\|^{3\mathfrak{E}}+\|\mathscr{V}_a\|^{3\mathfrak{E}}+\|\mathscr{W}_a\|^{3\mathfrak{E}}\big)\big\},\upsilon\big),\quad3\mathfrak{E}\neq1,\end{cases}\end{aligned}$$

$$
\begin{aligned}
&\mathfrak{B}_a\big(\mathfrak{Z}(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big)\\
&\quad\le\begin{cases}
\mathfrak{B}'_a(\mathcal{S},\upsilon),\\
\mathfrak{B}'_a\big(\mathcal{S}\big(\|\mathscr{X}_a\|^{\mathfrak{E}}+\|\mathscr{V}_a\|^{\mathfrak{E}}+\|\mathscr{W}_a\|^{\mathfrak{E}}\big),\upsilon\big), & \mathfrak{E}\neq 1,\\
\mathfrak{B}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{E}}\|\mathscr{W}_a\|^{\mathfrak{E}},\upsilon\big), & 3\mathfrak{E}\neq 1,\\
\mathfrak{B}'_a\big(\mathcal{S}\big\{\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{E}}\|\mathscr{W}_a\|^{\mathfrak{E}}+\big(\|\mathscr{X}_a\|^{3\mathfrak{E}}+\|\mathscr{V}_a\|^{3\mathfrak{E}}+\|\mathscr{W}_a\|^{3\mathfrak{E}}\big)\big\},\upsilon\big), & 3\mathfrak{E}\neq 1,
\end{cases}\\
&\mathfrak{C}_a\big(\mathfrak{Z}(\mathscr{X}_a,\mathscr{V}_a,\mathscr{W}_a),\upsilon\big)\\
&\quad\le\begin{cases}
\mathfrak{C}'_a(\mathcal{S},\upsilon),\\
\mathfrak{C}'_a\big(\mathcal{S}\big(\|\mathscr{X}_a\|^{\mathfrak{E}}+\|\mathscr{V}_a\|^{\mathfrak{E}}+\|\mathscr{W}_a\|^{\mathfrak{E}}\big),\upsilon\big), & \mathfrak{E}\neq 1,\\
\mathfrak{C}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{E}}\|\mathscr{W}_a\|^{\mathfrak{E}},\upsilon\big), & 3\mathfrak{E}\neq 1,\\
\mathfrak{C}'_a\big(\mathcal{S}\big\{\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{E}}\|\mathscr{W}_a\|^{\mathfrak{E}}+\big(\|\mathscr{X}_a\|^{3\mathfrak{E}}+\|\mathscr{V}_a\|^{3\mathfrak{E}}+\|\mathscr{W}_a\|^{3\mathfrak{E}}\big)\big\},\upsilon\big), & 3\mathfrak{E}\neq 1
\end{cases}
\end{aligned}
$$

such that

$$
\begin{aligned}
\mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)&\ge\begin{cases}
\mathfrak{A}'_a\Big(\dfrac{\mathcal{S}}{\mathfrak{T}_1},\dfrac{\mathfrak{T}_1}{1-\mathfrak{T}_1}\upsilon\Big),\\
\mathfrak{A}'_a\Big(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}},\dfrac{\mathfrak{T}_1}{\mathfrak{T}_1^{\mathfrak{E}}-\mathfrak{T}_1}\upsilon\Big),\\
\mathfrak{A}'_a\Big(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}},\dfrac{\mathfrak{T}_1}{\mathfrak{T}_1^{3\mathfrak{E}}-\mathfrak{T}_1}\upsilon\Big),\\
\mathfrak{A}'_a\Big(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\Big(\dfrac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}}+\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}},\dfrac{\mathfrak{T}_1}{\mathfrak{T}_1^{3\mathfrak{E}}-\mathfrak{T}_1}\upsilon\Big)\Big),
\end{cases}\\
\mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)&\le\begin{cases}
\mathfrak{B}'_a\Big(\dfrac{\mathcal{S}}{\mathfrak{T}_1},\dfrac{\mathfrak{T}_1}{1-\mathfrak{T}_1}\upsilon\Big),\\
\mathfrak{B}'_a\Big(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}},\dfrac{\mathfrak{T}_1}{\mathfrak{T}_1^{\mathfrak{E}}-\mathfrak{T}_1}\upsilon\Big),\\
\mathfrak{B}'_a\Big(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}},\dfrac{\mathfrak{T}_1}{\mathfrak{T}_1^{3\mathfrak{E}}-\mathfrak{T}_1}\upsilon\Big),\\
\mathfrak{B}'_a\Big(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\Big(\dfrac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}}+\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}},\dfrac{\mathfrak{T}_1}{\mathfrak{T}_1^{3\mathfrak{E}}-\mathfrak{T}_1}\upsilon\Big)\Big),
\end{cases}\qquad(4.14)\\
\mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a)-\mathcal{A}_1(\mathscr{X}_a),\upsilon\big)&\le\begin{cases}
\mathfrak{C}'_a\Big(\dfrac{\mathcal{S}}{\mathfrak{T}_1},\dfrac{\mathfrak{T}_1}{1-\mathfrak{T}_1}\upsilon\Big),\\
\mathfrak{C}'_a\Big(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}},\dfrac{\mathfrak{T}_1}{\mathfrak{T}_1^{\mathfrak{E}}-\mathfrak{T}_1}\upsilon\Big),\\
\mathfrak{C}'_a\Big(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}},\dfrac{\mathfrak{T}_1}{\mathfrak{T}_1^{3\mathfrak{E}}-\mathfrak{T}_1}\upsilon\Big),\\
\mathfrak{C}'_a\Big(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\Big(\dfrac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}}+\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}},\dfrac{\mathfrak{T}_1}{\mathfrak{T}_1^{3\mathfrak{E}}-\mathfrak{T}_1}\upsilon\Big)\Big).
\end{cases}
\end{aligned}
$$

