On Neutrosophic Metric Spaces Generated By Classical Metrics

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On Neutrosophic Metric Spaces Generated By Classical Metrics

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Abstract

This paper is dedicated to study the concept of symbolic neutrosophic metric spaces and refined neutrosophic metric spaces (X(I),d) which are generated by classical metrics, where many elementary properties will be discussed in terms of theorems and examples. Also, we study the topological properties of neutrosophic open balls, closed balls, and their topological metric complements, with many related examples that clarify the validity of our work.

Keywords: neutrosophic metric space; refined neutrosophic metric space; open ball; closed ball.

1. Introduction

Neutrosophy is a new kind of logic that generalizes the classical fuzzy approach into a new insight.
Neutrosophic structures were built over the idea of extending classical structures by adding a logical element I with the following property $I^2 = I$. This idea was used by many authors to define some generalizations of classical mathematical structures such as neutrosophic rings [3,6,8,12,13], neutrosophic spaces and modules [1,2,4,7], and neutrosophic functions [5,11].
In [14], the real inner product neutrosophic spaces were defined and studied with many related analytical concepts such as norms and distances. In [9-10], authors have presented many deep results in the theory of normed neutrosophic spaces generated by classical norms.
This has motivated us to study neutrosophic and refined neutrosophic metric spaces generated by classical metrics, where we give strict definitions for neutrosophic metrics generated by classical metrics, and we will provide many elementary properties of the novel class of metric spaces with many related examples that explain the analytical and topological properties of it.

2. Main Discussion

Definition:

Let $X \neq \emptyset$ be a non empty set, $X(I) = X + XI = \{a + bI; a, b \in X\}$ be the corresponding neutrosophic set.
Let $d_1, d_2: X \times X \to R$ be the two metric, we define the corresponding metric on $X(I)$ as follows:
$d_I: X(I) \times X(I) \to R$ such that:
$d_I(a + bI, c + dI) = d_1(a, c) + I[d_2(a + b, c + d) - d_1(a, c)]$.
$(X(I), d_I)$ is called neutrosophic metric space.

Theorem

Let $d_I: X(I) \times X(I) \to R$ be a neutrosophic metric, then:

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1). $d_I(a+bI, c+dI) \geq 0$.
2). $d_I(a+bI, c+dI) = 0 \Leftrightarrow a+bI = c+dI$.
3). $d_I(a+bI, c+dI) = d_I(c+dI, a+bI)$.
4). $d_I(a+bI, c+dI) \leq d(a+bI, m+nI) + d(m+nI, c+dI)$ for $m+nI \in X(I)$.

Proof.

1). Since d_1, d_2 are classical metrics, then:
$d_1(a,c) \geq 0, d_2(a+b, c+d) \geq 0$, thus $d_I(a+bI, c+dI) \geq 0$.
2). $d_I(a+bI, c+dI) = 0 \Leftrightarrow d_1(a,c) = d_2(a+b, c+d) = 0 \Leftrightarrow a = c, a+b = c+d, b = d$, hence $a + bI = c + dI$.
3). It is easy and clear.
4). $d_I(a+bI, m+nI) + d_I(m+nI, c+dI) = d_1(m,n) + d_1(m,n) + I[d_2(a+b, m+n) + d_2(m+n, c+d) - d_1(a,m) - d_1(m,n)] \geq d_1(a,c) + I[d_2(a+b, c+d) - d_1(a,c)] = d_I(a+bI, c+dI)$

That is because:
$d_1(a,c) \leq d_1(a,m) + d_1(m,n)$
$d_2(a+b, m+n) + d_2(m+n, c+d) \geq d_2(a+b, c+d)$
Where d_1, d_2 are two classical metrics.

Definition.

$d_I: X(I) \times X(I) \to R$ is called a neutrosophic metric if and only if the conditions (1),(2),(3),(4) from the previous theorem are true.
$(X(I), d_I)$ is called a neutrosophic metric space.

Remark.

A neutrosophic metric d_I is equivalent to two classical metrics $d_1, d_2: X \times X \to R$.

Example.

