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SOME RESULTS OF NEUTROSOPHIC NORMED SPACES VIA FIBONACCI MATRIX

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Recently, Fibonacci matrix was introduced and studied by Kara and Basarir [3]*. In the present paper, we introduce Fibonacci statistical convergence in neutrosophic normed space and examine some basic properties like Fibonacci statistical Cauchyness and Fibonacci statistical completeness.*

Keywords: Neutrosophic normed space, $t-$ norm, $t-$ conorm, Statistical convergence, Statistically completeness and Fibonacci Sequence.

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1. Introduction

Zadeh [17] introduced the notion of the fuzzy theory in 1965. Since then a large number of research papers have been published and fuzzification of many classical theories has also been made. Atanassov [16] generalized the fuzzy sets theory and studied the concepts of intuitionistic fuzzy sets (IFS). In 2004, Park [11] investigated the notion of intuitionistic fuzzy metric space, further Saadati, and Park [25] analyzed this concept in the norm.

The notion of neutrosophic sets (NS) was introduced by Smarandache [4].This set is an extension of IFS no matter if the sum of neutrosophic components is < 1, or > 1, or $= 1$. For the case when the sum of components is 1 (as in IFS), after applying the neutrosophic aggregation operators, one gets a different result than applying the intuitionistic fuzzy operators, since the intuitionistic fuzzy operators ignore the indeterminacy, while the neutrosophic aggregation operators take into consideration the indeterminacy at the same level as truth-membership and falsehood-nonmembership are taken. NS is also more flexible and effective because it handles, besides independent components, also partially independent and partially dependent components, while IFS cannot deal with these. Smarandache [5] examined the differences between neutrosophic logic, intuitionistic fuzzy logic, and the corresponding neutrosophic sets and intuitionistic fuzzy sets. Further, Smarandache [6, 7, 8] investigated neutroalgebra which is generalization of partial algebra , neutroalgebraic structures and antialgebraic structures. Moreover, Bera and Mahapatra [27] introduced the neutrosophic soft linear space. Bera and Mahapatra [28] studied convexity, metric, Cauchy

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sequence, and neutrosophic soft norm linear space (NSNLS).

The theory of statistical convergence of sequences of real numbers was introduced by H. Fast [9] and H. Steinhaus [10] independently. After that several researchers analyzed these concepts in different areas (see, [12, 20, 13, 24]). Moreover, Karakus [26] studied statistical convergence on probabilistic normed spaces , and then Mursaleen et. al [19] further generalized statistical convergence of double sequence in intuitionistic fuzzy normed spaces. Recently, Khan et al. [29, 30] investigated ideal convergence for single and double sequences in intuitionistic fuzzy normed spaces. Further, Kirişci [21] studied neutrosophic normed spaces and statistical convergence on it. Since the neutrosophic normed space is a natural generalization of the intuitionistic fuzzy normed space and statistical convergence using Fibonacci matrix has an important place in the theory of sequence spaces. In this paper, we generalized and studied the concepts given by Kirişci into neutrosophic normed spaces and we obtained some interesting results.

2. **Preliminaries**

In what follows, we collect relevant definitions needed in our subsequent discussions.

Definition 2.1. *The Fibonacci numbers are the terms of the sequence of numbers* (f_n) *for* $n = 1, 2, \ldots$ *defined by the linear recurrence equation*

$$f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2. \tag{1}$$

The first two terms are $f_0 = 0$ *and* $f_1 = 1$.

Recently, Kara and Basarir [3] used the Fibonacci sequence in the theory of sequence spaces. Later on the infinite matrix associated to the fibonacci numbers namely Fibonacci difference matrix F was initiated by Kara in [2]. Suppose for every $n \in \mathbb{N}$, f_n be the n^{th} Fibonacci number. Then, the infinite matrix $\hat{F} = (f_{n.k}), n, k = 0, 1, ...$ corresponding to the Fibonacci sequence (f_n) is defined by

$$\hat{f}_{n,k=0,1...} = \begin{cases} \frac{-f_{n+1}}{f_n} & (k = n-1) \\ \frac{f_n}{f_{n+1}} & (k = n) \\ 0, & (0 \leq k < n-1 \text{ or } k > n) \end{cases}$$

which aids in the formation of sequence spaces corresponding to the matrix domain of $\hat{F}$. Moreover, various researchers produced high-quality papers on the Fibonacci matrix [1, 18, 15, 22, 23].

Fibonacci numbers are strongly related to the golden ratio: Binet's formula expresses the nth Fibonacci number in terms of n and the golden ratio, and implies that the ratio of two consecutive Fibonacci numbers tends to the golden ratio as n increases. That is,

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \alpha \ \text{ (golden ratio)} \tag{2}$$

$$\sum_{k=0}^{n} f_k = f_{n+2} - 1, \ (n \in \mathbb{N}) \tag{3}$$

$$\sum_{k=0} \frac{1}{f_k} \text{ converges,} \tag{4}$$

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1}(n \geq 1) \text{ (Cassini formula)}$$

It provides $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$, if one substitutes for f_{n+1} in Cassini's formula.

