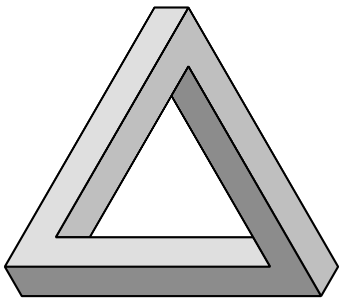
2018 International Conference on Topology and its Applications, July 7-11, 2018, Nafpaktos, Greece

Selected papers of the 2018 International Conference on Topology and its Applications



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Preface

The **2018 International Conference on Topology and its Applications** took place from July 7 to 11 in the 3rd High School of Nafpaktos, Nafpaktos, Greece. It covered all areas of Topology and its Applications (especially General Topology, Set-Theoretic Topology, Geometric Topology, Topological Groups, Dimension Theory, Dynamical Systems and Continua Theory, Computational Topology, History of Topology). This conference was attended by 241 participants from 46 countries and the program consisted by 167 talks.

The Organizing Committee consisted of S.D. Iliadis (Moscow State University (M.V. Lomonosov)), D.N. Georgiou (University of Patras), I.E. Kougias (Technological Educational Institute of Western Greece), A.C. Megaritis (Technological Educational Institute of Peloponnese), P. Loukopoulos (Mayor of the Municipality of Nafpaktia).

The Organizing Committee is very much indebted to the City of Nafpaktos for its hospitality and for its excellent support during the conference. The conference was sponsored by The Ministry of Education and Religious Affairs of Greece, University of Patras, Technological Educational Institute of Western Greece, Municipality of Nafpaktia, Moscow State University (M.V. Lomonosov), New Media Soft - Internet Solutions, Loux Marlafekas A.B.E.E., TAXYTYPO - TAXYEKTYPOSEIS GRAVANIS EPE., Agricultural Cooperatives Union Aeghion SA and Kintonis Winery (Aeghion).

This volume is a special volume under the title: "Selected papers of the 2018 International Conference on Topology and its Applications" which will be edited by the organizers (D.N. Georgiou, S.D. Iliadis, I.E. Kougias, and A.C. Megaritis) and published by the University of Patras. We thank the authors for their submissions and the referees for their help.

Editors

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On $\mathcal{I}$-open sets and $\mathcal{I}$-continuous functions in ideal Bitopological spaces

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Abstract

The aim of this paper is to introduce and characterize the concepts of $\mathcal{I}$-open sets and their related notions in ideal bitopological spaces.

Key words: Ideal bitopological spaces, (i, j)-$\mathcal{I}$-open sets, (i, j)-$\mathcal{I}$-closed sets.
1991 MSC: 54D10.

1. Introduction and Preliminaries

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [19] and Vaidyanathasamy [24]. Hamlett and Janković (see [12, 13, 17, 18]) used topological ideals to generalize many notions and properties in general topology. The research in this direction continued by many researchers such as M. E. Abd El-Monsef, A. Al-Omari, F. G. Arenas, M. Caldas, J. Dontchev, M. Ganster, D. N. Georgiou, T. R. Hamlett, E. Hatir, S. D. Iliadis, S. Jafari, D. Jankovic, E. F. Lashien, M. Maheswari, , H. Maki, A. C. Megaritis, F. I. Michael, A. A. Nasef, T. Noiri, B. K. Papadopoulos, M. Parimala, G. A. Prinos, M. L. Puertas, M. Rajamani, N. Rajesh, D. Rose, A. Selvakumar,

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Jun-Iti Umehara and many others (see [1, 2, 5, 7-11, 14, 15, 18, 21-23]). An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^*$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [24] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^*(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau | x \in U\}$. If $\mathcal{I}$ is an ideal on X, then $(X, \tau_1, \tau_2, \mathcal{I})$ is called an ideal bitopological space. Let A be a subset of a bitopological space (X, τ_1, τ_2). We denote the closure of A and the interior of A with respect to τ_i by τ_i-Cl(A) and τ_i-Int(A), respectively. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i,j)-preopen [16] if $A \subset \tau_i$-Int$(\tau_j$-Cl$(A))$, where $i, j = 1, 2$ and $i \neq j$. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be (i,j)-pre-$\mathcal{I}$-open [4] if $S \subset \tau_i$-Int$(\tau_j$-Cl$^*(S))$. A subset A of a bitopological space (X, τ_1, τ_2) is said to be (i,j)-preopen [16] (resp. (i,j)-semi-$\mathcal{I}$-open [3]) if $A \subset \tau_i$-Int$(\tau_j$-Cl$(A))$ (resp. $S \subset \tau_j$-Cl$^*(\tau_i$-Int$(S)))$, where $i, j = 1, 2$ and $i \neq j$. The complement of an (i,j)-semi-$\mathcal{I}$-open set is called an (i,j)-semi-$\mathcal{I}$-closed set. A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be (i,j)-pre-$\mathcal{I}$-continuous [4] if the inverse image of every σ_i-open set in (Y, σ_1, σ_2) is (i,j)-pre-$\mathcal{I}$-open in $(X, \tau_1, \tau_2, \mathcal{I})$, where $i \neq j$, i, j=1, 2.

2. (i,j)-$\mathcal{I}$-open sets

Definition 2.1. A subset A of an ideal bitopological space $(X, \tau_i, \tau_2, \mathcal{I})$ is said to be (i,j)-$\mathcal{I}$-open if $A \subset \tau_i$-Int(A_j^*).
The family of all (i,j)-$\mathcal{I}$-open subsets of $(X, \tau_i, \tau_2, \mathcal{I})$ is denoted by (i,j)-$\mathcal{I}O(X)$.

Remark 1. It is clear that $(1,2)$-$\mathcal{I}$-openness and τ_1-openness are independent notions.

Example 1. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then τ_1-Int$(\{a, b\}_2^*) = \tau_1$-Int$(\{b\}) = \emptyset \supsetneq \{a, b\}$. Therefore $\{a, b\}$ is a τ_1-open set but not $(1,2)$-$\mathcal{I}$-open.

Example 2. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then τ_1-Int$(\{a\}_2^*) = \tau_1$-Int$(X) = X \supset \{a\}$. Therefore, $\{a\}$ is $(1,2)$-$\mathcal{I}$-open set but not τ_1-open.

Remark 2. Similarly $(1,2)$-$\mathcal{I}$-openness and τ_2-openness are independent notions.

