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NEUTROSOPHIC $\mathcal{N}$-BI-IDEALS IN SEMIGROUPS

K. PORSELVI, B. ELAVARASAN, FLORENTIN SMARANDACHE, AND Y. B. JUN

Abstract. In this paper, we introduce the notion of neutrosophic $\mathcal{N}$-bi-ideal structure over a semigroup. We characterize semigroups, regular semigroups and intra-regular semigroups in terms of neutrosophic $\mathcal{N}$-bi-ideal structures. We also show that the intersection of neutrosophic $\mathcal{N}$-ideals and the neutrosophic $\mathcal{N}$-product of ideals will coincide for a regular semigroup.

1. Introduction

Throughout this paper, we take a semigroup X as the universe of discourse unless otherwise specified.

Let A and B be subsets of X. Then the multiplication of A and B is defined as $AB = \{ab \ : a \in A \text{ and } b \in B\}$. A nonempty subset A of X is said to be a subsemigroup of X if $A^2 \subseteq A$. A subsemigroup A of X is called a left (resp., right) ideal of X if $XA \subseteq A$ (resp., $AX \subseteq A$). If A is both a left and a right ideal of X, then A is called a two-sided ideal or ideal of X. For any $a \in X$, $L(a) = \{a, Xa\}$ (resp., $R(a) = \{a, aX\}$) is the principal left (resp., right) ideal of a semigroup X. A semigroup X is called left (resp., right) regular [3] if for each $a \in X$, there exists an element $x \in X$ such that $a = xa^2$ (resp., $a = a^2x$). A semigroup S is regular if for each $a \in X$ there exists $x \in X$ such that $a = axa$ [5]. A semigroup X is called intra-regular if for each $a \in X$, there exist $x, y \in X$ such that $a = xa^2y$ [5]. A subsemigroup A of X is said to be a bi-ideal of X if $AXA \subseteq A$. For any $a \in X$, $B(a) = \{a, a^2, aXa\}$ is the principal bi-ideal of a semigroup X.

L.A. Zadeh introduced the concept of fuzzy subsets of a well-defined set in his paper [8] for modeling the vague concepts in the real world. In order to deal with the negative meaning of information, Jun et al. [2] have introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. The concept of neutrosophic set has been developed by Smarandache in [6] and [7] as a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. M. Khan et al. [4] have introduced the notion of neutrosophic $\mathcal{N}$-subsemigroup in semigroup and several properties are investigated. They have shown that the homomorphic preimage of neutrosophic $\mathcal{N}$-subsemigroup is a neutrosophic $\mathcal{N}$-subsemigroup, and the onto homomorphic image of neutrosophic

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$\mathscr{N}$-subsemigroup is a neutrosophic $\mathscr{N}$-subsemigroup. In [1], Elavarasan et al. have introduced the notion of neutrosophic $\mathscr{N}$-ideals in semigroups and several properties are investigated.

In this paper, we introduce the notion of neutrosophic $\mathscr{N}$-bi-ideals over a semigroup X and characterize semigroups using neutrosophic $\mathscr{N}$-bi-ideals. We also discuss several equivalent conditions of neutrosophic $\mathscr{N}$-ideal structures and regular, intra-regular semigroups.

2. Neutrosophic $\mathscr{N}$-structures

In this section, we give some definitions of neutrosophic $\mathscr{N}$-stuctures of a semigroup X that we need in the sequel.

Denote by $\mathscr{F}(X,[-1,0])$ the collection of functions from a set X to $[-1,0]$. We say that an element of $\mathscr{F}(X,[-1,0])$ is a negative-valued function from X to $[-1,0]$ (briefly, $\mathscr{N}$-function on X). By an $\mathscr{N}$-structure, we mean an ordered pair (X,f) of X and an $\mathscr{N}$-function f on X.

Definition 2.1. [4] A neutrosophic $\mathscr{N}$-structure over X is defined to be the structure:

$$X_N := \frac{X}{T_N, I_N, F_N} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\}$$

where T_N, I_N and F_N are $\mathscr{N}$-functions on X which are called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively, on X.

It is clear that every neutrosophic $\mathscr{N}$-structure X_N over X satisfies the condition: $-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0$ for all $x \in X$.

Definition 2.2. [4] A neutrosophic $\mathscr{N}$-structure X_N over X is called a neutrosophic $\mathscr{N}$-subsemigroup of X if the following condition is valid:

$$(\forall x, y \in X) \begin{pmatrix} T_N(xy) \leq \bigvee\{T_N(x), T_N(y)\} \\ I_N(xy) \geq \bigwedge\{I_N(x), I_N(y)\} \\ F_N(xy) \leq \bigvee\{F_N(x), F_N(y)\} \end{pmatrix}.$$

Let X_N be a neutrosophic $\mathscr{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1,0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

$$T_N^{\alpha} := \{x \in X | T_N(x) \leq \alpha\}$$
$$I_N^{\beta} := \{x \in X | I_N(x) \geq \beta\}$$
$$F_N^{\gamma} := \{x \in X | F_N(x) \leq \gamma\}.$$

The set $X_N(\alpha,\beta,\gamma) := \{x \in X | T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$ is called (α,β,γ)-level set of X_N. Note that $X_N(\alpha,\beta,\gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$.

Definition 2.3. [1] A neutrosophic $\mathscr{N}$-structure X_N over X is called a neutrosophic $\mathscr{N}$-left (resp., right) ideal of X if the following condition is valid:

$$(\forall x, y \in X) \begin{pmatrix} T_N(xy) \leq T_N(y) \ (resp., T_N(xy) \leq T_N(x)) \\ I_N(xy) \geq I_N(y) \ (resp., I_N(xy) \geq I_N(x)) \\ F_N(xy) \leq F_N(y) \ (resp., F_N(xy) \leq F_N(x)) \end{pmatrix}.$$

A neutrosophic $\mathscr{N}$-structure X_N over X is called a neutrosophic $\mathscr{N}$-ideal of X if X_N is both a neutrosophic $\mathscr{N}$-left and neutrosophic $\mathscr{N}$-right ideal of X.

Definition 2.4. A neutrosophic $\mathcal{N}$-subsemigroup X_N of X is called a neutrosophic $\mathcal{N}$-bi-ideal of X if the following condition is valid:

$$(\forall x, y \in X) \begin{pmatrix} T_N(xyz) \leq T_N(x) \vee T_N(z) \\ I_N(xyz) \geq I_N(x) \wedge I_N(z) \\ F_N(xyz) \leq F_N(x) \vee F_N(z) \end{pmatrix}.$$

It is clear that every neutrosophic $\mathcal{N}$-left ideal (resp., right) of X is a neutrosophic $\mathcal{N}$-bi-ideal of X, but neutrosophic $\mathcal{N}$-bi-ideal of X need not be a neutrosophic $\mathcal{N}$-left (resp., right) ideal of X as can be seen by the following example.