Define $d_I^{(1)}: R(I) \times R(I) \to R(I)$ by:
$$d_I^{(1)}(a+bI, c+dI) = |a-c| + I[|a+b-c-d| - |a-c|]$$
Define $d_I^{(2)}: R^2(I) \times R^2(I) \to R(I)$ by:
$$d_I^{(2)}\big((a_0+b_0I, a_1+b_1I), (c_0+d_0I, c_1+d_1I)\big) = \sqrt{(a_0-c_0)^2 + (a_1-c_1)^2} + I\left[\sqrt{(a_0+b_0-c_0-d_0)^2 + (a_1+b_1-c_1-d_1)^2} - \sqrt{(a_0-c_0)^2 + (a_1-c_1)^2}\right].$$
We will prove that $d_I^{(1)}$ is a neutrosophic metric on $R(I)$.
Assume that $d_I^{(1)}(a+bI, c+dI) = 0$, then:
$$\begin{cases} |a-c| = 0 \\ |a+b-c-d| = 0 \end{cases} \Rightarrow \begin{cases} a = c \\ a+b = c+d \end{cases} \Rightarrow \begin{cases} a = c \\ b = d \end{cases} \Rightarrow a+bI = c+dI$$
Also, $|a-c| \geq 0, [|a+b-c-d| - |a-c|] + |a-c| = |a+b-c-d| \geq 0$, hence $d_I^{(1)}(a+bI, c+dI) \geq 0$.
$$d_I^{(1)}(a+bI, c+dI) = |a-c| + I[|a+b-c-d| - |a-c|] = |c-a| + I[|c+d-a-b| - |c-a|] = d_I^{(1)}(c+dI, a+bI)$$
$$d_I^{(1)}(a+bI, m+nI) + d_I^{(1)}(m+nI, c+dI) = |a-m| + |m-c| + I[|a+b-m-n| + |m+n-c-d| - |a-m| - |m-c|]$$
On the other hand, we have:
$|a-c| \leq |a-m| + |m-c|$
$|a+b-m-n| + |m+n-c-d| \geq |a+b-c-d|$
Hence:
$$|a-c| + I[|a+b-c-d| - |a-c|] \leq |a-m| + |m-c| + I[|a+b-m-n| + |m+n-c-d| - |a-m| - |m-c|]$$
Thus:
$$d_I^{(1)}(a+bI, c+dI) \leq d_I^{(1)}(a+bI, m+nI) + d_I^{(1)}(m+nI, c+dI)$$
And $\big(R(I), d_I^{(1)}\big)$ is a neutrosophic metric space.
For $d_I^{(2)}$, we can see:
$d_I^{(2)}\big((a_0+b_0I, a_1+b_1I), (c_0+d_0I, c_1+d_1I)\big) = 0$, hence:
$$\begin{cases} (a_0-c_0)^2 + (a_1-c_1)^2 = 0 \\ (a_0+b_0-c_0-d_0)^2 + (a_1+b_1-c_1-d_1)^2 = 0 \end{cases}$$

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Thus

$$\begin{cases} a_0 = c_0, a_1 = c_1 \\ b_0 = d_0, b_1 = d_1 \end{cases}$$

And $(a_0 + b_0 I, a_1 + b_1 I) = (c_0 + d_0 I, c_1 + d_1 I)$

$d_I^{(2)}\big((a_0 + b_0 I, a_1 + b_1 I), (c_0 + d_0 I, c_1 + d_1 I)\big) \geq 0$, that is because:

$$\begin{cases} \sqrt{(a_0 - c_0)^2 + (a_1 - c_1)^2} \geq 0 \\ \sqrt{(a_0 + b_0 - c_0 - d_0)^2 + (a_1 + b_1 - c_1 - d_1)^2} \geq 0 \end{cases}$$

It is clear that $d_I^{(2)}\big((a_0 + b_0 I, a_1 + b_1 I), (c_0 + d_0 I, c_1 + d_1 I)\big) = d_I^{(2)}\big((c_0 + d_0 I, c_1 + d_1 I), (a_0 + b_0 I, a_1 + b_1 I)\big)$

Also, we have:

$$\begin{aligned} d_I^{(2)}\big((a_0 + b_0 I, a_1 + b_1 I), (m_0 + n_0 I, m_1 + n_1 I)\big) &= \sqrt{(a_0 - m_0)^2 + (a_1 - m_1)^2} \\ &+ I\left[\sqrt{(a_0 + b_0 - m_0 - n_0)^2 + (a_1 + b_1 - m_1 - n_1)^2} - \sqrt{(a_0 - m_0)^2 + (a_1 - m_1)^2}\right] \end{aligned}$$

$$\begin{aligned} d_I^{(2)}\big((m_0 + n_0 I, m_1 + n_1 I), (c_0 + d_0 I, c_1 + d_1 I)\big) &= \sqrt{(m_0 - c_0)^2 + (m_1 - c_1)^2} \\ &+ I\left[\sqrt{(m_0 + n_0 - c_0 - d_0)^2 + (m_1 + n_1 - c_1 - d_1)^2} - \sqrt{(m_0 - c_0)^2 + (m_1 - c_1)^2}\right] \end{aligned}$$