Example 3. Let $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, \{c\},$

$\{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then τ_1-Int$(\{b,c\}_2^*) = \tau_1$-Int$(\{a,b\}) = \{a\} \supsetneq \{b,c\}$. Therefore, $\{b,c\}$ is a τ_2-open set but not $(1,2)$-$\mathcal{I}$-open.

Example 4. Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a,c\}\}$, $\tau_2 = \{\emptyset, \{b\}, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then τ_1-Int$(\{a\}_2^*) = \tau_1$-Int$(\{a\}) = \{a\} \supset \{a\}$. Therefore, $\{a\}$ is an $(1,2)$-$\mathcal{I}$-open set but not τ_2-open.

Proposition 2.2. Every (i,j)-$\mathcal{I}$-open set is (i,j)-pre-$\mathcal{I}$-open.

Proof. Let A be an (i,j)-$\mathcal{I}$-open set. Then $A \subset \tau_i$-Int$(A_j^*) \subset \tau_i$-Int$(A \cup A_j^*) = \tau_i$-Int$(\tau_j$-Cl$^*(A))$. Therefore, $A \in (i,j)$-$P\mathcal{I}O(X)$. ■

Example 5. Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}$, $\tau_2 = \{\emptyset, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the set $\{c\}$ is $(1,2)$-preopen but not $(1,2)$-$\mathcal{I}$-open.

Remark 3. The intersection of two (i,j)-$\mathcal{I}$-open sets need not be (i,j)-$\mathcal{I}$-open as shown in the following example.

Example 6. Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $\{a,b\}$, $\{a,c\} \in (1,2)$-$\mathcal{I}O(X)$ but $\{a,b\} \cap \{a,c\} = \{a\} \notin (1,2)$-$\mathcal{I}O(X)$.

Theorem 2.3. For an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ and $A \subset X$, we have:

(1) If $\mathcal{I} = \{\emptyset\}$, then $A_j^*(\mathcal{I}) = \tau_j$-Cl$(A)$ and hence each of (i,j)-$\mathcal{I}$-open set and (i,j)-preopen set are coincide.
(2) If $\mathcal{I} = \mathcal{P}(X)$, then $A_j^*(\mathcal{I}) = \emptyset$ and hence A is (i,j)-$\mathcal{I}$-open if and only if $A = \emptyset$.

Theorem 2.4. For any (i,j)-$\mathcal{I}$-open set A of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$, we have $A_j^* = (\tau_i$-Int$(A_j^*))_j^*$.

Proof. Since A is (i,j)-$\mathcal{I}$-open, $A \subset \tau_i$-Int(A_j^*). Then $A_j^* \subset (\tau_i$-Int$(A_j^*))_j^*$. Also we have τ_i-Int$(A_j^*) \subset A_j^*$, $(\tau_i$-Int$(A_j^*))^* \subset (A_j^*)^* \subset A_j^*$. Hence we have, $A_j^* = (\tau_i$-Int$(A_j^*))_j^*$. ■

Definition 2.5. A subset F of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is called (i,j)-$\mathcal{I}$-closed if its complement is (i,j)-$\mathcal{I}$-open.

Theorem 2.6. For $A \subset (X, \tau_1, \tau_2, \mathcal{I})$ we have $((\tau_i$-Int$(A))_j^*)^c \neq \tau_i$-Int$((A^c)_j^*)$ in general.

Example 7. Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a,b\}, X\}$, $\tau_2 = \{\emptyset, \{a,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then $((\tau_1$-Int$(\{a,b\}))_2^*)^c = (\{a,b\}_2^*)^c = X^c = \emptyset$ $(*)$ and τ_1-Int$((\{a,b\}^c)_2^*) = \tau_1$-Int$(\{c\}_2^*) = \tau_1$-Int$(X) = X$ $(**)$. Hence from $(*)$ and $(**)$, we get $((\tau_1$-Int$(\{a,b\}))_2^*)^c \neq \tau_1$-Int$((\{a,b\}^c)_2^*)$.

Theorem 2.7. If $A \subset (X, \tau_1, \tau_2, \mathcal{I})$ is (i,j)-$\mathcal{I}$-closed, then $A \supset (\tau_i$-Int$(A))_j^*$.

Proof. Let A be (i,j)-$\mathcal{I}$-closed. Then $B = A^c$ is (i,j)-$\mathcal{I}$-open. Thus, $B \subset \tau_i$-Int(B_j^*), $B \subset \tau_i$-Int$(\tau_j$-Cl$(B))$, $B^c \supset \tau_j$-Cl$(\tau_i$-Int$(B^c))$, $A \supset \tau_j$-Cl$(\tau_i$-Int$(A))$. That is, τ_j-Cl$(\tau_i$-Int$(A)) \subset A$, which implies that $(\tau_i$-Int$(A))_j^* \subset \tau_j$-Cl$(\tau_i$-Int$(A)) \subset A$. Therefore, $A \supset (\tau_i$-Int$(A))_j^*$. ■

Theorem 2.8. Let $A \subset (X, \tau_1, \tau_2, \mathcal{I})$ and $(X \backslash (\tau_i\text{-Int}(A))_j^*) = \tau_i\text{-Int}((X \backslash A)_j^*)$. Then A is (i,j)-$\mathcal{I}$-closed if and only if $A \supset (\tau_i$-Int$(A))_j^*$.

Proof. It is obvious. ■

Theorem 2.9. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space and $A, B \subset X$. Then:

(i) If $\{U_\alpha : \alpha \in \Delta\} \subset (i,j)$-$\mathcal{I}O(X)$, then $\bigcup\{U_\alpha : \alpha \in \Delta\} \in (i,j)$-$\mathcal{I}O(X)$.

(ii) If $A \in (i,j)$-$\mathcal{I}O(X)$, $B \in \tau_i$ and $A_j^* \cap B \subset (A \cap B)_j^*$, then $A \cap B \in (i,j)$-$\mathcal{I}O(X)$.

(iii) If $A \in (i,j)$-$\mathcal{I}O(X)$, $B \in \tau_i$ and $B \cap A_j^* = B \cap (B \cap A)_j^*$, then $A \cap B \subset \tau_i$-Int$(B \cap (B \cap A)_j^*)$.

