Arab. J. Math. (2022) 11:189–211
https://doi.org/10.1007/s40065-022-00367-z

Arabian Journal of Mathematics

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Numerical integration of neutrosophic valued function by Gaussian quadrature methods

Received: 28 August 2021 / Accepted: 10 January 2022 / Published online: 5 April 2022

Abstract In this article, different types of Gaussian quadrature methods have been presented to find the numerical integration of a neutrosophic valued function. A new definition of the distance between two neutrosophic number has been defined and it has been proved that the distance and the set of all neutrosophic number form a complete metric space. Also, the definition of neutrosophic continuity on a closed-bounded interval has been defined in the sense of (α, β, γ)-cut. This is the first time, when the Gauss–Legendre integration, Gauss–Chebyshev integration and Gauss–Laguerre integration rule have been discussed in neutrosophic environment. In the first test example, the comparison between one-point, two-point and three-point Gauss–Legendre integration rules have been presented in terms of tables and figures. Also, in the second example, the comparison between one-point, two-point Gauss–Chebyshev and Gauss–Laguerre integration rules have been discussed.

Mathematics Subject Classification (2000) 65D30 · 65D32 · 97I50 · 65G50

1 Introduction

Zadeh [18,38] discovered the concept of fuzzy sets. After the invention of fuzzy sets, so many generalizations have been done on fuzzy sets, like intuitionistic fuzzy sets [7], neutrosophic sets [26], etc. Then, there are so many authors who have been working on the development of the application of neutrosophic set theory in different fields of neutrosophic mathematics. Salama et al. [28] extended the concept of fuzzy topological space [16] and intuitionistic topological space [19] into the concept of neutrosophic topological space. Agboola et al. [6] presented the concept of neutrosophic groups and neutrosophic subgroups. Also, in the same article they have shown that the neutrosophic groups which are generated by neutrosophic element I and any neutrosophic group which is isomorphic to Klein's 4-group all are Lagrange neutrosophic groups. After that, Agboola et al. [5] presented neutrosophic rings and their properties. Also, they presented the structure of polynomial rings in neutrosophic environment. Agboola et al. [4] studied neutrosophic vector spaces, neutrosophic quotient spaces and homomorphism of neutrosophic vector spaces. Broumi et al. [15] proposed the idea of uniform single valued neutrosophic graph and they also developed an algorithm for computing the complement of single valued graph. After that, Broumi et al. [14] introduced some MATLAB computing procedures for computing the operational matrices in neutrosophic environment. Then, Broumi et al. [13] introduced a new score function for interval valued neutrosophic numbers and they solved the shortest path problem with the help of interval valued neutrosophic numbers. In the same article, Broumi et al. [13], introduced an algorithm

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to study the neutrosophic shortest path. Also, Broumi et al. [11] considered bellman algorithm to solve the shortest path problem in neutrosophic environment. Saranya et al. [27] introduced a computer application for finding the complement, union, intersection and inclusion of any neutrosophic sets. Recently, many authors are still working on the development of neutrosophic sets and its applications [17,20,21]. Haque et al. [20], proposed an innovative logarithmic operational law and aggregation operators for trapezoidal neutrosophic number and they also demonstrated a new type of multi-criteria group decision making (MCGDM) strategy for determining the most harmful viruses. After that, Haque et al. [21] defined a new type of exponential operational law for trapezoidal neutrosophic numbers and they introduced a new MCGDM method to measure pollution attributes in different mega-cities. Also, neutrosophic calculus is one of the most important aspects in neutrosophic mathematics and neutrosophic calculus is the generalization of fuzzy calculus. In the recent time, many authors are working on the development of fuzzy calculus [8,9,12]. In the next subsection we will discuss about the concept of neutrosophic calculus.

1.1 Neutrosophic calculus

In our literature survey, it has been seen that there are so many authors who have been working on the development of neutrosophic calculus. Smarandache [25] studied the neutrosophic calculus, where he discussed about neutrosophic limits, neutrosophic derivatives and neutrosophic integrals. Also in the same article [25] Smarandache introduced the concept of neutrosophic mereo-limit, mereo-continuity, mereo-derivative and mereo-integral. Son et al. [29] introduced some properties of granular calculus for single valued neutrosophic functions. They also defined some concept of limit, derivative and integration for single-valued neutrosophic functions. After that, Son et al. [30] introduced fractional derivative, Riemann–Liouville fractional derivatives and Caputo fractional derivatives in neutrosophic environment. Also, in the same article they have been investigated the numerical and exact solutions of some neutrosophic fractional differential equations and neutrosophic partial differential equations. Biswas et al. [10] developed the concept of neutrosophic Riemann integration. Also, they have defined some properties of neutrosophic set, neutrosophic number and neutrosophic function. Recently, Alhasan [1] studied neutrosophic integrals and integration methods. Alhasan also defined the concept of neutrosophic trigonometric identities. Neutrosophic differential equations plays a very important role in the field of neutrosophic calculus. Sumathi et al. [31] introduced the concept of first order neutrosophic ordinary differential equations. Also, Sumathi et al. [35] studied the solution of second-order linear neutrosophic differential equation with trapezoidal neutrosophic numbers as boundary conditions. Recently, Moi et al. [24] introduced second-order neutrosophic boundary-value problem with different types of first- and second-order derivatives. They also presented some properties of neutrosophic derivative and neutrosophic numbers.

1.2 Motivation

We have studied all the works cited above and it has been seen that no work has been done on the numerical integration of neutrosophic valued function. But lots of work have been done on the numerical integration of fuzzy valued function [2,3] and real valued function [32,37]. So, there are huge scope to develop this type of work in neutrosophic environment. Also, in future, we will try to work on the numerical solution of neutrosophic integral equations and neutrosophic integro-differential equations. In our literature survey, it has been observed that no work has been done on the numerical solution of neutrosophic integral equations and neutrosophic integro-differential equations. We know that it is necessary to work on numerical integration methods of neutrosophic valued functions for finding the numerical solution of neutrosophic integral equations and neutrosophic integro-differential equations. But no work has been done on the numerical integration of neutrosophic valued function. So, from there we are motivated to write this article and work on the numerical integration of neutrosophic valued function.

1.3 Novelty

The purpose of this article is as follows:

- To define a new definition of distance between two neutrosophic numbers.

- To proof that $(\mathcal{N}, D)$ is a complete metric space, where $\mathcal{N}$ is the set of all neutrosophic numbers and D is distance between two neutrosophic numbers.
- To define neutrosophic continuity in the sense of (α, β, γ)-cut.
- To introduce Gaussian-Quadrature integration rule for neutrosophic valued function.
- To prove that the convergences of $J(f_{T_1}; \alpha)$, $J(f_{T_2}; \alpha)$, $J(f_{I_1}; \beta)$, $J(f_{I_2}; \beta)$, $J(f_{F_1}; \gamma)$, $J(f_{F_2}; \gamma)$ to $I(f_{T_1}; \alpha)$, $I(f_{T_2}; \alpha)$, $I(f_{I_1}; \beta)$, $I(f_{I_2}; \beta)$, $I(f_{F_1}; \gamma)$, $I(f_{F_2}; \gamma)$ are uniform. The definition of these symbols are given in Sect. 4.
- To compare the numerical solutions of Gauss–Legendre integration rule, Gauss–Chebyshev integration rule and Gauss–Laguerre integration rule.

1.4 Structure of the article

The rest of the article is organized as follows. Some preliminaries for this article has been given in Sect. 2. Some important properties of neutrosophic calculus has been given in Sect. 3. In Sect. 4, the Gaussian quadrature rule for neutrosophic valued function has been presented. Section 5 contains some numerical examples which shows the validation of the proposed method. At last a brief conclusion of this article has been given in Sect. 6.

2 Preliminaries

Definition 2.1 [34] $N = \{\langle x, T_N(x), I_N(x), F_N(x)\rangle : x \in U\}$ is a single valued neutrosophic set over U if, $T_N : U \to [0, 1]$, $I_N : U \to [0, 1]$, $F_N : U \to [0, 1]$ such that $0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3, \forall x \in U$, where $T_N(x)$, $I_N(x)$ and $F_N(x)$ denote truth, indeterminacy and falsity membership function, respectively.

Definition 2.2 [35] The (α, β, γ)-cut of a neutrosophic set N is $N_{(\alpha,\beta,\gamma)} = \{\langle T_N(x), I_N(x), F_N(x)\rangle : x \in U, T_N(x) \geq \alpha, I_N(x) \leq \beta, F_N(x) \leq \gamma\}$, where $\alpha, \beta, \gamma \in [0, 1]$.

Proposition 2.3 [24] *Let us consider $\tilde{a}$ and $\tilde{b}$ be the two neutrosophic numbers, then*

1. $(\tilde{a} \odot \tilde{b})_{(\alpha,\beta,\gamma)} = \tilde{a}_{(\alpha,\beta,\gamma)} \odot \tilde{b}_{(\alpha,\beta,\gamma)}$, *where $\odot$ denotes any binary operation $'+', '-'$ and $'\times'$.*
2. $(\lambda\tilde{a})_{(\alpha,\beta,\gamma)} = \lambda\tilde{a}_{(\alpha,\beta,\gamma)}$, *here λ is any non zero real number.*

Definition 2.4 [23] $f : I \to \mathcal{N}$ be a neutrosophic function on I. Let $x_0 \in I$, then generalized neutrosophic derivative of the function $f(x)$ at x_0 is denoted by $f'(x_0)$ and defined by

1. $f'_{T\alpha} = \left[\min\left\{f'_{T_1}(x_0; \alpha), f'_{T_2}(x_0; \alpha)\right\}, \max\left\{f'_{T_1}(x_0; \alpha), f'_{T_2}(x_0; \alpha)\right\}\right]$, if $f'_{T_1}(x_0; \alpha)$, $f'_{T_2}(x_0; \alpha)$ exists.
2. $f'_{I\beta} = \left[\min\left\{f'_{I_1}(x_0; \beta), f'_{I_2}(x_0; \beta)\right\}, \max\left\{f'_{I_1}(x_0; \beta), f'_{I_2}(x_0; \beta)\right\}\right]$, if $f'_{I_1}(x_0; \beta)$, $f'_{I_2}(x_0; \beta)$ exists.
3. $f'_{F\gamma} = \left[\min\left\{f'_{F_1}(x_0; \gamma), f'_{F_2}(x_0; \gamma)\right\}, \max\left\{f'_{F_1}(x_0; \gamma), f'_{F_2}(x_0; \gamma)\right\}\right]$, if $f'_{F_1}(x_0; \gamma)$, $f'_{F_2}(x_0; \gamma)$ exists.

$f'(x)$ is called type-1 derivative at x_0 if $[f'(x_0)]_{(\alpha,\beta,\gamma)} = \langle\left[f'_{T_1}(x_0; \alpha), f'_{T_2}(x_0; \alpha)\right], \left[f'_{I_1}(x_0; \beta), f'_{I_2}(x_0; \beta)\right], [f'_{F_1}(x_0; \gamma), f'_{F_2}(x_0; \gamma)]\rangle$ and type-2 derivative at x_0 if $[f'(x_0)]_{(\alpha,\beta,\gamma)} = \langle[f'_{T_2}(x_0; \alpha), f'_{T_1}(x_0; \alpha)], [f'_{I_2}(x_0; \beta), f'_{I_1}(x_0; \beta)], [f'_{F_2}(x_0; \gamma), f'_{F_1}(x_0; \gamma)]\rangle$.