Example 2.1. Let $X = \{0, a, b, c\}$ be a semigroup with the following Cayley table:

.	0	a	b	c
0	0	0	0	0
a	0	0	0	b
b	0	0	0	b
c	b	b	b	c

Then $X_N = \{\frac{0}{(-0.9,-0.1,-0.7)}, \frac{a}{(-0.8,-0.2,-0.5)}, \frac{b}{(-0.7,-0.3,-0.3)}, \frac{c}{(-0.5,-0.4,-0.1)}\}$ is a neutrosophic $\mathcal{N}$-bi-ideal of X. It can be easily checked that X_N is neither a neutrosophic $\mathcal{N}$-left ideal nor neutrosophic $\mathcal{N}$-right ideal of X.

Definition 2.5. [4] For a subset A of X, consider neutrosophic $\mathcal{N}$-structure $\chi_A(X_N) = \frac{X}{(\chi_A(T)_N, \chi_A(I)_N, \chi_A(F)_N)}$, where

$$\chi_A(T)_N : X \to [-1, 0],\ x \mapsto \begin{cases} -1 & if\ x \in A \\ 0 & otherwise \end{cases}$$

$$\chi_A(I)_N : X \to [-1, 0],\ x \mapsto \begin{cases} 0 & if\ x \in A \\ -1 & otherwise \end{cases} \text{ and}$$

$$\chi_A(F)_N : X \to [-1, 0],\ x \mapsto \begin{cases} -1 & if\ x \in A \\ 0 & otherwise \end{cases}$$

which is called the characteristic neutrosophic $\mathcal{N}$-structure in X.

Definition 2.6. [4] Let $X_N := \frac{X}{(T_N, I_N, F_N)}$ and $X_M = \frac{X}{(T_M, I_M, F_M)}$ be neutrosophic $\mathcal{N}$-structures over X. Then

(i) We say that X_M is a neutrosophic $\mathcal{N}$-substructure over X, denoted by $X_N \subseteq X_M$, if it satisfies: $(\forall x \in X)(T_N(x) \geq T_M(x),\ I_N(x) \leq I_M(x),\ F_N(x) \geq F_M(x))$.

If $X_N \subseteq X_M$ and $X_M \subseteq X_N$, we say that $X_N = X_M$.

(ii) The neutrosophic $\mathcal{N}$-product of X_N and X_M is defined to be a neutrosophic $\mathcal{N}$-structure over X,

$X_N \odot X_M = \frac{X}{(T_{N\circ M}, I_{N\circ M}, F_{N\circ M})} = \{\frac{x}{(T_{N\circ M}(x), I_{N\circ M}(x), F_{N\circ M}(x))} \mid x \in X\}$, where

$$T_{N\circ M}(x) = \begin{cases} \bigwedge_{x=yz} \{T_N(y) \vee T_M(z)\} & if\ \exists\ y, z \in X\ such\ that\ x = yz \\ 0 & otherwise \end{cases}$$

$$I_{N\circ M}(x)=\begin{cases}\bigvee\limits_{x=yz}\{I_N(y)\wedge I_M(z)\} & if\ \exists\ y,z\in X\ such\ that\ x=yz\\ -1 & otherwise\end{cases}$$

and

$$F_{N\circ M}(x)=\begin{cases}\bigwedge\limits_{x=yz}\{F_N(y)\vee F_M(z)\} & if\ \exists\ y,z\in X\ such\ that\ x=yz\\ 0 & otherwise\end{cases}$$

For any $x\in X$, the element $\frac{x}{(T_{N\circ M}(x),I_{N\circ M}(x),F_{N\circ M}(x))}$ is simply denoted by

$$(X_N\odot X_M)(x):=(T_{N\circ M}(x),I_{N\circ M}(x),F_{N\circ M}(x))$$

for the sake of convenience.

We denote $T_{N\circ M}(x),I_{N\circ M}(x),F_{N\circ M}(x)$ by $(T_N\circ T_M)(x)$, $(I_N\circ I_M)(x)$ and $(F_N\circ F_M)(x)$ respectively.

(ii) The intersection of X_N and X_M is defined to be a neutrosophic $\mathscr{N}$-structure

$$X_N\cap X_M=X_{N\cap M}=(X;T_{N\cap M},I_{N\cap M},F_{N\cap M}),$$

where
$(T_N\cap T_M)(x)=T_{N\cap M}(x)=\bigvee\{T_N(x),T_M(x)\}$,
$(I_N\cap I_M)(x)=I_{N\cap M}(x)=\bigwedge\{I_N(x),I_M(x)\}$,
$(F_N\cap F_M)(x)=F_{N\cap M}(x)=\bigvee\{F_N(x),F_M(x)\}$ for all $x\in X$.

3. Neutrosophic $\mathscr{N}$-bi-ideals of a semigroup

In this section, we discuss various properties of Neutrosophic $\mathscr{N}$-bi-ideals over a semigroup X. It is clear that Neutrosophic $\mathscr{N}$-left (resp., right) ideal is a Neutrosophic $\mathscr{N}$-bi-ideal of X. But the converse is also true provided X is regular right duo. We also show that for a regular semigroup X, if every bi-ideal is a right ideal (resp., left) of X, then every neutrosophic $\mathscr{N}$-bi-ideal is a neutrosophic $\mathscr{N}$-right (resp., left) ideal of X.

Throughout this section, we consider X_M and X_N are Neutrosophic $\mathscr{N}$-structures over X.

Theorem 3.1. *For any non-empty subset A of X, the following conditions are equivalent:*

(i) A is a bi-ideal of X,

(ii) The characteristic neutrosophic $\mathscr{N}$-structure $\chi_A(X_N)$ is a neutrosophic $\mathscr{N}$-bi-ideal of X.

Proof. Assume that A is a bi-ideal of X. Let $x,y,z\in X$.

If $x\in A$ and $z\in A$, then $xyz\in A$, so $\chi_A(T)_N(xyz)=-1=\chi_A(T)_N(x)\vee\chi_A(T)_N(z)$; $\chi_A(I)_N(xyz)=0=\chi_A(I)_N(x)\wedge\chi_A(I)_N(z)$ and $\chi_A(F)_N(xyz)=-1=\chi_A(F)_N(x)\vee\chi_A(F)_N(z)$.