Proof Let

$$\begin{aligned}
&\mathfrak{A}'_a\big(K\big(\mathfrak{D}_i^n\mathscr{X}_a,\mathfrak{D}_i^n\mathscr{V}_a,\mathfrak{D}_i^n\mathscr{W}_a\big),\mathfrak{D}_i^{\mathscr{L}_1}\upsilon\big)\\
&\quad=\begin{cases}
\mathfrak{A}'_a\big(\mathcal{S},\mathfrak{D}_i^{k}\upsilon\big),\\
\mathfrak{A}'_a\big(\mathcal{S}\big(\|\mathscr{X}_a\|^{\mathfrak{E}}+\|\mathscr{V}_a\|^{\mathfrak{E}}+\|\mathscr{W}_a\|^{\mathfrak{E}}\big),\mathfrak{D}_i^{\mathscr{L}_1-\mathfrak{E}}\upsilon\big),\\
\mathfrak{A}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{E}}\|\mathscr{W}_a\|^{\mathfrak{E}},\mathfrak{D}_i^{\mathscr{L}_1-3\mathfrak{E}}\upsilon\big),\\
\mathfrak{A}'_a\big(\mathcal{S}\big\{\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{E}}\|\mathscr{W}_a\|^{\mathfrak{E}}+\big(\|\mathscr{X}_a\|^{3\mathfrak{E}}+\|\mathscr{V}_a\|^{3\mathfrak{E}}+\|\mathscr{W}_a\|^{3\mathfrak{E}}\big)\big\},\mathfrak{D}_i^{\mathscr{L}_1-3\mathfrak{E}}\upsilon\big)
\end{cases}\\
&\quad=\begin{cases}
\to 1 \quad \text{as } \mathscr{L}_1\to\infty,\\
\to 1 \quad \text{as } \mathscr{L}_1\to\infty,\\
\to 1 \quad \text{as } \mathscr{L}_1\to\infty,\\
\to 1 \quad \text{as } \mathscr{L}_1\to\infty
\end{cases}\\
&\mathfrak{B}'_a\big(K\big(\mathfrak{D}_i^n\mathscr{X}_a,\mathfrak{D}_i^n\mathscr{V}_a,\mathfrak{D}_i^n\mathscr{W}_a\big),\mathfrak{D}_i^{\mathscr{L}_1}\upsilon\big)\\
&\quad=\begin{cases}
\mathfrak{B}'_a\big(\mathcal{S},\mathfrak{D}_i^{\mathscr{L}_1}\upsilon\big),\\
\mathfrak{B}'_a\big(\mathcal{S}\big(\|\mathscr{X}_a\|^{\mathfrak{E}}+\|\mathscr{V}_a\|^{\mathfrak{E}}+\|\mathscr{W}_a\|^{\mathfrak{E}}\big),\mathfrak{D}_i^{\mathscr{L}_1-\mathfrak{E}}\upsilon\big),\\
\mathfrak{B}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{E}}\|\mathscr{W}_a\|^{\mathfrak{E}},\mathfrak{D}_i^{\mathscr{L}_1-3\mathfrak{E}}\upsilon\big),\\
\mathfrak{B}'_a\big(\mathcal{S}\big\{\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{E}}\|\mathscr{W}_a\|^{\mathfrak{E}}+\big(\|\mathscr{X}_a\|^{3\mathfrak{E}}+\|\mathscr{V}_a\|^{3\mathfrak{E}}+\|\mathscr{W}_a\|^{3\mathfrak{E}}\big)\big\},\mathfrak{D}_i^{\mathscr{L}_1-3\mathfrak{E}}\upsilon\big)
\end{cases}\\
&\quad=\begin{cases}
\to 0 \quad \text{as } \mathscr{L}_1\to\infty,\\
\to 0 \quad \text{as } \mathscr{L}_1\to\infty,\\
\to 0 \quad \text{as } \mathscr{L}_1\to\infty,\\
\to 0 \quad \text{as } \mathscr{L}_1\to\infty
\end{cases}\\
&\mathfrak{C}'_a\big(K\big(\mathfrak{D}_i^n\mathscr{X}_a,\mathfrak{D}_i^n\mathscr{V}_a,\mathfrak{D}_i^n\mathscr{W}_a\big),\mathfrak{D}_i^{\mathscr{L}_1}\upsilon\big)\\
&\quad=\begin{cases}
\mathfrak{C}'_a\big(\mathcal{S},\mathfrak{D}_i^{\mathscr{L}_1}\upsilon\big),\\
\mathfrak{C}'_a\big(\mathcal{S}\big(\|\mathscr{X}_a\|^{\mathfrak{E}}+\|\mathscr{V}_a\|^{\mathfrak{E}}+\|\mathscr{W}_a\|^{\mathfrak{E}}\big),\mathfrak{D}_i^{\mathscr{L}_1-\mathfrak{E}}\upsilon\big),\\