Proof. (i) Since $\{U_\alpha : \alpha \in \Delta\} \subset (i,j)$-$\mathcal{I}O(X)$, then $U_\alpha \subset \tau_i$-Int$((U_\alpha)_j^*)$, for every $\alpha \in \Delta$. Thus, $\bigcup(U_\alpha) \subset \bigcup(\tau_i$-Int$((U_\alpha)_j^*)) \subset \tau_i$-Int$(\bigcup(U_\alpha)_j^* \subset \tau_i$-Int$(\bigcup U_\alpha)_j^*$, for every $\alpha \in \Delta$. Hence $\bigcup\{U_\alpha : \alpha \in \Delta\} \in (i,j)$-$\mathcal{I}O(X)$.
(ii) Given $A \in (i,j)$-$\mathcal{I}O(X)$ and $B \in \tau_i$, that is $A \subset \tau_i$-Int(A_j^*). Then $A \cap B \subset \tau_i$-Int$(A_j^*) \cap B = \tau_i$-Int$(A_j^* \cap B)$. Since $B \in \tau_i$ and $A_j^* \cap B \subset (A \cap B)_j^*$, we have $A \cap B \subset \tau_i$-Int$((A \cap B)_j^*)$. Hence, $A \cap B \in (i,j)$-$\mathcal{I}O(X)$.
(iii) Given $A \in (i,j)$-$\mathcal{I}O(X)$ and $B \in \tau_i$, That is $A \subset \tau_i$-Int(A_j^*). We have to prove $A \cap B \subset \tau_i$-Int$(B \cap (B \cap A)_j^*)$. Thus, $A \cap B \subset \tau_i$-Int$(A_j^*) \cap B = \tau_i$-Int$(A_j^* \cap B) = \tau_i$-Int$(B \cap A_j^*)$. Since $B \cap A_j^* = B \cap (B \cap A)_j^*$. Hence $A \cap B \subset \tau_i$-Int$(B \cap (B \cap A)_j^*)$. ■

Corollary 2.10. The union of (i,j)-$\mathcal{I}$-closed set and τ_j-closed set is (i,j)-$\mathcal{I}$-closed.

Proof. It is obvious. ■

Theorem 2.11. If $A \subset (X, \tau_1, \tau_2, \mathcal{I})$ is (i,j)-$\mathcal{I}$-open and (i,j)-semiclosed, then $A = \tau_i$-Int(A_j^*).

Proof. Given A is (i,j)-$\mathcal{I}$-open. Then $A \subset \tau_i$-Int(A_j^*). Since (i,j)-semiclosed, τ_i-Int$(A_j^*) \subset \tau_i$-Int$(\tau_j$-Cl$(A)) \subset A$. Thus τ_i-Int$(A_j^*) \subset A$. Hence we have, $A = \tau_i$-Int(A_j^*). ■

Theorem 2.12. Let $A \in (i,j)$-$\mathcal{I}O(X)$ and $B \in (i,j)$-$\mathcal{I}O(Y)$, then $A \times B \in (i,j)$-$\mathcal{I}O(X \times Y)$, if $A_j^* \times B_j^* = (A \times B)_j^*$.

Proof. $A \times B \subset \tau_i$-Int$(A_j^*) \times \tau_i$-Int$(B_j^*) = \tau_i$-Int$(A_j^* \times B_j^*)$, from hypothesis. Then $A \times B = \tau_i$-Int$((A \times B)_j^*)$; hence, $A \times B \in (i,j)$-$\mathcal{I}O(X \times Y)$. ■

Theorem 2.13. If $(X, \tau_1, \tau_2, \mathcal{I})$ is an ideal bitopological space, $A \in \tau_i$ and $B \in (i,j)$-$\mathcal{I}O(X)$, then there exists a τ_i-open subset G of X such that $A \cap G = \emptyset$, implies $A \cap B = \emptyset$.

Proof. Since $B \in (i,j)$-$\mathcal{I}O(X)$, then $B \subset \tau_i$-Int(B_j^*). By taking $G = \tau_i$-Int(B_j^*) to be a τ_i-open set such that $B \subset G$. But $A \cap G = \emptyset$, then $G \subset X \backslash A$ implies that τ_i-Cl$(G) \subset X \backslash A$. Hence $B \subset (X \backslash A)$. Therefore, $A \cap B = \emptyset$. ■

Definition 2.14. A subset A of $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be:

(i) τ_i^*-closed if $A_i^* \subset A$.

(ii) τ_i-$*$-perfect $A_i^* = A$.

Theorem 2.15. For a subset $A \subset (X, \tau_1, \tau_2, \mathcal{I})$, we have

(i) If A is τ_j^*-closed and $A \in (i,j)$-$\mathcal{I}O(X)$, then τ_i-Int$(A) = \tau_i$-Int(A_j^*).

(ii) If A is τ_j-$*$-perfect, then $A = \tau_i$-Int(A_j^*) for every $A \in (i,j)$-$\mathcal{I}O(X)$.

Proof. (i) Let A be τ_j-$*$-closed and $A \in (i,j)$-$\mathcal{I}O(X)$. Then $A_j^* \subset A$ and $A \subset \tau_i$-Int(A_j^*). Hence $A \subset \tau_i$-Int$(A_j^*) \Rightarrow \tau_i$-Int$(A) \subset \tau_i$-Int$(\tau_i$-Int$(A_j^*)) \Rightarrow \tau_i$-Int$(A) \subset \tau_i$-Int$(A_j^*)$. Also, $A_j^* \subset A$. Then τ_i-Int$(A_j^*) \subset \tau_i$-Int(A). Hence τ_i-Int$(A) = \tau_i$-Int(A_j^*).
(ii) Let A be τ_j-$*$-perfect and $A \in (i,j)$-$\mathcal{I}O(X)$. We have, $A_j^* = A$, τ_i-Int$(A_j^*) = \tau_i$-Int(A), τ_i-Int$(A_j^*) \subset A$. Also we have $A \subset \tau_i$-Int(A_j^*). Hence we have, $A = \tau_i$-Int(A_j^*). ■

Definition 2.16. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space, S a subset of X and x be a point of X. Then

(i) x is called an (i,j)-$\mathcal{I}$-interior point of S if there exists $V \in (i,j)$-$\mathcal{I}O(X, \tau_1, \tau_2)$ such that $x \in V \subset S$.

(ii) the set of all (i,j)-$\mathcal{I}$-interior points of S is called (i,j)-$\mathcal{I}$-interior of S and is denoted by (i,j)-$\mathcal{I}\,$Int(S).

Theorem 2.17. Let A and B be subsets of $(X, \tau_1, \tau_2, \mathcal{I})$. Then the following properties hold:

(i) (i,j)-$\mathcal{I}\,$Int$(A) = \cup\{T : T \subset A$ and $A \in (i,j)$-$\mathcal{I}O(X)\}$.