Definition 2.2. [22] *A sequence $x = (x_j)$ is said to be Fibonacci statistically convergent (or $\hat{F}x_j$ - statistically convergent) if there is a number $\ell \in X$ such that for every $\epsilon > 0$ the set $K_\epsilon(\hat{F}) = \{j \leq n : |\hat{F}x_j - \ell| \geq \epsilon\}$ has the natural density zero, i.e., $\delta(K_\epsilon(\hat{F})) = 0$ That is*

$$\lim_{n\to\infty} \frac{1}{n}|\{j \leq n : |\hat{F}x_j - \ell| \geq \epsilon\}| = 0$$

In this case, we write $\delta(\hat{F})$ - lim $x_j = \ell$ or $x_j \to \ell(S_\delta(\hat{F}))$.

Definition 2.3. [22] *A sequence $x = (x_j)$ is called Fibonacci statistically Cauchy (or $\hat{F}x_j$-statistically Cauchy) if there exists a number $N = N(\epsilon)$ such that for each $\epsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{n}|\{j \leq n : |\hat{F}x_j - \hat{F}(x_N)| \geq \epsilon\}| = 0. \tag{5}$$

Definition 2.4. [16] *Suppose X be a universe of discourse Then the set $A_{IFS} \subseteq X$ by,*

$$A_{IFS} = \{< x, \mathcal{T}_A(x), \mathcal{H}_A(x) >: x \in X\}, \text{ is called intuitionistic fuzzy set.} \tag{6}$$

where $\mathcal{T}_A(x), \mathcal{H}_A(x) : X \to [0, 1]$ represent the degree of membership and degree of nonmembership respectively, with $\mathcal{T}_A(x) + \mathcal{H}_A(x) \leq 1$, and $\mathcal{J}_A(x) = 1 - \mathcal{J}_A(x) - \mathcal{H}_A(x)$ represents degree of hesitancy. The intuitionistic fuzzy components $\mathcal{T}_A(x), \mathcal{H}_A(x)$ and $\mathcal{J}_A(x)$ are dependent concerning each other.

Definition 2.5. [4] *Suppose X be a universe of discourse Then the set $A_{NS} \subseteq X$ by,*

$$A_{NS} = \{< x, \mathcal{T}_A(x), \mathcal{H}_A(x), \mathcal{J}_A(x) >: x \in X\}, \text{ is called neutrosophic set.} \tag{7}$$

where $\mathcal{T}_A(x), \mathcal{H}_A(x), \mathcal{J}_A(x) : X \to [0, 1]$ represent the degree of truth-membership, degree of indeterminacy-membership, and degree of false-nonmembership respectively, with $0 \leq \mathcal{T}_A(x) + \mathcal{H}_A(x) + \mathcal{J}_A(x) \leq 3$. The neutrosophic components $\mathcal{T}_A(x), \mathcal{H}_A(x)$ and $\mathcal{J}_A(x)$ are independent concerning each other.

Triangular norms (t-norms) were initiated by Menger [14]. Triangular conorms (t-conorms) are known as dual operations of t-norm. The t-norm and t-conorm are very significant for fuzzy operations (intersections and unions) which are defined as follows:

Definition 2.6. [14] *A binary operation $\star : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is said to be a continuous t-norm if it satisfies the following conditions:*
(a) $\star$ is associative and commutative,
(b) $\star$ is continuous,
(c) $a \star 1 = a$ for all $a \in [0, 1]$,
(d) $a \star b \leq c \star d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

Definition 2.7. [14] *A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions:*
(a) $\diamond$ is associative and commutative,
(b) $\diamond$ is continuous,
(c) $a \diamond 0 = a$ for all $a \in [0,1]$,
(d) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

From above definitions, we note that if we choose $0 < e_1, e_2 < 1$ with $e_1 > e_2$, then there exist $0 < e_3, e_4 < 1$ such that $e_1 * e_3 \geq e_2, e_1 \geq e_4 \diamond e_2$. Further, if we choose $e_5 \in (0,1)$, then there exist $e_6, e_7 \in (0,1)$ such that $e_6 * e_6 \geq e_5$ and $e_7 \diamond e_7 \leq e_5$.

Definition 2.8. [21] *Take X as a vector space and $\mathcal{M} = \{< x, \mathcal{T}(x), \mathcal{H}(x), \mathcal{J}(x) >: x \in X\}$ be a normed space such that $\mathcal{T}(x), \mathcal{H}(x), \mathcal{J}(x) : X \times R^+ \to [0,1]$. Assume $\star$ and $\diamond$ be the continuous t norm and continuous t-conorm respectively. The four-tuple $(X, \mathcal{M}, \star, \diamond)$ is said to be Neutrosophic normed space (NNS) if the subsequent conditions holds; for all $x, y, z \in X$ and $t, s > 0$*

(i) $0 \leq \mathcal{T}(x,t) \leq 1,\ 0 \leq \mathcal{H}(y,t) \leq 1,\ 0 \leq \mathcal{J}(z,t) \leq 1,\ t \in R^+$,

