Literal Neutrosophic and Plithogenic Marshall-Olkin Type II Class of Distributions with Application to Exponential Distribution

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Abstract

In this study, we introduce Marshall-Olkin type II class of distributions within the neutrosophic and plithogenic framework. We provide the formal expressions for the probability density function and derive the cumulative distribution function. As a specific instance, we examine the generalization of the exponential distribution in both neutrosophic and plithogenic forms according to this new class, presenting the probability density function and deriving the cumulative distribution function for this case. Furthermore, we propose an algorithm for generating random numbers based on this distribution. Additionally, we estimate its parameters using maximum likelihood and validate the results through a simulation study, demonstrating the efficiency of the calculated parameters. We also investigate the asymptotic properties, including unbiasedness and consistency.

Keywords: Marshall-Olkin Type II Class of Distributions; Neutrosophic; Plithogenic; AH Isometry; Maximum Likelihood Estimation; Random Numbers Generation.

1. Introduction

The examination of uncertainty in probability distributions holds great significance in practical situations as certain parameters are often non-existent. This issue has been addressed in numerous studies utilizing fuzzy logic and its extensions [1][2][3][4][5][6][7][8][9][10] [11]. The case of imprecise data has been explored in many studies as well [12][13] [14] studies in literature demonstrate that considering uncertainty in the parameters of probability distributions or random variables leads to improved modelling accuracy and efficiency [15][16][17]. Neutrosophic logic and its extensions have intrigued scientists, resulting in the generalization of several concepts and theories in probability to neutrosophic probability and its extensions.

The study of literal neutrosophic probability theory and the development of various probability distributions in this context were explored in [12][18][19][13][20][21]. Literal neutrosophic probability theory means that probabilities take the form $P_N = p_1 + p_2 I\ ; I^2$ where I is a symbol. The definition presented here offers a captivating mathematical extension of classical probability theory. Its beauty lies in the fact that this probability is isomorphic to $(p_1, p_1 + p_2)$ (see [13]). Another generalization of classical probability theory and neutrosophic probability theory was the theory of plithogenic probability. In [22] Zeina et. al. presented the concept of symbolic plithogenic probability theory and defined the symbolic plithogenic random variables and presented many theorems related to this new concept. This extension of probability theory has paved the way for the development of several other related theories including estimation theory, distributions theory, markov chains and

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stochastic modelling, etc.[23][24][25]. Marshall and Olkin introduced two classes of probability distributions, which have found extensive application in the modelling of various data types. These classes have served as the foundation for creating numerous specialized probability distributions, such as the Uniform Marshall-Olkin and Exponential Marshall-Olkin distribution. In [23] we have studied the neutrosophic and the plithogenic forms of Marshall-Olkin type I class of distributions and focused on the uniformly generated distribution based on it.

This work is dedicated to generalize Marshall-Olkin II class of distributions to neutrosophic and plithogenic forms to which is very important in modelling several types of data.

2. Preliminaries

Definition 2.1

Let $R(I) = \{a + bI\,; a, b \in R\ \&\ I^2 = I\}$ be the neutrosophic field of reals. AH-isometry is an algebraic isomorphism that preserves distances and it is defined as:

$$T: R(I) \to R^2;\ T(a + bI) = (a, a + b) \tag{1}$$

And its inverse is defined as:

$$T^{-1}: R^2 \to R(I);\ T^{-1}(a, b) = a + (b - a)I \tag{2}$$

Definition 2.2

Let $a_N = a_1 + a_2 I, b_N = b_1 + b_2 I \in R(I)$ be two neutrosophic numbers. We say that $a_N \geq_N b_N$ iff:

$$a_1 \geq b_1\,, a_1 + a_2 \geq b_1 + b_2$$

Definition 2.3

Let $f: R(I) \to R(I); f = f(x_N), x_N = \mathrm{x}_1 + \mathrm{x}_2\mathrm{I} \in R(I)$ then f is called a neutrosophic real function with one neutrosophic variable.

Definition 2.4

We say that $f(x_N)$ is integrable, continuous and differentiable on $R(I)$ iff $T[f(x_N)]$ is integrable, continuous and differentiable on R^2.

Definition 2.5

Let $R(P_1, P_2) = \{a_0 + a_1 P_1 + a_2 P_2\,; a_0, a_1, a_2 \in R,\ P_1^2 = P_1,\ P_2^2 = P_2,\ P_1 \cdot P_2 = P_2\}$ be the Plithogenic field of reals. B-isometry is an algebraic isomorphism that preserves distances and it is defined as follows:

$$T: R(P_1, P_2) \to R^3 :\ T(a_0 + a_1 P_1 + a_2 P_2) = (a_0\,, a_0 + a_1\,, a_0 + a_1 + a_2) \tag{3}$$

And its inverse is defined as:

$$T^{-1}: R^3 \to R(P_1, P_2) :\ T^{-1}(a_0, a_1, a_2) = a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2 \tag{4}$$

Definition 2.6

Let $a_P = a_0 + a_1 P_1 + a_2 P_2, b_P = b_0 + b_1 P_1 + b_2 P_2 \in R(P_1, P_2)$ be two plithogenic numbers. We say that $a_P \geq_P b_P$ iff:

$$a_0 \geq b_0\,, a_0 + a_1 \geq b_0 + b_1, a_0 + a_1 + a_2 \geq b_0 + b_1 + b_2$$

Definition 2.7

Let $f: R(P_1, P_2) \to R(P_1, P_2); f = f(x_P)\ , x_P = \mathrm{x}_0 + \mathrm{x}_1\mathrm{P}_1 + \mathrm{x}_2\mathrm{P}_2 \in R(P_1, P_2)$ then f is called a Plithogenic real function with one plithogenic variable.

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Definition 2.8

We say that $f(x_P)$ is integrable, continuous and differentiable on $R(P_1, P_2)$ iff $B[f(x_P)]$ is integrable, continuous and differentiable on R^3.

3. Neutrosophic Marshall-Olkin Type II Class of Distributions:

Definition 3.1

Probability density function of neutrosophic Marshall-Olkin type II class of distributions is defined as:

$$g(x_1 + x_2 I) = \beta_1 \alpha_1 \frac{[\alpha_1 \bar{F}(x_1)]^{\beta_1 - 1} f(x_1)}{[1 - \bar{\alpha}_1 \bar{F}(x_1)]^{\beta_1 + 1}} + I\left[(\beta_1 + \beta_2)(\alpha_1 + \alpha_2)\frac{\left[(\alpha_1 + \alpha_2)\left(\bar{F}(x_1 + x_2)\right)\right]^{\beta_1 + \beta_2 - 1}}{[1 - (\bar{\alpha}_1 + \bar{\alpha}_2)\bar{F}(x_1 + x_2)]^{\beta_1 + \beta_2 + 1}} - \beta_1 \alpha_1 \frac{[\alpha_1 \bar{F}(x_1)]^{\beta_1 - 1} f(x_1)}{[1 - \bar{\alpha}_1 \bar{F}(x_1)]^{\beta_1 + 1}}\right] \tag{5}$$

Where $x_1, x_1 + x_2 \in R$ & $\beta_1, \beta_1 + \beta_2, \alpha_1, \alpha_1 + \alpha_2 > 0$.

Remark

The neutrosophic probability density function (PDF) of Marshall-Olkin type II class of distributions defined in (5) is equivalent to definition in equation (6):

$$g(x_N) = \beta_N \alpha_N \frac{[\alpha_N \bar{F}(x_N)]^{\beta_N - 1}}{[1 - \bar{\alpha}_N \bar{F}(x_N)]^{\beta_N + 1}} f(x_N)\,; \quad x_N \in R(I)\ \&\ \alpha_N, \beta_N, \lambda_N >_N 0 \tag{6}$$

Theorem 3.1

Equation (5) presents a probability density function.

Proof

$$\int_{-\infty}^{+\infty} g(x_1 + x_2 I)\, d(x_1 + x_2 I)$$

$$= \int_{-\infty}^{+\infty} \beta_1 \alpha_1 \frac{[\alpha_1 \bar{F}(x_1)]^{\beta_1 - 1}}{[1 - \bar{\alpha}_1 \bar{F}(x_1)]^{\beta_1 + 1}} f(x_1)\, dx_1 + I\left[\int_{-\infty}^{+\infty} (\beta_1 + \beta_2)(\alpha_1 + \alpha_2)\frac{\left[(\alpha_1 + \alpha_2)\left(\bar{F}(x_1 + x_2)\right)\right]^{\beta_1 + \beta_2 - 1}}{[1 - (\bar{\alpha}_1 + \bar{\alpha}_2)\bar{F}(x_1 + x_2)]^{\beta_1 + \beta_2 + 1}} \cdot f(x_1 + x_2)\, d(x_1 + x_2) - \int_{-\infty}^{+\infty} \beta_1 \alpha_1 \frac{[\alpha_1 \bar{F}(x_1)]^{\beta_1 - 1}}{[1 - \bar{\alpha}_1 \bar{F}(x_1)]^{\beta_1 + 1}} \cdot f(x_1)\, dx_1\right]$$

$$= \left[1 - \left[\frac{\alpha_1 \bar{F}(x_1)}{1 - \bar{\alpha}_1 \bar{F}(x_1)}\right]^{\beta_1}\right]_{-\infty}^{+\infty} + I\left[\left[1 - \left[\frac{(\alpha_1 + \alpha_2)\bar{F}(x_1 + x_2)}{1 - (\bar{\alpha}_1 + \bar{\alpha}_2)\bar{F}(x_1 + x_2)}\right]^{\beta_1 + \beta_2}\right]_{-\infty}^{+\infty} - \left[1 - \left[\frac{\alpha_1 \bar{F}(x_1)}{1 - \bar{\alpha}_1 \bar{F}(x_1)}\right]^{\beta_1}\right]_{-\infty}^{+\infty}\right]$$