(ii) (i,j)-$\mathcal{I}\,$Int(A) is the largest (i,j)-$\mathcal{I}$-open subset of X contained in A.

(iii) A is (i,j)-$\mathcal{I}$-open if and only if $A = (i,j)$-$\mathcal{I}\,$Int(A).

(iv) (i,j)-$\mathcal{I}\,$Int$((i,j)$-$\mathcal{I}\,$Int$(A)) = (i,j)$-$\mathcal{I}\,$Int(A).

(v) If $A \subset B$, then (i,j)-$\mathcal{I}\,$Int$(A) \subset (i,j)$-$\mathcal{I}\,$Int(B).

(vi) (i,j)-$\mathcal{I}\,$Int$(A) \cup (i,j)$-$\mathcal{I}\,$Int$(B) \subset (i,j)$-$\mathcal{I}\,$Int$(A \cup B)$.

(vii) (i,j)-$\mathcal{I}\,$Int$(A \cap B) \subset (i,j)$-$\mathcal{I}\,$Int$(A) \cap (i,j)$-$\mathcal{I}\,$Int(B).

Proof. (i). Let $x \in \cup\{T : T \subset A$ and $A \in (i,j)$-$\mathcal{I}O(X)\}$. Then, there exists $T \in (i,j)$-$\mathcal{I}O(X,x)$ such that $x \in T \subset A$ and hence $x \in (i,j)$-$\mathcal{I}\,$Int(A). This shows that $\cup\{T : T \subset A$ and $A \in (i,j)$-$\mathcal{I}O(X)\} \subset (i,j)$-$\mathcal{I}\,Int(A)$. For the reverse inclusion, let $x \in (i,j)$-$\mathcal{I}\,$Int(A). Then there exists $T \in (i,j)$-$\mathcal{I}O(X,x)$ such that $x \in T \subset A$. we obtain $x \in \cup\{T : T \subset A$ and $A \in (i,j)$-$\mathcal{I}O(X)\}$. This shows that (i,j)-$\mathcal{I}\,$Int$(A) \subset \cup\{T : T \subset A$ and $A \in (i,j)$-$\mathcal{I}O(X)\}$. Therefore, we obtain (i,j)-$\mathcal{I}\,$Int$(A) = \cup\{T : T \subset A$ and $A \in (i,j)$-$\mathcal{I}O(X)\}$.

The proof of (ii)-(v) are obvious.
(vi). Clearly, (i,j)-$\mathcal{I}\operatorname{Int}(A) \subset (i,j)$-$\mathcal{I}\operatorname{Int}(A \cup B)$ and (i,j)-$\mathcal{I}\operatorname{Int}(B) \subset (i,j)$-$\mathcal{I}\operatorname{Int}(A \cup B)$. Then by (v) we obtain (i,j)-$\mathcal{I}\operatorname{Int}(A) \cup (i,j)$-$\mathcal{I}\operatorname{Int}(B) \subset (i,j)$-$\mathcal{I}\operatorname{Int}(A \cup B)$.
(vii). Since $A \cap B \subset A$ and $A \cap B \subset B$, by (v), we have (i,j)-$\mathcal{I}\operatorname{Int}(A \cap B) \subset (i,j)$-$\mathcal{I}\operatorname{Int}(A)$ and (i,j)-$\mathcal{I}\operatorname{Int}(A \cap B) \subset (i,j)$-$\mathcal{I}\operatorname{Int}(B)$. By (v) (i,j)-$\mathcal{I}\operatorname{Int}(A \cap B) \subset (i,j)$-$\mathcal{I}\operatorname{Int}(A) \cap (i,j)$-$\mathcal{I}\operatorname{Int}(B)$. ■

Definition 2.18. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space, S a subset of X and x be a point of X. Then

- (i) x is called an (i,j)-$\mathcal{I}$-cluster point of S if $V \cap S \neq \emptyset$ for every $V \in (i,j)$-$\mathcal{I}O(X, x)$.
- (ii) the set of all (i,j)-$\mathcal{I}$-cluster points of S is called (i,j)-$\mathcal{I}$-closure of S and is denoted by (i,j)-$\mathcal{I}\operatorname{Cl}(S)$.

Theorem 2.19. Let A and B be subsets of $(X, \tau_1, \tau_2, \mathcal{I})$. Then the following properties hold:

- (i) (i,j)-$\mathcal{I}\operatorname{Cl}(A) = \cap\{F : A \subset F$ and $F \in (i,j)$-$\mathcal{I}C(X)\}$.
- (ii) (i,j)-$\mathcal{I}\operatorname{Cl}(A)$ is the smallest (i,j)-$\mathcal{I}$-closed subset of X containing A.
- (iii) A is (i,j)-$\mathcal{I}$-closed if and only if $A = (i,j)$-$\mathcal{I}\operatorname{Cl}(A)$.
- (iv) (i,j)-$\mathcal{I}\operatorname{Cl}((i,j)$-$\mathcal{I}\operatorname{Cl}(A) = (i,j)$-$\mathcal{I}\operatorname{Cl}(A)$.
- (v) If $A \subset B$, then (i,j)-$\mathcal{I}\operatorname{Cl}(A) \subset (i,j)$-$\mathcal{I}\operatorname{Cl}(B)$.
- (vi) (i,j)-$\mathcal{I}\operatorname{Cl}(A \cup B) = (i,j)$-$\mathcal{I}\operatorname{Cl}(A) \cup (i,j)$-$\mathcal{I}\operatorname{Cl}(B)$.
- (vii) (i,j)-$\mathcal{I}\operatorname{Cl}(A \cap B) \subset (i,j)$-$\mathcal{I}\operatorname{Cl}(A) \cap (i,j)$-$\mathcal{I}\operatorname{Cl}(B)$.