Theorem 2.5 [22] *Let $g : [a_1, a_2] \to \mathbb{R}$ be a real valued continuous function. Then for any $\epsilon > 0$, $\exists$ a polynomial p on $[a_1, a_2]$ such that $|g - p| < \epsilon$.*

Definition 2.6 [36] Let $f(x)$ be a closed and bounded fuzzy valued function on [a,b]. Suppose $f_1(x; \alpha)$ and $f_2(x; \alpha)$ are Riemann integrable on [a,b]. Let

$$A_\alpha = \left[\int_a^b f_1(x; \alpha)\mathrm{d}x, \int_a^b f_2(x; \alpha)\mathrm{d}x\right]$$

Then we say that $f(x)$ is fuzzy Riemann integrable on [a,b]. The membership function of $\int_a^b f(x)\mathrm{d}x$ is defined by for $r \in A_0$

$$\xi_{\int_a^b f(x)\mathrm{d}x}(r) = \sup_{0\le\alpha\le 1} \alpha 1_{A_\alpha}(r)$$

Definition 2.7 [33] Let $f : [a, b] \to \mathcal{N}$ be a neutrosophic function which is continuous on [a, b]. Then neutrosophic integral of f on [a, b] defined as

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(C_i)\frac{b-a}{n},$$

where $C_i \in [x_{i-1}, x_i]$, for $i \in \{1, 2, \dots, n\}$ and $a = x_0 < x_1 < \cdots < x_{n-1} < x_n$ are the partitions of the interval $[a, b]$.

This definition exactly same as Riemann integral but $f(C_i)$ may be a real set or may have some indeterminacy.

3 Some important properties of neutrosophic calculus

Definition 3.1 Let $A, B \in \mathcal{N}$, where $\mathcal{N}$ is the set of all neutrosophic numbers and $D : \mathcal{N} \times \mathcal{N} \to \mathbb{R}^+ \cup \{0\}$ be a mapping defined by

$$D(A, B) = \frac{1}{3} \sup_{\alpha,\beta,\gamma} \{\max\{|A_{T_1}(\alpha) - B_{T_1}(\alpha)|, |A_{T_2}(\alpha) - B_{T_2}(\alpha)|\} + \max\{|A_{I_1}(\beta) - B_{I_1}(\beta)|, |A_{I_2}(\beta) - B_{I_2}(\beta)|\} + \max\{|A_{F_1}(\gamma) - B_{F_1}(\gamma)|, |A_{F_2}(\gamma) - B_{F_2}(\gamma)|\}\}.$$

Then D is called the distance function and the distance between two neutrosophic numbers A and B is defined by $D(A, B)$.

Theorem 3.2 Let D is the distance between two neutrosophic numbers which is defined in Definition 3.1 and $\mathcal{N}$ is the set of all neutrosophic numbers. Then,

1. D is metric on $\mathcal{N}$.
2. $(\mathcal{N}, D)$ is a complete metric space.

Proof 1. The proof of the first three axioms are obvious. So, we just show the triangle inequality.
Let $A, B, C \in \mathcal{N}$. Then we have to show that $D(A, C) \le D(A, B) + D(B, C)$.
Since

$$\begin{aligned}|A_{T_1}(\alpha) - C_{T_1}(\alpha)| &\le |A_{T_1}(\alpha) - B_{T_1}(\alpha)| + |B_{T_1}(\alpha) - C_{T_1}(\alpha)| \\ &\le \max\{|A_{T_1}(\alpha) - B_{T_1}(\alpha)|, |A_{T_2}(\alpha) - B_{T_2}(\alpha)|\} + \max\{|B_{T_1}(\alpha) \\ &\quad - C_{T_1}(\alpha)|, |B_{T_2}(\alpha) - C_{T_2}(\alpha)|\}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}|A_{T_2}(\alpha) - C_{T_2}(\alpha)| &\le \max\{|A_{T_1}(\alpha) - B_{T_1}(\alpha)|, |A_{T_2}(\alpha) - B_{T_2}(\alpha)|\} + \max\{|B_{T_1}(\alpha) \\ &\quad - C_{T_1}(\alpha)|, |B_{T_2}(\alpha) - C_{T_2}(\alpha)|\},\end{aligned}$$

we know that, if $a \le c$ and $b \le c$ then $\max(a, b) \le c$. Therefore,

$$\begin{aligned}\max\{|A_{T_1}(\alpha) - C_{T_1}(\alpha)|, |A_{T_2}(\alpha) - C_{T_2}(\alpha)|\} &\le \max\{|A_{T_1}(\alpha) - B_{T_1}(\alpha)|, |A_{T_2}(\alpha) - B_{T_2}(\alpha)|\} \\ &\quad + \max\{|B_{T_1}(\alpha) - C_{T_1}(\alpha)|, |B_{T_2}(\alpha) - C_{T_2}(\alpha)|\}.\end{aligned} \tag{1}$$

Similarly,

$$\begin{aligned}\max\{|A_{I_1}(\beta) - C_{I_1}(\beta)|, |A_{I_2}(\beta) - C_{I_2}(\beta)|\} &\le \max\{|A_{I_1}(\beta) - B_{I_1}(\beta)|, |A_{I_2}(\beta) - B_{I_2}(\beta)|\} \\ &\quad + \max\{|B_{I_1}(\beta) - C_{I_1}(\beta)|, |B_{I_2}(\beta) - C_{I_2}(\beta)|\} \\ \max\{|A_{F_1}(\gamma) - C_{F_1}(\gamma)|, |A_{F_2}(\gamma) - C_{F_2}(\gamma)|\} &\le \max\{|A_{F_1}(\gamma) - B_{F_1}(\gamma)|, |A_{F_2}(\gamma) - B_{F_2}(\gamma)|\}\end{aligned} \tag{2}$$

$$+\max\{|B_{F_1}(\gamma) - C_{F_1}(\gamma)|, |B_{F_2}(\gamma) - C_{F_2}(\gamma)|\}. \tag{3}$$

Add Eqs. (1)–(3). Taking supremum and multiplying both sides by $1/3$, we have

$$D(A, C) \leq D(A, B) + D(B, C), \quad \forall A, B, C \in \mathcal{N}.$$

Therefore, D is metric on $\mathcal{N}$.

2. Let $\{a_n\}$ be any Cauchy sequence in $\mathcal{N}$. So, for all $\epsilon > 0$, there exists $m \in \mathbb{N}$ such that for all $n \geq m$ and $p \geq 1$,

$$D(a_{n+p}, a_n) < \epsilon.$$

Then for any $k_1, k_2, k_3 > 0$, there exists $m \in \mathbb{N}$ such that for all $n \geq m$ and $p \geq 1$, we have

$$\max\{|a_{T_1 n+p}(\alpha) - a_{T_1 n}(\alpha)|, |a_{T_2 n+p}(\alpha) - a_{T_2 n}(\alpha)|\} < k_1 \tag{4}$$

$$\max\{|a_{I_1 n+p}(\beta) - a_{I_1 n}(\beta)|, |a_{I_2 n+p}(\beta) - a_{I_2 n}(\beta)|\} < k_2 \tag{5}$$

$$\max\{|a_{F_1 n+p}(\gamma) - a_{F_1 n}(\gamma)|, |a_{F_2 n+p}(\gamma) - a_{F_2 n}(\gamma)|\} < k_3. \tag{6}$$

Then Eq. (4) implies that, for any $k_1 > 0$, there exists $m \in \mathbb{N}$ such that for all $n \geq m$ and $p \geq 1$, we have

$$|a_{T_1 n+p}(\alpha) - a_{T_1 n}(\alpha)| < k_1 \text{ and } |a_{T_2 n+p}(\alpha) - a_{T_2 n}(\alpha)| < k_1$$

This shows that $\{a_{T_1 n}\}$ and $\{a_{T_2 n}\}$ are the Cauchy sequences of real numbers. Similarly, we can show that $\{a_{I_1 n}\}$, $\{a_{I_2 n}\}$, $\{a_{F_1 n}\}$ and $\{a_{F_2 n}\}$ are the Cauchy sequences of real numbers. Hence all the sequences are convergent. Then,

$$\lim_{n\to\infty} a_{T_1 n} = a_{T_1}, \quad \lim_{n\to\infty} a_{T_2 n} = a_{T_2} \tag{7}$$

$$\lim_{n\to\infty} a_{I_1 n} = a_{I_1}, \quad \lim_{n\to\infty} a_{I_2 n} = a_{I_2} \tag{8}$$

$$\lim_{n\to\infty} a_{F_1 n} = a_{F_1}, \quad \lim_{n\to\infty} a_{F_2 n} = a_{F_2}. \tag{9}$$

So, $\{a_n\} \in \mathcal{N}$ be any Cauchy sequence converges to a.
Therefore, $(\mathcal{N}, D)$ is a complete metric space.

□

Definition 3.3 A neutrosophic valued function $f : \mathbb{R} \to \mathcal{N}$ is said to be continuous, if for an arbitrary $x_0 \in \mathbb{R}$ and $\epsilon > 0$, there exists $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow D(f(x), f(x_0)) < \epsilon$.

Throughout this article, we consider the neutrosophic valued function on closed bounded interval $[a, b]$. Now, we define the continuity of neutrosophic valued function in the sense of (α, β, γ)-cut.

Definition 3.4 Let $f : [a, b] \to \mathcal{N}$ be a neutrosophic valued function. Let $[f(x)]_{(\alpha,\beta,\gamma)} = \langle [f_{T_1}(x; \alpha), f_{T_2}(x; \alpha)], [f_{I_1}(x; \beta), f_{I_2}(x; \beta)], [f_{F_1}(x; \gamma), f_{F_2}(x; \gamma)] \rangle$. Then $f(x)$ is said to be continuous on $[a, b]$, if $f_{T_1}(x; \alpha)$, $f_{T_2}(x; \alpha)$, $f_{I_1}(x; \beta)$, $f_{I_2}(x; \beta)$, $f_{F_1}(x; \gamma)$ and $f_{F_2}(x; \gamma)$ are all continuous on $[a, b]$.

Definition 3.5 [10] Let $f : [a, b] \to \mathcal{N}$ be a neutrosophic valued function. Let $f_{T_1}(x; \alpha)$, $f_{T_2}(x; \alpha)$, $f_{I_1}(x; \beta)$, $f_{I_2}(x; \beta)$, $f_{F_1}(x; \gamma)$ and $f_{F_2}(x; \gamma)$ are all integrable on $[a, b]$. Then $f(x)$ is said to be integrable on $[a, b]$. Furthermore,

$$\left[\int_a^b f(x)\mathrm{d}x\right]_{(\alpha,\beta,\gamma)} = \left\langle \left[\int_a^b f_{T_1}(x;\alpha)\mathrm{d}x, \int_a^b f_{T_2}(x;\alpha)\mathrm{d}x\right], \left[\int_a^b f_{I_1}(x;\beta)\mathrm{d}x, \int_a^b f_{I_2}(x;\beta)\mathrm{d}x\right], \left[\int_a^b f_{F_1}(x;\gamma)\mathrm{d}x, \int_a^b f_{F_2}(x;\gamma)\mathrm{d}x\right] \right\rangle$$

Theorem 3.6 *If $f : [a, b] \to \mathcal{N}$ is continuous on $[a, b]$. Then it is also integrable on $[a, b]$.*

Proof Proof of the theorem is obvious. □

4 Gaussian quadrature rule

Let f be a neutrosophic valued function, then for $n \in \mathbb{N}$ the Gaussian quadrature formula is

$$\int_a^b f(x)w(x)\mathrm{d}x = \sum_{i=0}^{n} w_i f(x_i) + E(f),\ i = 0, 1, \ldots, n \tag{10}$$

which gives the approximate value of the integral $\int_a^b f(x)w(x)\mathrm{d}x$.