$$= 1 + I[1 - 1] = 1$$

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And:

$$T[g(x_1+x_2I)] = T\left[\beta_1\alpha_1\frac{[\alpha_1\bar{F}(x_1)]^{\beta_1-1}}{[1-\bar{\alpha}_1\bar{F}(x_1)]^{\beta_1+1}}\cdot f(x_1)\right.$$
$$+I\left[(\beta_1+\beta_2)(\alpha_1+\alpha_2)\frac{[(\alpha_1+\alpha_2)(\bar{F}(x_1+x_2))]^{\beta_1+\beta_2-1}}{[1-(\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(x_1+x_2)]^{\beta_1+\beta_2+1}}\cdot f(x_1+x_2)\right.$$
$$\left.\left.-\beta_1\alpha_1\frac{[\alpha_1\bar{F}(x_1)]^{\beta_1-1}}{[1-\bar{\alpha}_1\bar{F}(x_1)]^{\beta_1+1}}f(x_1)\right]\right]$$
$$=\left(\beta_1\alpha_1\frac{[\alpha_1\bar{F}(x_1)]^{\beta_1-1}}{[1-\bar{\alpha}_1\bar{F}(x_1)]^{\beta_1+1}}f(x_1),(\beta_1+\beta_2)(\alpha_1+\alpha_2)\right.$$
$$\left.\cdot\frac{[(\alpha_1+\alpha_2)(\bar{F}(x_1+x_2))]^{\beta_1+\beta_2-1}}{[1-(\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(x_1+x_2)]^{\beta_1+\beta_2+1}}f(x_1+x_2)\right)=(g(x_1),g(x_1+x_2))$$

Where $g(x_1)$ and $g(x_1+x_2)$ are probability density functions of classical Marshall-Olkin type II distributions, then each of them is positive and continuous function on R, so $g(x_1+x_2I)$ is positive and continuous on $R(I)$.

Depending on previous results we can prove that equation (5) represents a probability density function.

Theorem 3.2

Cumulative distribution function of neutrosophic Marshall-Olkin type II class of distributions is:

$$G(x_1+x_2I) = 1-\left[\frac{\alpha_1\bar{F}(x_1)}{1-\bar{\alpha}_1\bar{F}(x_1)}\right]^{\beta_1}$$
$$+I\left[\left[\frac{\alpha_1\bar{F}(x_1)}{1-\bar{\alpha}_1\bar{F}(x_1)}\right]^{\beta_1}-\left[\frac{(\alpha_1+\alpha_2)\bar{F}(x_1+x_2)}{1-(\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(x_1+x_2)}\right]^{\beta_1+\beta_2}\right] \tag{7}$$

Where $x_1, x_1+x_2 \in R$ & $\beta_1, \beta_1+\beta_2, \alpha_1, \alpha_1+\alpha_2 > 0$

Proof

$$=\left[1-\left[\frac{\alpha_1\bar{F}(t_1)}{1-\bar{\alpha}_1\bar{F}(t_1)}\right]^{\beta_1}\right]_{-\infty}^{x_1}$$
$$+I\left[\left[1-\left[\frac{(\alpha_1+\alpha_2)\bar{F}(t_1+t_2)}{1-(\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(t_1+t_2)}\right]^{\beta_1+\beta_2}\right]_{-\infty}^{x_1+x_2}-\left[1-\left[\frac{\alpha_1\bar{F}(t_1)}{1-\bar{\alpha}_1\bar{F}(t_1)}\right]^{\beta_1}\right]_{-\infty}^{x_1}\right]$$

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$$\int_{-\infty}^{x_1+x_2I} g(t_1+t_2I)\,d(t_1+t_2I)$$
$$=\int_{-\infty}^{x_1}\beta_1\alpha_1\frac{[\alpha_1\bar{F}(t_1)]^{\beta_1-1}}{[1-\bar{\alpha}_1\bar{F}(t_1)]^{\beta_1+1}}\cdot f(t_1)\,d(t_1)$$
$$+I\left[\int_{-\infty}^{x_1+x_2}(\beta_1+\beta_2)(\alpha_1+\alpha_2)\frac{[(\alpha_1+\alpha_2)(\bar{F}(t_1+t_2))]^{\beta_1+\beta_2-1}}{[1-(\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(t_1+t_2)]^{\beta_1+\beta_2+1}}\right.$$
$$\left.\cdot f(t_1+t_2)d(t_1+t_2)-\int_{-\infty}^{x_1}\beta_1\alpha_1\frac{[\alpha_1\bar{F}(t_1)]^{\beta_1-1}}{[1-\bar{\alpha}_1\bar{F}(t_1)]^{\beta_1+1}}\cdot f(t_1)\,d(t_1)\right]$$

$$=1-\left[\frac{\alpha_1\bar{F}(x_1)}{1-\bar{\alpha}_1\bar{F}(x_1)}\right]^{\beta_1}+I\left[\left[\frac{\alpha_1\bar{F}(x_1)}{1-\bar{\alpha}_1\bar{F}(x_1)}\right]^{\beta_1}-\left[\frac{(\alpha_1+\alpha_2)\bar{F}(x_1+x_2)}{1-(\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(x_1+x_2)}\right]^{\beta_1+\beta_2}\right]$$

4. Neutrosophic Marshall-Olkin Type II Exponential Distribution:

Definition 4.1

The standard probability density function of neutrosophic Marshall-Olkin type II exponential distribution is defined as:

$$g(x_1+x_2I)=\beta_1\lambda_1\frac{{\alpha_1}^{\beta_1}e^{\lambda_1x_1}}{[e^{\lambda_1x_1}-\bar{\alpha}_1]^{\beta_1+1}}$$
$$+I\left[(\beta_1+\beta_2)(\lambda_1+\lambda_2)\frac{[(\alpha_1+\alpha_2)]^{\beta_1+\beta_2}e^{(\lambda_1+\lambda_2)(x_1+x_2)}}{[e^{(\lambda_1+\lambda_2)(x_1+x_2)}-(\bar{\alpha}_1+\bar{\alpha}_2)]^{\beta_1+\beta_2+1}}\right.$$
$$\left.-\beta_1\lambda_1\frac{{\alpha_1}^{\beta_1}e^{\lambda_1x_1}}{[e^{\lambda_1x_1}-\bar{\alpha}_1]^{\beta_1+1}}\right] \quad (8)$$

Where $x_1, x_1+x_2, \alpha_1, \alpha_1+\alpha_2, \beta_1, \beta_1+\beta_2,\ \lambda_1, \lambda_1+\lambda_2>0$.

Remark

The neutrosophic probability density function (PDF) of Marshall-Olkin type II exponential distribution is equivalent to the following equation:

$$g(x_N)=\beta_N\cdot\lambda_N\cdot\frac{\alpha_N^{\beta_N}}{[e^{\lambda_Nx_N}-\bar{\alpha}_N]^{\beta_N+1}}\cdot e^{\lambda_Nx_N}\quad;\quad x_N,\alpha_N,\beta_N,\lambda_N>_N 0 \quad (9)$$

Theorem 4.1

Function presented in (8) is a probability density function.

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Proof

$$\int_{-\infty}^{+\infty} g(x_1+x_2I)d(x_1+x_2I) = \int_0^{+\infty} \beta_1\lambda_1 \frac{{\alpha_1}^{\beta_1} e^{\lambda_1 x_1}}{[e^{\lambda_1 x_1}-\bar{\alpha}_1]^{\beta_1+1}}\,dx_1$$

$$+I\left[\int_0^{+\infty}(\beta_1+\beta_2)(\lambda_1+\lambda_2)\frac{[(\alpha_1+\alpha_2)]^{\beta_1+\beta_2}e^{(\lambda_1+\lambda_2)(x_1+x_2)}}{[e^{(\lambda_1+\lambda_2)(x_1+x_2)}-(\bar{\alpha}_1+\bar{\alpha}_2)]^{\beta_1+\beta_2+1}}d(x_1+x_2)\right.$$
$$\left.-\int_0^{+\infty}\beta_1\lambda_1\frac{{\alpha_1}^{\beta_1}}{[e^{\lambda_1 x_1}-\bar{\alpha}_1]^{\beta_1+1}}e^{\lambda_1 x_1}\,dx_1\right]$$

$$=\left[1-\left[\frac{\alpha_1}{e^{\lambda_1 x_1}-\bar{\alpha}_1}\right]^{\beta_1}\right]_0^{+\infty}$$

$$+I\left[\left[1-\left[\frac{(\alpha_1+\alpha_2)}{e^{(\lambda_1+\lambda_2)(x_1+x_2)}-(\bar{\alpha}_1+\bar{\alpha}_2)}\right]^{\beta_1+\beta_2}\right]_0^{+\infty}-\left[1-\left[\frac{\alpha_1}{e^{\lambda_1 x_1}-\bar{\alpha}_1}\right]^{\beta_1}\right]_0^{+\infty}\right]$$

$$=1+I[1-1]=1$$

Also:

$$T[g(x_1+x_2I)] = T\left[\beta_1\lambda_1\frac{{\alpha_1}^{\beta_1}}{[e^{\lambda_1 x_1}-\bar{\alpha}_1]^{\beta_1+1}}e^{\lambda_1 x_1}\right.$$
$$+I\left[(\beta_1+\beta_2)(\lambda_1+\lambda_2)\frac{[(\alpha_1+\alpha_2)]^{\beta_1+\beta_2}}{[e^{(\lambda_1+\lambda_2)(x_1+x_2)}-(\bar{\alpha}_1+\bar{\alpha}_2)]^{\beta_1+\beta_2+1}}e^{(\lambda_1+\lambda_2)(x_1+x_2)}\right.$$
$$\left.\left.-\beta_1\lambda_1\frac{{\alpha_1}^{\beta_1}}{[e^{\lambda_1 x_1}-\bar{\alpha}_1]^{\beta_1+1}}e^{\lambda_1 x_1}\right]\right]$$
$$=\left(\beta_1\lambda_1\frac{{\alpha_1}^{\beta_1}}{[e^{\lambda_1 x_1}-\bar{\alpha}_1]^{\beta_1+1}}e^{\lambda_1 x_1},(\beta_1+\beta_2)(\lambda_1+\lambda_2)\right.$$
$$\left.\cdot\frac{[(\alpha_1+\alpha_2)]^{\beta_1+\beta_2}}{[e^{(\lambda_1+\lambda_2)(x_1+x_2)}-(\bar{\alpha}_1+\bar{\alpha}_2)]^{\beta_1+\beta_2+1}}e^{(\lambda_1+\lambda_2)(x_1+x_2)}\right)=\left(g(x_1),g(x_1+x_2)\right)$$

Where $g(x_1)$ and $g(x_1+x_2)$ are probability density functions of classical Marshall-Olkin type II exponential distributions, then each of them is positive and continuous function on R, so $g(x_1+x_2I)$ is positive and continuous on $R(I)$.

Depending on previous results we can prove that equation (8) represents a probability density function.

Theorem 4.2

Cumulative distribution function of neutrosophic Marshall-Olkin type II exponential distribution is:

$$G(x_1+x_2I)=1-\left[\frac{\alpha_1}{e^{\lambda_1 x_1}-\bar{\alpha}_1}\right]^{\beta_1}$$
$$+I\left[\left[\frac{\alpha_1}{e^{\lambda_1 x_1}-\bar{\alpha}_1}\right]^{\beta_1}-\left[\frac{(\alpha_1+\alpha_2)}{e^{(\lambda_1+\lambda_2)(x_1+x_2)}-(\bar{\alpha}_1+\bar{\alpha}_2)}\right]^{\beta_1+\beta_2}\right] \quad (10)$$

Where $x_1, x_1+x_2, \alpha_1, \alpha_1+\alpha_2, \beta_1, \beta_1+\beta_2,\ \lambda_1, \lambda_1+\lambda_2>0$.

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Proof

$$\int_{-\infty}^{x_1+x_2I} g(t_1+t_2I)\, d(t_1+t_2I)$$

$$= \int_{-\infty}^{x_1} \beta_1 \cdot \lambda_1 \cdot \frac{{\alpha_1}^{\beta_1} \cdot e^{\lambda_1 t_1}}{[e^{\lambda_1 t_1} - \bar{\alpha}_1]^{\beta_1+1}} d(t_1)$$

$$+ I\left[\int_{-\infty}^{x_1+x_2} (\beta_1+\beta_2) \cdot (\lambda_1+\lambda_2) \cdot \frac{[(\alpha_1+\alpha_2)]^{\beta_1+\beta_2} \cdot e^{(\lambda_1+\lambda_2)(t_1+t_2)}}{[e^{(\lambda_1+\lambda_2)(t_1+t_2)} - (\bar{\alpha}_1+\bar{\alpha}_2)]^{\beta_1+\beta_2+1}} d(t_1+t_2)\right.$$

$$\left. - \int_{-\infty}^{x_1} \beta_1 \cdot \lambda_1 \cdot \frac{{\alpha_1}^{\beta_1} \cdot e^{\lambda_1 t_1}}{[e^{\lambda_1 t_1} - \bar{\alpha}_1]^{\beta_1+1}} d(t_1)\right]$$

$$= \left[1 - \left[\frac{\alpha_1}{e^{\lambda_1 t_1} - \bar{\alpha}_1}\right]^{\beta_1}\right]_{-\infty}^{x_1} + I\left[\left[\left[\frac{(\alpha_1+\alpha_2)}{e^{(\lambda_1+\lambda_2)(t_1+t_2)} - (\bar{\alpha}_1+\bar{\alpha}_2)}\right]^{\beta_1+\beta_2}\right]_{-\infty}^{x_1+x_2} - \left[1 - \left[\frac{\alpha_1}{e^{\lambda_1 t_1} - \bar{\alpha}_1}\right]^{\beta_1}\right]_{-\infty}^{x_1}\right]$$

$$= 1 - \left[\frac{\alpha_1}{e^{\lambda_1 x_1} - \bar{\alpha}_1}\right]^{\beta_1} + I\left[\left[\frac{\alpha_1}{e^{\lambda_1 x_1} - \bar{\alpha}_1}\right]^{\beta_1} - \left[\frac{(\alpha_1+\alpha_2)}{e^{(\lambda_1+\lambda_2)(x_1+x_2)} - (\bar{\alpha}_1+\bar{\alpha}_2)}\right]^{\beta_1+\beta_2}\right]$$

Remark

The cumulative distribution function of neutrosophic Marshall-Olkin type II exponential distribution can be written as follows:

$$G(x_N) = 1 - \left[\frac{\alpha_N}{e^{\lambda_N x_N} - \bar{\alpha}_N}\right]^{\beta_N}; \quad x_N, \alpha_N, \beta_N, \lambda_N >_N 0 \tag{11}$$

4.1 Parameters' estimation using neutrosophic maximum likelihood method

Let $\mathbb{X}_N$ be a neutrosophic random sample drawn from the probability density function of neutrosophic Marshall-Olkin type II exponential distribution given in equation (9), then the neutrosophic likelihood function will be:

$$L(\mathbb{X}_N; \Theta_N) = \prod_{i=1}^{n} g(X_{iN}; \alpha_N, \beta_N, \lambda_N) = \prod_{i=1}^{n} \beta_N \cdot \lambda_N \cdot \alpha_N^{\beta_N} \frac{1}{[e^{\lambda_N X_{iN}} - \bar{\alpha}_N]^{\beta_N+1}} \cdot e^{\lambda_N x_N}$$

$$= \beta_N^n \cdot \lambda_N^n \cdot \alpha_N^{n\beta_N} \prod_{i=1}^{n} [e^{\lambda_N X_{iN}} - \bar{\alpha}_N]^{-(\beta_N+1)} \cdot e^{\lambda_N \sum_{i=1}^{n} x_N}$$

So, the loglikelihood function is:

$$\mathcal{L}_N = ln\, L(\mathbb{X}_N; \Theta_N)$$

$$= n\, ln(\beta_N) + n\, ln(\lambda_N) + n\beta_N\, ln(\alpha_N) - (\beta_N+1) \sum_{i=1}^{n} ln[e^{\lambda_N X_{iN}} - \bar{\alpha}_N] + \lambda_N \sum_{i=1}^{n} x_N \tag{12}$$

Taking partial derivatives with respect to the parameters, we obtain the following equations:

$$\frac{\partial}{\partial \alpha_N} \mathcal{L}_N = \frac{n\beta_N}{\alpha_N} - (\beta_N+1) \sum_{i=1}^{n} \left(\frac{1}{e^{\lambda_N X_{iN}} - \bar{\alpha}_N}\right) \tag{13}$$

$$\frac{\partial}{\partial \beta_N} \mathcal{L}_N = \frac{n}{\beta_N} + n\, ln\, \alpha_N - \sum_{i=1}^{n} ln[e^{\lambda_N X_{iN}} - \bar{\alpha}_N] \tag{14}$$

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$$\frac{\partial}{\partial\lambda_N}\mathcal{L}_N = \frac{n}{\lambda_N} - (\beta_N + 1)\sum_{i=1}^{n}\left(\frac{X_{iN}}{1 - \bar{\alpha}_N e^{-\lambda_N X_{iN}}}\right) + n\bar{X}_N \tag{15}$$

Using AH-Isometry:

$$\begin{cases} \frac{\partial}{\partial\alpha_1}\mathcal{L}_1 = \frac{n\beta_1}{\alpha_1} - (\beta_1 + 1)\sum_{i=1}^{n}\left(\frac{1}{e^{\lambda_1 X_{i1}} - \bar{\alpha}_1}\right) \\ \frac{\partial}{\partial(\alpha_1+\alpha_2)}(\mathcal{L}_1 + \mathcal{L}_2) = \frac{n(\beta_1+\beta_2)}{(\alpha_1+\alpha_2)} - (\beta_1 + \beta_2 + 1)\sum_{i=1}^{n}\left(\frac{1}{e^{(\lambda_1+\lambda_2)(X_{i1}+X_{i2})} - (\bar{\alpha}_1 + \bar{\alpha}_2)}\right) \end{cases} \tag{16}$$

$$\begin{cases} \frac{\partial}{\partial\beta_1}\mathcal{L}_1 = \frac{n}{\beta_1} + n\ln\alpha_1 - \sum_{i=1}^{n}\ln\left[e^{\lambda_1 X_{i1}} - \bar{\alpha}_1\right] \\ \frac{\partial}{\partial(\beta_1+\beta_2)}(\mathcal{L}_1 + \mathcal{L}_2) = \frac{n}{(\beta_1+\beta_2)} + n\ln(\alpha_1+\alpha_2) - \sum_{i=1}^{n}\ln\left[e^{(\lambda_1+\lambda_2)(X_{i1}+X_{i2})} - (\bar{\alpha}_1 + \bar{\alpha}_2)\right] \end{cases} \tag{17}$$

$$\begin{cases} \frac{\partial}{\partial\lambda_1}\mathcal{L}_1 = \frac{n}{\lambda_1} - (\beta_1 + 1)\sum_{i=1}^{n}\left(\frac{X_{i1}}{1 - \bar{\alpha}_1 e^{-\lambda_1 X_{i1}}}\right) + n\bar{X}_1 \\ \frac{\partial}{\partial(\lambda_1+\lambda_2)}(\mathcal{L}_1 + \mathcal{L}_2) = \frac{n}{(\lambda_1+\lambda_2)} - (\beta_1 + \beta_2 + 1)\sum_{i=1}^{n}\left(\frac{X_{i1}+X_{i2}}{1 - (\bar{\alpha}_1 + \bar{\alpha}_2)e^{-(\lambda_1+\lambda_2)(X_{i1}+X_{i2})}}\right) + n(\bar{X}_1 + \bar{X}_2) \end{cases} \tag{18}$$

Solving the equations (16-18) numerically leads to the desired estimators.