(ii) $\mathcal{T}(x,t) + \mathcal{H}(x,t) + \mathcal{J}(x,t) \leq 3$, *for* $t \in R^+$,

(iii) $\mathcal{T}(x,t) = 1$ *for* $t > 0$ *iff* $x = 0$

(iv) $\mathcal{T}(\alpha x, t) = \mathcal{T}(x, \frac{t}{|\alpha|})$,

(v) $\mathcal{T}(x,t) \star \mathcal{T}(y,s) \leq \mathcal{T}(x+y, t+s)$,

(vi) $\mathcal{T}(x, \star)$ *is continuous non-decreasing function*

(vii) $\lim_{t \to \infty} \mathcal{T}(x,t) = 1$

(viii) $\mathcal{H}(y,t) = 0$ *for* $t > 0$ *iff* $x = 0$

(ix) $\mathcal{H}(\alpha y, t) = \mathcal{H}(y, \frac{t}{|\alpha|})$,

(x) $\mathcal{H}(y,t) \diamond \mathcal{H}(z,t) \geq \mathcal{H}(y+z, t+s)$,

(xi) $\mathcal{H}(y, \diamond)$ *is continuous non-increasing function,*

(xii) $\lim_{t \to \infty} \mathcal{H}(x,t) = 0$,

(xiii) $\mathcal{J}(x,t) = 0$ *for* $t > 0$ *iff* $x = 0$

(xiv) $\mathcal{J}(\alpha x, t) = \mathcal{J}(x, \frac{t}{|\alpha|})$,

(xv) $\mathcal{J}(z,t) \diamond \mathcal{J}(x,s) \geq \mathcal{J}(z+x, t+s)$,

(xvi) $\mathcal{J}(z,.)$ *is continuous non-increasing function,*

(xvii) $\lim_{t\to\infty} \mathcal{J}(z,t) = 0$,

(xviii) *If* $t \leq 0$, *then* $\mathcal{T}(x,t) = 0$, $\mathcal{H}(y,t) = 1$, $\mathcal{J}(z,t) = 1$.
In such case, $\mathcal{M} = (\mathcal{T}, \mathcal{H}, \mathcal{J})$ *is said to be neutrosophic normed* (NN).

Example 2.1. [21] *Suppose* $(X, \| \,.\, \|)$ *be a NNS. Give the operations as* $x \star y = x + y - xy$ *and* $x \diamond y = \min(x,y)$. *For* $t > ||y||$,

$$\mathcal{T}(y,t) = \frac{t}{t + ||y||},\ \mathcal{H}(y,t) = \frac{y}{t + ||y||},\ \mathcal{J}(y,t) = \frac{||y||}{t} \tag{8}$$

for all $x, y \in \mathcal{M}$ *and* $t > 0$. *If we take* $t \leq ||y||$, *then*

$$\mathcal{T}(y,t) = 0, \mathcal{H}(y,t) = 1 \text{ and } \mathcal{J}(y,t) = 1.$$

Hence, $(X, \mathcal{M}, \star, \diamond)$ *is Neutrosophic normed space such that* $\mathcal{M} : X \times R^+ \to [0,1]$.

Definition 2.9. [21] *Let* $(X, \mathcal{M}, \star, \diamond)$ *be a Neutrosophic normed space. A sequence* $x = (x_j)$ *is said to be convergent to* ℓ *with respect to* $\mathcal{M}$, *if for every* $0 < \epsilon < 1$ *and* $t > 0$, *there exists* $J \in \mathbb{N}$ *such that* $\mathcal{T}(x_j - \ell, t) > 1 - \epsilon$, $\mathcal{H}(x_j - \ell, t) < \epsilon$ *and* $\mathcal{J}(x_j - \ell, t) < \epsilon$ *That is, for all* $t > 0$, *we have*

$$\lim_{j\to\infty} \mathcal{T}(x_j - \ell, t) = 1, \lim_{j\to\infty} \mathcal{H}(x_j - \ell, t) = 0 \text{ and } \lim_{j\to\infty} \mathcal{J}(x_j - \ell, t) = 0. \tag{9}$$

The convergence in $(X, \mathcal{M}, \star, \diamond)$ *is denoted by* $\mathcal{M} - \lim x_j = \ell$.

Definition 2.10. [21] *Suppose* $(X, \mathcal{M}, \star, \diamond)$ *be a Neutrosophic normed space. A sequence* $x = (x_j)$ *is said to be Cauchy sequence with respect to* $\mathcal{M}$, *if for every* $0 < \epsilon < 1$ *and* $t > 0$, *there exists* $J \in \mathbb{N}$ *such that* $\mathcal{T}(x_j - y_k, t) > 1 - \epsilon$, $\mathcal{H}(x_j - y_k, t) < \epsilon$ *and* $\mathcal{J}(x_j - y_k, t) < \epsilon$ *for all* $j, k \in J$