If $x\notin A$ or $z\notin A$, then $\chi_A(T)_N(xyz)\leq 0=\chi_A(T)_N(x)\vee\chi_A(T)_N(z)$; $\chi_A(I)_N(xyz)\geq -1=\chi_A(I)_N(x)\wedge\chi_A(I)_N(z)$ and $\chi_A(F)_N(xyz)\leq 0=\chi_A(F)_N(x)\vee\chi_A(F)_N(z)$.

Therefore $\chi_A(X_N)$ is a neutrosophic $\mathscr{N}$-bi-ideal of X.

Conversely, assume that $\chi_A(X_N)$ is a neutrosophic $\mathscr{N}$-bi-ideal of X. Let $x,z\in A$ and $y\in X$. Then $\chi_A(T)_N(xyz)\leq\chi_A(T)_N(x)\vee\chi_A(T)_N(z)=-1$; $\chi_A(I)_N(xyz)\geq$

$\chi_A(I)_N(x) \wedge \chi_A(I)_N(z) = 0$ and $\chi_A(F)_N(xyz) \leq \chi_A(F)_N(x) \vee \chi_A(F)_N(z) = -1$ which imply $xyz \in A$. □

Theorem 3.2. *Let N_N be a neutrosophic $\mathscr{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If X_N is a neutrosophic $\mathscr{N}$-bi-ideal of X, then the (α, β, γ)-level set of X_N is a neutrosophic bi-ideal of X whenever it is non-empty.*

Proof. Assume that $X_N(\alpha, \beta, \gamma) \neq \phi$ for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha+\beta+\gamma \leq 0$. Let X_N be a neutrosophic $\mathscr{N}$-bi-ideal of X and let $x, y, z \in X_N(\alpha, \beta, \gamma)$. Then $T_N(xyz) \leq T_N(x) \vee T_N(z) \leq \alpha$; $I_N(xyz) \geq I_N(x) \wedge I_N(z) \geq \beta$ and $F_N(xyz) \leq F_N(x) \vee F_N(z) \leq \gamma$ which imply $xyz \in X_N(\alpha, \beta, \gamma)$. Therefore $X_N(\alpha, \beta, \gamma)$ is a neutrosophic $\mathscr{N}$-bi-ideal of X. □

Theorem 3.3. *Let X_M be a neutrosophic $\mathscr{N}$-structure over X. Then the following conditions are equivalent:*

(i) X_M is a neutrosophic $\mathscr{N}$-bi-ideal of X,

(ii) $X_M \odot X_M \subseteq X_M$ and $X_M \odot \chi_X(X_N) \odot X_M \subseteq X_M$ for any neutrosophic $\mathscr{N}$-structure X_N.

Proof. Assume that (i) holds. Then by Theorem 4.6 of [4], $X_M \odot X_M \subseteq X_M$. Let X_M be a neutrosophic $\mathscr{N}$-structure over X and $x \in X$. Assume that there exist $a, b, p, q \in X$ such that $x = ab$ and $a = pq$. Then

$$\begin{aligned}(T_M \circ \chi_X(T)_N \circ T_M)(x) &= \bigwedge_{x=ab} \{(T_M \circ \chi_X(T)_N)(a) \vee T_M(b)\} \\ &= \bigwedge_{x=ab} \{ \bigwedge_{a=pq} \{T_M(p) \vee \chi_X(T)_N(q)\} \vee T_M(b)\} \\ &= \bigwedge_{x=ab} \{ \bigwedge_{a=pq} \{T_M(p)\} \vee T_M(b)\} \\ &= \bigwedge_{x=ab} T_M(p_i) \vee T_M(b) \text{ for some } p_i \in X \text{ with } a = p_i q_i \\ &\geq \bigwedge_{x=p_i q_i b} T_M(p_i q_i b) = T_M(x), \text{ as } X_M \text{ is a neutro-}\end{aligned}$$

sophic $\mathscr{N}$-bi-ideal of X,

$$\begin{aligned}(I_M \circ \chi_X(I)_N \circ I_M)(x) &= \bigvee_{x=ab} \{(I_M \circ \chi_X(I)_N)(a) \wedge I_M(b)\} \\ &= \bigvee_{x=ab} \{ \bigvee_{a=pq} \{I_M(p) \wedge \chi_X(I)_N(q)\} \wedge I_M(b)\} \\ &= \bigvee_{x=ab} \{ \bigvee_{a=pq} \{I_M(p)\} \wedge I_M(b)\} \\ &= \bigvee_{x=ab} \{I_M(p_i) \wedge I_M(b)\} \text{ for some } p_i \in X \text{ with } a = p_i q_i \\ &\leq \bigvee_{x=p_i q_i b} I_M(p_i q_i b) = I_M(x), \text{ as } X_M \text{ is a neutrosophic}\end{aligned}$$

$\mathscr{N}$-bi-ideal of X,

and $(F_M \circ \chi_X(F)_N \circ F_M)(x) = \bigwedge_{x=ab} \{(F_M \circ \chi_X(F)_N)(a) \vee F_M(b)\}$

$$= \bigwedge_{x=ab} \{ \bigwedge_{a=pq} \{F_M(p) \vee \chi_X(F)_N(q)\} \vee F_M(b)\}$$

$$= \bigwedge_{x=ab} \{ \bigwedge_{a=pq} \{F_M(p)\} \vee F_M(b)\}$$

$$= \bigwedge_{x=ab} F_M(p_i) \vee F_M(b) \text{ for some } p_i \in X \text{ with } a = p_i q_i$$

$$\geq \bigwedge_{x=p_i q_i b} F_M(p_i q_i b) = F_M(x),$$ as X_M is a neutrosophic $\mathcal{N}$-bi-ideal of X.

Otherwise $x \neq ab$ or $a \neq pq$ for all $a, b, p, q \in X$. Then $(T_M \circ \chi_X(T)_N \circ T_M)(x) = 0 \geq T_M(x)$, $(I_M \circ \chi_X(I)_N \circ I_M)(x) = -1 \leq I_M(x)$ and $(F_M \circ \chi_X(F)_N \circ F_M)(x) = 0 \geq F_M(x)$.

Therefore $X_M \odot \chi_X(X_N) \odot X_M \subseteq X_M$ for any neutrosophic $\mathcal{N}$-structure X_N over X.

Conversely, assume that (ii) holds. Then by Theorem 4.6 of [4], X_M is a neutrosophic $\mathcal{N}$-subsemigroup of X. Let $x, y, z \in X$ and let $a = xyz$.