\mathfrak{C}'_a\big(\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{E}}\|\mathscr{W}_a\|^{\mathfrak{E}},\mathfrak{D}_i^{\mathscr{L}_1-3\mathfrak{E}}\upsilon\big),\\
\mathfrak{C}'_a\big(\mathcal{S}\big\{\|\mathscr{X}_a\|^{\mathfrak{E}}\|\mathscr{V}_a\|^{\mathfrak{E}}\|\mathscr{W}_a\|^{\mathfrak{E}}+\big(\|\mathscr{X}_a\|^{3\mathfrak{E}}+\|y\|^{3\mathfrak{E}}+\|z\|^{3\mathfrak{E}}\big)\big\},\mathfrak{D}_i^{\mathscr{L}_1-3\mathfrak{E}}\upsilon\big)
\end{cases}\\
&\quad=\begin{cases}
\to 0 \quad \text{as } \mathscr{L}_1\to\infty,\\
\to 0 \quad \text{as } \mathscr{L}_1\to\infty,\\
\to 0 \quad \text{as } \mathscr{L}_1\to\infty,\\
\to 0 \quad \text{as } \mathscr{L}_1\to\infty.
\end{cases}
\end{aligned}$$

Here (4.1) exists, then

$$
\left\{
\begin{aligned}
\mathfrak{A}'_a\left(\frac{1}{\mathfrak{T}_1}K\left(\frac{\mathscr{X}_a}{\mathfrak{T}_1},\frac{\mathscr{X}_a}{\mathfrak{T}_1},\frac{\mathscr{X}_a}{\mathfrak{T}_1}\right),\upsilon\right) &= \begin{cases}
\mathfrak{A}'_a\left(\dfrac{\mathcal{S}}{\mathfrak{T}_1},\upsilon\right),\\
\mathfrak{A}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}},\upsilon\right),\\
\mathfrak{A}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}},\upsilon\right),\\
\mathfrak{A}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\left(\dfrac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}}+\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}\right),\upsilon\right),
\end{cases}\\
\mathfrak{B}'_a\left(\frac{1}{\mathfrak{T}_1}K\left(\frac{\mathscr{X}_a}{\mathfrak{T}_1},\nu,\frac{\mathscr{X}_a}{\mathfrak{T}_1}\right),\upsilon\right) &= \begin{cases}
\mathfrak{B}'_a\left(\dfrac{\mathcal{S}}{\mathfrak{T}_1},\upsilon\right),\\
\mathfrak{B}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}},\upsilon\right),\\
\mathfrak{B}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}},\upsilon\right),\\
\mathfrak{B}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\left(\dfrac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}}+\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}\right),\upsilon\right),
\end{cases}\\
\mathfrak{C}'_a\left(\frac{1}{\mathfrak{T}_1}K\left(\frac{\mathscr{X}_a}{\mathfrak{T}_1},\frac{\mathscr{X}_a}{\mathfrak{T}_1},\frac{\mathscr{X}_a}{\mathfrak{T}_1}\right),\upsilon\right) &= \begin{cases}
\mathfrak{C}'_a\left(\dfrac{\mathcal{S}}{\mathfrak{T}_1},\upsilon\right),\\
\mathfrak{C}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}},\upsilon\right),\\
\mathfrak{C}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}},\upsilon\right),\\
\mathfrak{C}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\left(\dfrac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}}+\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}\right),\upsilon\right).
\end{cases}
\end{aligned}
\right\}
$$