Hence:

$$\begin{aligned} &d_I^{(2)}\big((a_0 + b_0 I, a_1 + b_1 I), (m_0 + n_0 I, m_1 + n_1 I)\big) + d_I^{(2)}\big((m_0 + n_0 I, m_1 + n_1 I), (c_0 + d_0 I, c_1 + d_1 I)\big) \\ &= \sqrt{(a_0 - m_0)^2 + (a_1 - m_1)^2} + \sqrt{(m_0 - c_0)^2 + (m_1 - c_1)^2} \\ &+ I\left[\sqrt{(a_0 + b_0 - m_0 - n_0)^2 + (a_1 + b_1 - m_1 - n_1)^2} - \sqrt{(a_0 - m_0)^2 + (a_1 - m_1)^2}\right] \\ &+ I\left[\sqrt{(m_0 + n_0 - c_0 - d_0)^2 + (m_1 + n_1 - c_1 - d_1)^2} - \sqrt{(m_0 - c_0)^2 + (m_1 - c_1)^2}\right] \\ &\geq \sqrt{(a_0 - c_0)^2 + (a_1 - c_1)^2} \\ &+ I\left[\sqrt{(a_0 + b_0 - c_0 - d_0)^2 + (a_1 + b_1 - c_1 - d_1)^2} - \sqrt{(a_0 - c_0)^2 + (a_1 - c_1)^2}\right] \\ &= d_I^{(2)}\big((a_0 + b_0 I, a_1 + b_1 I), (c_0 + d_0 I, c_1 + d_1 I)\big) \end{aligned}$$

Definition.

Let $(X(I), d_I)$ be a neutrosophic metric space, and let $r = r_1 + r_2 I;\ r_1 > 0, r_1 + r_2 > 0$, we define:

1). For $c + dI \in X(I)$:

The set $\{a + bI \in X(I);\ d_I(a + bI, c + dI) < r_1 + r_2 I\}$ is called a neutrosophic open ball with center at $c + dI$ and radius $r = r_1 + r_2 I$.

We denote it by $B(c + dI, r_1 + r_2 I)$.

2). The set $\bar{B}(c + dI, r_1 + r_2 I) = \{a + bI \in X(I);\ d_I(a + bI, c + dI) \leq r_1 + r_2 I\}$ is called a neutrosophic closed ball.

3). The set $S(c + dI, r_1 + r_2 I) = \{a + bI \in X(I);\ d_I(a + bI, c + dI) = r_1 + r_2 I\}$ is called a neutrosophic ball.

Definition.

Let $B_1(c + dI, r_1 + r_2 I), B_2(c + dI, r_1 + r_2 I)$ be two neutrosophic open balls, we define:

$B_s(c + dI, s_1 + s_2 I), B_k(c + dI, k_1 + k_2 I)$ such that:

$s_1 = min(r_1, t_1), s_2 = min(r_1 + r_2, t_1 + t_2) - min(r_1, t_1)$

$k_1 = max(r_1, t_1), k_2 = max(r_1 + r_2, t_1 + t_2) - max(r_1, t_1)$

We denote that by $B_s = min(B_1, B_2)$ the minimum of B_1, B_2.

And $B_k = max(B_1, B_2)$ is called the minimum of B_1, B_2.

B_s is called the minimal open ball of B_1, B_2, and B_k is maximal open ball of B_1, B_2.

Definition.

Let $\overline{B_1}(c + dI, r_1 + r_2 I), \overline{B_2}(c + dI, t_1 + t_2 I)$ be two neutrosophic closed balls, we define:

$\overline{B_s}(c + dI, s_1 + s_2 I), \overline{B_k}(c + dI, k_1 + k_2 I)$ by a similar way of the previous definition.

$\overline{B_s}$ is called the minimal closed ball of B_1, B_2, and $\overline{B_k}$ is maximal closed ball of B_1, B_2.