3. Main Results

Definition 3.1. *Let* $(X, \mathcal{M}, \star, \diamond)$ *be a Neutrosophic normed space. A sequence* $x = (x_j)$ *is said to be Fibonacci statistical* $(FS)-$ *convergent to* $\ell \in X$ *with respect* $\mathcal{M}$ *if for every* $\epsilon, t > 0$

$$\delta(\{j \in \mathbb{N} : \mathcal{T}(\hat{F}x_j - \ell, t) \leq 1 - \epsilon,\ \mathcal{H}(\hat{F}x_j - \ell, t) \geq \epsilon \ \text{ and } \mathcal{J}(\hat{F}x_j - \ell, t) \geq \epsilon\}) = 0. \tag{10}$$

or equivalently

$$\lim_n \frac{1}{n} |\{j \leq n : \mathcal{T}(\hat{F}x_j - \ell, t) \leq 1 - \epsilon, \mathcal{H}(\hat{F}x_j - \ell, t) \geq \epsilon \text{ and } \mathcal{J}(\hat{F}x_j - \ell, t) \geq \epsilon\}| = 0. \tag{11}$$

In this case we write $\delta(\hat{F})_{NN} - \lim x_j = \ell$. *The set of all Fibonacci statistical convergent sequences, denoted by* $FSC - NN$ *will be denoted by* $\mathcal{M}(\hat{F})_{NN}$. *In case* $\ell = 0$, *we will write* $\mathcal{M}_0(\hat{F})_{NN}$

Lemma 3.1. *Let $(X, \mathcal{M}, \star, \diamond)$ be a Neutrosophic normed space. Then, for every $\epsilon, t > 0$. the subsequent statements are equivalent:*

(1) $\delta(\hat{F})_{NN} - \lim x_j = \ell$

(2) $\delta(\{j \in \mathbb{N} : \mathcal{T}(\hat{F}x_j - \ell, t) \leq 1 - \epsilon = \mathcal{H}(\hat{F}x_j - \ell, t) \geq \epsilon = \mathcal{J}(\hat{F}x_j - \ell, t) \geq \epsilon\}) = 0$

(3) $\delta(\{j \in \mathbb{N} : \mathcal{T}(\hat{F}x_j - \ell, t) > 1 - \epsilon, \mathcal{H}(\hat{F}x_j - \ell, t) < \epsilon$ *and* $\mathcal{J}(\hat{F}x_j - \ell, t) < \epsilon\}) = 1.$

(4) $\delta(\{j \in \mathbb{N} : \mathcal{T}(\hat{F}x_j - \ell, t) > 1 - \epsilon = \delta(\{\mathcal{H}(\hat{F}x_j - \ell, t) < \epsilon = \delta(\{\mathcal{J}(\hat{F}x_j - \ell, t) < \epsilon\}) = 1$

(5) $\mathcal{M} - \lim \mathcal{T}(\hat{F}x_j - \ell, t) = 1$, $\mathcal{M} - \lim \mathcal{H}(\hat{F}x_j - \ell, t) = 0$ *and* $\mathcal{M} - \lim \mathcal{J}(\hat{F}x_j - \ell, t) = 0$.

Theorem 3.1. *Let $(X, \mathcal{M}, \star, \diamond)$ be a Neutrosophic normed space. If a sequence $x = (x_j)$ is Fibonacci statistically convergent with respect to the norm $\mathcal{M}$ then the $\delta(\hat{F})_{NN}-$ limit is unique.*

Proof. Let $\delta(\hat{F})_{NN} - \lim x_j = \ell_1$, $\delta(\hat{F})_{NN} - \lim x_j = \ell_2$ and $\ell_1 \neq \ell_2$. Given $\epsilon, s > 0$ such that $s \diamond s < \epsilon$ and $(1 - s) \star (1 - s) > 1 - \epsilon$. Then, for all $t > 0$;

$$K_{\mathcal{T},1}(s,t) = \{j \in \mathbb{N} : \mathcal{T}(\hat{F}x_j - \ell_1, \frac{t}{2}) \leq 1 - s\}$$

$$K_{\mathcal{T},2}(s,t) = \{j \in \mathbb{N} : \mathcal{T}(\hat{F}x_j - \ell_2, \frac{t}{2}) \leq 1 - s\}$$

$$K_{\mathcal{H},1}(s,t) = \{j \in \mathbb{N} : \mathcal{H}(\hat{F}x_j - \ell_1, \frac{t}{2}) \geq s\}$$

$$K_{\mathcal{H},2}(s,t) = \{j \in \mathbb{N} : \mathcal{H}(\hat{F}x_j - \ell_2, \frac{t}{2}) \geq s\}$$

$$K_{\mathcal{J},1}(s,t) = \{j \in \mathbb{N} : \mathcal{J}(\hat{F}x_j - \ell_1, \frac{t}{2}) \geq s\}$$

$$K_{\mathcal{J},2}(s,t) = \{j \in \mathbb{N} : \mathcal{J}(\hat{F}x_j - \ell_2, \frac{t}{2}) \geq s\}$$

Since $\delta(\hat{F})_{NN} - \lim x_j = \ell_1$. Then

$$\delta(K_{\mathcal{T},1}(\epsilon,t)) = \delta(K_{\mathcal{H},1}(\epsilon,t)) = \delta(K_{\mathcal{J},1}(\epsilon,t)) = 0.$$