Then $T_M(xyz) \leq (T_M \circ \chi_X(T)_N \circ T_M)(xyz)$

$$= \bigwedge_{a=xyz} \{(T_M \circ \chi_X(T)_N)(xy) \vee T_M(z)\}$$

$$= \bigwedge_{a=bz} \{ \bigwedge_{b=xy} \{T_M(x) \vee \chi_X(T)_N(y)\} \vee T_M(z)\}$$

$$= \bigwedge \{T_M(x) \vee T_M(z)\} \leq T_M(x) \vee T_M(z),$$

$$I_M(xyz) \geq (I_M \circ \chi_X(I)_N \circ I_M)(xyz)$$

$$= \bigvee_{a=xyz} \{(I_M \circ \chi_X(I)_N)(xy) \wedge I_M(z)\}$$

$$= \bigvee_{a=bz} \{ \bigvee_{b=xy} \{I_M(x) \wedge \chi_X(I)_N(y)\} \wedge I_M(z)\}$$

$$= \bigvee_{a=xyz} \{I_M(x) \wedge I_M(z)\} \geq I_M(x) \wedge I_M(z),$$

and

$$F_M(xyz) \leq (F_M \circ \chi_X(F)_N \circ F_M)(xyz)$$

$$= \bigwedge_{a=xyz} \{(F_M \circ \chi_X(F)_N)(xy) \vee F_M(z)\}$$

$$= \bigwedge_{a=bz} \{ \bigwedge_{b=xy} \{F_M(x) \vee \chi_X(F)_N(y)\} \vee F_M(z)\}$$

$$= \bigwedge_{a=xyz} \{F_M(x) \vee F_M(z)\} \leq F_M(x) \vee F_M(z).$$

Therefore X_M is a neutrosophic $\mathcal{N}$-bi-ideal of X. □

Definition 3.1. A neutrosophic $\mathscr{N}$-structure over X is called neutrosophic $\mathscr{N}$-left (resp., right) duo over X if every neutrosophic $\mathscr{N}$-left (resp., right) ideal of X is a neutrosophic $\mathscr{N}$-ideal of X. A semigroup X is called neutrosophic $\mathscr{N}$-duo if it is both a neutrosophic $\mathscr{N}$-left duo and a neutrosophic $\mathscr{N}$-right duo.

Theorem 3.4. *Let X be a regular left duo (resp., right duo, duo) of a semigroup. Then the following conditions are equivalent:*

(i) X_M is a neutrosophic $\mathscr{N}$-bi-ideal of X,

(ii) X_M is a neutrosophic $\mathscr{N}$-right ideal (resp., left ideal, ideal) of X.

Proof. $(i) \Rightarrow (ii)$ Assume that X_M is a neutrosophic $\mathscr{N}$-bi-ideal of X, and let $a, b \in X$. Since X is regular, we have $a = ata \in aX \cap Xa$ for some $t \in X$ which implies $ab \in (aX \cap Xa)X \subseteq aX \cap Xa$ as X is left duo. So $ab = as$ and $ab = s'a$ for some $s, s' \in X$. Since X is regular, there exists $r \in X$ such that $ab = abrab = asrs'a = a(srs')a$. Since X_M is a neutrosophic $\mathscr{N}$-bi-ideal, we have

$T_M(ab) = T_M(a(srs')a) \leq T_M(a) \vee T_M(a) = T_M(a)$,

$I_M(ab) = I_M(a(srs')a) \geq I_M(a) \wedge I_M(a) = I_M(a)$ and

$F_M(ab) = F_M(a(srs')a) \leq F_M(a) \vee F_M(a) = F_M(a)$.

Therefore X_M is a neutrosophic $\mathscr{N}$-right ideal of X.

$(ii) \Rightarrow (i)$ Assume that X_M is a neutrosophic $\mathscr{N}$-right ideal (resp., left ideal, ideal) of X. Let $x, y, z \in X$. Then $T_M(xyz) \leq T_M(x) \leq T_M(x) \vee T_M(z)$,
$I_M(xyz) \geq I_M(x) \geq I_M(x) \wedge I_M(z)$ and
$F_M(xyz) \leq F_M(x) \leq F_M(x) \vee F_M(z)$.

Therefore X_M is a neutrosophic $\mathscr{N}$-bi-ideal of X. □

Theorem 3.5. *Let X be a regular semigroup. Then the following conditions are equivalent:*

(i) X is left duo (resp., right duo, duo),

(ii) X_M is neutrosophic $\mathscr{N}$-left duo (resp., right duo, duo).

Proof. $(i) \Rightarrow (ii)$ Assume that X is left duo. Then for any $a, b \in X$, we have $ab \in (aXa)b \subseteq a(Xa)X \subseteq Xa$ as Xa is a left ideal of X. Since x is regular, there exists $t \in X$ such that $ab = ta$. Let X_M be a neutrosophic $\mathscr{N}$-left ideal of X. Then $T_M(ab) = T_M(ta) \leq T_M(a)$, $I_M(ab) = I_M(ta) \geq I_M(a)$ and $F_M(ab) = F_M(ta) \leq F_M(a)$. Thus X_M is a neutrosophic $\mathscr{N}$-right ideal of X and hence X_M is a neutrosophic $\mathscr{N}$-left duo.

$(ii) \Rightarrow (i)$ Assume that X_M is a neutrosophic $\mathscr{N}$-left duo and let A be any left ideal of X. Then by Theorem 3.5 of [1], $\chi_A(X_M)$ is a neutrosophic $\mathscr{N}$-left ideal of X. By assumption, $\chi_A(X_M)$ is a neutrosophic $\mathscr{N}$-ideal of X. Again by Theorem 3.5 of [1], A is a right ideal of X and hence X is left duo. □

Theorem 3.6. *Let X be a regular semigroup. Then the following conditions are equivalent:*

(i) Every bi-ideal of X is a right ideal (resp., left ideal, ideal) of X,

(ii) Every neutrosophic $\mathscr{N}$-bi-ideal is a neutrosophic $\mathscr{N}$-right ideal (resp., left ideal, ideal) of X.

Proof. $(i) \Rightarrow (ii)$ Assume that (i) holds. Let X_M be a neutrosophic $\mathscr{N}$-bi-ideal of X and let $a, b \in X$. Then aXa is a bi-ideal of X. By assumption, we have aXa is a right ideal of X. Since X is regular, we have $a \in aXa$. So $ab \in (aXa)X \subseteq aXa$ implies $ab = axa$ for some $x \in X$. Now $T_M(ab) = T_M(axa) \leq T_M(a) \vee T_M(a) =$

$T_M(a)$, $I_M(ab) = I_M(axa) \geq I_M(a) \wedge I_M(a) = I_M(a)$ and $F_M(ab) = F_M(axa) \leq F_M(a) \vee F_M(a) = F_M(a)$. Therefore X_M is a neutrosophic $\mathcal{N}$-right ideal of X.