From (4.5),

$$\left.\begin{array}{l}
\mathfrak{A}'_a\left(\dfrac{\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a)}{\mathfrak{D}_i},\upsilon\right) = \begin{cases}
\mathfrak{A}'_a\left(\mathcal{S},\mathfrak{D}_i\upsilon\right), \\
\mathfrak{A}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}},\mathfrak{D}_i{}^{1-\mathfrak{E}}\upsilon\right), \\
\mathfrak{A}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}},\mathfrak{D}_i{}^{1-3\mathfrak{E}}\upsilon\right), \\
\mathfrak{A}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\left(\dfrac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}}+\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}\right),\mathfrak{D}_i{}^{1-3\mathfrak{E}}\upsilon\right),
\end{cases} \\
\mathfrak{B}'_a\left(\dfrac{\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a)}{\mathfrak{D}_i},\upsilon\right) = \begin{cases}
\mathfrak{B}'_a\left(\mathcal{S},\mathfrak{D}_i\upsilon\right), \\
\mathfrak{B}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}},\mathfrak{D}_i{}^{1-\mathfrak{E}}\upsilon\right), \\
\mathfrak{B}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}},\mathfrak{D}_i{}^{1-3\mathfrak{E}}\upsilon\right), \\
\mathfrak{B}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\left(\dfrac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}}+\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}\right),\mathfrak{D}_i{}^{1-3\mathfrak{E}}\upsilon\right),
\end{cases} \\
\mathfrak{C}'_a\left(\dfrac{\mathscr{A}_1(\mathfrak{D}_i\mathscr{X}_a)}{\mathfrak{D}_i},\upsilon\right) = \begin{cases}
\mathfrak{B}'_a\left(\mathcal{S},\mathfrak{D}_i\upsilon\right), \\
\mathfrak{C}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}},\mathfrak{D}_i{}^{1-\mathfrak{E}}\upsilon\right), \\
\mathfrak{C}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}},\mathfrak{D}_i{}^{1-3\mathfrak{E}}\upsilon\right), \\
\mathfrak{C}'_a\left(\dfrac{\mathcal{S}\|\mathscr{X}_a\|^{3\mathfrak{E}}}{\mathfrak{T}_1}\left(\dfrac{3}{|\mathfrak{T}_1|^{3\mathfrak{E}}}+\dfrac{1}{|\mathfrak{T}_1|^{3\mathfrak{E}}}\right),\mathfrak{D}_i{}^{1-3\mathfrak{E}}\upsilon\right).
\end{cases}
\end{array}\right\}$$

Hence (4.6), we have

	L	$\mathfrak{E}, i = 0$	L	$\mathfrak{E}, i = 1$
1.	$\mathfrak{T}_1$	0	$\mathfrak{T}_1^{-1}$	0
2.	$\mathfrak{T}_1^{1-\mathfrak{E}}$	$\mathfrak{E} < 1$	$\mathfrak{T}_1^{\mathfrak{E}-1}$	$\mathfrak{E} > 1$
3.	$\mathfrak{T}_1^{1-3\mathfrak{E}}$	$3\mathfrak{E} < 1$	$\mathfrak{T}_1^{3\mathfrak{E}-1}$	$3\mathfrak{E} > 1$
4.	$\mathfrak{T}_1^{3\mathfrak{E}-1}$	$3\mathfrak{E} < 1$	$\mathfrak{T}_1^{3\mathfrak{E}-1}$	$3\mathfrak{E} > 1.$