Also, using $\delta(\hat{F})_{NN} - \lim x_j = \ell_2$, one get

$$\delta(K_{\mathcal{T},2}(\epsilon,t)) = \delta(K_{\mathcal{H},2}(\epsilon,t)) = \delta(K_{\mathcal{J},2}(\epsilon,t)) = 0.$$

Let us denote

$$K_{(\mathcal{M},t)} = [K_{\mathcal{T},1}(s,t) \cup K_{\mathcal{T},2}(s,t)] \cap [K_{\mathcal{H},1}(s,t) \cup K_{\mathcal{H},2}(s,t)] \cap [K_{\mathcal{J},1}(s,t) \cup K_{\mathcal{J},2}(s,t)].$$

It can be easily seen that $\delta(K_{(\mathcal{M},t)}) = 0$ which implies that $\delta(\mathbb{N} \backslash K_{(\mathcal{M},t)}) = 1$

If $j \in \mathbb{N} \backslash K_{(\mathcal{M},t)}$, then we have three possible cases:

(a) $(\{j \in \mathbb{N} \backslash K_{\mathcal{T},1}(\epsilon,t) \cup K_{\mathcal{T},2}(\epsilon,t)\})$

(b) $(\{j \in \mathbb{N} \backslash K_{\mathcal{H},1}(\epsilon,t) \cup K_{\mathcal{H},2}(\epsilon,t)\})$

(c) $(\{j \in \mathbb{N} \backslash K_{\mathcal{J},1}(\epsilon,t) \cup K_{\mathcal{J},2}(\epsilon,t)\})$

Now, consider (a), one has

$$\begin{aligned}\mathcal{T}(\ell_1-\ell_2,t) &\geq \mathcal{T}(\hat{F}x_j-\ell_1,\frac{t}{2}) \star \mathcal{T}(\hat{F}x_j-\ell_2,\frac{t}{2})\\ &> (1-s)\star(1-s)\\ &> 1-\epsilon\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we get $\mathcal{T}(\ell_1-\ell_2,t)=1$ for all $t>0$, which yields $\ell_1=\ell_2$.

Consider (b), if $j \in \mathbb{N}\backslash K_{\mathcal{H},1}(\epsilon,t)\cup K_{\mathcal{H},2}(\epsilon,t)$. Then, one write

$$\begin{aligned}\mathcal{H}(\ell_1-\ell_2,t) &\leq \mathcal{H}(\hat{F}x_j-\ell_1,\frac{t}{2}) \diamond \mathcal{H}(\hat{F}x_j-\ell_2,\frac{t}{2})\\ &< s\diamond s\\ &< \epsilon\end{aligned}$$

So, we have $\mathcal{H}(\ell_1-\ell_2,t)=0$ for all $t>0$, which implies $\ell_1=\ell_2$.

and consider (c) if $j \in \mathbb{N}\backslash K_{\mathcal{J},1}(\epsilon,t)\cup K_{\mathcal{J},2}(\epsilon,t)$. Then

$$\begin{aligned}\mathcal{J}(\ell_1-\ell_2,t) &\leq \mathcal{J}(\hat{F}x_j-\ell_1,\frac{t}{2}) \diamond \mathcal{J}(\hat{F}x_j-\ell_2,\frac{t}{2})\\ &< s\diamond s\\ &< \epsilon\end{aligned} \tag{12}$$

It follows that $\mathcal{J}(\ell_1-\ell_2,t)=0$ for all $t>0$, which implies $\ell_1=\ell_2$.
This completes the proof of the theorem. □

Theorem 3.2. *Let $(X,\mathcal{M},\star,\diamond)$ be a Neutrosophic normed space. If $\mathcal{M}-\lim x_j=\ell$ then $\delta(\hat{F})_{NN}-\lim x_j=\ell$ but the converse need not be true.*

Proof. Let $\mathcal{M}-\lim x_j=\ell$. Then for each $\epsilon>0$ and $t>0$, there is a number $J\in\mathbb{N}$ so that $\mathcal{T}(\hat{F}x_j-\ell,t)>1-\epsilon$ or $\mathcal{H}(\hat{F}x_j-\ell,t)<\epsilon$, $\mathcal{J}(\hat{F}x_j-\ell,t)<\epsilon$ for all $j\geq J$.
Hence the set

$$\Big\{j\in\mathbb{N}:\mathcal{T}(\hat{F}x_j-\ell,t)\leq 1-\epsilon \text{ or } \mathcal{H}(\hat{F}x_j-\ell,t)\geq\epsilon, \mathcal{J}(\hat{F}x_j-\ell,t)\geq\epsilon\Big\}$$

has at most a finite number of terms. Since every finite subset of $\mathbb{N}$ has natural density zero. it follows that,
$\delta(\{j\in\mathbb{N}:\mathcal{T}(\hat{F}x_j-\ell,t)\leq 1-\epsilon \text{ or } \mathcal{H}(\hat{F}x_j-\ell,t)\geq\epsilon, \mathcal{J}(\hat{F}x_j-\ell,t)\geq\epsilon\})=0$.
That is, $\delta(\hat{F})_{NN}-\lim x_j=\ell$.
For converse, we construct the following example: □

Example 3.1. *Let $(X,\|\,.\,\|)$ be a normed space and let $a\star b=ab$ and $a\diamond b=\min\{a+b,1\}$ for all $a,b\in[0,1]$. For all $x\in X$ and every $t>0$, consider*

$$\mathcal{T}(x,t)=\frac{t}{t+||x||},\ \mathcal{H}(y,t)=\frac{y}{t+||y||},\ \mathcal{J}(x,t)=\frac{||x||}{t} \tag{13}$$

Then $(X, \mathcal{M}, \star, \diamond)$ is a Neutrosophic normed space.