$(ii) \Rightarrow (i)$ Assume that (ii) holds and let A be any bi-ideal of X. Then by Theorem 3.1, $\chi_A(X_M)$ is a neutrosophic $\mathcal{N}$-bi-ideal of X for a neutrosophic $\mathcal{N}$-structure X_M over X. By assumption, $\chi_A(X_M)$ is a neutrosophic $\mathcal{N}$-right ideal of X. By Theorem 3.5 of [1], A is a right ideal of X. □

Theorem 3.7. *Let X be a semigroup. Then the following conditions are equivalent:*

(i) X is regular,

(ii) $X_M \cap X_N = X_M \odot X_N \odot X_M$ for every neutrosophic $\mathcal{N}$-bi-ideal X_M and every neutrosophic $\mathcal{N}$-ideal X_N of X.

Proof. $(i) \Rightarrow (ii)$ Assume that X is regular, and let X_M be a neutrosophic $\mathcal{N}$-bi-ideal and X_N a neutrosophic $\mathcal{N}$-ideal of X. Then by Theorem 3.3, we have $X_M \odot X_N \odot X_M \subseteq X_M$ and $X_M \odot X_N \odot X_M \subseteq X_N$. So $X_M \odot X_N \odot X_M \subseteq X_M \cap X_N$.

Let $a \in X$. Since X is regular, there exists $x \in S$ such that $a = axa = axaxa$.

$$\begin{aligned}
\text{Now } T_{M\circ N\circ M}(a) &= \bigwedge_{a=uv} \{T_M(u) \vee T_{N\circ M}(v)\} \\
&= \bigwedge_{a=av} \{T_M(a) \vee \{ \bigwedge_{v=xaxa} \{T_N(xax) \vee T_M(a)\}\} \\
&\leq \bigwedge_{a=av} \{T_M(a) \vee T_N(a)\} \\
&\leq T_M(a) \vee T_N(a) = T_{M\cap N}(a),
\end{aligned}$$

$$\begin{aligned}
I_{M\circ N\circ M}(a) &= \bigvee_{a=uv} \{I_M(u) \wedge I_{N\circ M}(v)\} \\
&= \bigvee_{a=av} \{I_M(a) \wedge \{ \bigvee_{v=xaxa} \{I_N(xax) \wedge I_M(a)\}\} \\
&\geq \bigvee_{a=av} \{I_M(a) \wedge I_N(a)\} \\
&\geq I_M(a) \wedge I_N(a) = I_{M\cap N}(a) \text{ and}
\end{aligned}$$

$$\begin{aligned}
F_{M\circ N\circ M}(a) &= \bigwedge_{a=uv} \{F_M(u) \vee F_{N\circ M}(v)\} \\
&= \bigwedge_{a=av} \{F_M(a) \vee \{ \bigwedge_{v=xaxa} \{F_N(xax) \vee F_M(a)\}\} \\
&\leq \bigwedge_{a=av} \{F_M(a) \vee F_N(a)\} \\
&\leq F_M(a) \vee F_N(a) = F_{M\cap N}(a).
\end{aligned}$$

Thus $X_{M\cap N} \subseteq X_M \odot X_N \odot X_M$ and hence $X_{M\cap N} = X_M \odot X_N \odot X_M$ for every neutrosophic $\mathcal{N}$-bi-ideal X_M and every neutrosophic $\mathcal{N}$-ideal X_N of X.

$(ii) \Rightarrow (i)$ Assume that (ii) holds. Then $X_M \cap \chi_X(X_N) = X_M \odot \chi_X(X_N) \odot X_M$. But $X_M \cap \chi_X(X_N) = X_M$, so $X_M = X_M \odot \chi_X(X_N) \odot X_M$ for every neutrosophic $\mathcal{N}$-bi-ideal X_M of X. Let $a \in X$. Then by Theorem 3.1, $\chi_{B(a)}(X_M)$ is a neutrosophic $\mathcal{N}$-bi-ideal of X. So $\chi_{B(a)}(T)_M = \chi_{B(a)}(T)_M \circ \chi_X(T)_N \circ \chi_{B(a)}(T)_M = \chi_{B(a)XB(a)}(T)_M$, $\chi_{B(a)}(I)_M = \chi_{B(a)}(I)_M \circ \chi_X(I)_N \circ \chi_{B(a)}(I)_M = \chi_{B(a)XB(a)}(I)_M$ and $\chi_{B(a)}(F)_M = \chi_{B(a)}(F)_M \circ \chi_X(F)_N \circ \chi_{B(a)}(F)_M = \chi_{B(a)XB(a)}(F)_M$. Since $a \in B(a)$, we have $\chi_{B(a)XB(a)}(T)_M(a) = \chi_{B(a)}(T)_M(a) = -1$, $\chi_{B(a)XB(a)}(I)_M(a) = \chi_{B(a)}(I)_M(a) = 0$ and $\chi_{B(a)XB(a)}(F)_M(a) = \chi_{B(a)}(F)_M(a) = -1$ which imply $a \in B(a)XB(a)$. Therefore X is regular. □

Theorem 3.8. *Let X be a semigroup. Then the following conditions are equivalent:*

(i) X is regular,

(ii) $X_M \cap X_N \subseteq X_M \odot X_N$ for every neutrosophic $\mathscr{N}$-bi-ideal X_M and every neutrosophic $\mathscr{N}$-left ideal X_N of X.

Proof. $(i) \Rightarrow (ii)$ Assume that X is regular and let X_M be a neutrosophic $\mathscr{N}$-bi-ideal and X_N a neutrosophic $\mathscr{N}$-left ideal of X. Let $a \in X$. Then there exists $x \in X$ such that $a = axa$.

$$\text{Now } T_{M\circ N}(a) = \bigwedge_{a=uv} \{T_M(u) \vee T_N(v)\} \leq T_M(a) \vee T_N(xa) \leq T_M(a) \vee T_N(a) = T_{M\cap N}(a),$$

$$I_{M\circ N}(a) = \bigvee_{a=uv} \{I_M(u) \wedge I_N(v)\} \geq I_M(a) \wedge I_N(xa) \geq I_M(a) \wedge I_N(a) = I_{M\cap N}(a), \text{ and}$$

$$F_{M\circ N}(a) = \bigwedge_{a=uv} \{F_M(u) \vee F_N(v)\} \leq F_M(a) \vee F_N(xa) \leq F_M(a) \vee F_N(a) = F_{M\cap N}(a).$$

Therefore $X_{M\cap N} \subseteq X_M \odot X_N$.