Now, taking $(\alpha, \beta, \gamma)-$cut of Eq. (10) we have

$$\int_a^b f_{T_j}(x;\alpha)w(x)\mathrm{d}x = \sum_{i=0}^{n} w_i f_{T_j}(x_i;\alpha) + E(f_{T_j};\alpha) \tag{11}$$

$$\int_a^b f_{I_j}(x;\beta)w(x)\mathrm{d}x = \sum_{i=0}^{n} w_i f_{I_j}(x_i;\beta) + E(f_{I_j};\beta) \tag{12}$$

$$\int_a^b f_{F_j}(x;\gamma)w(x)\mathrm{d}x = \sum_{i=0}^{n} w_i f_{F_j}(x_i;\gamma) + E(f_{F_j};\gamma) \tag{13}$$

w_i be the weights of real numbers, with the property that $\sum_{i=1}^{n} w_i = \int_a^b w(x)\mathrm{d}x$. This will hold when we take $f_{T_j}(x;\alpha) = f_{I_j}(x;\beta) = f_{F_j}(x;\gamma) = 1$.

Let

$$J(f_{T_j};\alpha) = \sum_{i=1}^{n} w_i f_{T_j}(x_i;\alpha),$$

$$J(f_{I_j};\beta) = \sum_{i=1}^{n} w_i f_{I_j}(x_i;\beta),$$

$$J(f_{F_j};\gamma) = \sum_{i=1}^{n} w_i f_{F_j}(x_i;\gamma).$$

Then from Eqs. (11)–(13), we have

$$I(f_{T_j};\alpha) = \int_a^b f_{T_j}(x;\alpha)w(x)\mathrm{d}x = J(f_{T_j};\alpha) + E(f_{T_j};\alpha)$$

$$I(f_{I_j};\beta) = \int_a^b f_{I_j}(x;\beta)w(x)\mathrm{d}x = J(f_{I_j};\beta) + E(f_{I_j};\beta)$$

$$I(f_{F_j};\gamma) = \int_a^b f_{F_j}(x;\gamma)w(x)\mathrm{d}x = J(f_{F_j};\gamma) + E(f_{F_j};\gamma).$$

The approximate error is

$$E(f_{T_j};\alpha) = \frac{f_{T_j}^{(2n)}(\delta_{T_j};\alpha)}{(2n)!}\int_a^b w(x)p^2(x)\mathrm{d}x = Af_{T_j}^{(2n)}(\delta_{T_j};\alpha),\ \delta_{T_j} \in (a,b)$$

$$E(f_{I_j};\beta) = \frac{f_{I_j}^{(2n)}(\delta_{I_j};\beta)}{(2n)!}\int_a^b w(x)p^2(x)\mathrm{d}x = Af_{I_j}^{(2n)}(\delta_{I_j};\beta),\ \delta_{I_j} \in (a,b)$$

$$E(f_{F_j};\gamma) = \frac{f_{F_j}^{(2n)}(\delta_{F_j};\gamma)}{(2n)!}\int_a^b w(x)p^2(x)\mathrm{d}x = Af_{F_j}^{(2n)}(\delta_{F_j};\gamma),\ \delta_{F_j} \in (a,b),$$

where $j = 1, 2$ denotes left and right branches, respectively, of truth, indeterminacy and falsity membership function. Also, $A = \dfrac{1}{(2n)!}\int_a^b w(x)p^2(x)\mathrm{d}x$ and $p(x) = \prod_{i=0}^{n}(x - x_i)$, x_i are the roots of Legendre, Chebyshev and Laguerre polynomials, where $i = 0, 1, \ldots, n$.

Theorem 4.1 *If $f(x)$ is continuous neutrosophic valued function. Then the convergence of $J(f_{T_1};\alpha)$, $J(f_{T_2};\alpha)$, $J(f_{I_1};\beta)$, $J(f_{I_2};\beta)$, $J(f_{F_1};\gamma)$, $J(f_{F_2};\gamma)$ to $I(f_{T_1};\alpha)$, $I(f_{T_2};\alpha)$, $I(f_{I_1};\beta)$, $I(f_{I_2};\beta)$, $I(f_{F_1};\gamma)$, $I(f_{F_2};\gamma)$, respectively, is uniform.*

Proof Let $\epsilon > 0$ and p be a polynomial in $\mathcal{N}$. Then,

$$J(p_{T_j};\alpha) - J(f_{T_j};\alpha) = \sum_{i=1}^{n} w_i p_{T_j}(x_i;\alpha) - \sum_{i=1}^{n} w_i f_{T_j}(x_i;\alpha) = \sum_{i=1}^{n} w_i(p_{T_j}(x_i;\alpha) - f_{T_j}(x_i;\alpha))$$
$$J(p_{I_j};\beta) - J(f_{I_j};\beta) = \sum_{i=1}^{n} w_i p_{I_j}(x_i;\beta) - \sum_{i=1}^{n} w_i f_{I_j}(x_i;\beta) = \sum_{i=1}^{n} w_i(p_{I_j}(x_i;\beta) - f_{I_j}(x_i;\beta))$$
$$J(p_{F_j};\gamma) - J(f_{F_j};\gamma) = \sum_{i=1}^{n} w_i p_{F_j}(x_i;\gamma) - \sum_{i=1}^{n} w_i f_{F_j}(x_i;\gamma) = \sum_{i=1}^{n} w_i(p_{F_j}(x_i;\gamma) - f_{F_j}(x_i;\gamma)),$$

where $j = 1, 2$ denotes left and right branches, respectively, of truth, indeterminacy and falsity membership function.
Also,

$$I(p_{T_j};\alpha) - I(f_{T_j};\alpha) = \int_a^b p_{T_j}(x;\alpha)w(x)\mathrm{d}x - \int_a^b f_{T_j}(x;\alpha)w(x)\mathrm{d}x = \int_a^b (p_{T_j}(x;\alpha) - f_{T_j}(x;\alpha))w(x)\mathrm{d}x$$
$$I(p_{I_j};\beta) - I(f_{I_j};\beta) = \int_a^b p_{I_j}(x;\beta)w(x)\mathrm{d}x - \int_a^b f_{I_j}(x;\beta)w(x)\mathrm{d}x = \int_a^b (p_{I_j}(x;\beta) - f_{I_j}(x;\beta))w(x)\mathrm{d}x$$
$$I(p_{F_j};\gamma) - I(f_{F_j};\gamma) = \int_a^b p_{F_j}(x;\gamma)w(x)\mathrm{d}x - \int_a^b f_{F_j}(x;\gamma)w(x)\mathrm{d}x = \int_a^b (p_{F_j}(x;\gamma) - f_{F_j}(x;\gamma))w(x)\mathrm{d}x,$$

where $j = 1, 2$ denotes left and right branches, respectively, of truth, indeterminacy and falsity membership function.

Since $J(p) = I(p)$, then we have

$$\begin{aligned}
D(J(f), I(f)) &\le D(J(f), J(p)) + D(I(p), I(f)) \\
&\le \frac{1}{3}\sup_{\alpha,\beta,\gamma}\{\max\{|J(f_{T_1};\alpha) - J(p_{T_1};\alpha)|, |J(f_{T_2};\alpha) - J(p_{T_2};\alpha)|\} + \max\{|J(f_{I_1};\beta) - J(p_{I_1};\beta)|, \\
&\quad |J(f_{I_2};\beta) - J(p_{I_2};\beta)|\} + \max\{|J(f_{F_1};\gamma) - J(p_{F_1};\gamma)|, |J(f_{F_2};\gamma) - J(p_{F_2};\gamma)|\}\} \\
&\quad + \frac{1}{3}\sup_{\alpha,\beta,\gamma}\{\max\{|I(f_{T_1};\alpha) - I(p_{T_1};\alpha)|, |I(f_{T_2};\alpha) - I(p_{T_2};\alpha)|\} + \max\{|I(f_{I_1};\beta) - I(p_{I_1};\beta)|, \\
&\quad |I(f_{I_2};\beta) - I(p_{I_2};\beta)|\} + \max\{|I(f_{F_1};\gamma) - I(p_{F_1};\gamma)|, |I(f_{F_2};\gamma) - I(p_{F_2};\gamma)|\}\} \\
&\le \epsilon\int_a^b w(x)\mathrm{d}x + \epsilon\sum_{i=0}^{n} w_i \qquad \text{(by Theorem 2.1)} \\
&= 2\epsilon\int_a^b w(x)\mathrm{d}x.
\end{aligned}$$

This completes the proof. □

4.1 Gauss–Legendre integration rule

In this integration rule, we have $w(x) = 1$ and $[a, b] = [-1, 1]$. Then,

$$\int_{-1}^{1} f_{T_j}(x; \alpha) = \sum_{i=0}^{n} w_i f_{T_j}(x; \alpha) + E(f_{T_j}; \alpha) \tag{14}$$

$$\int_{-1}^{1} f_{I_j}(x; \beta) = \sum_{i=0}^{n} w_i f_{I_j}(x; \beta) + E(f_{I_j}; \beta) \tag{15}$$

$$\int_{-1}^{1} f_{F_j}(x; \gamma) = \sum_{i=0}^{n} w_i f_{F_j}(x; \gamma) + E(f_{F_j}; \gamma), \tag{16}$$

where $E(f_{T_j}; \alpha) = \dfrac{f_{T_j}^{(2n)}(\delta_{T_j}; \alpha)}{(2n)!} \int_{-1}^{1} p^2(x)\mathrm{d}x$, $E(f_{I_j}; \beta) = \dfrac{f_{I_j}^{(2n)}(\delta_{I_j}; \beta)}{(2n)!} \int_{-1}^{1} p^2(x)\mathrm{d}x$ and $E(f_{F_j}; \gamma) = \dfrac{f_{F_j}^{(2n)}(\delta_{F_j}; \gamma)}{(2n)!} \int_{-1}^{1} p^2(x)\mathrm{d}x$, $\delta_{T_J}, \delta_{I_J}, \delta_{F_J} \in (-1, 1)$. Here $j = 1, 2$ denotes left and right branch respectively of truth, indeterminacy and falsity membership function. $p(x) = \prod_{i=0}^{n}(x - x_i)$, $i = 0, 1, \ldots, n$ are the roots of the Legendre polynomial and $\sum_{i=1}^{n} w_i = 2$.

4.1.1 One-point Gauss–Legendre integration rule

$$I(f_{T_j}; \alpha) = 2f_{T_j}(0; \alpha) + E(f_{T_j}; \alpha),$$
$$I(f_{I_j}; \beta) = 2f_{I_j}(0; \beta) + E(f_{I_j}; \beta),$$
$$I(f_{F_j}; \gamma) = 2f_{F_j}(0; \gamma) + E(f_{F_j}; \gamma),$$

where $E(f_{T_j}; \alpha) = \dfrac{1}{3} f_{T_j}^{''}(\delta_{T_j}; \alpha)$, $E(f_{I_j}; \beta) = \dfrac{1}{3} f_{I_j}^{''}(\delta_{I_j}; \beta)$ and $E(f_{F_j}; \gamma) = \dfrac{1}{3} f_{F_j}^{''}(\delta_{F_j}; \gamma)$. Here $\delta_{T_J}, \delta_{I_j}, \delta_{F_j} \in (-1, 1)$ and $j = 1, 2$.

4.1.2 Two-point Gauss–Legendre integration rule

$$I(f_{T_j}; \alpha) = f_{T_j}\left(-\frac{1}{\sqrt{3}}; \alpha\right) + f_{T_j}\left(\frac{1}{\sqrt{3}}; \alpha\right) + E(f_{T_j}; \alpha)$$
$$I(f_{I_j}; \beta) = f_{I_j}\left(-\frac{1}{\sqrt{3}}; \beta\right) + f_{I_j}\left(\frac{1}{\sqrt{3}}; \beta\right) + E(f_{I_j}; \beta)$$
$$I(f_{F_j}; \gamma) = f_{F_j}\left(-\frac{1}{\sqrt{3}}; \gamma\right) + f_{F_j}\left(\frac{1}{\sqrt{3}}; \gamma\right) + E(f_{F_j}; \gamma)$$

where $E(f_{T_j}; \alpha) = \dfrac{1}{135} f_{T_j}^{(4)}(\delta_{T_j}; \alpha)$, $E(f_{I_j}; \beta) = \dfrac{1}{135} f_{I_j}^{(4)}(\delta_{I_j}; \beta)$ and $E(f_{F_j}; \gamma) = \dfrac{1}{135} f_{F_j}^{(4)}(\delta_{T_j}; \gamma)$. Here $\delta_{T_j}, \delta_{I_j}, \delta_{F_j} \in (-1, 1)$ and $j = 1, 2$.