Proof. (i). Suppose that $x \notin (i,j)$-$\mathcal{I}\operatorname{Cl}(A)$. Then there exists $F \in (i,j)$-$\mathcal{I}O(X)$ such that $V \cap S \neq \emptyset$. Since $X \backslash V$ is (i,j)-$\mathcal{I}$-closed set containing A and $x \notin X \backslash V$, we obtain $x \notin \cap\{F : A \subset F$ and $F \in (i,j)$-$\mathcal{I}C(X)\}$. Then there exists $F \in (i,j)$-$\mathcal{I}C(X)$ such that $A \subset F$ and $x \notin F$. Since $X \backslash V$ is (i,j)-$\mathcal{I}$-closed set containing x, we obtain $(X \backslash F) \cap A = \emptyset$. This shows that $x \notin (i,j)$-$\mathcal{I}\operatorname{Cl}(A)$. Therefore, we obtain (i,j)-$\mathcal{I}\operatorname{Cl}(A) = \cap\{F : A \subset F$ and $F \in (i,j)$-$\mathcal{I}C(X)$.
The other proofs are obvious. ■

Theorem 2.20. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space and $A \subset X$. A point $x \in (i,j)$-$\mathcal{I}\operatorname{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for every $U \in (i,j)$-$\mathcal{I}O(X, x)$.

Proof. Suppose that $x \in (i,j)$-$\mathcal{I}\operatorname{Cl}(A)$. We shall show that $U \cap A \neq \emptyset$ for every $U \in (i,j)$-$\mathcal{I}O(X, x)$. Suppose that there exists $U \in (i,j)$-$\mathcal{I}O(X, x)$ such that $U \cap A = \emptyset$. Then $A \subset X \backslash U$ and $X \backslash U$ is (i,j)-$\mathcal{I}$-closed. Since $A \subset X \backslash U$, (i,j)-$\mathcal{I}\operatorname{Cl}(A) \subset (i,j)$-$\mathcal{I}\operatorname{Cl}(X \backslash U)$. Since $x \in (i,j)$-$\mathcal{I}\operatorname{Cl}(A)$, we have $x \in (i,j)$-$\mathcal{I}\operatorname{Cl}(X \backslash U)$. Since $X \backslash U$ is (i,j)-$\mathcal{I}$-closed, we have $x \in X \backslash U$; hence $x \notin U$, which is a contradiction that $x \in U$. Therefore, $U \cap A \neq \emptyset$. Conversely, suppose that $U \cap A \neq \emptyset$ for every $U \in (i,j)$-$\mathcal{I}O(X, x)$. We shall show that $x \in (i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A)$. Suppose that $x \notin (i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A)$. Then there exists $U \in (i,j)\text{-}\mathcal{I}O(X,x)$ such that $U \cap A = \emptyset$. This is a contradiction to $U \cap A \neq \emptyset$; hence $x \in (i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A)$. ■

Theorem 2.21. Let $(X, \tau_1, \tau_2, \mathcal{I})$ be an ideal bitopological space and $A \subset X$. Then the following propeties hold:

(i) $(i,j)\text{-}\mathcal{I}\,\mathrm{Int}(X\backslash A) = X\backslash(i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A)$;

(ii) $(i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(X\backslash A) = X\backslash(i,j)\text{-}\mathcal{I}\,\mathrm{Int}(A)$.

Proof. (i). Let $x \in (i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A)$. There exists $V \in (i,j)\text{-}\mathcal{I}O(X,x)$ such that $V \cap A \neq \emptyset$; hence we obtain $x \in (i,j)\text{-}\mathcal{I}\,\mathrm{Int}(X\backslash A)$. This shows that $X\backslash(i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A) \subset (i,j)\text{-}\mathcal{I}\,\mathrm{Int}(X\backslash A)$. Let $x \in (i,j)\text{-}\mathcal{I}\,\mathrm{Int}(X\backslash A)$. Since $(i,j)\text{-}\mathcal{I}\,\mathrm{Int}(X\backslash A) \cap A = \emptyset$, we obtain $x \notin (i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A)$; hence $x \in X\backslash(i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A)$. Therefore, we obtain $(i,j)\text{-}\mathcal{I}\,\mathrm{Int}(X\backslash A) = X\backslash(i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A)$.
(ii). Follows from (i). ■

Definition 2.22. A subset B_x of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is said to be an (i,j)-$\mathcal{I}$-neighbourhood of a point $x \in X$ if there exists an (i,j)-$\mathcal{I}$-open set U such that $x \in U \subset B_x$.

Theorem 2.23. A subset of an ideal bitopological space $(X, \tau_1, \tau_2, \mathcal{I})$ is (i,j)-$\mathcal{I}$-open if and only if it is an (i,j)-$\mathcal{I}$-neighbourhood of each of its points.

Proof. Let G be an (i,j)-$\mathcal{I}$-open set of X. Then by definition, it is clear that G is an (i,j)-$\mathcal{I}$-neighbourhood of each of its points, since for every $x \in G$, $x \in G \subset G$ and G is (i,j)-$\mathcal{I}$-open. Conversely, suppose G is an (i,j)-$\mathcal{I}$-neighbourhood of each of its points. Then for each $x \in G$, there exists $S_x \in (i,j)\text{-}\mathcal{I}O(X)$ such that $S_x \subset G$. Then $G = \bigcup\{S_x : x \in G\}$. Since each S_x is (i,j)-$\mathcal{I}$-open and arbitrary union of (i,j)-$\mathcal{I}$-open sets is (i,j)-$\mathcal{I}$-open, G is (i,j)-$\mathcal{I}$-open in $(X, \tau_1, \tau_2, \mathcal{I})$. ■

3. (i,j)-$\mathcal{I}$-continuous functions

Definition 3.1. A function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is said to be (i,j)-$\mathcal{I}$-continuous if for every $V \in \sigma_i$, $f^{-1}(V) \in (i,j)\text{-}\mathcal{I}O(X)$.

Remark 4. Every (i,j)-$\mathcal{I}$-continuous function is (i,j)-precontinuous but the converse is not true, in general.

Example 8. Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}$, $\tau_2 = \{\emptyset, \{b,c\}, X\}$, $\sigma_1 = \mathcal{P}(X)$, $\sigma_2 = \{\emptyset, \{a\}, \{a,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \sigma_1, \sigma_2)$ is $(1,2)$-precontinuous but not $(1,2)$-$\mathcal{I}$-continuous, because $\{c\} \in \sigma_1$, but $f^{-1}(\{c\}) = \{c\} \notin (1,2)\text{-}\mathcal{I}O(X)$.

Remark 5. It is clear that $(1,2)$-$\mathcal{I}$-continuity and τ_1-continuity (resp. τ_2-continuity) are independent notions.

Example 9. Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{b\}, X\}$, $\tau_2 = \{\emptyset, \{a,b\}, X\}$, $\sigma_1 = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$, $\sigma_2 = \{\emptyset, \{a\}, \{a,b\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \sigma_1, \sigma_2)$ is τ_1-continuous but not $(1,2)$-$\mathcal{I}$-continuous, because $\{b\} \in \sigma_1$, but $f^{-1}(\{b\}) = \{b\} \notin (1,2)$-$\mathcal{I}O(X)$.