4.2 Generating random numbers following neutrosophic Marshall-Olkin type II exponential distribution

Solving equation (11), with respect to x_N, we find:

$$x_N = \frac{1}{\lambda_N}\ln\left(\frac{\alpha_N}{\sqrt[\beta_N]{1 - y_N}} + \bar{\alpha}_N\right)\ ;\ y_N = G(x_N) \sim U_N[0,1] \tag{19}$$

Taking AH-isometry, we obtain the following two equations:

$$\begin{cases} x_1 = \frac{1}{\lambda_1}\ln\left(\frac{\alpha_1}{\sqrt[\beta_1]{1 - y_1}} + \bar{\alpha}_1\right) \\ x_1 + x_2 = \frac{1}{(\lambda_1+\lambda_2)}\ln\left(\frac{(\alpha_1+\alpha_2)}{\sqrt[(\beta_1+\beta_2)]{1 - (y_1+y_2)}} + (\bar{\alpha}_1 + \bar{\alpha}_2)\right) \end{cases} \tag{20}$$

Where the above two equations can be used to generate random numbers following classical Marshall-Olkin type II exponential distribution relying on fixed parameters- then, we use T^{-1} to obtain the desired numbers.

4.3 Monte Carlo simulation

This simulation is done using Maple software with total number of iterations $N = 1000$ and sample sizes of $10, 30$ and 75 with fixed parameters $\alpha_N = 3 + 1.5I, \beta_N = 2 - I$ and $\lambda_N = 0.5 + 3.1I$. Results of simulation are presented in table 1 where the efficiency of estimation is shown and clear.

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To confirm the goodness of our estimations, we use mean square error of the estimators and bias of it using the following equations:

$$MSE = \frac{\sum_{i=1}^{n}\left(\hat{\theta}_{iN} - \theta_N\right)^2}{n} \tag{21}$$

$$Bias = \frac{\sum_{i=1}^{n}\left|\hat{\theta}_{iN} - \theta_N\right|}{n} \tag{22}$$

Table 1: Simulation results for neutrosophic case

$\boldsymbol{\alpha_N = 3 + 1.5I}$			
n	$\hat{\alpha}_N$	$MSE\ \hat{\alpha}_N$	$Bias\ \hat{\alpha}_N$
10	$3.63387 + 0.99057I$	$0.97244 + 0.09715I$	$0.90817 - 0.05072I$
30	$3.36561 + 1.25139I$	$0.88970 + 0.33614I$	$0.85008 - 0.10083I$
75	$3.13722 + 1.50350I$	$0.72098 + 0.28065I$	$0.75082 + 0.12365I$
$\boldsymbol{\beta_N = 2 - I}$			
n	$\hat{\beta}_N$	$MSE\ \hat{\beta}_N$	$Bias\ \hat{\beta}_N$
10	$1.81066 - 0.61241I$	$0.17650 - 0.01421I$	$0.26397 - 0.05690I$
30	$1.78481 - 0.62930I$	$0.19439 - 0.04580I$	$0.25145 + 0.05610I$
75	$1.81035 - 0.66224I$	$0.16443 - 0.02989I$	$0.22532 + 0.06274I$
$\boldsymbol{\lambda_N = 0.5 + 3.1I}$			
n	$\hat{\lambda}_N$	$MSE\ \hat{\lambda}_N$	$Bias\ \hat{\lambda}_N$
10	$0.66725 + 3.22284I$	$0.11015 + 0.64734I$	$0.21453 + 0.48228I$
30	$0.61063 + 3.28654I$	$0.05120 + 0.68916I$	$0.13918 + 0.53513I$
75	$0.56328 + 3.30079I$	$0.01370 + 0.74090I$	$0.08637 + 0.60421I$

5. Plithogenic Marshall-Olkin Type II Class of Distributions:

Definition 5.1

Probability density function of Plithogenic Marshall-Olkin type II class of distributions is:

$$\begin{aligned} g(x_0 + x_1P_1 + x_2P_2) &= \beta_0\alpha_0 \frac{[\alpha_0\bar{F}(x_0)]^{\beta_0-1}}{[1-\bar{\alpha}_0\bar{F}(x_0)]^{\beta_0+1}} f(x_0) \\ &+P_1\left[(\beta_0+\beta_1)(\alpha_0+\alpha_1)\frac{\left[(\alpha_0+\alpha_1)\left(\bar{F}(x_0+x_1)\right)\right]^{\beta_0+\beta_1-1}}{[1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)]^{\beta_0+\beta_1+1}} f(x_0+x_1)\right. \\ &\left. - \beta_0\alpha_0\frac{[\alpha_0\bar{F}(x_0)]^{\beta_0-1}}{[1-\bar{\alpha}_0\bar{F}(x_0)]^{\beta_0+1}} f(x_0)\right] \end{aligned}$$

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$$+P_2\left[(\beta_0+\beta_1+\beta_2)(\alpha_0+\alpha_1+\alpha_2)\frac{[(\alpha_0+\alpha_1+\alpha_2)\bar{F}(x_0+x_1+x_2)]^{\beta_0+\beta_1+\beta_2-1}}{[1-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(x_0+x_1+x_2)]^{\beta_0+\beta_1+\beta_2+1}}\cdot f(x_0+x_1+x_2) - (\beta_0+\beta_1)(\alpha_0+\alpha_1)\frac{\left[(\alpha_0+\alpha_1)\left(\bar{F}(x_0+x_1)\right)\right]^{\beta_0+\beta_1-1}\cdot f(x_0+x_1)}{[1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)]^{\beta_0+\beta_1+1}}\right] \quad (23)$$

Where $x_0, x_0+x_1, x_0+x_1+x_2 \in R$ & $\alpha_0, \alpha_0+\alpha_1, \alpha_0+\ \alpha_1+\alpha_2, \beta_0, \beta_0+\beta_1, \beta_0+\beta_1+\beta_2 > 0$

Remark

Equation (23) is equivalent to the following equation:

$$g(x_P)=\beta_P\cdot\alpha_P\cdot\frac{[\alpha_P\bar{F}(x_P)]^{\beta_P-1}}{[1-\bar{\alpha}_P\bar{F}(x_P)]^{\beta_P+1}}\cdot f(x_P)\,;\ \ x_P\in R(P_1,P_2)\ \&\ \alpha_P,\beta_P,\lambda_P >_P 0 \qquad (24)$$

Theorem 5.1

Equation (23) is probability density function.

Proof

$$\int_{-\infty}^{+\infty} g(x_0+x_1P_1+x_2P_2)d(x_0+x_1P_1+x_2P_2)=\int_{-\infty}^{+\infty}\beta_0\alpha_0\frac{[\alpha_0\bar{F}(x_0)]^{\beta_0-1}f(x_0)}{[1-\bar{\alpha}_0\bar{F}(x_0)]^{\beta_0+1}}dx_0$$
$$+P_1\left[\int_{-\infty}^{+\infty}(\beta_0+\beta_1)(\alpha_0+\alpha_1)\frac{\left[(\alpha_0+\alpha_1)\left(\bar{F}(x_0+x_1)\right)\right]^{\beta_0+\beta_1-1}f(x_0+x_1)}{[1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)]^{\beta_0+\beta_1+1}}d(x_0+x_1) - \int_{-\infty}^{+\infty}\beta_0\alpha_0\frac{[\alpha_0\bar{F}(x_0)]^{\beta_0-1}}{[1-\bar{\alpha}_0\bar{F}(x_0)]^{\beta_0+1}}f(x_0)\,dx_0\right]$$
$$+P_2\left[\int_{-\infty}^{+\infty}(\beta_0+\beta_1+\beta_2)(\alpha_0+\alpha_1+\alpha_2)\frac{[(\alpha_0+\alpha_1+\alpha_2)\bar{F}(x_0+x_1+x_2)]^{\beta_0+\beta_1+\beta_2-1}}{[1-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(x_0+x_1+x_2)]^{\beta_0+\beta_1+\beta_2+1}}\cdot f(x_0+x_1+x_2)d(x_0+x_1+x_2) - \int_{-\infty}^{+\infty}(\beta_0+\beta_1)(\alpha_0+\alpha_1)\frac{\left[(\alpha_0+\alpha_1)\left(\bar{F}(x_0+x_1)\right)\right]^{\beta_0+\beta_1-1}}{[1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)]^{\beta_0+\beta_1+1}}\cdot f(x_0+x_1)\,d(x_0+x_1)\right]$$

$$=\left[1-\left[\frac{\alpha_0\bar{F}(x_0)}{1-\bar{\alpha}_0\bar{F}(x_0)}\right]^{\beta_0}\right]_{-\infty}^{+\infty}$$
$$+P_1\left[\left[1-\left[\frac{(\alpha_0+\alpha_1)\bar{F}(x_0+x_1)}{1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)}\right]^{\beta_0+\beta_1}\right]_{-\infty}^{+\infty}-\left[1-\left[\frac{\alpha_0\bar{F}(x_0)}{1-\bar{\alpha}_0\bar{F}(x_0)}\right]^{\beta_0}\right]_{-\infty}^{+\infty}\right]$$
$$+P_2\left[\left[1-\left[\frac{(\alpha_0+\alpha_1+\alpha_2)\bar{F}(x_0+x_1+x_2)}{1-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(x_0+x_1+x_2)}\right]^{\beta_0+\beta_1+\beta_2}\right]_{-\infty}^{+\infty} - \left[1-\left[\frac{(\alpha_0+\alpha_1)\bar{F}(x_0+x_1)}{1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)}\right]^{\beta_0+\beta_1}\right]_{-\infty}^{+\infty}\right]=1+P_1[1-1]+P_2[1-1]=1$$