Method: 1 For $i = 0$,

$$\left.\begin{array}{l}
\mathfrak{A}_a\left(\mathscr{A}_1(\mathscr{X}_a) - \mathcal{A}_1(\mathscr{X}_a),\upsilon\right) \ge \mathfrak{A}'_a\left(\mathscr{A}_1(\mathscr{X}_a), \dfrac{\mathfrak{T}_1^{1-0}}{1-\mathfrak{T}_1}\upsilon\right) = \mathfrak{A}'_a\left(\dfrac{\mathcal{S}}{\mathfrak{T}_1}, \dfrac{\mathfrak{T}_1}{1-\mathfrak{T}_1}\upsilon\right), \\
\mathfrak{B}_a\left(\mathscr{A}_1(\mathscr{X}_a) - \mathcal{A}_1(\mathscr{X}_a),\upsilon\right) \le \mathfrak{B}'_a\left(\mathscr{A}_1(\mathscr{X}_a), \dfrac{\mathfrak{T}_1^{1-0}}{1-\mathfrak{T}_1}\upsilon\right) = \mathfrak{B}'_a\left(\dfrac{\mathcal{S}}{\mathfrak{T}_1}, \dfrac{\mathfrak{T}_1}{1-\mathfrak{T}_1}\upsilon\right), \\
\mathfrak{C}_a\left(\mathscr{A}_1(\mathscr{X}_a) - \mathcal{A}_1(\mathscr{X}_a),\upsilon\right) \le \mathfrak{C}'_a\left(\mathscr{A}_1(\mathscr{X}_a), \dfrac{\mathfrak{T}_1^{1-0}}{1-\mathfrak{T}_1}\upsilon\right) = \mathfrak{C}'_a\left(\dfrac{\mathcal{S}}{\mathfrak{T}_1}, \dfrac{\mathfrak{T}_1}{1-\mathfrak{T}_1}\upsilon\right).
\end{array}\right\}$$

Method: 1 For $i = 1$,

$$\left.\begin{aligned}
&\mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a) - \mathcal{A}_1(\mathscr{X}_a), \upsilon\big) \geq \mathfrak{A}'_a\left(\mathscr{A}_1(\mathscr{X}_a), \frac{((\mathfrak{T}_1)^{-1})^{1-1}}{1-(\mathfrak{T}_1)^{-1}}\upsilon\right) = \mathfrak{A}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1 - 1}\upsilon\right),\\
&\mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a) - \mathcal{A}_1(\mathscr{X}_a), \upsilon\big) \leq \mathfrak{B}'_a\left(\mathscr{A}_1(\mathscr{X}_a), \frac{((\mathfrak{T}_1)^{-1})^{1-1}}{1-(\mathfrak{T}_1)^{-1}}\upsilon\right) = \mathfrak{B}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1 - 1}\upsilon\right),\\
&\mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a) - \mathcal{A}_1(\mathscr{X}_a), \upsilon\big) \leq \mathfrak{C}'_a\left(\mathscr{A}_1(\mathscr{X}_a), \frac{((\mathfrak{T}_1)^{-1})^{1-1}}{1-(\mathfrak{T}_1)^{-1}}\upsilon\right) = \mathfrak{C}'_a\left(\frac{\mathcal{S}}{\mathfrak{T}_1}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1 - 1}\upsilon\right).
\end{aligned}\right\}$$

Method: 2 For $i = 0$,

$$\left.\begin{aligned}
&\mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a) - \mathcal{A}_1(\mathscr{X}_a), \upsilon\big) \geq \mathfrak{A}'_a\left(\mathscr{A}_1(\mathscr{X}_a), \frac{(\mathfrak{T}_1^{1-\mathfrak{E}})^{1-0}}{1-(\mathfrak{T}_1^{1-\mathfrak{E}})}\upsilon\right)\\
&\qquad = \mathfrak{A}'_a\left(\frac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\frac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^{\mathfrak{E}} - \mathfrak{T}_1^{k}}\upsilon\right),\\
&\mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a) - \mathcal{A}_1(\mathscr{X}_a), \upsilon\big) \leq \mathfrak{B}'_a\left(\mathscr{A}_1(\mathscr{X}_a), \frac{(\mathfrak{T}_1^{1-\mathfrak{E}})^{1-0}}{1-(\mathfrak{T}_1^{1-\mathfrak{E}})}\upsilon\right)\\
&\qquad = \mathfrak{B}'_a\left(\frac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\frac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^{\mathfrak{E}} - \mathfrak{T}_1}\upsilon\right),\\
&\mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a) - \mathcal{A}_1(\mathscr{X}_a), \upsilon\big) \leq \mathfrak{C}'_a\left(\mathscr{A}_1(\mathscr{X}_a), \frac{(\mathfrak{T}_1^{1-\mathfrak{E}})^{1-0}}{1-(\mathfrak{T}_1^{1-\mathfrak{E}})}\upsilon\right)\\
&\qquad = \mathfrak{C}'_a\left(\frac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\frac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^{\mathfrak{E}} - \mathfrak{T}_1}\upsilon\right).
\end{aligned}\right\}$$