Now, we determine the $\hat{F}x_j = (f_{j+1}^2) = (1, 2^2, 3^2, 5^2 \dots)$. Since $f_{j+1}^2 \to \infty$ as $j \to \infty$ and $\hat{F}x = (1, 0, 0, \dots)$, therefore $\hat{F}x \in \mathcal{M}$. For $\epsilon \in (0,1)$ and for any $t > 0$, consider

$$K_n(\epsilon, t) = \Big\{ j \leq n : \ \mathcal{T}(\hat{F}x_j - \ell, t) \leq 1 - \epsilon \text{ or } \mathcal{H}(\hat{F}x_j - \ell, t) \geq \epsilon \\ \text{ or } \mathcal{J}(\hat{F}x_j - \ell, t) \geq \epsilon \Big\}.$$

When n becomes sufficiently large, the quantity $\mathcal{T}(\hat{F}x_j - \ell, t)$ becomes less than $1 - \epsilon$, $\mathcal{H}(\hat{F}x_j - \ell, t)$ and $\mathcal{J}(\hat{F}x_j - \ell, t)$ become greater than ϵ. Therefore, for $\epsilon > 0$ and $t > 0$, $K_\epsilon(\hat{F}) = 0$.

Theorem 3.3. *Let $(X, \mathcal{M}, \star, \diamond)$ be a Neutrosophic normed space and $\delta(\hat{F})_{NN} - \lim x_j = \ell$ if and only if there exists a subset $K = \{k_1 < k_2 < k_3 < \dots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $\mathcal{M} - \lim_{n\to\infty} x_{k_n} = \ell$.*

Proof. Suppose that $\delta(\hat{F})_{NN} - \lim x_j = \ell$. For $\alpha = 1, 2, \dots$ and for each $t > 0$,

$$S_{\mathcal{M}}(\alpha, t) = \Big\{ j \in \mathbb{N} : \ \mathcal{T}(\hat{F}x_j - \ell, t) > 1 - \tfrac{1}{\alpha} \text{ and } \mathcal{H}(\hat{F}x_j - \ell, t) < \tfrac{1}{\alpha}, \\ \mathcal{J}(\hat{F}x_j - \ell, t) < \tfrac{1}{\alpha} \Big\}.$$

$$R_{\mathcal{M}}(\alpha, t) = \Big\{ j \in \mathbb{N} : \ \mathcal{T}(\hat{F}x_j - \ell, t) \leq 1 - \tfrac{1}{\alpha} \text{ or } \mathcal{H}(\hat{F}x_j - \ell, t) \geq \tfrac{1}{\alpha}, \\ \mathcal{J}(\hat{F}x_j - \ell, t) \geq \tfrac{1}{\alpha} \Big\}$$

Therefore, $\delta(R_{\mathcal{M}}(s, t)) = 0$. Since $\delta(\hat{F})_{NN} - \lim x_j = \ell$. Additionally, for $\alpha = 1, 2 \dots$ and for all $t > 0$,

$S_{\mathcal{M}}(\alpha + 1, t) \subset S_{\mathcal{M}}(\alpha, t)$ and therefore

$$\delta(S_{\mathcal{M}}(\alpha, t)) = 1. \tag{14}$$

Now, we have to prove that $j \in S_{\mathcal{M}}(\alpha, t)$, $\mathcal{M} - \lim x_j = \ell$. Suppose that $\mathcal{M} - \lim x_j \neq \ell$ for some $j \in S_{\mathcal{M}}(\alpha, t)$. Therefore there exists $\beta > 0$ and a positive integer J such that

$\mathcal{T}(\hat{F}x_j - \ell, t) \leq 1 - \beta$ or $\mathcal{H}(\hat{F}x_j - \ell, t) \geq \beta$, $\mathcal{J}(\hat{F}x_j - \ell, t) \geq \beta$, for all $j \geq J$. Let $\mathcal{T}(\hat{F}x_j - \ell, t) > 1 - \beta$, $\mathcal{H}(\hat{F}x_j - \ell, t) < \beta$ and $\mathcal{J}(\hat{F}x_j - \ell, t) < \beta$, for all $j < J$. Then

$$\delta\left(\Big\{\mathcal{T}(\hat{F}x_j - \ell, t) > 1 - \beta, \mathcal{H}(\hat{F}x_j - \ell, t) < \beta \text{ and } \mathcal{J}(\hat{F}x_j - \ell, t) < \beta\Big\}\right) = 0$$

Since $\beta > \frac{1}{\alpha}$, we have $\delta(M_{\mathcal{M}}(\alpha, t)) = 0$. which is a contradiction of equation (14). Therefore $\mathcal{M} - \lim x_j = \ell$.