4.1.3 Three-point Gauss–Legendre integration rule

$$I(f_{T_j};\alpha)=\frac{5}{9}f_{T_j}\left(-\sqrt{\frac{3}{5}};\alpha\right)+\frac{8}{9}f_{T_j}(0;\alpha)+\frac{5}{9}f_{T_j}\left(\sqrt{\frac{3}{5}};\alpha\right)+E(f_{T_j};\alpha)$$
$$I(f_{I_j};\beta)=\frac{5}{9}f_{I_j}\left(-\sqrt{\frac{3}{5}};\beta\right)+\frac{8}{9}f_{I_j}(0;\beta)+\frac{5}{9}f_{I_j}\left(\sqrt{\frac{3}{5}};\beta\right)+E(f_{I_j};\beta)$$
$$I(f_{F_j};\gamma)=\frac{5}{9}f_{F_j}\left(-\sqrt{\frac{3}{5}};\gamma\right)+\frac{8}{9}f_{F_j}(0;\gamma)+\frac{5}{9}f_{F_j}\left(\sqrt{\frac{3}{5}};\gamma\right)+E(f_{F_j};\gamma),$$

where $E(f_{T_j};\alpha)=\frac{1}{15750}f_{T_j}^{(6)}(\delta_{T_j};\alpha)$, $E(f_{I_j};\beta)=\frac{1}{15750}f_{I_j}^{(6)}(\delta_{I_j};\beta)$ and $E(f_{F_j};\gamma)=\frac{1}{15750}f_{F_j}^{(6)}(\delta_{F_j};\gamma)$. Here $\delta_{T_j},\delta_{I_j},\delta_{F_j}\in(-1,1)$ and $j=1,2$.

4.2 Gauss–Chebyshev integration rule

In this integration rule, we have $w(x)=\frac{1}{\sqrt{1-x^2}}$ and $[a,b]=[-1,1]$. Then,

$$\int_{-1}^{1}\frac{f_{T_j}(x;\alpha)}{\sqrt{1-x^2}}=\sum_{i=0}^{n}w_i f_{T_j}(x;\alpha)+E(f_{T_j};\alpha) \tag{17}$$
$$\int_{-1}^{1}\frac{f_{I_j}(x;\beta)}{\sqrt{1-x^2}}=\sum_{i=0}^{n}w_i f_{I_j}(x;\beta)+E(f_{I_j};\beta) \tag{18}$$
$$\int_{-1}^{1}\frac{f_{F_j}(x;\gamma)}{\sqrt{1-x^2}}=\sum_{i=0}^{n}w_i f_{F_j}(x;\gamma)+E(f_{F_j};\gamma), \tag{19}$$

where $E(f_{T_j};\alpha)=\frac{f_{T_j}^{(2n)}(\delta_{T_j};\alpha)}{(2n)!}\int_{-1}^{1}\frac{p^2(x)}{\sqrt{1-x^2}}\mathrm{d}x$, $E(f_{I_j};\beta)=\frac{f_{I_j}^{(2n)}(\delta_{I_j};\beta)}{(2n)!}\int_{-1}^{1}\frac{p^2(x)}{\sqrt{1-x^2}}\mathrm{d}x$ and $E(f_{F_J};\gamma)=\frac{f_{F_J}^{(2n)}(\delta_{F_J};\gamma)}{(2n)!}\int_{-1}^{1}\frac{p^2(x)}{\sqrt{1-x^2}}\mathrm{d}x$. Here $\delta_{T_J},\delta_{I_J},\delta_{F_J}\in(-1,1)$ and $j=1,2$. $p(x)=\prod_{i=0}^{n}(x-x_i)$, $i=0,1,\ldots,n$ are the roots of the Chebyshev polynomials and $\sum_{i=1}^{n}w_i=\pi$.

4.2.1 One-point Gauss–Chebyshev integration rule

$$I(f_{T_J};\alpha)=\pi f_{T_J}(0;\alpha)+E(f_{T_J};\alpha)$$
$$I(f_{I_J};\beta)=\pi f_{I_J}(0;\beta)+E(f_{I_J};\beta)$$
$$I(f_{F_J};\gamma)=\pi f_{F_J}(0;\gamma)+E(f_{F_J};\gamma)$$

where $E(f_{T_J};\alpha)=\frac{1}{2}f_{T_J}''(\delta_{T_J};\alpha)\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}$, $E(f_{I_J};\beta)=\frac{1}{2}f_{I_J}''(\delta_{I_J};\beta)\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}$ and $E(f_{F_j};\gamma)=\frac{1}{2}f_{F_j}''(\delta_{F_j};\gamma)\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}$. Here $j=1,2$ and $\delta_{T_j},\delta_{I_j},\delta_{F_j}\in(-1,1)$.

4.2.2 Two-point Gauss–Chebyshev integration rule

$$I(f_{T_j};\alpha)=\frac{\pi}{2}f_{T_j}\left(-\frac{1}{\sqrt{2}};\alpha\right)+\frac{\pi}{2}f_{T_j}\left(\frac{1}{\sqrt{2}};\alpha\right)+E(f_{T_j};\alpha)$$
$$I(f_{I_j};\beta)=\frac{\pi}{2}f_{I_j}\left(-\frac{1}{\sqrt{2}};\beta\right)+\frac{\pi}{2}f_{I_j}\left(\frac{1}{\sqrt{2}};\beta\right)+E(f_{I_j};\beta)$$
$$I(f_{F_j};\gamma)=\frac{\pi}{2}f_{F_j}\left(-\frac{1}{\sqrt{2}};\gamma\right)+\frac{\pi}{2}f_{F_j}\left(\frac{1}{\sqrt{2}};\gamma\right)+E(f_{F_j};\gamma)$$

where $E(f_{T_j};\alpha)=\frac{1}{4!}f_{T_j}^{(4)}\left(\delta_{T_j};\alpha\right)\int_{-1}^{1}\frac{\left(x^2-\frac{1}{2}\right)^2}{\sqrt{1-x^2}}$, $E(f_{I_j};\beta)=\frac{1}{4!}f_{I_j}^{(4)}(\delta_{I_j};\beta)\int_{-1}^{1}\frac{\left(x^2-\frac{1}{2}\right)^2}{\sqrt{1-x^2}}$ and $E(f_{F_j};\gamma)=\frac{1}{4!}f_{F_j}^{(4)}\left(\delta_{F_j};\gamma\right)\int_{-1}^{1}\frac{(x^2-\frac{1}{2})^2}{\sqrt{1-x^2}}$. Here $j=1,2$ and $\delta_{T_j},\delta_{I_j},\delta_{F_j}\in(-1,1)$.

4.3 Gauss–Laguerre integration rule

In this integration rule, we have $w(x)=e^{-x}$ and $[a,b]=[0,\infty)$. Then,

$$\int_0^\infty f_{T_j}(x;\alpha)e^{-x}=\sum_{i=0}^n w_i f_{T_j}(x;\alpha)+E(f_{T_j};\alpha) \tag{20}$$
$$\int_0^\infty f_{I_j}(x;\beta)e^{-x}=\sum_{i=0}^n w_i f_{I_j}(x;\beta)+E(f_{I_j};\beta) \tag{21}$$
$$\int_0^\infty f_{F_j}(x;\gamma)e^{-x}=\sum_{i=0}^n w_i f_{F_j}(x;\gamma)+E(f_{F_j};\gamma), \tag{22}$$

where $E(f_{T_j};\alpha)=\frac{f_{T_j}^{(2n)}(\delta_{T_j};\alpha)}{(2n)!}\int_0^\infty p^2(x)e^{-x}\mathrm{d}x$, $E(f_{I_j};\beta)=\frac{f_{I_j}^{(2n)}(\delta_{I_j};\beta)}{(2n)!}\int_0^\infty p^2(x)e^{-x}\mathrm{d}x$ and $E(f_{F_j};\gamma)=\frac{f_{F_j}^{(2n)}(\delta_{F_j};\gamma)}{(2n)!}\int_0^\infty p^2(x)e^{-x}\mathrm{d}x$. Here $\delta_{T_j},\delta_{I_j},\delta_{F_j}\in(-1,1)$ and $j=1,2$. $p(x)=\prod_{i=0}^{n}(x-x_i)$, $i=0,1,\ldots,n$ are the roots of the Laguerre polynomial and $\sum_{i=1}^{n}w_i=1$.

4.3.1 One-point Gauss–Laguerre integration rule

$$I(f_{T_j};\alpha)=f_{T_j}(1;\alpha)+E(f_{T_j};\alpha)$$
$$I(f_{I_j};\beta)=f_{I_j}(1;\beta)+E(f_{I_j};\beta)$$
$$I(f_{F_j};\gamma)=f_{F_j}(1;\gamma)+E(f_{F_j};\gamma)$$

where $E(f_{T_j};\alpha)=\frac{1}{2}f_{T_j}^{''}(\delta_{T_j};\alpha)\int_0^\infty(x-1)^2e^{-x}\mathrm{d}x$, $E(f_{I_j};\beta)=\frac{1}{2}f_{I_j}^{''}(\delta_{I_j};\beta)\int_0^\infty(x-1)^2e^{-x}\mathrm{d}x$ and $E(f_{F_j};\gamma)=\frac{1}{2}f_{F_j}^{''}(\delta_{F_j};\gamma)\int_0^\infty(x-1)^2e^{-x}\mathrm{d}x$, here $j=1,2$ and $\delta_{T_J},\delta_{I_J},\delta_{F_J}\in(-1,1)$.

4.3.2 Two-point Gauss–Laguerre integration rule

$$I(f_{T_j};\alpha) = 0.853553 f_{T_j}(x_0;\alpha) + 0.146447 f_{T_j}(x_1;\alpha) + E(f_{T_j};\alpha)$$
$$I(f_{I_j};\beta) = 0.8535533 f_{I_j}(x_0;\beta) + 0.146447 f_{I_j}(x_1;\beta) + E(f_{I_j};\beta),$$
$$I(f_{F_j};\gamma) = 0.853553 f_{F_j}(x_0;\gamma) + 0.146447 f_{F_j}(x_1;\gamma) + E(f_{F_j};\gamma),$$

where $E(f_{T_j};\alpha) = \frac{1}{4!} f_{T_j}^{(4)}(\delta_{T_j};\alpha) \int_0^\infty (x-x_0)^2(x-x_1)^2 e^{-x}\mathrm{d}x$, $E(f_{I_j};\beta) = \frac{1}{4!} f_{T_j}^{(4)}(\delta_{I_j};\beta) \int_0^\infty (x-x_0)^2(x-x_1)^2 e^{-x}\mathrm{d}x$ and $E(f_{F_j};\gamma) = \frac{1}{4!} f_{F_j}^{(4)}(\delta_{F_j};\gamma) \int_0^\infty (x-x_0)^2(x-x_1)^2 e^{-x}\mathrm{d}x$. Here $\delta_{T_j}, \delta_{I_j}, \delta_{F_j} \in (-1,1)$ and $j = 1, 2$. Also, here $x_0 = 0.585786$ and $x_1 = 3.414213$.

4.4 Limitations of Gaussian quadrature rule

The Gaussian quadrature method has some limitations.

1. The main limitation of the Gaussian quadrature method is that it is very difficult to calculate the node points and weights.
2. The nodes are the roots of a polynomial. The roots of the polynomial are computed by the traditional root finding method and these methods are not always give accurate results. So, there always involved some errors in the computed nodes.