$(ii) \Rightarrow (i)$ Assume that (ii) holds, and let X_M be a neutrosophic $\mathscr{N}$-right ideal and X_N a neutrosophic $\mathscr{N}$-left ideal of X. Since every neutrosophic $\mathscr{N}$-right ideal of X is a neutrosophic $\mathscr{N}$-bi-ideal of X, X_M is a neutrosophic $\mathscr{N}$-bi-ideal of X. Then by assumption, we have $X_{M\cap N} \subseteq X_M \odot X_N$. By Theorem 3.8 and Theorem 3.9 of [1], we can get $X_M \odot X_N \subseteq X_N$ and $X_M \odot X_N \subseteq X_M$ and so $X_M \odot X_N \subseteq X_M \cap X_N = X_{M\cap N}$. Therefore $X_M \odot X_N = X_{M\cap N}$.

Let K be a right ideal and L be a left ideal of X and $a \in K \cap L$. Then $\chi_K(X_M) \odot \chi_L(X_M) = \chi_K(X_M) \cap \chi_L(X_M)$ which implies $\chi_{KL}(X_M) = \chi_{K\cap L}(X_M)$. Since $a \in K \cap L$, we have $\chi_{K\cap L}(T)_M(a) = -1 = \chi_{KL}(T)_M(a)$, $\chi_{K\cap L}(I)_M(a) = 0 = \chi_{KL}(I)_M(a)$ and $\chi_{K\cap L}(F)_M(a) = -1 = \chi_{KL}(F)_M(a)$ which imply $a \in KL$. Thus $K \cap L \subseteq KL \subseteq K \cap L$ and hence $K \cap L = KL$. Therefore X is regular. □

Theorem 3.9. *Let X be a semigroup. Then the following conditions are equivalent:*

(i) X is regular,

(ii) $X_{M\cap N} \subseteq X_M \odot X_N$ for every neutrosophic $\mathscr{N}$-bi-ideal X_M and every neutrosophic $\mathscr{N}$-right ideal X_N of X.

Proof. It is similar to the proof of Theorem 3.8. □

Theorem 3.10. *Let X be a semigroup. Then the following conditions are equivalent:*

(i) X is regular,

(ii) $X_L \cap X_M \cap X_N \subseteq X_L \odot X_M \odot X_N$ for every neutrosophic $\mathscr{N}$-right ideal X_L, every neutrosophic $\mathscr{N}$-bi-ideal X_M and every neutrosophic $\mathscr{N}$-left ideal X_N of X.

Proof. $(i) \Rightarrow (ii)$ Assume that X is regular and let X_L be a neutrosophic $\mathcal{N}$-right ideal, X_N a neutrosophic $\mathcal{N}$-left ideal and X_M a neutrosophic $\mathcal{N}$-bi-ideal of X. Let $a \in X$. Then there exists $x \in X$ such that $a = axa = axaxa$.

$$\begin{aligned}\text{Now } T_{L\circ M\circ N}(a) &= \bigwedge_{a=uv} \{T_L(u) \vee T_{M\circ N}(v)\} \leq T_L(ax) \vee T_{M\circ N}(axa)\\ &\leq T_L(a) \vee \{T_M(a) \vee T_N(xa)\}\\ &\leq T_L(a) \vee T_M(a) \vee T_N(a) = T_{L\cap M\cap N},\end{aligned}$$

$$\begin{aligned}I_{L\circ M\circ N}(a) &= \bigvee_{a=uv} \{I_L(u) \wedge I_{M\circ N}(v)\} \geq I_L(ax) \wedge I_{M\circ N}(axa)\\ &\geq I_L(a) \wedge \{I_M(a) \wedge I_N(xa)\}\\ &\geq I_L(a) \wedge I_M(a) \wedge I_N(a) = I_{L\cap M\cap N},\end{aligned}$$

$$\begin{aligned}\text{and } F_{L\circ M\circ N}(a) &= \bigwedge_{a=uv} \{F_L(u) \vee F_{M\circ N}(v)\} \leq F_L(ax) \vee F_{M\circ N}(axa)\\ &\leq F_L(a) \vee \{F_M(a) \vee F_N(xa)\}\\ &\leq F_L(a) \vee F_M(a) \vee F_N(a) = F_{L\cap M\cap N}.\end{aligned}$$

Therefore $X_L \cap X_M \cap X_N \subseteq X_L \odot X_M \odot X_N$.

$(ii) \Rightarrow (i)$ Assume that (ii) holds, and let X_L and X_N be a neutrosophic $\mathcal{N}$-right, left ideal respectively, and X_M a neutrosophic $\mathcal{N}$-structure over X. Then by Theorem 3.1, $\chi_X(X_M)$ is a neutrosophic $\mathcal{N}$-bi-ideal of X. Then $X_L \cap X_N = X_L \cap \chi_X(X_M) \cap X_N \subseteq X_L \odot \chi_X(X_M) \odot X_N \subseteq X_L \odot X_N$. Again by Theorem 3.8 and 3.9 of [1], $X_L \odot X_N \subseteq X_L \cap X_N$ and so $X_L \odot X_N = X_L \cap X_N$.

Let K, L be a right, left ideal of X respectively and let $a \in K \cap L$. Then $\chi_K(X_M) \odot \chi_L(X_M) = \chi_K(X_M) \cap \chi_L(X_M)$. By Theorem 3.6 of [1], we have $\chi_{KL}(X_M) = \chi_{K\cap L}(X_M)$. Since $a \in K\cap L$, we get $\chi_{KL}(T)_M(a) = \chi_{K\cap L}(T)_M(a) = -1$, $\chi_{KL}(I)_M(a) = \chi_{K\cap L}(I)_M(a) = 0$ and $\chi_{KL}(F)_M(a) = \chi_{K\cap L}(F)_M(a) = -1$ which imply $a \in KL$. Thus $K \cap L \subseteq KL \subseteq K \cap L$ and hence $K \cap L = KL$. Therefore X is regular. □

Theorem 3.11. *Let X be a semigroup. Then the following conditions are equivalent:*

(i) X is regular and intra-regular,

(ii) $X_M \cap X_N \subseteq X_M \odot X_N$ for every neutrosophic $\mathcal{N}$-bi-ideals X_M and X_N of X.