Conversely, suppose that there exists a subset $K = \{k_1 < k_2 < k_3 < \dots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $\mathcal{M} - \lim_{n\to\infty} y_{j_n} = \ell$.Then for every $\beta, t > 0$, there exists $J \in \mathbb{N}$ such

that $\mathcal{T}(\hat{F}x_j - \ell, t) > 1 - \beta$, $\mathcal{H}(\hat{F}x_j - \ell, t) < \beta$ and $\mathcal{J}(\hat{F}x_j - \ell, t) < \beta$.
Now
$K_{\mathcal{M}}(\beta, t) = \{j \in \mathbb{N} : \mathcal{T}(\hat{F}x_j - \ell, t) \leq 1 - \beta$ or $\mathcal{H}(\hat{F}x_j - \ell, t) \geq \beta, \mathcal{J}(\hat{F}x_j - \ell, t) \geq \beta\} \subseteq \mathbb{N} - \{j_{K+1} < j_{K+2} < j_{K+3} < \dots\}$.
Therefore $\delta(K_{\mathcal{M}}(\beta, t)) \leq 1 - 1 = 0$. Hence $\delta(\hat{F})_{NN} - \lim x_j = \ell$. This completes the proof of the theorem. □

4. **Fibonacci statistically complete NNS**

Definition 4.1. *Let $(X, \mathcal{M}, \star, \diamond)$ be a Neutrosophic normed space. A sequence $x = (x_j)$ is said to be Fibonacci statistically Cauchy with respect to norm $\mathcal{M}$ if for every $\epsilon, t > 0$, there exists $l = l(\epsilon)$ such that*

$$\delta\left(\left\{j \in \mathbb{N} : \mathcal{T}(\hat{F}x_j - \hat{F}y_l, t) \leq 1 - \epsilon \text{ or } \mathcal{H}(\hat{F}x_j - \hat{F}y_l, t) \geq \epsilon, \mathcal{J}(\hat{F}x_j - \hat{F}y_l, t) \geq \epsilon\right\}\right) = 0.$$

Theorem 4.1. *Every Fibonacci statistically convergent sequence in $(X, \mathcal{M}, \star, \diamond)$ is Fibonacci Cauchy.*

Proof. Assume that the sequence $x = (x_j)$ be Fibonacci statistically convergent to ℓ with respect to the norm $\mathcal{M}$, i.e., $\delta(\hat{F})_{NN} - \lim x_j = \ell$. Given $\epsilon > 0$ select $\gamma > 0$ in such way that $(1 - \epsilon) \star (1 - \epsilon) > 1 - \gamma$ and $\epsilon \diamond \epsilon < \gamma$. Therefore, for all $t > 0$, one obtain

$$\delta(G(\epsilon, t)) = \delta(\{j \in \mathbb{N} : \mathcal{T}(\hat{F}x_j - \ell, \frac{t}{2}) \leq 1 - \epsilon \text{ or } \mathcal{H}(\hat{F}x_j - \ell, \frac{t}{2}) \geq \epsilon, \mathcal{J}(\hat{F}x_j - \ell, \frac{t}{2}) \geq \epsilon\}) = 0 \tag{15}$$

which implies
$\delta(G^c(\epsilon, t)) = \delta(\{j \in \mathbb{N} : \mathcal{T}(\hat{F}x_j - \ell, \frac{t}{2}) > 1 - \epsilon$ and $\mathcal{H}(\hat{F}x_j - \ell, \frac{t}{2}) < \epsilon, \mathcal{J}(\hat{F}x_j - \ell, \frac{t}{2}) < \epsilon\}) = 1$.
Suppose that $k \in G^c(\epsilon, t)$. Then
$\mathcal{T}(\hat{F}(y_k) - \ell, t) > 1 - \epsilon$, $\mathcal{H}(\hat{F}(y_k) - \ell, t) < \epsilon$ and $\mathcal{J}(\hat{F}(y_k) - \ell, t) < \epsilon$.
Now, suppose that,

$$H(\gamma, t) = \left\{j \in \mathbb{N} : \begin{array}{l} \mathcal{T}(\hat{F}x_j - \hat{F}(y_k), t) \leq 1 - \gamma \text{ or } \mathcal{H}(\hat{F}x_j - \hat{F}(y_k), t) \geq \gamma, \\ \mathcal{J}(\hat{F}x_j - \hat{F}(y_k), t) \geq \gamma \end{array}\right\}.$$

We need to prove that $H(\gamma, t) \subset G(\epsilon, t)$. Assume $j \in H(\gamma, t) \backslash G(\epsilon, t)$. Then

$$\mathcal{T}(\hat{F}x_j - \hat{F}(y_k), t) \leq 1 - \gamma \text{ and } \mathcal{T}(\hat{F}(y_k) - \ell, t) > 1 - \epsilon,$$