Example 10. Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a,b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a,b\}, X\}$, $\sigma_1 = \{\emptyset, \{b\}, \{b,c\}, X\}$, $\sigma_2 = \{\emptyset, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \sigma_1, \sigma_2)$ is $(1,2)$-$\mathcal{I}$-continuous but not τ_1-continuous, because $f^{-1}(\{a\}) = \{a\} \in (1,2)$-$\mathcal{I}O(X)$, but $\{a\} \notin \sigma_1$.

Example 11. Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{a,c\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$, $\sigma_1 = \{\emptyset, \{b,c\}, X\}$, $\sigma_2 = \{\emptyset, \{b\}, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \sigma_1, \sigma_2)$ is τ_2-continuous but not $(1,2)$-$\mathcal{I}$-continuous, because $\{b\} \in \sigma_2$ but $f^{-1}(\{b\}) = \{b\} \notin (1,2)$-$\mathcal{I}O(X)$.

Example 12. Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, \{b,c\}, X\}$, $\sigma_1 = \{\emptyset, \{a,c\}, X\}$, $\sigma_2 = \{\emptyset, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then the identity function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \sigma_1, \sigma_2)$ is $(1,2)$-$\mathcal{I}$-continuous but not τ_2-continuous, because $\{a\} \notin \sigma_2$ but $f^{-1}(\{a\}) = \{a\} \in (1,2)$-$\mathcal{I}O(X)$.

Theorem 3.2. For a function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:

(i) f is pairwise $\mathcal{I}$-continuous;

(ii) For each point x in X and each σ_j-open set F in Y such that $f(x) \in F$, there is a (i,j)-$\mathcal{I}$-open set A in X such that $x \in A$, $f(A) \subset F$;

(iii) The inverse image of each σ_j-closed set in Y is (i,j)-$\mathcal{I}$-closed in X;

(iv) For each subset A of X, $f((i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A)) \subset \sigma_j\text{-}\mathrm{Cl}(f(A))$;

(v) For each subset B of Y, $(i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_j\text{-}\mathrm{Cl}(B))$;

(vi) For each subset C of Y, $f^{-1}(\sigma_j\text{-}\mathrm{Int}(C)) \subset (i,j)\text{-}\mathcal{I}\,\mathrm{Int}(f^{-1}(C))$.

Proof. (i)$\Rightarrow$(ii): Let $x \in X$ and F be a σ_j-open set of Y containing $f(x)$. By (i), $f^{-1}(F)$ is (i,j)-$\mathcal{I}$-open in X. Let $A = f^{-1}(F)$. Then $x \in A$ and $f(A) \subset F$.
(ii)$\Rightarrow$(i): Let F be σ_j-open in Y and let $x \in f^{-1}(F)$. Then $f(x) \in F$. By (ii), there is an (i,j)-$\mathcal{I}$-open set U_x in X such that $x \in U_x$ and $f(U_x) \subset F$. Then $x \in U_x \subset f^{-1}(F)$. Hence $f^{-1}(F)$ is (i,j)-$\mathcal{I}$-open in X.
(i)$\Leftrightarrow$(iii): This follows due to the fact that for any subset B of Y, $f^{-1}(Y \backslash B) = X \backslash f^{-1}(B)$.
(iii)$\Rightarrow$(iv): Let A be a subset of X. Since $A \subset f^{-1}(f(A))$ we have $A \subset f^{-1}(\sigma_j\text{-}\mathrm{Cl}(f(A)))$. Now, $(i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(f(A))$ is σ_j-closed in Y and hence $f^{-1}(\sigma_j\text{-}\mathrm{Cl}(A)) \subset f^{-1}(\sigma_j\text{-}\mathrm{Cl}(f(A)))$, for $(i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A)$ is the smallest (i,j)-$\mathcal{I}$-closed set containing A. Then $f((i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A)) \subset \sigma_j\text{-}\mathrm{Cl}(f(A))$.
(iv)$\Rightarrow$(iii): Let F be any (i,j)-pre-$\mathcal{I}$-closed subset of Y. Then

$f((i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(f^{-1}(F))) \subset (i,j)\text{-}\sigma_i\text{-Cl}(f(f^{-1}(F))) = (i,j)\text{-}\sigma_i\text{-Cl}(F) = F$. Therefore, $(i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(f^{-1}(F)) \subset f^{-1}(F)$. Consequently, $f^{-1}(F)$ is (i,j)-$\mathcal{I}$-closed in X.
(iv)⇒(v): Let B be any subset of Y. Now,
$f((i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(f^{-1}(B))) \subset \sigma_i\text{-Cl}(f(f^{-1}(B))) \subset \sigma_i\text{-Cl}(B)$.
Consequently, $(i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-Cl}(B))$.
(v)⇒(iv): Let $B = f(A)$ where A is a subset of X. Then,
$(i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A) \subset (i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(f^{-1}(B)) \subset f^{-1}(\sigma_i\text{-Cl}(B)) = f^{-1}(\sigma_i\text{-Cl}(f(A)))$.
This shows that $f((i,j)\text{-}\mathcal{I}\,\mathrm{Cl}(A)) \subset \sigma_i\text{-Cl}(f(A))$.
(i)⇒(vi): Let B be a σ_j-open set in Y. Clearly, $f^{-1}(\sigma_i\text{-Int}(B)$ is (i,j)-$\mathcal{I}$-open and we have $f^{-1}(\sigma_i\text{-Int}(B)) \subset (i,j)\text{-}\mathcal{I}\,\mathrm{Int}(f^{-1}\sigma_i\text{-Int}(B)) \subset (i,j)\text{-}\mathcal{I}\,\mathrm{Int}(f^{-1}B)$.
(vi)⇒(i): Let B be a σ_j-open set in Y. Then
$\sigma_i\text{-Int}(B) = B$ and $f^{-1}(B)\backslash f^{-1}(\sigma_i\text{-Int}(B)) \subset (i,j)\text{-}\mathcal{I}\,\mathrm{Int}(f^{-1}(B))$.
Hence we have $f^{-1}(B) = (i,j)\text{-}\mathcal{I}\,\mathrm{Int}(f^{-1}(B))$. This shows that $f^{-1}(B)$ is (i,j)-$\mathcal{I}$-open in X. ■

Theorem 3.3. Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ be (i,j)-$\mathcal{I}$-continuous and σ_i-open function, then the inverse image of each (i,j)-$\mathcal{I}$-open set in Y is (i,j)-preopen in X.