Theorem.

Let $(X(I), d_I)$ be a neutrosophic metric pace, then:

1). $min(B_1, B_2) \subseteq B_1 \cap B_2 \subseteq max(B_1, B_2)$.

2). $min(\overline{B_1}, \overline{B_2}) \subseteq \overline{B_1} \cap \overline{B_2} \subseteq max(\overline{B_1}, \overline{B_2})$

Proof.

1). Let $a + bI \in min(B_1, B_2)$, then:

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$d_I(a+bI,c+dI) < s_1+s_2I$, hene:
$d_1(a,c)+I[d_2(a+b,c+d)-d_1(a,c)] < s_1+s_2I$, so that:
$d_1(a,c) < s_1 = min(r_1,t_1) \leq r_1,t_1$.
$[d_2(a+b,c+d)-d_1(a,c)]+d_1(a,c) = d_2(a+b,c+d) < s_1+s_2 = min(r_1+r_2,t_1+t_2) \leq r_1+r_2,t_1+t_2$, thus:
$d_I(a+bI,c+dI) \in B_1 \cap B_2$.
On other hand, let $a+bI \in B_1 \cap B_2$, then:
$d_I(a+bI,c+dI) < r_1+r_2I,t_1+t_2I$.
Also, $r_1+r_2I \leq max(r_1,t_1)+I[max(r_1+r_2,t_1+t_2)-max(r_1,t_1)] = k_1+k_2I$
And $t_1+t_2I \leq k_1+k_2I$, thus $d_I(a+bI,c+dI) < k_1+k_2I$ and $a+bI \in min(B_1,B_2)$, which implies:
$min(B_1,B_2) \subseteq B_1 \cap B_2 \subseteq max(B_1,B_2)$.
2). It can be proved by a similar argument.

Definition.

Let $B(,c+dI,r_1+r_2I)$ be a neutrosophic open ball, we define:
$\sim B = \{a+bI \in X(I);\ d_I(a+bI,c+dI) \geq r_1+r_2I\}$
Also, $\sim \bar{B} = \{a+bI \in X(I);\ d_I(a+bI,c+dI) > r_1+r_2I\}$
It is clear that:

$$\begin{cases} B \cap \sim B = \emptyset \\ \bar{B} \cap \sim B = S(c+dI,r_1+r_2I) \\ \bar{B} \cap \sim \bar{B} = \emptyset \end{cases}$$

Theorem.

Let $B_1(c+dI,r_1+r_2I), B_2(c+dI,t_1+t_2I)$ be two neutrosophic open balls, then:
1). $\sim max(B_1,B_2) \subseteq \sim B_1 \cap \sim B_2 \subseteq \sim min(B_1,B_2)$
2). $\sim max(\overline{B_1},\overline{B_2}) \subseteq \sim \overline{B_1} \cap \sim \overline{B_2} \subseteq \sim min(B_1,B_2)$

Proof.

1). Let $a+bI \in \sim max(B_1,B_2)$, then:
$d_I(a+bI,c+dI) \geq max(r_1,t_1)+I[max(r_1+r_2,t_1+t_2)-max(r_1,t_1)] \geq r_1+r_2I,t_1+t_2I$, thus
$a+bI \in \sim B_1 \cap \sim B_2$.
Now, let $a+bI \in\in \sim B_1 \cap \sim B_2$, hence:
$d_I(a+bI,c+dI) \geq r_1+r_2I,t_1+t_2I \geq min(r_1,t_1)+I[min(r_1+r_2,t_1+t_2)-min(r_1,t_1)]$, thus
$a+bI \in \sim min(B_1,B_2)$
So that, $\sim max(B_1,B_2) \subseteq \sim B_1 \cap \sim B_2 \subseteq \sim min(B_1,B_2)$
2). It can be proved by the same.

Refined neutrosophic metric space.

Definition.