And:

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$$B[g(x_0+x_1P_1+x_2P_2)]$$
$$= B\Bigg[\beta_0\alpha_0\frac{[\alpha_0\bar{F}(x_0)]^{\beta_0-1}}{[1-\bar{\alpha}_0\bar{F}(x_0)]^{\beta_0+1}}f(x_0)$$
$$+P_1\Bigg[(\beta_0+\beta_1)(\alpha_0+\alpha_1)\frac{\left[(\alpha_0+\alpha_1)\left(\bar{F}(x_0+x_1)\right)\right]^{\beta_0+\beta_1-1}}{[1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)]^{\beta_0+\beta_1+1}}f(x_0+x_1)$$
$$-\beta_0\alpha_0\frac{[\alpha_0\bar{F}(x_0)]^{\beta_0-1}}{[1-\bar{\alpha}_0\bar{F}(x_0)]^{\beta_0+1}}f(x_0)\Bigg]$$
$$+P_2\Bigg[(\beta_0+\beta_1+\beta_2)(\alpha_0+\alpha_1$$
$$+\alpha_2)\frac{[(\alpha_0+\alpha_1+\alpha_2)\bar{F}(x_0+x_1+x_2)]^{\beta_0+\beta_1+\beta_2-1}}{[1-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(x_0+x_1+x_2)]^{\beta_0+\beta_1+\beta_2+1}}f(x_0+x_1+x_2)$$
$$-(\beta_0+\beta_1)(\alpha_0+\alpha_1)\frac{\left[(\alpha_0+\alpha_1)\left(\bar{F}(x_0+x_1)\right)\right]^{\beta_0+\beta_1-1}}{[1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)]^{\beta_0+\beta_1+1}}f(x_0+x_1)\Bigg]\Bigg]$$

$$=\Bigg(\beta_0\alpha_0\frac{[\alpha_0\bar{F}(x_0)]^{\beta_0-1}}{[1-\bar{\alpha}_0\bar{F}(x_0)]^{\beta_0+1}}f(x_0),(\beta_0+\beta_1)(\alpha_0+\alpha_1)\frac{\left[(\alpha_0+\alpha_1)\left(\bar{F}(x_0+x_1)\right)\right]^{\beta_0+\beta_1-1}}{[1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)]^{\beta_0+\beta_1+1}}$$
$$\cdot f(x_0+x_1),(\beta_0+\beta_1+\beta_2)(\alpha_0+\alpha_1+\alpha_2)$$
$$\cdot\frac{[(\alpha_0+\alpha_1+\alpha_2)\bar{F}(x_0+x_1+x_2)]^{\beta_0+\beta_1+\beta_2-1}}{[1-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(x_0+x_1+x_2)]^{\beta_0+\beta_1+\beta_2+1}}f(x_0+x_1+x_2)\Bigg)$$
$$=\left(g(x_0),g(x_0+x_1),g(x_0+x_1+x_2)\right)$$

Where $g(x_0)$, $g(x_0+x_1)$ and $g(x_0+x_1+x_2)$ are probability density functions of classical Marshall-Olkin type II exponential distributions, then each of them is positive and continuous function on R, so $g(x_0+x_1P_1+x_2P_2)$ is positive and continuous on $R(P_1,P_2)$.

Depending on previous results we can prove that equation (23) represents a probability density function.

Theorem 5.2

Cumulative distribution function of Plithogenic Marshall-Olkin type II class of distributions is:

$$G(x_0+x_1P_1+x_2P_2)=1-\left[\frac{\alpha_0\bar{F}(x_0)}{1-\bar{\alpha}_0\bar{F}(x_0)}\right]^{\beta_0}$$
$$+P_1\left[\left[\frac{\alpha_0\bar{F}(x_0)}{1-\bar{\alpha}_0\bar{F}(x_0)}\right]^{\beta_0}-\left[\frac{(\alpha_0+\alpha_1)\bar{F}(x_0+x_1)}{1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)}\right]^{\beta_0+\beta_1}\right]$$
$$+P_2\left[\left[\frac{(\alpha_0+\alpha_1+\alpha_2)\bar{F}(x_0+x_1+x_2)}{1-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(x_0+x_1+x_2)}\right]^{\beta_0+\beta_1+\beta_2}-\left[\frac{(\alpha_0+\alpha_1)\bar{F}(x_0+x_1)}{1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)}\right]^{\beta_0+\beta_1}\right] \quad (25)$$

Where $x_0, x_0+x_1, x_0+x_1+x_2 \in R$ & $\alpha_0, \alpha_0+\alpha_1, \alpha_0+\alpha_1+\alpha_2, \beta_0, \beta_0+\beta_1, \beta_0+\beta_1+\beta_2 > 0$.

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Proof

$$\int_{-\infty}^{x_0+x_1P_1+x_2P_2} g(t_0+t_1P_1+t_2P_2)\,d(t_0+t_1P_1+t_2P_2) = \int_{-\infty}^{x_0} \beta_0\alpha_0 \frac{[\alpha_0\bar{F}(t_0)]^{\beta_0-1}}{[1-\bar{\alpha}_0\bar{F}(t_0)]^{\beta_0+1}} f(t_0)\,dt_0$$
$$+P_1\left[\int_{-\infty}^{x_0+x_1} (\beta_0+\beta_1)(\alpha_0+\alpha_1)\frac{[(\alpha_0+\alpha_1)(\bar{F}(t_0+t_1))]^{\beta_0+\beta_1-1}}{[1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(t_0+t_1)]^{\beta_0+\beta_1+1}} f(t_0+t_1)\,d(t_0+t_1) - \int_{-\infty}^{x_0}\beta_0\alpha_0\frac{[\alpha_0\bar{F}(t_0)]^{\beta_0-1}}{[1-\bar{\alpha}_0\bar{F}(t_0)]^{\beta_0+1}} f(t_0)\,dt_0\right]$$
$$+P_2\left[\int_{-\infty}^{x_0+x_1+x_2}(\beta_0+\beta_1+\beta_2)(\alpha_0+\alpha_1+\alpha_2)\frac{[(\alpha_0+\alpha_1+\alpha_2)\bar{F}(t_0+t_1+t_2)]^{\beta_0+\beta_1+\beta_2-1}}{[1-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(t_0+t_1+t_2)]^{\beta_0+\beta_1+\beta_2+1}} \cdot f(t_0+t_1+t_2)\,d(t_0+t_1+t_2) - \int_{-\infty}^{x_0+x_1}(\beta_0+\beta_1)(\alpha_0+\alpha_1)\frac{[(\alpha_0+\alpha_1)(\bar{F}(t_0+t_1))]^{\beta_0+\beta_1-1}}{[1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(t_0+t_1)]^{\beta_0+\beta_1+1}} \cdot f(t_0+t_1)\,d(t_0+t_1)\right] =$$

$$= \left[1-\left[\frac{\alpha_0\bar{F}(t_0)}{1-\bar{\alpha}_0\bar{F}(t_0)}\right]^{\beta_0}\right]_{-\infty}^{x_0}$$
$$+P_1\left[\left[1-\left[\frac{(\alpha_0+\alpha_1)\bar{F}(x_0+x_1)}{1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)}\right]^{\beta_0+\beta_1}\right]_{-\infty}^{x_0+x_1} - \left[1-\left[\frac{\alpha_0\bar{F}(t_0)}{1-\bar{\alpha}_0\bar{F}(t_0)}\right]^{\beta_0}\right]_{-\infty}^{x_0}\right]$$
$$+P_2\left[\left[1-\left[\frac{(\alpha_0+\alpha_1+\alpha_2)\bar{F}(x_0+x_1+x_2)}{1-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(x_0+x_1+x_2)}\right]^{\beta_0+\beta_1+\beta_2}\right]_{-\infty}^{x_0+x_1+x_2} - \left[1-\left[\frac{(\alpha_0+\alpha_1)\bar{F}(x_0+x_1)}{1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)}\right]^{\beta_0+\beta_1}\right]_{-\infty}^{x_0+x_1}\right]$$

$$= 1-\left[\frac{\alpha_0\bar{F}(x_0)}{1-\bar{\alpha}_0\bar{F}(x_0)}\right]^{\beta_0} + P_1\left[\left[\frac{\alpha_0\bar{F}(x_0)}{1-\bar{\alpha}_0\bar{F}(x_0)}\right]^{\beta_0} - \left[\frac{(\alpha_0+\alpha_1)\bar{F}(x_0+x_1)}{1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)}\right]^{\beta_0+\beta_1}\right]$$
$$+P_2\left[\left[\frac{(\alpha_0+\alpha_1+\alpha_2)\bar{F}(x_0+x_1+x_2)}{1-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)\bar{F}(x_0+x_1+x_2)}\right]^{\beta_0+\beta_1+\beta_2} - \left[\frac{(\alpha_0+\alpha_1)\bar{F}(x_0+x_1)}{1-(\bar{\alpha}_0+\bar{\alpha}_1)\bar{F}(x_0+x_1)}\right]^{\beta_0+\beta_1}\right]$$