Method: 2 For $i = 1$,

$$\left.\begin{aligned}
&\mathfrak{A}_a\big(\mathscr{A}_1(\mathscr{X}_a) - \mathcal{A}_1(\mathscr{X}_a), \upsilon\big) \geq \mathfrak{A}'_a\left(\mathscr{A}_1(\mathscr{X}_a), \frac{(\mathfrak{T}_1^{A-1})^{1-1}}{1-(\mathfrak{T}_1^{A-1})}\upsilon\right)\\
&\qquad = \mathfrak{A}'_a\left(\frac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\frac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1^{k} - \mathfrak{T}_1^{a}}\upsilon\right),\\
&\mathfrak{B}_a\big(\mathscr{A}_1(\mathscr{X}_a) - \mathcal{A}_1(\mathscr{X}_a), \upsilon\big) \leq \mathfrak{B}'_a\left(\mathscr{A}_1(\mathscr{X}_a), \frac{(\mathfrak{T}_1^{A-1})^{1-1}}{1-(\mathfrak{T}_1^{A-1})}\upsilon\right)\\
&\qquad = \mathfrak{B}'_a\left(\frac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\frac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1 - \mathfrak{T}_1^{a}}\upsilon\right),\\
&\mathfrak{C}_a\big(\mathscr{A}_1(\mathscr{X}_a) - \mathcal{A}_1(\mathscr{X}_a), \upsilon\big) \leq \mathfrak{C}'_a\left(\mathscr{A}_1(\mathscr{X}_a), \frac{(\mathfrak{T}_1^{A-1})^{1-1}}{1-(\mathfrak{T}_1^{A-1})}\upsilon\right)\\
&\qquad = \mathfrak{C}'_a\left(\frac{\mathcal{S}\|\mathscr{X}_a\|^{\mathfrak{E}}}{\mathfrak{T}_1}\frac{3}{|\mathfrak{T}_1|^{\mathfrak{E}}}, \frac{\mathfrak{T}_1}{\mathfrak{T}_1 - \mathfrak{T}_1^{a}}\upsilon\right).
\end{aligned}\right\}$$

□

5 Conclusion

This paper has presented a novel approach for analyzing the stability of the Euler-Lagrange additive FE within neutrosophic normed spaces, addressing a critical need in the realm of uncertainty modeling and functional analysis. The investigation delves into the existence of solutions for this equation and explores its Ulam-Hyers stability within neutrosophic normed spaces. The results are obtained by applying two distinct approaches: direct and fixed point techniques. The findings presented in this article establish the relationship between four distinct areas of research: FEs, neutrosophic normed spaces, Ulam-Hyers stability, and fixed point theory. The stability analysis of this equation in neutrosophic normed spaces is unique, given the absence of prior research on the stability of equations employing neutrosophic concepts. This paper contributes to the broader advancement of neutrosophic mathematics by offering new perspectives and methodologies for analyzing functional equations in nonclassical settings. Our work opens up avenues for further research and exploration, inviting interdisciplinary collaboration and innovation in the fields of mathematics, uncertainty modeling, and system analysis.

Acknowledgements
The authors A. Aloqaily and N. Mlaiki would like to thank Prince Sultan University for paying the APC and for the support through the TAS research lab.

Author contributions
A.A., P.A., K.J., S.A. and N.M. wrote the main manuscript text. All authors reviewed the manuscript.

Funding
This research did not receive any external funding.

Data Availability
No datasets were generated or analysed during the current study.

Declarations

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare no competing interests.

Author details
[1]Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia. [2]School of Computer, Data and Mathematical Sciences, Western Sydney University, Sydney, 2150, Australia. [3]Department of Mathematics, St. Joseph's College of Engineering, OMR, Chennai, 600 119, TamilNadu, India. [4]Department of Mathematics, School of Engineering, Presidency University, Bengaluru, 560 064, TamilNadu, India.