Let $d_r(I_1,I_2): X(I_1,I_2) \times X(I_1,I_2) \to R(I_1,I_2)$ is called a refined neutrosophic metric if and only if:
1). $d_r(a_0+b_0I_1+c_0I_2, a_1+b_1I_1+c_1I_2) \geq 0$.
2). $d_r(a_0+b_0I_1+c_0I_2, a_1+b_1I_1+c_1I_2) = 0 \Leftrightarrow a_0+b_0I_1+c_0I_2 = a_1+b_1I_1+c_1I_2$
3). $d_r(a_0+b_0I_1+c_0I_2, a_1+b_1I_1+c_1I_2) = d_r(a_1+b_1I_1+c_1I_2, a_0+b_0I_1+c_0I_2)$
4). $d_r(a_0+b_0I_1+c_0I_2, a_1+b_1I_1+c_1I_2) + d_r(a_1+b_1I_1+c_1I_2, a_2+b_2I_1+c_2I_2) \geq d_r(a_0+b_0I_1+c_0I_2, a_2+b_2I_1+c_2I_2)$
$(X(I_1,I_2),d_r)$ is called refined neutrosophic metric space.

Theorem.

Let $d_1,d_2,d_3: X \times X \to R$ be three metric on X, then there exists a refined neutrosophic metric
$d_r(I_1,I_2): X(I_1,I_2) \times X(I_1,I_2) \to R(I_1,I_2)$, such that:

$$\begin{aligned} d_r(a+bI_1+cI_2, d+eI_1+kI_2) &= d_1(a,d)+I_1[d_2(a+b+c,d+e+k)-d_3(a+c,d+k)] \\ &+ I_2[d_3(a+c,d+k)-d_1(a,d)] \end{aligned}$$

Proof.

$d_1(a,d) \geq 0$, $\quad d_2(a+b+c,d+e+k)-d_3(a+c,d+k)+d_1(a,d)+d_3(a+c,d+k)-d_1(a,d) = d_2(a+b+c,d+e+k) \geq 0$
$d_3(a+c,d+k)-d_1(a,d)+d_1(a,d) = d_3(a+c,d+k) \geq 0$, hence $d_r(a+bI_1+cI_2,d+eI_1+kI_2) \geq 0$.

$$\begin{aligned} d_r(a+bI_1+cI_2,d+eI_1+kI_2) = 0 &\Leftrightarrow \begin{cases} d_1(a,d)=0 \\ d_2(a+b+c,d+e+k)=0 \\ d_3(a+c,d+k)=0 \end{cases} \Leftrightarrow \begin{cases} a=d \\ c=k \\ b=e \end{cases} \Leftrightarrow a+bI_1+cI_2 \\ &= d+eI_1+kI_2 \end{aligned}$$

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It is clear that $d_r(a_0+b_0I_1+c_0I_2,a_1+b_1I_1+c_1I_2)=d_r(a_1+b_1I_1+c_1I_2,a_0+b_0I_1+c_0I_2)$.
Also, $d_r(a+bI_1+cI_2,d+eI_1+kI_2)=d_1(a,d)+I_1[d_2(a+b+c,d+e+k)-d_3(a+c,d+k)]+I_2[d_3(a+c,d+k)-d_1(a,d)]$, we have:

$$\begin{cases} d_1(a,d)\le d_1(a,x)+d_1(x,d); x\in X \\ d_2(a+b+c,d+e+k)\le d_2(a+b+c,x+y+x)+d_2(x+y+z,d+e+k) \\ d_3(a+c,d+k)\le d_3(a+c,x+z)+d_3(x+z,d+k)\end{cases}$$

So that:

$$\begin{aligned} d_r(a+bI_1+cI_2,d+eI_1+kI_2) &= d_1(a,d)+I_1[d_2(a+b+c,d+e+k)-d_3(a+c,d+k)] \\ &\quad +I_2[d_3(a+c,d+k)-d_1(a,d)] \\ &\le d_1(a,x)+d_1(x,d) \\ &\quad +I_1[d_2(a+b+c,x+y+x)+d_2(x+y+z,d+e+k)-d_1(a,x)-d_1(x,d)] \\ &\quad +I_2[d_3(a+c,x+z)+d_3(x+z,d+k)-d_1(a,x)-d_1(x,d)] \\ &= d_r(a+bI_1+cI_2,x+yI_1+zI_2)+d_r(x+yI_1+zI_2,a+bI_1+cI_2)\end{aligned}$$

Example.