5 Examples

In this section, some examples and their numerical results have been discussed. The numerical computations have been done by Wolfram Mathematica 9.0 and the figures have been drawn by MATLAB R2018a.

Example 5.1 Let us consider the neutrosophic integral as follows:

$$\int_{-1}^{1} f(x)\mathrm{d}x,\ \text{where } f(x) = ax^2 + \sin^{-1}(x) + b\cos^{-1}(x),\ a = \langle(-1,1,3); 0.6, 0.4, 0.2\rangle \text{ and } b = \langle(0,1,2); 0.6, 0.4, 0.2\rangle$$

The exact solution of this integral is $\left\langle \left[\frac{(20+15\pi)\alpha - 6}{9}, \frac{18(\pi+1) - (20+15\pi)\alpha}{9}\right], \left[\frac{14+15\pi - (20+15\pi)\beta}{9}, \frac{3\pi - 2 + (20+15\pi)\beta}{9}\right], \left[\frac{12+15\pi - (20+15\pi)\gamma}{12}, \frac{4+9\pi + (20+15\pi)\gamma}{12}\right]\right\rangle$

Now, we calculate the integral with the help of one-point Gauss–Legendre integration rule, two-point Gauss–Legendre integration rule and three-point Gauss–Legendre integration rule.

One-point Gauss–Legendre integration rule

The errors for the One-Point Gauss–Legendre integration rules are

$$E(f_{T_1};\alpha) = \frac{1}{9}\left(\frac{(3-5\alpha)\delta_{T_1}}{(1-\delta_{T_1}^2)^{3/2}} + 20\alpha - 6\right),\ E(f_{T_2};\alpha) = \frac{1}{9}\left(\frac{(5\alpha-3)\delta_{T_1}}{(1-\delta_{T_1}^2)^{3/2}} + 18 - 20\alpha\right),$$
$$E(f_{I_1};\beta) = \frac{1}{9}\left(\frac{(5\beta-2)\delta_{I_1}}{(1-\delta_{I_1}^2)^{3/2}} + 14 - 20\beta\right),\ E(f_{I_2};\beta) = \frac{1}{9}\left(\frac{(2-5\beta)\delta_{I_2}}{(1-\delta_{I_2}^2)^{3/2}} + 20\beta - 2\right),$$

$$E(f_{F_1};\gamma)=\frac{1}{3}\left(\frac{(5\gamma-1)\delta_{F_1}}{(4(1-\delta_{F_1}^2)^{3/2}}+3-5\gamma\right),\ E(f_{F_2};\gamma)=\frac{1}{3}\left(\frac{(1-5\gamma)\delta_{F_2}}{(4(1-\delta_{F_2}^2)^{3/2}}+5\gamma+1\right),$$

where $\delta_{T_j}, \delta_{I_j}, \delta_{F_j} \in (-1, 1)$ and $j = 1, 2$ denotes left and right branches, respectively, of truth, indeterminacy and falsity membership function.

Two-point Gauss–Legendre integration rule

The errors for the two-point Gauss–Legendre integration rules are

$$E(f_{T_1};\alpha)=\frac{(3-5\alpha)\delta_{T_1}\left(2\delta_{T_1}^2+3\right)}{135(1-\delta_{T_1}^2)^{7/2}},\ E(f_{T_2};\alpha)=\frac{(5\alpha-3)\delta_{T_2}\left(2\delta_{T_2}^2+3\right)}{135(1-\delta_{T_2}^2)^{7/2}},$$

$$E(f_{I_1};\beta)=\frac{(5\beta-2)\delta_{I_1}\left(2\delta_{I_1}^2+3\right)}{135(1-\delta_{I_1}^2)^{7/2}},$$

$$E(f_{I_2};\beta)=\frac{(2-5\beta)\delta_{I_2}\left(2\delta_{I_2}^2+3\right)}{135\left(1-\delta_{I_2}^2\right)^{7/2}},$$

$$E(f_{F_1};\gamma)=\frac{3(5\gamma-1)\delta_{F_1}\left(2\delta_{F_1}^2+3\right)}{540\left(1-\delta_{F_1}^2\right)^{7/2}},\ E(f_{F_2};\gamma)=\frac{3(1-5\gamma)\delta_{F_2}(2\delta_{F_2}^2+3)}{540(1-\delta_{F_2}^2)^{7/2}},$$

where $\delta_{T_j}, \delta_{I_j}, \delta_{F_j} \in (-1, 1)$ and $j = 1, 2$ denotes left and right branch respectively of truth, indeterminacy and falsity membership function.

Three-point Gauss–Legendre integration rule

The errors for the three-point Gauss–Legendre integration rules are

$$E(f_{T_1};\alpha)=\frac{(3-5\alpha)\delta_{T_1}\left(8\delta_{T_1}^4+40\delta_{T_1}^2+15\right)}{3150\left(1-\delta_{T_1}^2\right)^{11/2}},\ E(f_{T_2};\alpha)=\frac{(5\alpha-3)\delta_{T_2}\left(8\delta_{T_2}^4+40\delta_{T_2}^2+15\right)}{3150\left(1-\delta_{T_2}^2\right)^{11/2}},$$

$$E(f_{I_1};\beta)=\frac{(5\beta-2)\delta_{I_1}\left(8\delta_{I_1}^4+40\delta_{I_1}^2+15\right)}{3150\left(1-\delta_{I_1}^2\right)^{11/2}},\ E(f_{I_2};\beta)=\frac{(2-5\beta)\delta_{I_2}\left(8\delta_{I_2}^4+40\delta_{I_2}^2+15\right)}{3150\left(1-\delta_{I_2}^2\right)^{11/2}},$$

$$E(f_{F_1};\gamma)=\frac{(5\gamma-1)\delta_{F_1}\left(8\delta_{F_1}^4+40\delta_{F_1}^2+15\right)}{4200(1-\delta_{F_1}^2)^{11/2}},\ E(f_{F_2};\gamma)=\frac{(1-5\gamma)\delta_{F_2}\left(8\delta_{F_2}^4+40\delta_{F_2}^2+15\right)}{4200(1-\delta_{F_2}^2)^{11/2}},$$

where $\delta_{T_j}, \delta_{I_j}, \delta_{F_j} \in (-1, 1)$ and $j = 1, 2$ denotes left and right branches, respectively, of truth, indeterminacy and falsity membership function.

Comparison of the errors of different Gauss–Legendre integration rules for Example 5.1 has been investigated in Table 1. From Table 1, it has been seen that three-point Gauss–Legendre integration rule gives less error than one-point and two-point Gauss–Legendre integration rules. The graphical presentations of the approximate solutions for one-point, two-point and three-point Gauss–Legendre integration rules have been seen in Figs.1, 2 and 3. Also, from Figs. 1, 2 and 3, it has been noted that three-point Gauss–Legendre integration rule gives better approximate solution than one-point and two-point Gauss–Legendre integration rules. The graphical representation of the neutrosophic valued function $f(x)$ for Example 5.1 has been presented in Fig. 4 and this figure shows that the neutrosophic function $f(x)$ is continuous in (− 1,1). The comparison of the errors of one-point and two-point Gauss–Legendre integration rule for the different values of δ_{K_j} have been shown in Table 2, where $j = 1, 2$ and $k = T, I, F$. From Table 2, it has been seen that when the value

Table 1 Comparison of the errors of different Gauss–Legendre integration rules for Example 5.1, where$\delta_{T_j} = \delta_{I_j} = \delta_{F_j} = 0.2$, $j = 1, 2$

Methods	α	$E(f_{T_1}; \alpha)$	$E(f_{T_2}; \alpha)$	β	$E(f_{I_1}; \beta)$	$E(f_{I_2}; \beta)$	γ	$E(f_{F_1}; \gamma)$	$E(f_{F_2}; \gamma)$
One-Point	0	− 0.5958	1.9291	0.4	0.6667	0.6667	0.2	0.6667	0.6667
Gauss–Legendre	0.2	− 0.1750	1.5083	0.6	0.2459	1.0874	0.4	0.3511	0.9823
integration	0.4	0.2459	1.0875	0.8	− 0.1750	1.5083	0.6	0.0354	1.2979
	0.6	0.6667	0.6667	1	− 0.5958	1.9291	1	− 0.5958	1.9291
Two-Point	0	0.0158	− 0.0158	0.4	0.0000	0.0000	0.2	0.0000	0.0000
Gauss–Legendre	0.2	0.0105	− 0.0105	0.6	0.0053	− 0.0053	0.4	0.0039	− 0.0039
integration	0.4	0.0053	− 0.0053	0.8	0.0153	− 0.0153	0.6	0.0079	− 0.0079
	0.6	0.000	0.000	1	0.0158	− 0.0158	1	0.0158	− 0.0158
Three-Point	0	0.0040	− 0.0040	0.4	0.0000	0.0000	0.2	0.0000	0.0000
Gauss–Legendre	0.2	0.0026	− 0.0026	0.6	0.0013	− 0.0013	0.4	0.0009	− 0.0009
integration	0.4	0.0013	− 0.0013	0.8	0.0026	− 0.0026	0.6	0.0020	− 0.0020
	0.6	0.0000	0.0000	1	0.0040	− 0.0040	1	0.0040	− 0.0040

Table 2 Comparison of the errors of different Gauss–Legendre integration rules for Example 5.1, where $\alpha = 0.5$, $\beta = 0.7$ and $\gamma = 0.8$. Here $j = 1,2$ and $K = T, I, F$

Methods	δ_{K_j}	$E(f_{T_1}; \alpha)$	$E(f_{T_2}; \alpha)$	$E(f_{I_1}; \beta)$	$E(f_{I_2}; \beta)$	$E(f_{F_1}; \gamma)$	$E(f_{F_2}; \gamma)$
One-Point	− 0.8	0.2387	1.0947	− 0.6173	1.9506	− 1.2593	2.5926
Gauss–Legendre	− 0.2	0.4326	0.9007	− 0.0354	1.3688	− 0.3865	1.7198
integration	0.0	0.4444	0.8889	0.0000	1.3333	− 0.3333	1.6667
	0.2	0.4563	0.8771	0.0354	1.2979	− 0.2802	1.6135
	0.8	0.6502	0.6831	0.6173	0.7160	0.5926	0.7407
Three-Point	− 0.8	− 1.5357	1.5357	− 4.6073	4.6073	− 6.9109	6.9109
Gauss–Legendre	− 0.2	− 0.0007	0.0007	− 0.0020	− 0.0020	− 0.0030	0.0030
integration	0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.2	0.0007	− 0.0007	0.0020	− 0.0020	0.0030	− 0.0030
	0.8	1.5357	− 1.5357	4.6073	− 4.6073	6.9109	− 6.9109

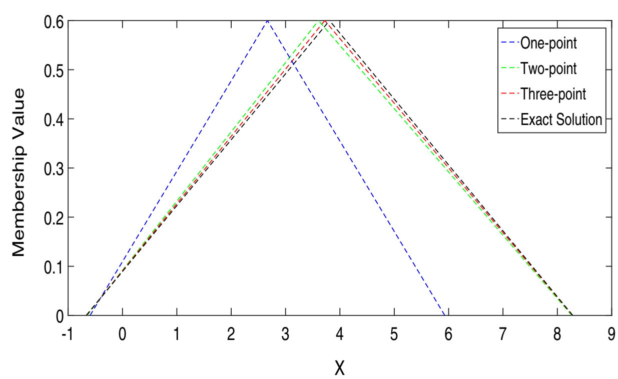

Fig. 1 Truth membership function for exact solution, one-point, two-point and three-point Gauss–Legendre integration rule of Example 5.1

of δ_{K_j} increases then the errors for left branch of truth, indeterminacy and falsity functions increases. Also, when the value of δ_{K_j} decreases then the errors for left branch of truth, indeterminacy and falsity functions decrease and the graphical representation of these results have been shown in Figs. 5 and 6 for one-point and three-point rules, respectively. Also, from Figs. 7, 8 and 9, it has been seen that three-point Gauss–Legendre integration rule gives less error than two-point Gauss–Legendre integration rule in between (− 0.6, 0.6).