Proof. $(i) \Rightarrow (ii)$ Assume that X is regular and intra-regular, and $X_M,\ X_N$ be neutrosophic $\mathcal{N}$-bi-ideals of X. Let $a \in X$. Since X is regular, there exists $x \in X$ such that $a = axa = axaxa$. Again since X is intra-regular, there exist $y, z \in X$ such that $a = ya^2z$. Then $a = axyaazxa$. Now

$$\begin{aligned}T_{M\circ N}(a) &= \bigwedge_{a=uv} \{T_M(u) \vee T_N(v)\} \leq T_M(axya) \vee T_N(azxa)\\ &\leq \{T_M(a) \vee T_M(a)\} \vee \{T_N(a) \vee T_N(a)\}\\ &\leq T_M(a) \vee T_N(a) = T_{M\cap N}(a),\end{aligned}$$

$$\begin{aligned}I_{M\circ N}(a) &= \bigvee_{a=uv} \{I_M(u) \wedge I_N(v)\} \geq I_M(axya) \wedge I_N(azxa)\\ &\geq \{I_M(a) \wedge I_M(a)\} \wedge \{I_N(a) \wedge I_N(a)\}\end{aligned}$$

$$\geq I_M(a) \wedge I_N(a) = I_{M\cap N}(a), \text{ and}$$

$$F_{M\circ N}(a) = \bigwedge_{a=uv} \{F_M(u) \vee F_N(v)\} \leq F_M(axya) \vee F_N(azxa)$$

$$\leq \{F_M(a) \vee F_M(a)\} \vee \{F_N(a) \vee F_N(a)\}$$

$$\leq F_M(a) \vee F_N(a) = F_{M\cap N}(a).$$

Therefore $X_M \cap X_N \subseteq X_M \odot X_N$ for every neutrosophic $\mathcal{N}$-bi-ideals X_M and X_N of X.

$(ii) \Rightarrow (i)$ Assume that (ii) holds, and let X_M be a neutrosophic $\mathcal{N}$-right ideal and X_N a neutrosophic $\mathcal{N}$-left ideal of X. Then X_M and X_N are neutrosophic $\mathcal{N}$-bi-ideals of X. Then by assumption, we have $X_{M\cap N} \subseteq X_M \odot X_N$. By Theorem 3.8 and Theorem 3.9 of [1], we can get $X_M \odot X_N \subseteq X_N$ and $X_M \odot X_N \subseteq X_M$ and so $X_M \odot X_N \subseteq X_M \cap X_N$. Therefore $X_M \odot X_N = X_M \cap X_N$.

Let P be a right ideal and Q a left ideal of X and let $a \in P\cap Q$. Then $\chi_P(X_M)\cap \chi_Q(X_M) = \chi_P(X_M) \odot \chi_Q(X_M)$. By Theorem 3.6 of [1], $\chi_{P\cap Q}(X_M) = \chi_{PQ}(X_M)$. Since $a \in P \cap Q$, we have $\chi_{P\cap Q}(T)_M(a) = -1 = \chi_{PQ}(T)_M(a)$, $\chi_{P\cap Q}(I)_M(a) = 0 = \chi_{PQ}(I)_M(a)$ and $\chi_{P\cap Q}(F)_M(a) = -1 = \chi_{PQ}(F)_M(a)$ which imply $a \in PQ$. Thus $P \cap Q \subseteq PQ \subseteq P \cap Q$ and hence $P \cap Q = PQ$. Therefore X is regular.

Also, for $a \in X$, $\chi_{B(a)}(X_M) \cap \chi_{B(a)}(X_M) = \chi_{B(a)}(X_M) \odot \chi_{B(a)}(X_M)$. By Theorem 3.8 and Theorem 3.9 of [1], we can get $\chi_{B(a)}(X)_M = \chi_{B(a)B(a)}(X)_M$. Since $\chi_{B(a)}(T)_M(a) = -1 = \chi_{B(a)}(F)_M(a)$ and $\chi_{B(a)}(I)_M(a) = 0$, we can get $\chi_{B(a)B(a)}(T)_M(a) = -1 = \chi_{B(a)B(a)}(F)_M(a)$ and $\chi_{B(a)B(a)}(I)_M(a) = 0$ which imply $a \in B(a)B(a)$. Therefore X is intra-regular. □

Theorem 3.12. *Let X be a semigroup. Then the following conditions are equivalent:*

(i) X is regular and intra-regular,

(ii) $X_M \cap X_N \subseteq (X_M \odot X_N) \cap (X_N \odot X_M)$ for every neutrosophic $\mathcal{N}$-bi-ideals X_M and X_N of X.

Proof. $(i) \Rightarrow (ii)$ Assume that X is regular and intra-regular, and let $X_M,\ X_N$ be neutrosophic $\mathcal{N}$-bi-ideals of X. Then by Theorem 3.11, $X_M \cap X_N \subseteq X_M \odot X_N$. Similarly we can prove that $X_N \cap X_M \subseteq X_N \odot X_M$. Therefore $X_M \cap X_N \subseteq (X_M \odot X_N) \cap (X_N \odot X_M)$ for every neutrosophic $\mathcal{N}$-bi-ideals X_M and X_N of X.

$(ii) \Rightarrow (i)$ Assume that (ii) holds, and let X_M and X_N be neutrosophic $\mathcal{N}$-bi-ideals of X. Then $X_M \cap X_N \subseteq X_M \odot X_N$. By Theorem 3.11, X is regular and intra-regular. □

Theorem 3.13. *Let X be a semigroup. Then the following conditions are equivalent:*

(i) X is regular and intra-regular,

(ii) $X_M \cap X_N \subseteq X_M \odot X_N \odot X_M$ for every neutrosophic $\mathcal{N}$-bi-ideals X_M and X_N of X.

Proof. $(i) \Rightarrow (ii)$ Assume that X is regular and intra-regular, and let $X_M,\ X_N$ be neutrosophic $\mathcal{N}$-bi-ideals of X. Let $a \in X$. Since X is regular, there exists $x \in X$ such that $a = axa = axaxaxa$. Again since X is intra-regular, there exist $y, z \in X$ such that $a = ya^2z = (axya)(azxya)(azxa)$. Now

$$\begin{aligned}
T_{M\circ N\circ M}(a) &= \bigwedge_{a=uv}\{T_M(u)\vee T_{N\circ M}(v)\}\\
&= \bigwedge_{a=(axya)v}\{T_M(axya)\vee\{\bigwedge_{v=pq}\{T_N(p)\vee T_M(q)\}\}\\
&\le T_M(axya)\vee T_N(azxya)\vee T_M(azxa)\\
&\le T_M(a)\vee T_N(a)\vee T_M(a)=T_M(a)\vee T_N(a) \;=\; T_{M\cap N}(a),\\
I_{M\circ N\circ M}(a) &= \bigvee_{a=uv}\{I_M(u)\wedge I_{N\circ M}(v)\}\\
&= \bigvee_{a=(axya)v}\{I_M(axya)\wedge\{\bigvee_{v=pq}\{I_N(p)\wedge I_M(q)\}\}\\
&\ge I_M(axya)\wedge I_N(azxya)\wedge I_M(azxa)\\
&\ge I_M(a)\wedge I_N(a)\wedge I_M(a)=I_M(a)\wedge I_N(a) \;=\; I_{M\cap N}(a), \text{ and}\\
F_{M\circ N\circ M}(a) &= \bigwedge_{a=uv}\{F_M(u)\vee F_{N\circ M}(v)\}\\
&= \bigwedge_{a=(axya)v}\{F_M(axya)\vee\{\bigwedge_{v=pq}\{F_N(p)\vee F_M(q)\}\}\\
&\le F_M(axya)\vee F_N(azxya)\vee F_M(azxa)\\
&\le F_M(a)\vee F_N(a)\vee F_M(a)=F_M(a)\vee F_N(a) \;=\; F_{M\cap N}(a).
\end{aligned}$$

Therefore $X_M \cap X_N \subseteq X_M \odot X_N \odot X_M$ for every neutrosophic $\mathscr{N}$-bi-ideals X_M and X_N of X.