Define $d_r(I_1,I_2): R(I_1,I_2)\times R(I_1,I_2)\to R(I_1,I_2)$, such that:

$$d_r(a+bI_1+cI_2,d+eI_1+kI_2) = |a-d|+I_1[|a+b+c-d-e-k|-|a+c-d-k|]+I_2[|a+c-d-k|-|a-d|]$$

d_r is a refined neutrosophic metric, that is because:

1). $d_r(a+bI_1+cI_2,d+eI_1+kI_2)\ge 0$, that is because:

$$\begin{cases} |a-d|\ge 0 \\ |a+b+c-d-e-k|\ge 0 \\ |a+c-d-k|\ge 0\end{cases}$$

2). $d_r(a+bI_1+cI_2,d+eI_1+kI_2)=0 \Leftrightarrow \begin{cases} |a-d|=0\Leftrightarrow a=d \\ |a+b+c-d-e-k|=0\Leftrightarrow a+b+c=d+e+k \\ |a+c-d-k|=0\Leftrightarrow a+c=d+k\end{cases}$

Hence

$\begin{cases} a=d \\ b=e \\ c=k\end{cases}$, thus $a+bI_1+cI_2=d+eI_1+kI_2$

3). It is clear that, $d_r(a+bI_1+cI_2,d+eI_1+kI_2)=d_r(d+eI_1+kI_2,a+bI_1+cI_2)$

4). We have:

$$\begin{cases} |a-d|\le |a-x|+|x-d| \\ |a+b+c-d-e-k|\le |a+b+c-x-y-z|+|x+y+z-d-e-k| \\ |a+c-d-k|\le |a+c-x-z|+|x+z-d-k|\end{cases}$$

Thus:

$$\begin{aligned} |a-d|+I_1[|a+b+c-d-e-k|-|a+c-d-k|]+I_2[|a+c-d-k|-|a-d|] \\ \le |a-x|+|x-d| \\ +I_1[|a+b+c-x-y-z|+|x+y+z-d-e-k|-|a+c-x-z| \\ -|x+z-d-k|]++I_2[|a+c-x-z|+|x+z-d-k|-|a-x|-|x-d|]\end{aligned}$$

Hence,

$$d_r(a+bI_1+cI_2,d+eI_1+kI_2)\le d_r(a+bI_1+cI_2,x+yI_1+zI_2)+d_r(x+yI_1+zI_2,d+eI_1+kI_2)$$

Definition.

1). $B_r(a+bI_1+cI_2,r_0+r_1I_1+r_2I_2)=\{d+eI_1+kI_2\in X(I_1,I_2)\}$
$d_r(a+bI_1+cI_2,d+eI_1+kI_2)<r_0+r_1I_1+r_2I_2\in R(I_1,I_2)$
Is called refined neutrosophic open ball with radius $r_0+r_1I_1+r_2I_2$ and center at $a+bI_1+cI_2$.

2). $\overline{B_r}(a+bI_1+cI_2,r_0+r_1I_1+r_2I_2)=\{d+eI_1+kI_2\in X(I_1,I_2)\}$
$d_r(a+bI_1+cI_2,d+eI_1+kI_2)\le r_0+r_1I_1+r_2I_2\in R(I_1,I_2)$
Is called refined neutrosophic closed ball.

3). For $B_r^{(1)}(a+bI_1+cI_2,r_0+r_1I_1+r_2I_2), B_r^{(2)}(a+bI_1+cI_2,k_0+k_1I_1+k_2I_2)$, we define:
$min(B_r^{(1)},B_r^{(2)})=\{e+dI_1+lI_2\in X(I_1,I_2); d_r(a+bI_1+cI_2,e+dI_1+lI_2)<s_0+s_1I_1+s_2I_2\}$, where
$s_0=min(r_0,k_0), s_1=min(r_0+r_1+r_2,k_0+k_1+k_2)-min(r_0+r_2,k_0+k_2), s_2 = min(r_0+r_2,k_0+k_2)-min(r_0,k_0)$

4). $max(B_r^{(1)},B_r^{(2)})=\{e+dI_1+lI_2\in X(I_1,I_2); d_r(a+bI_1+cI_2,e+dI_1+lI_2)<t_0+t_1I_1+t_2I_2\}$, where
$t_0=max(r_0,k_0), t_1=max(r_0+r_1+r_2,k_0+k_1+k_2)-max(r_0+r_2,k_0+k_2), t_2 = max(r_0+r_2,k_0+k_2)-max(r_0,k_0)$

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By a similar argument, we define:

$min\left(\overline{B_r^{(1)}}, \overline{B_r^{(2)}}\right), max\left(\overline{B_r^{(1)}}, \overline{B_r^{(2)}}\right)$

Theorem.