Proof. Let A be (i,j)-$\mathcal{I}$-open. Then $A \subset \tau_i\text{-Int}(A_j^*)$. We have to prove $f^{-1}(A)$ is (i,j)-preopen which implies $f^{-1}(A) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}(f^{-1}(A)))$. For this, $f(A) = f(\tau_i\text{-Int}(A_j^*)) = \tau_i\text{-Int}(f(\tau_i\text{-Int}(A_j^*))) \subset \tau_i\text{-Int}(f(A_j^*))$, $A \subset f^{-1}(\tau_i\text{-Int}(f(A_j^*))) \subset \tau_i\text{-Int}(f^{-1}(\tau_i\text{-Int}(f(A_j^*))))_j^* \subset \tau_i\text{-Int}(A_j^*)_j^* \subset \tau_i\text{-Int}(A_j^*) \subset \tau_i\text{-Int}(A \cup A_j^*) = \tau_i\text{-Int}(\tau_j\text{-Cl}^*(A))$. Hence $f^{-1}(A) \subset \tau_i\text{-Int}(\tau_j\text{-Cl}^*(f^{-1}(A)))$. Therefore, $f^{-1}(A)$ is (i,j)-preopen in X. ■

Theorem 3.4. Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ be (i,j)-$\mathcal{I}$-continuous and $f^{-1}(V_j^*) \subset (f^{-1}(V))_j^*$, for each $V \subset Y$. Then the inverse image of each (i,j)-$\mathcal{I}$-open set is (i,j)-$\mathcal{I}$-open.

Remark 6. The composition of two (i,j)-$\mathcal{I}$-continuous functions need not be (i,j)-$\mathcal{I}$-continuous, in general.

Example 13. Let $X = \{a, b, c\}$, $\tau_i = \{\emptyset, \{a, b\}, X\}$, $\tau_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\sigma_1 = \{\emptyset, \{b\}, \{b, c\}, X\}$, $\sigma_2 = \{\emptyset, \{b, c\}, X\}$, $\gamma_1 = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$, $\gamma_2 = \{\emptyset, \{b, c\}, X\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$, $\mathcal{J} = \{\emptyset, \{c\}\}$ and let the function $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ is defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$ and $g : (Y, \sigma_1, \sigma_2, \mathcal{J}) \to (Z, \gamma_1, \gamma_2)$ is defined by $g(a) = c$, $g(b) = a$ and $g(c) = a$. It is clear that both f and g are $(1,2)$-$\mathcal{I}$-continuous. However, the composition function $g \circ f$ is not $(1,2)$-$\mathcal{I}$-continuous, because $\{a\} \in \gamma_1$, but $(g \circ f)^{-1}(\{a\}) = \{c\} \notin (1,2)\text{-}\mathcal{I}O(X)$.

Theorem 3.5. Let $f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)$ and $g : (Y, \sigma_1, \sigma_2, \mathcal{J}) \to (Z, \mu_1, \mu_2)$. Then $g \circ f$ is (i,j)-$\mathcal{I}$-continuous, if f is (i,j)-$\mathcal{I}$-continuous and g is σ_j-continuous.

Proof. Let $V \in \mu_j$. Since g is μ_j-continuous, then $g^{-1}(V) \in \sigma_j$. On the other hand, since f is (i,j)-$\mathcal{I}$-continuous, we have $f^{-1}(g^{-1}(V)) \in (i,j)$-$\mathcal{I}O(X)$. Since $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$, we obtain that $g \circ f$ is (i,j)-$\mathcal{I}$-continuous. ■

4. (i,j)-$\mathcal{I}$-open and (i,j)-$\mathcal{I}$-closed functions

Definition 4.1. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$ is said to be:

(i) pairwise $\mathcal{I}$-open if $f(U)$ is a (i,j)-$\mathcal{I}$-open set of Y for every τ_i-open set U of X.
(ii) pairwise $\mathcal{I}$-closed if $f(U)$ is a (i,j)-$\mathcal{I}$-closed set of Y for every τ_i-closed set U of X.

Proposition 4.2. Every (i,j)-$\mathcal{I}$-open function is (i,j)-preopen function but the converse is not true in general.

Example 14. Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b,c\}, X\}$, $\tau_2 = \{\emptyset, \{b\}, \{a,b\}, \{b,c\}, X\}$, $\sigma_1 = \{\emptyset, \{a\}, X\}$, $\sigma_2 = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2, \mathcal{I})$ is defined by $f(a) = b$, $f(b) = a$ and $f(c) = c$ is $(1,2)$-preopen but not $(1,2)$-$\mathcal{I}$-open, because $\{a\} \notin \tau_1$, but $f(\{a\}) = \{b\} \notin (1,2)$-$\mathcal{I}O(Y)$.

Remark 7. Each of (i,j)-$\mathcal{I}$-open function and τ_i-open function are independent.

Example 15. Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{b\}, \{b,c\}, X\}$, $\tau_2 = \{\emptyset, \{b,c\}, X\}$, $\sigma_1 = \{\emptyset, \{a\}, \{a,b\}, X\}$, $\sigma_2 = \{\emptyset, \{a\}, \{a,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$ on Y. Then the identity function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2, \mathcal{I})$ is $(1,2)$-$\mathcal{I}$-open function but not τ_1-open, because $\{a\} \notin \tau_1$, but $f(\{a\}) = \{a\} \in (1,2)$-$\mathcal{I}O(Y)$.

Example 16. Let $X = \{a,b,c\}$, $\tau_1 = \{\emptyset, \{a\}, \{b,c\}, X\}$, $\tau_2 = \{\emptyset, \{b,c\}, X\}$, $\sigma_1 = \{\emptyset, \{a\}, \{c\}, \{a,c\}, X\}$, $\sigma_2 = \{\emptyset, \{b\}, \{c\}, \{b,c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$ on Y. Then the identity function $f : (X, \tau_1, \tau_2) \to (X, \sigma_1, \sigma_2, \mathcal{I})$ is defined by $f(a) = b = f(b)$ and $f(c) = c$ is τ_1-open but not $(1,2)$-$\mathcal{I}$-open function, because $\{a\} \in \tau_1$, but $f(\{a\}) = \{b\} \notin (1,2)$-$\mathcal{I}O(Y)$.