$(ii) \Rightarrow (i)$ Assume that (ii) holds, and let $a \in X$. Then $(\chi_{B(a)}(X_M) \cap \chi_{B(a)}(X_M) \subseteq \chi_{B(a)}(X_M) \odot \chi_{B(a)}(X_M) \odot \chi_{B(a)}(X_M)$.

So $\chi_{B(a)}(X_M) \subseteq \chi_{B(a)B(a)B(a)}(X_M)$.

Therefore $(\chi_{B(a)}(T)_M)(a) \geq (\chi_{B(a)B(a)B(a)}(T)_M)(a)$,
$(\chi_{B(a)}(I)_M)(a) \leq (\chi_{B(a)B(a)B(a)}(I)_M)(a)$ and
$(\chi_{B(a)}(F)_M)(a) \geq (\chi_{B(a)B(a)B(a)}(F)_M)(a)$.

Since $\chi_{B(a)}(T)_M(a) = -1 = \chi_{B(a)}(F)_M(a)$ and $\chi_{B(a)}(I)_M(a) = 0$, we get $\chi_{B(a)B(a)B(a)}(T)_M(a) = -1 = \chi_{B(a)B(a)B(a)}(F)_M(a)$, $\chi_{B(a)B(a)B(a)}(I)_M(a) = 0$ which imply $a \in B(a)B(a)B(a)$. Thus X is regular and intra-regular. □

Theorem 3.14. *Let X be a semigroup. Then the following conditions are equivalent:*

(i) X is intra-regular,

(ii) For each neutrosophic $\mathscr{N}$-ideal X_M of X, we have $X_M(a) = X_M(a^2)$ for all $a \in X$.

Proof. $(i) \Rightarrow (ii)$ Assume that X is intra-regular, and X_M is a neutrosophic $\mathscr{N}$-ideal of X and $a \in X$. Then there exist $y, z \in X$ such that $a = ya^2z$. Now $T_M(a) = T_M(ya^2z) \leq T_M(a^2z) \leq T_M(a^2) \leq T_M(a)$ and so $T_M(a) = T_M(a^2)$, $I_M(a) = I_M(ya^2z) \geq I_M(a^2z) \geq I_M(a^2) \geq I_M(a)$ and so $I_M(a) = I_M(a^2)$, and $F_M(a) = F_M(ya^2z) \leq F_M(a^2z) \leq F_M(a^2) \leq F_M(a)$ and so $F_M(a) = F_M(a^2)$. Therefore $X_M(a) = X_M(a^2)$ for all $a \in X$.

$(ii) \Rightarrow (i)$ Assume that (ii) holds and $a \in X$. Then $I(a^2)$ is an ideal of X. By Theorem 3.5 of [1], $\chi_{I(a^2)}(X_M)$ is a neutrosophic $\mathscr{N}$-ideal of X. By assumption, $\chi_{I(a^2)}(X_M)(a) = \chi_{I(a^2)}(X_M)(a^2)$. Since $\chi_{I(a^2)}(T)_M(a^2) = -1 = \chi_{I(a^2)}(F)_M(a^2)$

and $\chi_{I(a^2)}(I)_M(a^2) = 0$, we get $\chi_{I(a^2)}(T)_M(a) = -1 = \chi_{I(a^2)}(F)_M(a)$ and $\chi_{I(a^2)}(I)_M(a) = 0$ which imply $a \in I(a^2)$ and so X is intra-regular. □

Theorem 3.15. *Let X be a semigroup. Then the following conditions are equivalent:*

(i) X is left (resp., right) regular,

(ii) For each neutrosophic $\mathscr{N}$-left (resp., right) ideal X_M, we have $X_M(a) = X_M(a^2)$ for all $a \in X$.

Proof. $(i) \Rightarrow (ii)$ Assume that X is left regular. Then there exist $y \in X$ such that $a = ya^2$. Let X_M be a neutrosophic $\mathscr{N}$-left ideal of X. Then $T_M(a) = T_M(ya^2) \leq T_M(a)$ and so $T_M(a) = T_M(a^2)$, $I_M(a) = I_M(ya^2) \geq I_M(a)$ and so $I_M(a) = I_M(a^2)$, and $F_M(a) = F_M(ya^2) \leq F_M(a)$ and so $F_M(a) = F_M(a^2)$. Thus $X_M(a) = X_M(a^2)$ for all $a \in X$.

$(ii) \Rightarrow (i)$ Assume that (ii) holds and let X_M be a neutrosophic $\mathscr{N}$-left ideal of X. Then for any $a \in X$, we have $\chi_{L(a^2)}(T)_M(a) = \chi_{L(a^2)}(T)_M(a^2) = -1$, $\chi_{L(a^2)}(I)_M(a) = \chi_{L(a^2)}(I)_M(a^2) = 0$ and $\chi_{L(a^2)}(F)_M(a) = \chi_{L(a^2)}(F)_M(a^2) = -1$ which imply $a \in L(a^2)$ and hence X is left regular. □

Corollary 3.1. *Let X be a regular right duo (resp., left duo) semigroup. Then the following conditions are equivalent:*

(i) X is left regular,

(ii) For each neutrosophic $\mathscr{N}$-bi-ideal X_M, we have $X_M(a) = X_M(a^2)$ for all $a \in X$.

Proof. It follows from Theorem 3.4 and Theorem 3.15. □

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K. Porselvi

Department of Mathematics, Karunya Institute of Technology and Sciences, Coimbatore - 641 114, Tamilnadu, India.

E-mail address: `porselvi94@yahoo.co.in`

B. Elavarasan

Department of Mathematics, Karunya Institute of Technology and Sciences, Coimbatore - 641 114, Tamilnadu, India.
E-mail address: belavarasan@gmail.com

Florentin Smarandache
Department of Mathematics, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA.
E-mail address: fsmarandache@gmail.com

Y. B. Jun
Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Korea.
E-mail address: skywine@gmail.com