Let $B_r^{(1)}(a + bI_1 + cI_2, r_0 + r_1I_1 + r_2I_2), B_r^{(2)}(a + bI_1 + cI_2, k_0 + k_1I_1 + k_2I_2)$ be two refined neutrosophic open balls, then:

$min(B_r^{(1)}, B_r^{(2)}) \subseteq B_r^{(1)} \cap B_r^{(2)} \subseteq max(B_r^{(1)}, B_r^{(2)})$.

Proof.

Let $e + dI_1 + lI_2 \in min(B_r^{(1)}, B_r^{(2)})$, then:

$d_r(a + bI_1 + cI_2, d + eI_1 + kI_2) < s_0 + s_1I_1 + s_2I_2 \leq r_0 + r_1I_1 + r_2I_2$

$d_r(a + bI_1 + cI_2, d + eI_1 + kI_2) < s_0 + s_1I_1 + s_2I_2 \leq k_0 + k_1I_1 + k_2I_2$

Hence, $e + dI_1 + lI_2 \in B_r^{(1)} \cap B_r^{(2)}$.

Let $e + dI_1 + lI_2 \in B_r^{(1)} \cap B_r^{(2)}$, then:

$\begin{cases} d_r(a + bI_1 + cI_2, d + eI_1 + kI_2) < r_0 + r_1I_1 + r_2I_2 \\ d_r(a + bI_1 + cI_2, d + eI_1 + kI_2) < k_0 + k_1I_1 + k_2I_2 \end{cases}$

Thus, $d_r(a + bI_1 + cI_2, d + eI_1 + kI_2) < t_0 + t_1I_1 + t_2I_2$, hence $+dI_1 + lI_2 \in max(B_r^{(1)}, B_r^{(2)})$, which implies that $min(B_r^{(1)}, B_r^{(2)}) \subseteq B_r^{(1)} \cap B_r^{(2)} \subseteq max(B_r^{(1)}, B_r^{(2)})$.

Remark.

$min\left(\overline{B_r^{(1)}}, \overline{B_r^{(2)}}\right) \subseteq \overline{\overline{B_r^{(1)}}} \cap B_r^{(2)} \subseteq max\left(\overline{B_r^{(1)}}, \overline{B_r^{(2)}}\right)$

Definition.

Let $B_r(a + bI_1 + cI_2, r_0 + r_1I_1 + r_2I_2)$ be a refined neutrosophic open ball, we define:

1). $\sim B_r = \{e + dI_1 + lI_2 \in X(I_1, I_2); d_r(a + bI_1 + cI_2, e + dI_1 + lI_2) \geq r_0 + r_1I_1 + r_2I_2\}$.

2). $\sim \overline{B_r} = \{e + dI_1 + lI_2 \in X(I_1, I_2); d_r(a + bI_1 + cI_2, e + dI_1 + lI_2) \geq r_0 + r_1I_1 + r_2I_2\}$.

Remark.

$\sim B_r \cap B_r = \emptyset, \sim \overline{B_r} \cap \overline{B_r} = \emptyset$.

Theorem.

Let $B_r^{(1)}(a + bI_1 + cI_2, r_0 + r_1I_1 + r_2I_2)$ be a refined neutrosophic open ball, $B_r^{(2)}(a + bI_1 + cI_2, k_0 + k_1I_1 + k_2I_2)$ is another refined neutrosophic open ball, then:

$\sim max(B_1, B_2) \subseteq \sim B_1 \cap \sim B_2 \subseteq \sim min(B_1, B_2)$

Proof.

Let $d + eI_1 + kI_2 \in \sim max(B_1, B_2)$, then:

$d_r(a + bI_1 + cI_2, d + eI_1 + kI_2) > t_0 + t_1I_1 + t_2I_2$, where:

$t_0 = max(r_0, k_0), t_1 = max(r_0 + r_1 + r_2, k_0 + k_1 + k_2) - max(r_0 + r_2, k_0 + k_2), t_2 = max(r_0 + r_2, k_0 + k_2) - max(r_0, k_0)$.

3. Conclusion

In this work, we have defined for the first time the concept of neutrosophic metric space and refined neutrosophic metric space. Also, we have studied the elementary properties of these new classes, where open and closed balls are defined and handled by many theorems. The relationships between these spaces and the corresponding classical metric spaces are obtained.

In the future, we aim to study symbolic n-plithogenic metric spaces, and n-refined neutrosophic metric spaces and their connection with classical well known metrics.

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