In particular $\mathcal{T}(\hat{F}(y_k) - \ell, \frac{t}{2}) > 1 - \epsilon$. Then
$1 - \gamma \geq \mathcal{T}(\hat{F}x_j - \hat{F}(y_k), t) \geq \mathcal{T}(\hat{F}x_j - \ell, \frac{t}{2}) \star \mathcal{T}(\hat{F}(y_k) - \ell, \frac{t}{2}) > (1 - \epsilon) \star (1 - \epsilon) > 1 - \gamma$,

which is impossible. At the same time,

$$\mathcal{H}(\hat{F}x_j - \hat{F}(y_k), t) \geq \gamma \text{ and } \mathcal{H}(\hat{F}x_j - \ell, \frac{t}{2}) < \epsilon,$$

In particular $\mathcal{H}(\hat{F}x_j - \ell, \frac{t}{2}) < \epsilon$. Then,
$\gamma \leq \mathcal{H}(\hat{F}x_j - \hat{F}(y_k), t) \leq \mathcal{H}(\hat{F}x_j - \ell, \frac{t}{2}) \star \mathcal{H}(\hat{F}(y_k) - \ell, \frac{t}{2}) < \epsilon \star \epsilon < \gamma$,

which is impossible.

In a similar way,
$\mathcal{J}(\hat{F}x_j - \hat{F}(y_k), t) \geq \gamma$ and $\mathcal{J}(\hat{F}x_j - \ell, \frac{t}{2}) < \epsilon$,
In particular $\mathcal{J}(\hat{F}x_j - \ell, \frac{t}{2}) < \epsilon$. Then
$\gamma \leq \mathcal{J}(\hat{F}x_j - \hat{F}(y_k), t) \leq \mathcal{J}(\hat{F}x_j - \ell, \frac{t}{2}) \star \mathcal{J}(\hat{F}(y_k) - \ell, \frac{t}{2}) < \epsilon \star \epsilon < \gamma$,
which is impossible. Therefore $H(\gamma, t) \subset G(\epsilon, t)$. Hence, by equation (15), $\delta(H(\gamma, t)) = 0$. Then $x = (x_j)$ is a Fibonacci statistically Cauchy sequence with respect to norm $\mathcal{M}$.

□

Definition 4.2. *A neutrosophic normed space $(X, \mathcal{M}, \star, \diamond)$ is known as Fibonacci Statistically complete, if every Cauchy sequence with respect to the norm $\mathcal{M}$ is Fibonacci Statistically convergent with respect to the same norm.*

Theorem 4.2. *A neutrosophic normed space $(X, \mathcal{M}, \star, \diamond)$ is Fibonacci Statistically complete.*

Proof. Suppose $x = (x_j)$ be Fibonacci Statistically Cauchy sequence but not Fibonacci Statistically convergent with respect to norm $\mathcal{M}$. Given $\epsilon > 0$ and $t > 0$, select $s > 0$ in such a manner that $\epsilon \diamond \epsilon < s$ and $(1 - \epsilon) \star (1 - \epsilon) > 1 - s$. Now

$$\begin{aligned}\mathcal{T}(\hat{F}x_j - \hat{F}(x_l), t) &\geq \mathcal{T}(\hat{F}x_j - \ell, \frac{t}{2}) \star \mathcal{T}(\hat{F}(x_l) - \ell, \frac{t}{2})\\ &> (1 - \epsilon) \star (1 - \epsilon) \star (1 - \epsilon)\\ &> 1 - s\end{aligned}$$

and

$$\begin{aligned}\mathcal{H}(\hat{F}x_j - \hat{F}(x_l), t) &\leq \mathcal{H}(\hat{F}x_j - \ell, \frac{t}{2}) \star \mathcal{H}(\hat{F}(x_l) - \ell, \frac{t}{2})\\ &< \epsilon \star \epsilon\\ &< s\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}(\hat{F}x_j - \hat{F}(x_l), t) &\leq \mathcal{J}(\hat{F}x_j - \ell, \frac{t}{2}) \star \mathcal{J}(\hat{F}(x_l) - \ell, \frac{t}{2})\\ &< \epsilon \star \epsilon\\ &< s\end{aligned}$$

Since sequence $x = (x_j)$ is not Fibonacci statistically convergent. Then $\delta(Q^c(\epsilon, t)) = 0$, where

$Q(\epsilon, t) = \{j \leq l : \mathcal{H}_{(\hat{F}x_j - \hat{F}(x_l))}(\epsilon) \leq s\}$, therefore $\delta(Q(\epsilon, t)) = 1$,
which contradicts our assumption. Therefore, $x = (x_j)$ must be Fibonacci statistically convergent. □

5. **Conclusions**

In this study, we have studied the concept of statistical convergence using Fibonacci matrix which has an important place in the literature. The statistical convergence is a generalization of the usual convergence. We have defined the Fibonacci type statistical convergence and investigated basic properties. These are illustrated by suitable examples. Their related properties and structural characteristics have been discussed.

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