6. Plithogenic Marshall-Olkin Type II Exponential Distribution:

Definition 6.1

Probability density function of Plithogenic Marshall-Olkin type II exponential distribution is:

$$g(x_0+x_1P_1+x_2P_2) = \beta_0\lambda_0\frac{(\alpha_0)^{\beta_0}}{[e^{\lambda_0x_0}-\bar{\alpha}_0]^{\beta_0+1}}e^{\lambda_0x_0}$$
$$+P_1\left[(\beta_0+\beta_1)(\lambda_0+\lambda_1)\frac{(\alpha_0+\alpha_1)^{\beta_0+\beta_1}e^{(\lambda_0+\lambda_1)(x_0+x_1)}}{[e^{(\lambda_0+\lambda_1)(x_0+x_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)]^{\beta_0+\beta_1+1}} - \beta_0\lambda_0\frac{(\alpha_0)^{\beta_0}}{[e^{\lambda_0x_0}-\bar{\alpha}_0]^{\beta_0+1}}e^{\lambda_0x_0}\right]$$

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$$+P_2\left[(\beta_0+\beta_1+\beta_2)(\lambda_0+\lambda_1+\lambda_2)\frac{(\alpha_0+\alpha_1+\alpha_2)^{\beta_0+\beta_1+\beta_2}\cdot e^{(\lambda_0+\lambda_1+\lambda_2)(x_0+x_1+x_2)}}{[e^{(\lambda_0+\lambda_1+\lambda_2)(x_0+x_1+x_2)}-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)]^{\beta_0+\beta_1+\beta_2+1}} - (\beta_0+\beta_1)(\lambda_0+\lambda_1)\frac{(\alpha_0+\alpha_1)^{\beta_0+\beta_1}\cdot e^{(\lambda_0+\lambda_1)(x_0+x_1)}}{[e^{(\lambda_0+\lambda_1)(x_0+x_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)]^{\beta_0+\beta_1+1}}\right] \quad (26)$$

Where $x_0, x_0+x_1, x_0+x_1+x_2, \alpha_0, a_0+a_1, \alpha_0+\alpha_1+\alpha_2, \beta_0, \beta_0+\beta_1, \beta_0+\beta_1+\beta_2, \lambda_0, \lambda_0+\lambda_1, \lambda_0+\lambda_1+\lambda_2 > 0$.

Remark

The Plithogenic probability density function (PDF) of Marshall-Olkin type II exponential distribution can also be written as follows:

$$g(x_P) = \beta_P\lambda_P\frac{\alpha_P^{\beta_P}}{[e^{\lambda_P x_P}-\bar{\alpha}_P]^{\beta_P+1}}\, e^{\lambda_P x_P} \quad ; \quad x_P, \alpha_P, \beta_P, \lambda_P >_P 0 \quad (27)$$

Theorem 6.1

Equation (26) is probability density function.

Proof

$$\int_{-\infty}^{+\infty} g(x_0+x_1P_1+x_2P_2)d(x_0+x_1P_1+x_2P_2) = \int_0^{+\infty}\beta_0\lambda_0\frac{(\alpha_0)^{\beta_0}}{[e^{\lambda_0x_0}-\bar{\alpha}_0]^{\beta_0+1}}e^{\lambda_0x_0}\,dx_0$$
$$+P_1\left[(\beta_0+\beta_1)(\lambda_0+\lambda_1)\frac{(\alpha_0+\alpha_1)^{\beta_0+\beta_1}\cdot e^{(\lambda_0+\lambda_1)(x_0+x_1)}}{[e^{(\lambda_0+\lambda_1)(x_0+x_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)]^{\beta_0+\beta_1+1}}d(x_0+x_1) - \int_0^{+\infty}\beta_0\lambda_0\frac{(\alpha_0)^{\beta_0}\cdot e^{\lambda_0x_0}}{[e^{\lambda_0x_0}-\bar{\alpha}_0]^{\beta_0+1}}dx_0\right]$$
$$+P_2\left[\int_0^{+\infty}(\beta_0+\beta_1+\beta_2)(\lambda_0+\lambda_1+\lambda_2)\cdot\frac{(\alpha_0+\alpha_1+\alpha_2)^{\beta_0+\beta_1+\beta_2}\cdot e^{(\lambda_0+\lambda_1+\lambda_2)(x_0+x_1+x_2)}}{[e^{(\lambda_0+\lambda_1+\lambda_2)(x_0+x_1+x_2)}-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)]^{\beta_0+\beta_1+\beta_2+1}}d(x_0+x_1+x_2) - \int_0^{+\infty}(\beta_0+\beta_1)(\lambda_0+\lambda_1)\frac{(\alpha_0+\alpha_1)^{\beta_0+\beta_1}\cdot e^{(\lambda_0+\lambda_1)(x_0+x_1)}}{[e^{(\lambda_0+\lambda_1)(x_0+x_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)]^{\beta_0+\beta_1+1}}\cdot d(x_0+x_1)\right]$$
$$=\left[1-\left[\frac{\alpha_0}{e^{\lambda_0x_0}-\bar{\alpha}_0}\right]^{\beta_0}\right]_0^{+\infty}$$
$$+P_1\left[\left[1-\left[\frac{(\alpha_0+\alpha_1)}{e^{(\lambda_0+\lambda_1)(x_0+x_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)}\right]^{\beta_0+\beta_1}\right]_0^{+\infty} - \left[1-\left[\frac{\alpha_0}{e^{\lambda_0x_0}-\bar{\alpha}_0}\right]^{\beta_0}\right]_0^{+\infty}\right]$$
$$+P_2\left[\left[1-\left[\frac{(\alpha_0+\alpha_1+\alpha_2)}{e^{(\lambda_0+\lambda_1+\lambda_2)(x_0+x_1+x_2)}-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)}\right]^{\beta_0+\beta_1+\beta_2}\right]_0^{+\infty} - \left[1-\left[\frac{(\alpha_0+\alpha_1)}{e^{(\lambda_0+\lambda_1)(x_0+x_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)}\right]^{\beta_0+\beta_1}\right]_0^{+\infty}\right]$$
$$=1+P_1[1-(1-0)]+P_2[1-(1-0)]=1$$

And:

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$$
\begin{aligned}
&B[g(x_0+x_1P_1+x_2P_2)]\\
&\quad = B\Bigg[\beta_0\lambda_0\frac{(\alpha_0)^{\beta_0}}{[e^{\lambda_0x_0}-\bar{\alpha}_0]^{\beta_0+1}}e^{\lambda_0x_0}\\
&\quad + P_1\Bigg[(\beta_0+\beta_1)(\lambda_0+\lambda_1)\frac{(\alpha_0+\alpha_1)^{\beta_0+\beta_1}\cdot e^{(\lambda_0+\lambda_1)(x_0+x_1)}}{[e^{(\lambda_0+\lambda_1)(x_0+x_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)]^{\beta_0+\beta_1+1}}\\
&\quad - \beta_0\lambda_0\frac{(\alpha_0)^{\beta_0}}{[e^{\lambda_0x_0}-\bar{\alpha}_0]^{\beta_0+1}}e^{\lambda_0x_0}\Bigg]\\
&\quad + P_2\Bigg[(\beta_0+\beta_1+\beta_2)(\lambda_0+\lambda_1+\lambda_2)\\
&\quad \cdot\frac{(\alpha_0+\alpha_1+\alpha_2)^{\beta_0+\beta_1+\beta_2}\cdot e^{(\lambda_0+\lambda_1+\lambda_2)(x_0+x_1+x_2)}}{[e^{(\lambda_0+\lambda_1+\lambda_2)(x_0+x_1+x_2)}-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)]^{\beta_0+\beta_1+\beta_2+1}}\\
&\quad - (\beta_0+\beta_1)(\lambda_0+\lambda_1)\frac{(\alpha_0+\alpha_1)^{\beta_0+\beta_1}\cdot e^{(\lambda_0+\lambda_1)(x_0+x_1)}}{[e^{(\lambda_0+\lambda_1)(x_0+x_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)]^{\beta_0+\beta_1+1}}\Bigg]\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \Bigg(&\beta_0\lambda_0\frac{(\alpha_0)^{\beta_0}}{[e^{\lambda_0x_0}-\bar{\alpha}_0]^{\beta_0+1}}e^{\lambda_0x_0}, (\beta_0+\beta_1)(\lambda_0+\lambda_1)\frac{(\alpha_0+\alpha_1)^{\beta_0+\beta_1}e^{(\lambda_0+\lambda_1)(x_0+x_1)}}{[e^{(\lambda_0+\lambda_1)(x_0+x_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)]^{\beta_0+\beta_1+1}}, (\beta_0\\
&+\beta_1+\beta_2)(\lambda_0+\lambda_1+\lambda_2)\frac{(\alpha_0+\alpha_1+\alpha_2)^{\beta_0+\beta_1+\beta_2}\cdot e^{(\lambda_0+\lambda_1+\lambda_2)(x_0+x_1+x_2)}}{[e^{(\lambda_0+\lambda_1+\lambda_2)(x_0+x_1+x_2)}-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)]^{\beta_0+\beta_1+\beta_2+1}}\Bigg)\\
&= \big(g(x_0), g(x_0+x_1), g(x_0+x_1+x_2)\big)
\end{aligned}
$$

Where $g(x_0)$, $g(x_0+x_1)$ and $g(x_0+x_1+x_2)$ are probability density functions of classical Marshall-Olkin type II exponential distributions, then each of them is positive and continuous function on R, so $g(x_0+x_1P_1+x_2P_2)$ is positive and continuous on $R(P_1,P_2)$.