Received: 11 February 2024 Accepted: 22 March 2024 Published online: 29 March 2024

References

1. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**(3), 338–353 (1965)
2. Atanassov, K.: Intuitionistic fuzzy sets. Fuzzy Sets Syst. **20**, 87–96 (1986)
3. Atanassov, K.T.: More on intuitionistic fuzzy sets. Fuzzy Sets Syst. **33**(1), 37–45 (1989)
4. Smarandache, F.: Neutrosophy: Neutrosophic Probability Set and Logic: Analytic Synthesis and Synthetic Analysis (1998)
5. Smarandache, F.: A unifying field in logics: neutrosophic logic. In: Philosophy, pp. 1–141. American Research Press (1999)
6. Katsaras, A.K.: Fuzzy topological vector spaces. Fuzzy Sets Syst. **12**, 143–154 (1984)
7. Saadati, R., Park, J.H.: On the intuitionistic fuzzy topological spaces. Chaos Solitons Fractals **27**, 331–344 (2006)
8. Bera, T., Mahapatra, N.K.: On neutrosophic soft linear spaces. Fuzzy Inf. Eng. **9**(3), 299–324 (2017)
9. Bera, T., Mahapatra, N.K.: Neutrosophic soft normed linear spaces. Neutrosophic Sets Syst. **23**, 52–71 (2018)
10. Agboola, A.A.A., Akwu, A.D., Oyebo, Y.T.: Neutrosophic groups and subgroups. Int. J. Math. Combin. **3**, 1–9 (2012)
11. Agboola, A.A.A., Akinleye, S.A.: Neutrosophic vector spaces. Neutrosophic Sets Syst. **4**, 9–17 (2014)
12. Vasantha, W.B., Smarandache, F.: Neutrosophic Rings. Hexis, Phoenix (2006). ISBN 978-1-931233-20-9
13. Abobala, M.: Classical homomorphisms between refined neutrosophic rings and neutrosophic rings. Int. J. Neutrosophic Sci. **5**, 72–75 (2020)
14. Abobala, M.: On the representation of neutrosophic matrices by neutrosophic linear transformations. J. Math. **2021**, Article ID 5591576 (2021)
15. Abobala, M.: Partial foundation of neutrosophic number theory. Neutrosophic Sets Syst. **39**, 120–132 (2021)
16. Vasantha, K., Ilanthenral, K., Smarandache, F.: Neutrosophic graphs: a new dimension to graph theory. Infin. Study (2015)
17. Smarandache, F.: Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic probability. Infin. Study (2013)
18. Broumi, S., Sundareswaran, R., Shanmugapriya, M., Singh, P.K., Voskoglou, M., Talea, M.: Faculty performance evaluation through multi-criteria decision analysis using interval-valued fermatean neutrosophic sets. Mathematics **11**, 3817 (2023)