Example 5.2 Let us consider the neutrosophic integral as follows:

$$\int_c^d f(x)\mathrm{d}x, \text{ where } f(x) = ae^x + \frac{bx}{\sqrt{1-x^2}}, \; a = \langle(0, 1, 2); 0.6, 0.4, 0.2\rangle \text{ and } b = \langle(-1, 1, 3); 0.6, 0.4, 0.2\rangle.$$

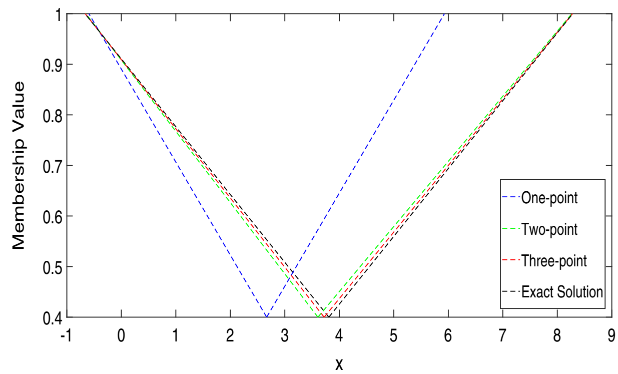

Fig. 2 Indeterminacy membership function for exact solution, one-point, two-point and three-point Gauss–Legendre integration rule of Example 5.1

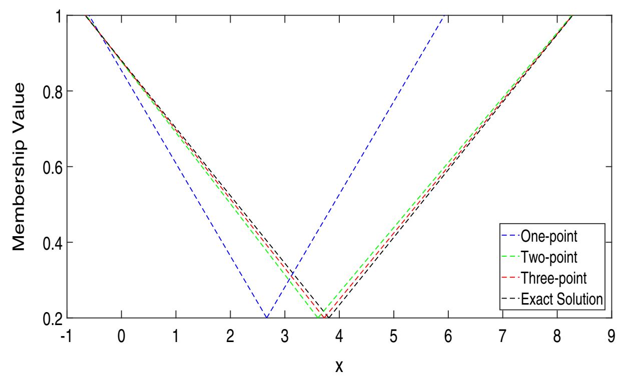

Fig. 3 Falsity membership function for exact solution, one-point, two-point and three-point Gauss–Legendre integration rule of Example 5.1

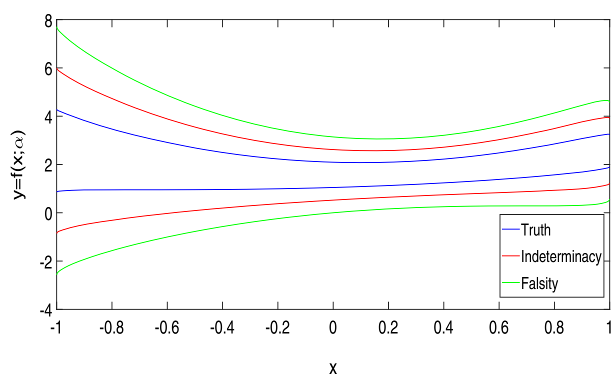

Fig. 4 Graphical representation of neutrosophic function $f(x)$ of Example 5.1

The exact solution of this integral is $a(e-e^{-1})$, i.e $\left\langle \left[\frac{5\alpha}{3}, \frac{6-5\alpha}{3}\right](e-e^{-1}), \left[\frac{5(1-\beta)}{3}, \frac{5\beta+1}{3}\right](e-e^{-1}), \left[\frac{5(1-\gamma)}{4}, \frac{5\gamma+3}{4}\right](e-e^{-1})\right\rangle$ when $[c,d]=[-1,1]$.

Now, we solve the integral with the help of Gauss–Chebyshev integration rule and Gauss–Laguerre integration rule, where we take $[c,d]=[-1,1]$ and $[c,d]=[0,\infty)$ respectively.

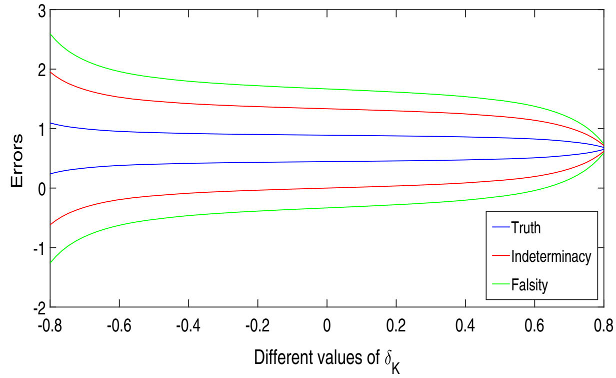

Fig. 5 Graphical representation of error for the one-point Gauss–Legendre integration rule of Example 5.1, where $K = T, I, F$

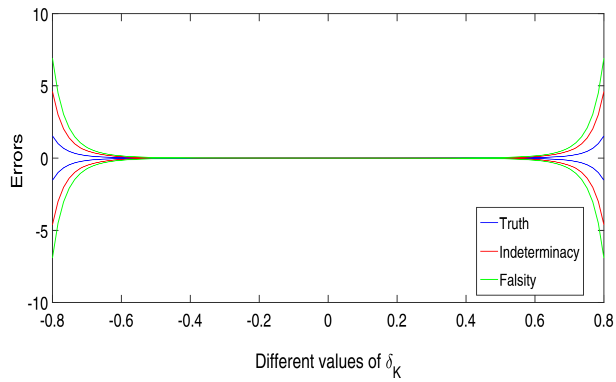

Fig. 6 Graphical representation of error for the three-point Gauss–Legendre integration rule of Example 5.1, where $K = T, I, F$

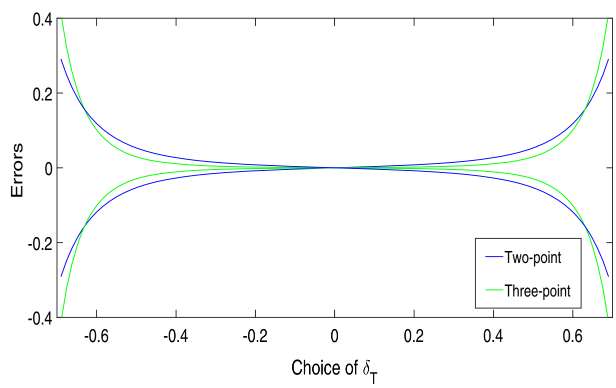

Fig. 7 Truth membership function for two-point and three-point Gauss–Legendre integration rule for different values of δ_T of Example 5.1, where $\alpha = 0.3$

One-point Gauss–Chebyshev integration rule

The errors for the One-Point Gauss–Chebyshev integration rule are

$$E(f_{T_1}; \alpha) = \frac{5\pi\alpha(1-\delta_{T_1}^2)^{5/2}e^{\delta_{T_1}} + 3\pi(10\alpha-3)\delta_{T_1}}{12(1-\delta_{T_1}^2)^{5/2}},$$

$$E(f_{T_2}; \alpha) = \frac{(6-5\alpha)\pi(1-\delta_{T_2}^2)^{5/2}e^{\delta_{T_2}} + 3\pi(9-10\alpha)\delta_{T_2}}{12(1-\delta_{T_2}^2)^{5/2}},$$

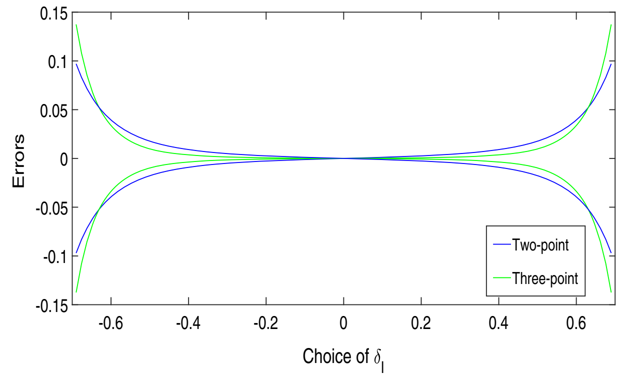

Fig. 8 Indeterminacy membership function for two-point and three-point Gauss–Legendre integration rule for different values of δ_I of Example 5.1, where $\beta = 0.5$

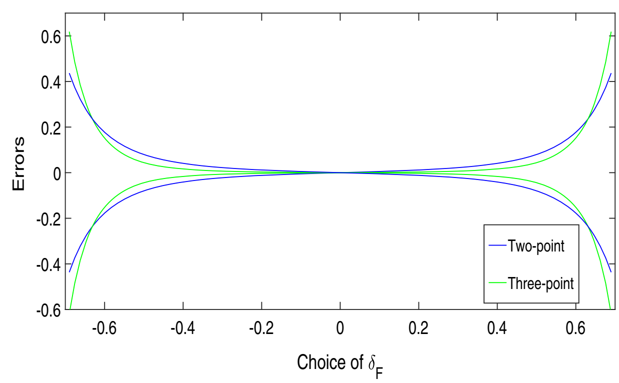

Fig. 9 Falsity membership function for two-point and three-point Gauss–Legendre integration rule for different values of δ_F of Example 5.1, where $\gamma = 0.8$

$$E(f_{I_1};\beta) = \frac{5\pi(1-\beta)(1-\delta_{I_1}^2)^{5/2}e^{\delta_{I_1}} + 3\pi(7-10\beta)\delta_{I_1}}{12(1-\delta_{I_1}^2)^{5/2}},$$

$$E(f_{I_2};\beta) = \frac{(5\beta+1)\pi(1-\delta_{I_2}^2)^{5/2}e^{\delta_{I_2}} + 3\pi(10\beta-1)\delta_{I_2}}{12(1-\delta_{I_2}^2)^{5/2}},$$

$$E(f_{F_1};\gamma) = \frac{5\pi(1-\gamma)(1-\delta_{F_1}^2)^{5/2}e^{\delta_{F_1}} + 6\pi(3-5\gamma)\delta_{F_1}}{16(1-\delta_{F_1}^2)^{5/2}},$$

$$E(f_{F_2};\gamma) = \frac{\pi(5\gamma+3)(1-\delta_{F_2}^2)^{5/2}e^{\delta_{F_2}} + 6\pi(5\gamma+1)\delta_{F_2}}{16(1-\delta_{F_2}^2)^{5/2}},$$

where $\delta_{T_j}, \delta_{I_j}, \delta_{F_j} \in (-1, 1)$ and $j = 1, 2$ denotes left and right branches, respectively, of truth, indeterminacy and falsity membership function.