Depending on previous results we can prove that equation (26) represents a probability density function.

Theorem 6.2

Cumulative distribution function of Plithogenic Marshall-Olkin type II exponential distribution is:

$$
\begin{aligned}
&G(x_0+x_1P_1+x_2P_2)\\
&\quad = 1-\left[\frac{\alpha_0}{e^{\lambda_0x_0}-\bar{\alpha}_0}\right]^{\beta_0}+P_1\left[\left[\frac{\alpha_0}{e^{\lambda_0x_0}-\bar{\alpha}_0}\right]^{\beta_0}-\left[\frac{(\alpha_0+\alpha_1)}{e^{(\lambda_0+\lambda_1)(x_0+x_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)}\right]^{\beta_0+\beta_1}\right]\\
&\quad + P_2\left[\left[\frac{(\alpha_0+\alpha_1)}{e^{(\lambda_0+\lambda_1)(x_0+x_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)}\right]^{\beta_0+\beta_1}\right.\\
&\quad \left.-\left[\frac{(\alpha_0+\alpha_1+\alpha_2)}{e^{(\lambda_0+\lambda_1+\lambda_2)(x_0+x_1+x_2)}-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)}\right]^{\beta_0+\beta_1+\beta_2}\right] \qquad (28)
\end{aligned}
$$

Where $x_0, x_0+x_1, x_0+x_1+x_2, \alpha_0, a_0+a_1, \alpha_0+\alpha_1+\alpha_2, \beta_0, \beta_0+\beta_1, \beta_0+\beta_1+\beta_2, \lambda_0, \lambda_0+\lambda_1, \lambda_0+\lambda_1+\lambda_2 > 0$.

Proof

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$$\int_{-\infty}^{x_0+x_1P_1+x_2P_2} g(t_0+t_1P_1+t_2P_2)\,d(t_0+t_1P_1+t_2P_2) = \int_{-\infty}^{x_0} \beta_0\lambda_0 \frac{(\alpha_0)^{\beta_0}\cdot e^{\lambda_0 t_0}}{[e^{\lambda_0 t_0}-\bar{\alpha}_0]^{\beta_0+1}}\,dt_0$$

$$+P_1\left[\int_{-\infty}^{x_0+x_1}(\beta_0+\beta_1)(\lambda_0+\lambda_1)\frac{(\alpha_0+\alpha_1)^{\beta_0+\beta_1}\cdot e^{(\lambda_0+\lambda_1)(t_0+t_1)}}{[e^{(\lambda_0+\lambda_1)(t_0+t_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)]^{\beta_0+\beta_1+1}}\,d(t_0+t_1) - \int_{-\infty}^{x_0}\beta_0\lambda_0\frac{(\alpha_0)^{\beta_0}\cdot e^{\lambda_0 t_0}}{[e^{\lambda_0 t_0}-\bar{\alpha}_0]^{\beta_0+1}}\,dt_0\right]$$

$$+P_2\left[\int_{-\infty}^{x_0+x_1+x_2}(\beta_0+\beta_1+\beta_2)(\lambda_0+\lambda_1+\lambda_2)\cdot\frac{(\alpha_0+\alpha_1+\alpha_2)^{\beta_0+\beta_1+\beta_2}\cdot e^{(\lambda_0+\lambda_1+\lambda_2)(t_0+t_1+t_2)}}{[e^{(\lambda_0+\lambda_1+\lambda_2)(t_0+t_1+t_2)}-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)]^{\beta_0+\beta_1+\beta_2+1}}\,d(t_0+t_1+t_2) - \int_{-\infty}^{x_0+x_1}(\beta_0+\beta_1)(\lambda_0+\lambda_1)\frac{(\alpha_0+\alpha_1)^{\beta_0+\beta_1}\cdot e^{(\lambda_0+\lambda_1)(t_0+t_1)}}{[e^{(\lambda_0+\lambda_1)(t_0+t_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)]^{\beta_0+\beta_1+1}}\,d(t_0+t_1)\right]$$

$$=\left[1-\left[\frac{\alpha_0}{e^{\lambda_0 t_0}-\bar{\alpha}_0}\right]^{\beta_0}\right]_{-\infty}^{x_0} + P_1\left[\left[1-\left[\frac{(\alpha_0+\alpha_1)}{e^{(\lambda_0+\lambda_1)(t_0+t_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)}\right]^{\beta_0+\beta_1}\right]_{-\infty}^{x_0+x_1} - \left[1-\left[\frac{\alpha_0}{e^{\lambda_0 t_0}-\bar{\alpha}_0}\right]^{\beta_0}\right]_{-\infty}^{x_0}\right]$$
$$+P_2\left[\left[1-\left[\frac{(\alpha_0+\alpha_1+\alpha_2)}{e^{(\lambda_0+\lambda_1+\lambda_2)(t_0+t_1+t_2)}-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)}\right]^{\beta_0+\beta_1+\beta_2}\right]_{-\infty}^{x_0+x_1+x_2} - \left[1-\left[\frac{(\alpha_0+\alpha_1)}{e^{(\lambda_0+\lambda_1)(t_0+t_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)}\right]^{\beta_0+\beta_1}\right]_{-\infty}^{x_0+x_1}\right]$$

$$=1-\left[\frac{\alpha_0}{e^{\lambda_0 x_0}-\bar{\alpha}_0}\right]^{\beta_0} + P_1\left[\left[\frac{\alpha_0}{e^{\lambda_0 x_0}-\bar{\alpha}_0}\right]^{\beta_0} - \left[\frac{(\alpha_0+\alpha_1)}{e^{(\lambda_0+\lambda_1)(x_0+x_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)}\right]^{\beta_0+\beta_1}\right]$$
$$+P_2\left[\left[\frac{(\alpha_0+\alpha_1)}{e^{(\lambda_0+\lambda_1)(x_0+x_1)}-(\bar{\alpha}_0+\bar{\alpha}_1)}\right]^{\beta_0+\beta_1} - \left[\frac{(\alpha_0+\alpha_1+\alpha_2)}{e^{(\lambda_0+\lambda_1+\lambda_2)(x_0+x_1+x_2)}-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)}\right]^{\beta_0+\beta_1+\beta_2}\right]$$

Remark

The cumulative distribution function of neutrosophic Marshall-Olkin type II exponential distribution is defined as following:

$$G(x_P) = 1-\left[\frac{\alpha_P}{e^{\lambda_P x_P}-\bar{\alpha}_P}\right]^{\beta_P};\ x_P,\alpha_P,\beta_P,\lambda_P >_P 0 \quad (29)$$

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6.1 Parameters' Estimation using Plithogenic loglikelihood method

Let $\mathbb{X}_P$ be a Plithogenic random sample drawn from the probability density function of Plithogenic Marshall-Olkin type II exponential distribution given in equation (27), then the Plithogenic likelihood function is:

$$L(\mathbb{X}_P;\Theta_P)=\prod_{i=1}^{n} g(X_{iP};\alpha_P,\beta_P,\lambda_P)=\prod_{i=1}^{n}\beta_P\lambda_P\alpha_P^{\beta_P}\frac{e^{\lambda_P x_P}}{[e^{\lambda_P X_{iP}}-\bar{\alpha}_P]^{\beta_P+1}}$$

$$=\beta_P^n\cdot\lambda_P^n\cdot\alpha_P^{n\beta_P}\cdot e^{\lambda_P\sum_{i=1}^{n}x_P}\prod_{i=1}^{n}\left[e^{\lambda_P X_{iP}}-\bar{\alpha}_P\right]^{-(\beta_P+1)}$$

So, the loglikelihood function is:

$$\mathcal{L}_P=\ln L(\mathbb{X}_P;\Theta_\mathrm{P})=n\ln(\beta_P)+n\ln(\lambda_P)+n\beta_P\ln(\alpha_P)$$
$$+\lambda_P\sum\nolimits_{i=1}^{n}x_P-(\beta_P+1)\sum_{i=1}^{n}\ln\left[e^{\lambda_P X_{iP}}-\bar{\alpha}_P\right] \qquad (30)$$

Taking partial derivatives with respect to the parameters, we obtain the following equations:

$$\frac{\partial}{\partial\alpha_P}\mathcal{L}_P=\frac{n\beta_P}{\alpha_P}-(\beta_P+1)\sum_{i=1}^{n}\left(\frac{1}{e^{\lambda_P X_{iP}}-\bar{\alpha}_P}\right) \qquad (31)$$

$$\frac{\partial}{\partial\beta_P}\mathcal{L}_P=\frac{n}{\beta_P}+n\ln\alpha_P-\sum_{i=1}^{n}\ln\left[e^{\lambda_P X_{iP}}-\bar{\alpha}_P\right] \qquad (32)$$

$$\frac{\partial}{\partial\lambda_P}\mathcal{L}_P=\frac{n}{\lambda_P}-(\beta_P+1)\sum_{i=1}^{n}\left(\frac{X_{iP}}{1-\bar{\alpha}_P e^{-\lambda_P X_{iP}}}\right)+n\bar{X}_P \qquad (33)$$