19. Riaz, M., Farid, H.M.A., Antucheviciene, J., Demir, G.: Efficient decision making for sustainable energy using single-valued neutrosophic prioritized interactive aggregation operators. Mathematics **11**, 2186 (2023)
20. Boloş, M.-I., Bradea, I.-A., Delcea, C.: Optimization of financial asset neutrosophic portfolios. Mathematics **9**, 1162 (2021)
21. Zavadskas, E.K., Bausys, R., Lescauskiene, I., Usovaite, A.: MULTIMOORA under interval-valued neutrosophic sets as the basis for the quantitative heuristic evaluation methodology HEBIN. Mathematics **9**, 66 (2021)
22. Amin, B., Salama, A.A., El-Henawy, I.M., Mahfouz, K., Gafar, M.G.: Intelligent neutrosophic diagnostic system for cardiotocography data. Comput. Intell. Neurosci. **2021**, 6656770 (2021)
23. Zhang, S., Ye, J.: Multiple attribute group decision-making models using single-valued neutrosophic and linguistic neutrosophic hybrid element aggregation algorithms. J. Math. **2022**, Article ID 1021280 (2022)
24. Fu, Z., Liu, C., Ruan, S., Chen, K.: Design of neutrosophic self-tuning PID controller for AC permanent magnet synchronous motor based on neutrosophic theory. Math. Probl. Eng. **2021**, Article ID 5548184 (2021)
25. Ishtiaq, U., Javed, K., Uddin, F., de la Sen, M., Ahmed, K., Ali, M.U.: Fixed point results in orthogonal neutrosophic metric spaces. Complexity **2021**, Article ID 2809657 (2021)
26. Jeyaraman, M., Sowndrarajan, S.: Common fixed point results in neutrosophic metric spaces. Neutrosophic Sets Syst. **42**, 208–220 (2021)
27. Salama, A.A., Alblowi, S.A.: Neutrosophic set and neutrosophic topological spaces. IOSR J. Math. **3**(4), 31–35 (2012)
28. Al-Omeri, W.F., Jafari, S., Smarandache, F.: (Φ, Ψ)-Weak contractions in neutrosophic cone metric spaces via fixed point theorems. Math. Probl. Eng. **2020**, Article ID 9216805 (2020)
29. Riaz, M., Ishtiaq, U., Park, C., Ahmad, K., Uddin, F.: Some fixed point results for ξ-chainable neutrosophic and generalized neutrosophic cone metric spaces with application. AIMS Math. **7**(8), 14756–14784 (2022)
30. Sharma, A., Kumar, V.: Some remarks on generalised summability using difference operators on neutrosophic normed spaces. J. Ramanujan Soc. Math. Math. Sci. **9**, 2 (2022)
31. Ulam, S.M.: Problems in Modern Mathematics, science editions. Wiley, New York (1964)
32. Hyers, D.H.: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222–224 (1941)
33. Aoki, T.: On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64–66 (1950)
34. Rassias, T.M.: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297–300 (1978)
35. Găvruţă, P.: A generalization of the Ulam-Hyers-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431–436 (1994)
36. Rassias, J.M.: On approximately of approximately linear mappings by linear mappings. J. Funct. Anal. **46**, 126–130 (1982)
37. Ravi, K., Arunkumar, M., Rassias, J.M.: On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation. Int. J. Math. Sci. **3**(08), 36–47 (2008)
38. Aczel, J., Dhombres, J.: Functional Equations in Several Variables. Cambridge University Press, Cambridge (1989)
39. Czerwik, S.: Functional Equations and Inequalities in Several Variables. World Scientific, River Edge (2002)
40. Jung, S.M.: Ulam-Hyers-Rassias Stability of Functional Equations in Mathematical Analysis. Hadronic Press, Palm Harbor (2001)
41. Pasupathi, A., Konsalraj, J., Fatima, N., Velusamy, V., Mlaiki, N., Souayah, N.: Direct and fixed-point stability-instability of additive functional equation in Banach and quasi-beta normed spaces. Symmetry **2022**, 14 (1700)
42. Agilan, P., Julietraja, K., Mlaiki, N., Mukheimer, A.: Intuitionistic fuzzy stability of an Euler-Lagrange symmetry additive functional equation via direct and fixed point technique (FPT). Symmetry **14**, 2454 (2022)
43. Agilan, P., Almazah, M.A.A., Julietraja, K., Alsinai, A.: Classical and fixed point approach to the stability analysis of a bilateral symmetric additive functional equation in fuzzy and random normed spaces. Mathematics **11**, 681 (2023)
44. Agilan, P., Julietraja, K., Almazah, M.A.A., Alsinai, A.: Stability analysis of a new class of series type additive functional equation in Banach spaces: direct and fixed point techniques. Mathematics **11**, 887 (2023). https://doi.org/10.3390/math11040887
45. Marin, M., Hobiny, A., Abbas, I.: The effects of fractional time derivatives in porothermoelastic materials using finite element method. Mathematics **9**, 1606 (2021)
46. Shannon, A., Dubeau, F., Uysal, M., Özkan, E.: A difference equation model of infectious disease. Int. J. Bioautom. **26**, 339–352 (2022)
47. Marin, M., Seadawy, A., Vlase, S., Chirila, A.: On mixed problem in thermoelasticity of type III for Cosserat media. J. Taibah Univ. Sci. **16**(1), 1264–1274 (2022)
48. Marin, M., Öchsner, A., Vlase, S., Grigorescu, D.O., Tuns, I.: Some results on eigenvalue problems in the theory of piezoelectric porous dipolar bodies. Contin. Mech. Thermodyn. **35**, 1969–1979 (2023)
49. Margolis, B., Diaz, J.B.: A fixed point theorem of the alternative for contractions on a generalized complete metric space. Bull. Am. Math. Soc. **74**, 305–309 (1968)

Publisher's Note