Two-point Gauss–Chebyshev integration rule

The errors for the two-point Gauss–Chebyshev integration rule are

$$E(f_{T_1};\alpha) = \frac{5\pi\alpha(1-\delta_{T_1}^2)^{9/2}e^{\delta_{T_1}} + 60\pi(10\alpha-3)\delta_{T_1}^3 + 45\pi(10\alpha-3)\delta_{T_1}}{576(1-\delta_{T_1}^2)^{9/2}},$$

$$E(f_{T_2};\alpha) = \frac{(6-5\alpha)\pi(1-\delta_{T_2}^2)^{9/2}e^{\delta_{T_2}} + 60\pi(9-10\alpha)\delta_{T_2}^3 + 45\pi(9-10\alpha)\delta_{T_2}}{576(1-\delta_{T_2}^2)^{9/2}},$$

$$E(f_{I_1};\beta)=\frac{5\pi(1-\beta)(1-\delta_{I_1}^2)^{9/2}e^{\delta_{I_1}}+60\pi(7-10\beta)\delta_{I_1}^3+45\pi(7-10\beta)\delta_{I_1}}{576(1-\delta_{I_1}^2)^{9/2}},$$
$$E(f_{I_2};\beta)=\frac{(5\beta+1)\pi(1-\delta_{I_2}^2)^{9/2}e^{\delta_{I_2}}+60\pi(10\beta-1)\delta_{I_2}^3+45\pi(10\beta-1)\delta_{I_2}}{576(1-\delta_{I_2}^2)^{9/2}},$$
$$E(f_{F_1};\gamma)=\frac{5\pi(1-\gamma)(1-\delta_{F_1}^2)^{9/2}e^{\delta_{F_1}}+120\pi(3-5\gamma)\delta_{F_1}^3+90\pi(3-5\gamma)\delta_{F_1}}{768(1-\delta_{F_1}^2)^{9/2}},$$
$$E(f_{F_2};\gamma)=\frac{\pi(5\gamma+3)(1-\delta_{F_2}^2)^{9/2}e^{\delta_{F_2}}+120\pi(5\gamma+1)\delta_{F_2}^3+90\pi(5\gamma+1)\delta_{F_2}}{768(1-\delta_{F_2}^2)^{9/2}},$$

where $\delta_{T_j},\delta_{I_j},\delta_{F_j}\in(-1,1)$ and $j=1,2$ denotes left and right branches, respectively, of truth, indeterminacy and falsity membership function.

One-point Gauss–Laguerre integration rule

The errors for the One-Point Gauss–Laguerre integration rule is

$$E(f_{T_1};\alpha)=\frac{5\alpha(1-\delta_{T_1}^2)^{5/2}e^{\delta_{T_1}}+3(10\alpha-3)\delta_{T_1}}{6(1-\delta_{T_1}^2)^{5/2}},$$
$$E(f_{T_2};\alpha)=\frac{(6-5\alpha)(1-\delta_{T_2}^2)^{5/2}e^{\delta_{T_2}}+3(9-10\alpha)\delta_{T_2}}{6(1-\delta_{T_2}^2)^{5/2}},$$
$$E(f_{I_1};\beta)=\frac{5(1-\beta)(1-\delta_{I_1}^2)^{5/2}e^{\delta_{I_1}}+3(7-10\beta)\delta_{I_1}}{6(1-\delta_{I_1}^2)^{5/2}},$$
$$E(f_{I_2};\beta)=\frac{(5\beta+1)(1-\delta_{I_2}^2)^{5/2}e^{\delta_{I_2}}+(10\beta-1)\delta_{I_2}}{6(1-\delta_{I_2}^2)^{5/2}},$$
$$E(f_{F_1};\gamma)=\frac{5(1-\gamma)(1-\delta_{F_1}^2)^{5/2}e^{\delta_{F_1}}+6(3-5\gamma)\delta_{F_1}}{8(1-\delta_{F_1}^2)^{5/2}},$$
$$E(f_{F_2};\gamma)=\frac{(5\gamma+3)(1-\delta_{F_2}^2)^{5/2}e^{\delta_{F_2}}+6(5\gamma+1)\delta_{F_2}}{8(1-\delta_{F_2}^2)^{5/2}}$$

,

where $\delta_{T_j},\delta_{I_j},\delta_{F_j}\in(-1,1)$ and $j=1,2$ denotes left and right branches, respectively, of truth, indeterminacy and falsity membership function.

Two-point Gauss–Laguerre integration rule

The errors for the Two-Point Gauss–Laguerre integration rule is

$$E(f_{T_1};\alpha)=\frac{5\alpha\left(1-\delta_{T_1}^2\right)^{9/2}e^{\delta_{T_1}}+(10\alpha-3)\left(60\delta_{T_1}^3+45\delta_{T_1}\right)}{18(1-\delta_{T_1}^2)^{9/2}},$$
$$E(f_{T_2};\alpha)=\frac{(6-5\alpha)\left(1-\delta_{T_2}^2\right)^{9/2}e^{\delta_{T_2}}+(9-10\alpha)(60\delta_{T_2}^3+45\delta_{T_2})}{18\left(1-\delta_{T_2}^2\right)^{9/2}},$$
$$E(f_{I_1};\beta)=\frac{5(1-\beta)\left(1-\delta_{I_1}^2\right)^{9/2}e^{\delta_{I_1}}+(7-10\beta)\left(60\delta_{I_1}^3+45\delta_{I_1}\right)}{18\left(1-\delta_{I_1}^2\right)^{9/2}},$$

Table 3 Comparison of the errors of different Gauss–Chebyshev and Gauss–Laguerre integration rules for Example 2, where $\delta_{T_j} = \delta_{I_j} = \delta_{F_j} = 0.2$

Methods	α	$E(f_{T_1};\alpha)$	$E(f_{T_2};\alpha)$	β	$E(f_{I_1};\beta)$	$E(f_{I_2};\beta)$	γ	$E(f_{F_1};\gamma)$	$E(f_{F_2};\gamma)$
One-Point	0	– 0.5219	3.4842	0.4	1.4812	1.4812	0.2	1.4812	1.4812
Gauss–Chebyshev	0.2	0.1458	2.8165	0.6	0.8135	2.1488	0.4	0.9804	1.9819
integration	0.4	0.8135	2.1488	0.8	0.1458	2.8165	0.6	0.4796	2.4827
	0.6	1.4812	1.4812	1	– 0.5219	3.4842	1	– 0.5219	3.4842
Two-Point	0	– 0.1864	0.5992	0.4	0.2064	0.2064	0.2	0.2064	0.2064
Gauss–Chebyshev	0.2	– 0.0555	0.4682	0.6	0.0755	0.3373	0.4	0.1082	0.3046
integration	0.4	0.0755	0.3373	0.8	– 0.0555	0.4682	0.6	0.01	0.4028
	0.6	0.2064	0.2064	1	– 0.1864	0.5992	1	– 0.1864	0.5992
One-Point	0	– 0.3322	2.2181	0.4	0.9429	0.9429	0.2	0.9429	0.9429
Gauss–Laguerre	0.2	0.0928	1.7930	0.6	0.5179	1.3680	0.4	0.6241	1.2617
integration	0.4	0.5179	1.3680	0.8	0.0928	1.7930	0.6	0.3054	1.5805
	0.6	0.9429	0.9429	1	– 0.3322	2.2181	1	– 0.3322	2.21812
Two-Point	0	– 1.8986	6.1030	0.4	2.1022	2.1022	0.2	2.1022	2.1022
Gauss–Laguerre	0.2	– 0.5650	4.7694	0.6	0.7686	3.4358	0.4	1.1020	3.1024
integration	0.4	0.7686	3.4358	0.8	– 0.5650	4.7694	0.6	0.1018	4.1026
	0.6	2.1022	2.1022	1	– 1.8986	6.1030	1	– 1.8986	6.1030

Table 4 Comparison of the errors of different Gauss–Chebyshev and Gauss–Laguerre integration rules for Example 1, where $\alpha = 0.3, \beta = 0.5\ \gamma = 0.7$, $j = 1, 2$ and $K = T, I, F$

Methods	δ_{K_j}	$E(f_{T_1};\alpha)$	$E(f_{T_2};\alpha)$	$E(f_{I_1};\beta)$	$E(f_{I_2};\beta)$	$E(f_{F_1};\gamma)$	$E(f_{F_2};\gamma)$
Two-Point	– 0.8	0.0037	– 216.65	– 72.213	– 144.43	27.08	– 243.73
Gauss–Chebyshev	– 0.2	0.0067	– 0.3527	– 0.1131	– 0.2329	0.0516	– 0.3976
integration	0.0	0.0082	0.0245	0.0136	0.0191	0.0061	0.0266
	0.2	0.0100	0.4028	0.1409	0.2718	– 0.0391	0.4519
	0.8	0.0182	217.71	72.25	144.48	– 27.07	243.80
Two-Point	– 0.8	0.0374	– 2206.7	– 735.56	– 1471.43	130.35	– 2482.59
Gauss–Laguerre	– 0.2	0.0682	– 3.5925	– 1.1520	– 2.3723	0.0852	-4.0501
integration	0.0	0.0833	0.2500	0.1389	0.1944	0.0625	0.2708
	0.2	0.1018	4.1026	1.4354	2.7690	0.0423	4.6027
	0.8	0.1855	2207.4	735.93	1471.67	– 130.18	2483.31

$$E(f_{I_2};\beta) = \frac{(5\beta+1)\left(1-\delta_{I_2}^2\right)^{9/2} e^{\delta_{I_2}} + (10\beta-1)(60\delta_{I_2}^3 + 45\delta_{I_2})}{18\left(1-\delta_{I_2}^2\right)^{9/2}},$$

$$E(f_{F_1};\gamma) = \frac{5(1-\gamma)\left(1-\delta_{F_1}^2\right)^{9/2} e^{\delta_{F_1}} + (6-10\gamma)\left(60\delta_{F_1}^3 + 45\delta_{F_1}\right)}{24(1-\delta_{F_1}^2)^{9/2}},$$

$$E(f_{F_2};\gamma) = \frac{(5\gamma+3)\left(1-\delta_{F_2}^2\right)^{9/2} e^{\delta_{F_2}} + (10\gamma+2)\left(60\delta_{F_2}^3 + 45\delta_{F_2}\right)}{24(1-\delta_{F_2}^2)^{9/2}}$$

where $\delta_{T_j}, \delta_{I_j}, \delta_{F_j} \in (-1, 1)$ and $j = 1, 2$ denotes left and right branch respectively of truth, indeterminacy and falsity membership function.

In Table 3, the comparison of the errors of different Gauss–Chebyshev and Gauss–Laguerre integration rules of Example 5.2 have been investigated. From Table 3, it has been found that Gauss–Chebyshev integration rule gives less error than Gauss–Laguerre integration rule for Example 5.2. The Comparison of one-point and two-point Gauss–Chebyshev integration rule with the exact solution for Example 5.2 has been shown in Figs. 10, 11 and 12 and from these figures, it has been seen that two-point Gauss–Chebyshev integration rule gives better result than one-point Gauss–Chebyshev rule. Also, from Fig. 13 it has been shown that the solution of Two-point Gauss–Chebyshev integration rule gives a triangular neutrosophic number when the parameters are taken as triangular neutrosophic numbers. The errors of two-point Gauss–Chebyshev and Gauss–Laguerre integration rule for the different values of δ_{K_j} have been shown in Table 4, where $K = T, I, F$

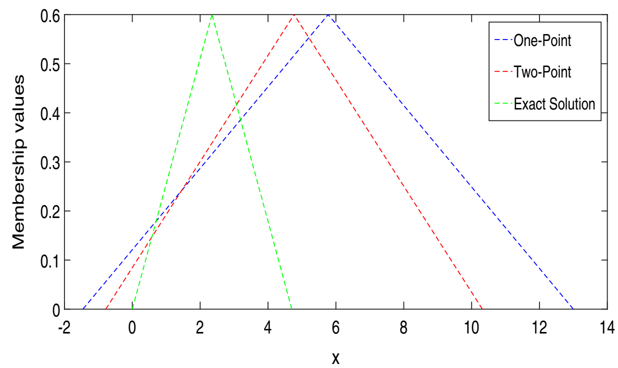

Fig. 10 Truth membership function for exact solution, one-point and two-point Gauss–Chebyshev integration rule of Example 5.2