Theorem 4.3. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2, \mathcal{I})$, the following statements are equivalent:

(i) f is pairwise $\mathcal{I}$-open;
(ii) $f(\tau_i\text{-Int}(U)) \subset (i,j)\text{-}\mathcal{I}\,\text{Int}(f(U))$ for each subset U of X;
(iii) $\tau_i\text{-Int}(f^{-1}(V)) \subset f^{-1}((i,j)\text{-}\mathcal{I}\,\text{Int}(V))$ for each subset V of Y.

Proof. (i)⇒(ii): Let U be any subset of X. Then τ_i-Int(U) is a τ_i-open set of

X. Then $f(\tau_i\text{-Int}(U))$ is a (i,j)-$\mathcal{I}$-open set of Y. Since $f(\tau_i\text{-Int}(U)) \subset f(U)$, $f(\tau_i\text{-Int}(U)) = (i,j)\text{-}\mathcal{I}\,\text{Int}(f(\tau_i\text{-Int}(U))) \subset (i,j)\text{-}\mathcal{I}\,\text{Int}(f(U))$.
(ii)⇒(iii): Let V be any subset of Y. Then $f^{-1}(V)$ is a subset of X. Hence $f(\tau_i\text{-Int}(f^{-1}(V))) \subset (i,j)\text{-}\mathcal{I}\,\text{Int}(f(f^{-1}(V))) \subset (i,j)\text{-}\mathcal{I}\,\text{Int}(V))$. Then $\tau_i\text{-Int}(f^{-1}(V)) \subset f^{-1}(f(\tau_i\text{-Int}(f^{-1}(V)))) \subset f^{-1}((i,j)\text{-}\mathcal{I}\,\text{Int}(V))$.
(iii)⇒(i): Let U be any τ_i-open set of X. Then $\tau_i\text{-Int}(U) = U$ and $f(U)$ is a subset of Y. Now, $V = \tau_i\text{-Int}(V) \subset \tau_i\text{-Int}(f^{-1}(f(V))) \subset f^{-1}((i,j)\text{-}\mathcal{I}\,\text{Int}(f(V)))$. Then $f(V) \subset f(f^{-1}((i,j)\text{-}\mathcal{I}\,\text{Int}(f(V)))) \subset (i,j)\text{-}\mathcal{I}\,\text{Int}(f(V))$ and $(i,j)\text{-}\mathcal{I}\,\text{Int}(f(V)) \subset f(V)$. Hence $f(V)$ is a (i,j)-$\mathcal{I}$-open set of Y; hence f is pairwise $\mathcal{I}$-open. ■

Theorem 4.4. Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2,\mathcal{I})$ be a function. Then f is a pairwise $\mathcal{I}$-closed function if and only if for each subset V of X, $(i,j)\text{-}\mathcal{I}\,\text{Cl}(f(V)) \subset f(\tau_i\,\text{Cl}(V))$.

Proof. Let f be a pairwise $\mathcal{I}$-closed function and V any subset of X. Then $f(V) \subset f(\tau_i\text{-Cl}(V))$ and $f(\tau_i\text{-Cl}(V))$ is a (i,j)-$\mathcal{I}$-closed set of Y. We have $(i,j)\text{-}\mathcal{I}\,\text{Cl}(f(V)) \subset (i,j)\text{-}\mathcal{I}\,\text{Cl}(f(\tau_i\text{-Cl}(V))) = f(\tau_i\text{-Cl}(V))$. Conversely, let V be a τ_i-open set of X. Then $f(V) \subset (i,j)\text{-}\mathcal{I}\,\text{Cl}(f(V)) \subset f(\tau_i\text{-Cl}(V)) = f(V)$; hence $f(V)$ is a (i,j)-$\mathcal{J}$-closed subset of Y. Therefore, f is a pairwise $\mathcal{I}$-closed function. ■

Theorem 4.5. Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2,\mathcal{I})$ be a function. Then f is a pairwise $\mathcal{I}$-closed function if and only if for each subset V of Y, $f^{-1}((i,j)\text{-}\mathcal{I}\,\text{Cl}(V)) \subset \tau_i\text{-Cl}(f^{-1}(V))$.

Proof. Let V be any subset of Y. Then by Theorem 4.4,
$(i,j)\text{-}\mathcal{I}\,\text{Cl}(V) \subset f(\tau_i\text{-Cl}(f^{-1}(V)))$. Since f is bijection,
$f^{-1}((i,j)\text{-}\mathcal{I}\,\text{Cl}(V)) = f^{-1}((i,j)\text{-}\mathcal{I}\,\text{Cl}(f(f^{-1}(V)))) \subset f^{-1}(f(\tau_i\text{-Cl}(f^{-1}(V)))) = \tau_i\text{-Cl}(f^{-1}(V))$.
Conversely, let U be any subset of X. Since f is bijection,
$(i,j)\text{-}\mathcal{I}\,\text{Cl}(f(U)) = f(f^{-1}((i,j)\text{-}\mathcal{I}\,\text{Cl}(f(U))) \subset f(\tau_i\text{-Cl}(f^{-1}(f(U))))$
$= f(\tau_i\text{-Cl}(U))$. Therefore, by Theorem 4.4, f is a pairwise $\mathcal{I}$-closed function. ■

Theorem 4.6. Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2,\mathcal{I})$ be a pairwise $\mathcal{I}$-open function. If V is a subset of Y and U is a τ_i-closed subset of X containing $f^{-1}(V)$, then there exists a (i,j)-$\mathcal{I}$-closed set F of Y containing V such that $f^{-1}(F) \subset U$.

Proof. Let V be any subset of Y and U a τ_i-closed subset of X containing $f^{-1}(V)$, and let $F = Y\backslash(f(X\backslash V))$. Then $f(X\backslash V) \subset f(f^{-1}(X\backslash V)) \subset X\backslash V$ and $X\backslash U$ is a τ_i-open set of X. Since f is pairwise $\mathcal{I}$-open, $f(X\backslash U)$ is a (i,j)-$\mathcal{I}$-open set of Y. Hence F is an (i,j)-$\mathcal{I}$-closed set of Y and $f^{-1}(F) = f^{-1}(Y\backslash(f(X\backslash U)) \subset U$. ■

Theorem 4.7. Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2,\mathcal{I})$ be a pairwise $\mathcal{I}$-closed function. If V is a subset of Y and U is a open subset of X containing $f^{-1}(V)$, then there exists (i,j)-$\mathcal{I}$-open set F of Y containing V such that $f^{-1}(F) \subset U$.

Proof. The proof is similar to the Theorem 4.6. ■

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