Using B-Isometry:

$$\begin{cases}\frac{\partial\mathcal{L}_0}{\partial\alpha_0}=\frac{n\beta_0}{\alpha_0}-(\beta_0+1)\sum_{i=1}^{n}\left(\frac{1}{e^{\lambda_0 X_{i0}-\bar{\alpha}_0}}\right)\\ \frac{\partial(\mathcal{L}_0+\mathcal{L}_1)}{\partial(\alpha_0+\alpha_1)}=\frac{n(\beta_0+\beta_1)}{(\alpha_0+\alpha_1)}-(\beta_0+\beta_1+1)\sum_{i=1}^{n}\left(\frac{1}{e^{(\lambda_0+\lambda_1)(X_{i0}+X_{i1})-(\bar{\alpha}_0+\bar{\alpha}_1)}}\right)\\ \frac{\partial(\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2)}{\partial(\alpha_0+\alpha_1+\alpha_2)}=\frac{n(\beta_0+\beta_1+\beta_2)}{(\alpha_0+\alpha_1+\alpha_2)}-(\beta_0+\beta_1+\beta_2+1)\sum_{i=1}^{n}\left(\frac{1}{e^{(\lambda_0+\lambda_1+\lambda_2)(X_{i0}+X_{i1}+X_{i2})-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)}}\right)\end{cases} \qquad (34)$$

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$$\begin{cases} \dfrac{\partial \mathcal{L}_0}{\partial \beta_0} = \dfrac{n}{\beta_0} + n\ln\alpha_0 - \sum_{i=1}^{n} \ln\left[e^{\lambda_0 X_{i0}} - \bar{\alpha}_0\right] \\ \dfrac{\partial(\mathcal{L}_0+\mathcal{L}_1)}{\partial(\beta_0+\beta_1)} = \dfrac{n}{(\beta_0+\beta_1)} + n\ln(\alpha_0+\alpha_1) - \sum_{i=1}^{n} \ln\left[e^{(\lambda_0+\lambda_1)(X_{i0}+X_{i1})} - (\bar{\alpha}_0+\bar{\alpha}_1)\right] \\ \dfrac{\partial(\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2)}{\partial(\beta_0+\beta_1+\beta_2)} = \dfrac{n}{(\beta_0+\beta_1+\beta_2)} + n\ln(\alpha_0+\alpha_1+\alpha_2) \\ \qquad - \sum_{i=1}^{n} \ln\left[e^{(\lambda_0+\lambda_1+\lambda_2)(X_{i0}+X_{i1}+X_{i2})} - (\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)\right] \end{cases} \quad (35)$$

$$\begin{cases} \dfrac{\partial \mathcal{L}_0}{\partial \lambda_0} = \dfrac{n}{\lambda_0} - (\beta_0+1)\sum_{i=1}^{n}\left(\dfrac{X_{i0}}{1-\bar{\alpha}_0 e^{-\lambda_0 X_{i0}}}\right) + n\bar{X}_0 \\ \dfrac{\partial(\mathcal{L}_0+\mathcal{L}_1)}{\partial(\lambda_0+\lambda_1)} = \dfrac{n}{(\lambda_0+\lambda_1)} - (\beta_0+\beta_1+1)\sum_{i=1}^{n}\left(\dfrac{X_{i0}+X_{i1}}{1-(\bar{\alpha}_0+\bar{\alpha}_1)e^{-(\lambda_0+\lambda_1)(X_{i0}+X_{i1})}}\right) + n(\bar{X}_0+\bar{X}_1) \\ \dfrac{\partial(\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2)}{\partial(\lambda_0+\lambda_1+\lambda_2)} = \dfrac{n}{(\lambda_0+\lambda_1+\lambda_2)} - (\beta_0+\beta_1+\beta_2+1) \\ \qquad \cdot \sum_{i=1}^{n}\left(\dfrac{X_{i0}+X_{i1}+X_{i2}}{1-(\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)e^{-(\lambda_0+\lambda_1+\lambda_2)(X_{i0}+X_{i1}+X_{i2})}}\right) + n(\bar{X}_0+\bar{X}_1+\bar{X}_2) \end{cases} \quad (36)$$

Solving the equations (34-36) numerically, we obtain the desired estimations.

6.2 Generating random numbers following Plithogenic Marshall-Olkin type II exponential distribution

Solving equation (29), with respect to x_P, we find:

$$x_P = \frac{1}{\lambda_P}\ln\left(\frac{\alpha_P}{\sqrt[\beta_P]{1-y_P}} + \bar{\alpha}_P\right) \; ; \; y_N = G(x_P) \sim U_P[0,1] \quad (37)$$

Taking B-isometry, we obtain the following equations:

$$\begin{cases} x_0 = \dfrac{1}{\lambda_0}\ln\left(\dfrac{\alpha_0}{\sqrt[\beta_0]{1-y_0}} + \bar{\alpha}_0\right) \\ x_0 + x_1 = \dfrac{1}{(\lambda_0+\lambda_1)}\ln\left(\dfrac{(\alpha_0+\alpha_1)}{\sqrt[(\beta_0+\beta_1)]{1-(y_0+y_1)}} + (\bar{\alpha}_0+\bar{\alpha}_1)\right) \\ x_0+x_1+x_2 = \dfrac{1}{(\lambda_0+\lambda_1+\lambda_2)}\ln\left(\dfrac{(\alpha_0+\alpha_1+\alpha_2)}{\sqrt[(\beta_0+\beta_1+\beta_2)]{1-(y_0+y_1+y_2)}} + (\bar{\alpha}_0+\bar{\alpha}_1+\bar{\alpha}_2)\right) \end{cases} \quad (38)$$

Where the above equations can be used to generate random numbers following classical Marshall-Olkin type II exponential distribution relying on fixed parameters- then, we use B^{-1} to obtain the desired numbers.

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6.3 Monte Carlo simulation

This simulation is done using Maple software with $N = 1000$ iterations and sample sizes of $10, 30$ and 75, in addition to fixed parameters $\alpha_P = 2 - P_1 - 0.5P_2, \beta_P = 3 - 1.7P_1 + 2P_2$ and $\lambda_P = 1 + 2P_1 - 1.5P_2$.

To confirm the goodness of our estimations, we use mean square error of the estimators and bias of it taking in consideration the plithogenic form of the estimators and results of simulation are given in table 2 where the efficiency of the estimators is clear based on MSE and Bias.

Table 2: Simulation results for plithogenic case

	$\boldsymbol{\alpha_P = 2 - P_1 - 0.5P_2}$		
n	$\widehat{\alpha}_P$	$MSE\ \widehat{\alpha}_P$	$Bias\ \widehat{\alpha}_P$
10	$2.854 - 1.236P_1 - 0.151P_2$	$1.699 - 0.745P_1 + 0.412$	$1.152 - 0.342P_1 + 0.202P_2$
30	$2.600 - 1.135P_1 - 0.069P_2$	$1.095 - 0.379P_1 + 0.539P_2$	$0.931 - 0.226P_1 + 0.244P_2$
75	$2.461 - 1.216P_1 + 0.059P_2$	$0.731 - 0.397P_1 + 0.651P_2$	$0.786 - 0.326P_1 + 0.370P_2$
	$\boldsymbol{\beta_P = 3 - 1.7P_1 + 2P_2}$		
n	$\widehat{\beta}_P$	$MSE\ \widehat{\beta}_P$	$Bias\ \widehat{\beta}_P$
10	$2.866 - 1.082P_1 + 1.522P_2$	$0.319 - 0.168P_1 - 0.369P_2$	$0.431 - 0.149P_1 + 0.327P_2$
30	$2.938 - 1.270P_1 + 1.571P_2$	$0.205 + 0.129P_1 - 0.259P_2$	$0.295 + 0.172P_1 - 0.269P_2$
75	$2.992 - 1.438P_1 + 1.661P_2$	$0.101 + 0.075P_1 - 0.130P_2$	$0.176 + 0.146P_1 - 0.172P_2$
	$\boldsymbol{\lambda_P = 1 + 2P_1 - 1.5P_2}$		
n	$\widehat{\lambda}_P$	$MSE\ \widehat{\lambda}_P$	$Bias\ \widehat{\lambda}_P$
10	$1.424 + 1.687P_1 - 1.467P_2$	$0.512 - 0.134P_1 - 0.013P_2$	$0.552 - 0.097P_1 + 0.011P_2$
30	$1.236 + 1.725P_1 - 1.427P_2$	$0.231 - 0.020P_1 + 0.157P_2$	$0.375 - 0.036P_1 + 0.144P_2$
75	$1.142 + 1.683P_1 - 1.383P_2$	$0.114 + 0.029P_1 + 0.142P_2$	$0.287 - 0.008P_1 + 0.143P_2$

6. Conclusion

We have successfully generalized the well-known class of distributions, Marshall-Olkin type II, to handle literal neutrosophic data and symbolic plithogenic data. Parameters of this class of distribution were estimated successfully using maximum likelihood estimation and the efficiency of the driven estimators were tested using Monte Carlo simulation. As a special case, exponential Marshall Olkin generalized distribution is well studied and this special case was supported with suitable simulation study.

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