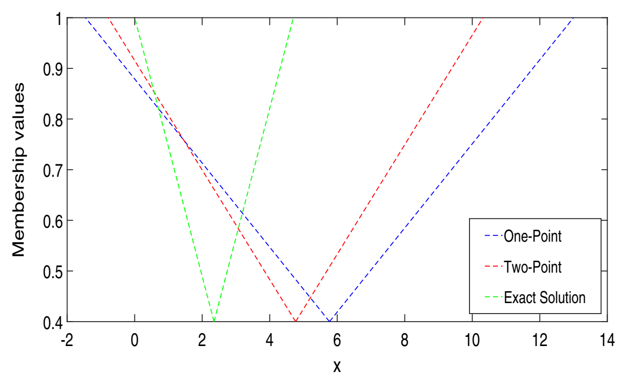

Fig. 11 Indeterminacy membership function for exact solution, one-point and two-point Gauss–Chebyshev integration rule of Example 5.2

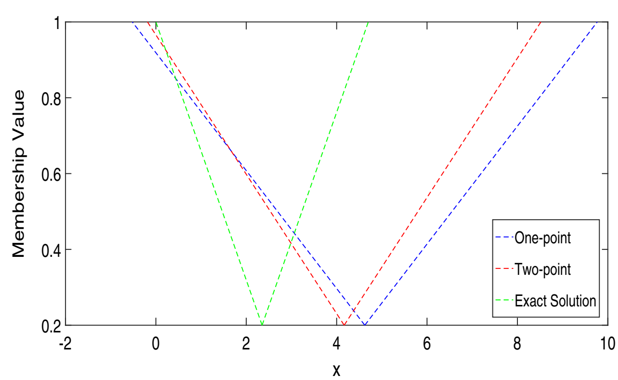

Fig. 12 Falsity membership function for exact solution, one-point and two-point Gauss–Chebyshev integration rule of Example 5.2

and $j = 1, 2$. From Table 4, it has been seen that when the value of δ_{K_j} increases then the errors for truth and indeterminacy functions increase. Also, when the value of δ_{K_j} decreases then the errors for left branch of falsity functions decrease and right branch of falsity function increases. The graphical representation of error for One-point Gauss–Chebyshev and One-point Gauss–Laguerre integration rule have been shown in Figs. 14 and 15 respectively. These figures have shown similar types of results as shown in Table 4. Also, from Figs. 16, 17 and 18, it has been seen that three-point Gauss–Chebyshev integration rule gives less error than two-point Gauss–Chebyshev integration rule in between $(-0.2, 0.4)$.

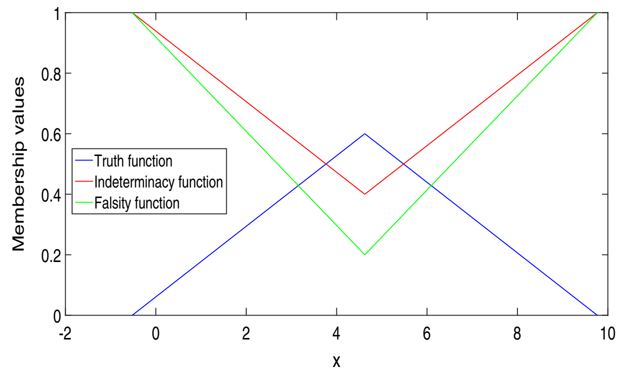

Fig. 13 Solution for two-point Gauss–Chebyshev integration rule of Example 5.2

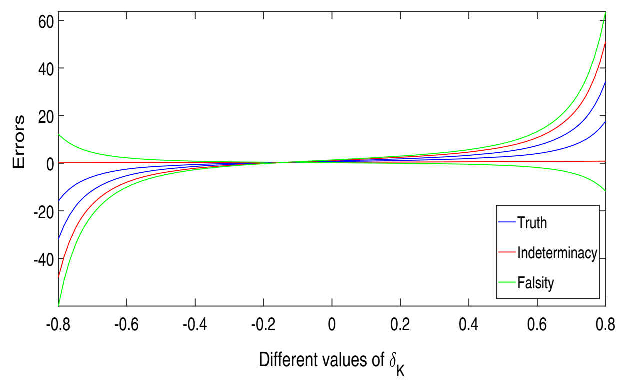

Fig. 14 Graphical representation of error for the one-point Gauss–Chebyshev integration rule of Example 5.2, where $\alpha = 0.5$, $\beta = 0.7$, $\gamma = 0.8$ and $K = T, I, F$

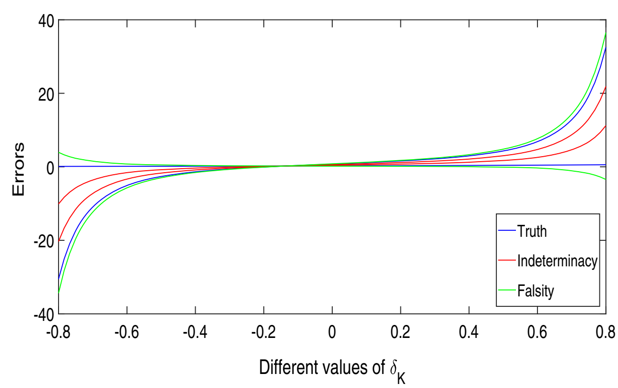

Fig. 15 Graphical representation of error for the one-point Gauss–Laguerre integration rule of Example 5.2, where $\alpha = 0.3$, $\beta = 0.5$, $\gamma = 0.7$ and $K = T, I, F$

6 Conclusion

For the first time, different types of Gaussian quadrature rules for numerical integration of neutrosophic valued function have been discussed in this article. A new type of distance function has been defined in neutrosophic environment. From Theorem 3.2, it can be concluded that $(\mathcal{N}, D)$ forms a complete metric space. Also, the proof of the Theorem 4.1 can be concluded that $J(f_{T_1}; \alpha)$, $J(f_{T_2}; \alpha)$, $J(f_{I_1}; \beta)$, $J(f_{I_2}; \beta)$, $J(f_{F_1}; \gamma)$, $J(f_{F_2}; \gamma)$ are uniformly convergent to $I(f_{T_1}; \alpha)$, $I(f_{T_2}; \alpha)$, $I(f_{I_1}; \beta)$, $I(f_{I_2}; \beta)$, $I(f_{F_1}; \gamma)$, $I(f_{F_2}; \gamma)$, respectively. Our study shows that the Gaussian-quadrature method is very simple and provides highly accurate numerical results using small number of quadrature points. In the first test example, one-point, two-point, three-point Gauss–Legendre integration rules have been investigated. From this investigation, we have seen that the three-point

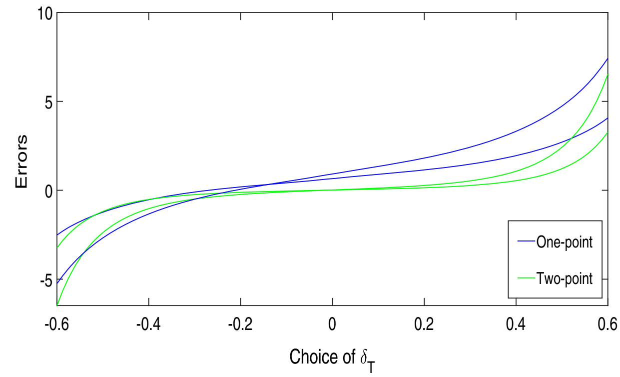

Fig. 16 Truth membership function for one-point and two-point Gauss–Chebyshev integration rule for different values of δ_T of Example 5.2, where $\alpha = 0.5$

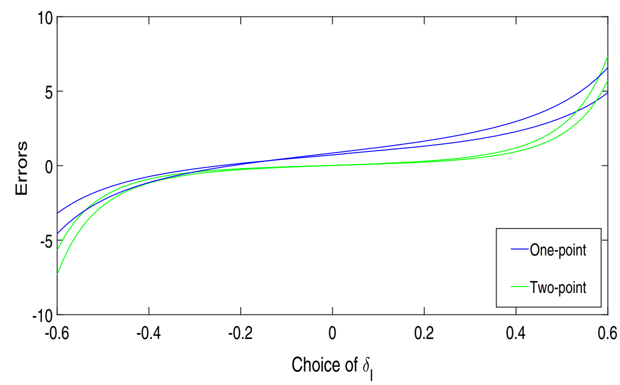

Fig. 17 Indeterminacy membership function for one-point and two-point Gauss–Chebyshev integration rule for different values of δ_I of Example 5.2, where $\beta = 0.45$

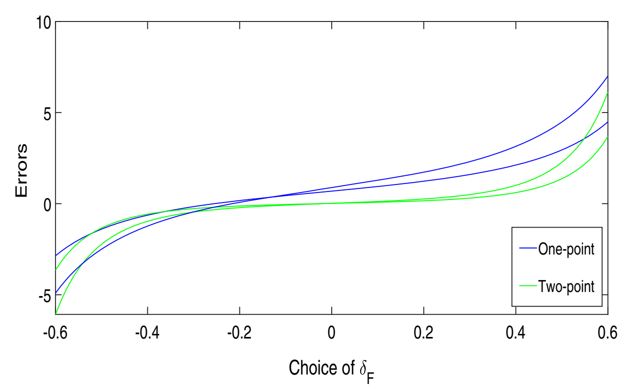

Fig. 18 Falsity membership function for one-point and two-point Gauss–Chebyshev integration rule for different values of δ_F of Example 5.2, where $\gamma = 0.3$

Gauss–Legendre integration rule gives less error than one-point and two-point Gauss–Legendre integration rule. Also, from Figs. 1, 2 and 3, it has been observed that three-point Gauss–Legendre integration rule gives more accurate numerical results than one-point and two-point Gauss–Legendre rules. Based on the results of Table 2, Figs. 5 and 6, it can be concluded that the errors of Gauss–Legendre integration rule depend on the suitable choice of δ_{T_j}, δ_{I_j} and δ_{F_j}, where $j = 1, 2$. Also, it can be concluded from Table 2 that when the value of δ_{T_j}, δ_{I_j}, δ_{F_j} come close to zero then error for Three-point Gauss–Legendre integration also moves close to zero. In case of Example 5.1, from Figs. 7, 8 and 9, it has been observed that if we choose δ_{T_j}, δ_{I_j} and δ_{F_j} in between $(-0.6, 0.6)$ then Gauss–Legendre rule gives more accurate result than any other choice of δ_{T_j}, δ_{I_j} and δ_{F_j}. In the second test example, one-point, two-point Gauss–Chebyshev and Gauss–Laguerre integration rules have been discussed. This discussion shows that Gauss–Chebyshev integration rule gives less error than Gauss–Laguerre integration rule when the quadrature points are increased. In case of Example 5.2, it can be concluded from Figs. 10, 11 and 12 that the two-point Gauss–Chebyshev integration rule gives more accurate results than one-point Gauss–Chebyshev integration rule which has been shown the validity and accuracy of our proposed method. Also, from Table 4, Figs. 14 and 15, it has been concluded that the errors of Gauss–Chebyshev and Gauss–Laguerre integration rule depend on the proper choice of δ_{T_j}, δ_{I_j} and δ_{F_j}, where and $j = 1, 2$. Also, it can be concluded from Table 4 that when the value of δ_{T_j}, δ_{I_j}, δ_{F_j} come close to zero then errors for Two-point Gauss–Chebyshev integration and Two-point Gauss–Laguerre integration also moves close to zero. From Figs. 16, 17 and 18, it has been observed that if we choose δ_{T_j}, δ_{I_j} and δ_{F_j} in between $(-\ 0.2, 0.4)$ then Gauss–Chebyshev integration rule gives suitable results for Example 2. Finally, it can be concluded that the proposed method is easy to understand and within few number of quadrature steps this method gives extremely high accurate approximate results. In the future, we will apply this method to solve other different types of neutrosophic integral equations and integro-differential equations.

Acknowledgements In this article, the study of Sandip Moi is funded by Council of Scientific and Industrial Research (CSIR), Government of India (File No. 08/003(0135)/2019-EMR-I).

Funding This research has received funding from the Council of Scientific and Industrial Research (CSIR), Government of India (File No.- 08/003(0135)/2019-EMR-I). The support of the CSIR is greatly acknowledged.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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