Neutrosophic permeable values and energetic subsets with applications in BCK/BCI-algebras

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Abstract The concept of $(\in, \in)$-neutrosophic ideal is introduced, and its characterizations are established. The notions of neutrosophic permeable values are introduced, and related properties are investigated. Conditions for the neutrosophic level sets to be energetic, right stable and right vanished are discussed. Relations between neutrosophic permeable S-value and neutrosophic permeable I-value are considered.

Keywords: $(\in, \in)$-neutrosophic subalgebra; $(\in, \in)$-neutrosophic ideal; neutrosophic (anti-) permeable S-value; neutrosophic (anti-) permeable I-value; S-energetic set; I-energetic set.

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1 Introduction

The notion of neutrosophic set theory developed by Smarandache (see [1] and [2]) is a more general platform which extends the concepts of classic and fuzzy set, intuitionistic fuzzy set and interval valued (intuitionistic) fuzzy set, and is applied to various parts. Smarandache [2] mentioned a cloud is a neutrosophic set, because its borders are ambiguous, and each element (water drop) belongs with a neutrosophic probability to the set (e.g. there are a kind of separated water drops, around a compact mass of water drops, that we don't know how to consider them: in or out of the cloud). Also, we are not sure where the cloud ends nor where it begins, neither if some elements are or are not in the set. That's why the percent of indeterminacy is required and the neutrosophic probability (using subsets - not numbers - as components) should be used for better modeling: it is a more organic, smooth, and especially accurate estimation. Indeterminacy is the zone of ignorance of a propositions value, between truth and falsehood.

Algebraic structures play an important role in mathematics with wide ranging applications in several disciplines such as coding theory, information sciences, computer sciences, control engineering, theoretical physics etc. Neutrosophic set theory is also applied to several algebraic structures. In particular, Jun et al. applied it to BCK/BCI-algebras (see [3], [4], [5], [6], [7], [8]). Jun et al. [4] introduced the notions of energetic subset, right vanished subset, right stable subset, and (anti) permeable values in BCK/BCI-algebras, and investigated relations between these sets.

In this paper, we introduced the notions of neutrosophic permeable S-value, neutrosophic permeable I-value, $(\in, \in)$-neutrosophic ideal, neutrosophic anti-permeable S-value and neutrosophic anti-permeable I-value, and investigate their properties. We consider characterizations of $(\in, \in)$-neutrosophic ideal. We discuss conditions for the lower (resp. upper) neutrosophic $\in_\Phi$-subsets to be S-energetic and I-energetic. We provide conditions for a triple (α, β, γ) of numbers to be a neutrosophic (anti-) permeable S-value and a neutrosophic (anti-) permeable I-value. We consider conditions for the upper (resp. lower) neutrosophic $\in_\Phi$-subsets to be right stable (resp. right vanished) subset. We establish relations between neutrosophic (anti-) permeable S-value and neutrosophic (anti-) permeable I-value.

2 Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)$ $(((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)$ $((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)$ $(x * x = 0)$,

(IV) $(\forall x, y \in X)$ $(x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)$ $(0 * x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; x * z \leq y * z,\, z * y \leq z * x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in I, \tag{2.5}$$

$$(\forall x, y \in X)\,(x * y \in I,\, y \in I \;\rightarrow\; x \in I)\,. \tag{2.6}$$

We refer the reader to the books [9] and [10] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} = \sup\{a_i \mid i \in \Lambda\}$$

and

$$\bigwedge\{a_i \mid i \in \Lambda\} = \inf\{a_i \mid i \in \Lambda\}.$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \{1,2\}\}$ and $\bigwedge\{a_i \mid i \in \{1,2\}\}$, respectively.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [1]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0,1]$ is a truth membership function, $A_I : X \to [0,1]$ is an indeterminate membership function, and $A_F : X \to [0,1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

A subset A of a BCK/BCI-algebra X is said to be S-energetic (see [4]) if it satisfies:

$$(\forall x, y \in X)\,(x * y \in A \;\Rightarrow\; \{x, y\} \cap A \neq \emptyset)\,. \tag{2.7}$$

A subset A of a BCK/BCI-algebra X is said to be *I-energetic* (see [4]) if it satisfies:

$$(\forall x, y \in X)\,(y \in A \;\Rightarrow\; \{x, y * x\} \cap A \neq \emptyset)\,. \tag{2.8}$$

A subset A of a BCK/BCI-algebra X is said to be *right vanished* (see [4]) if it satisfies:

$$(\forall x, y \in X)\,(x * y \in A \;\Rightarrow\; x \in A)\,. \tag{2.9}$$

A subset A of a BCK/BCI-algebra X is said to be *right stable* (see [4]) if $A * X := \{a * x \mid a \in A,\, x \in X\} \subseteq A$.

3 Neutrosophic permeable values

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets:

$$U_T^{\in}(A;\alpha) = \{x \in X \mid A_T(x) \geq \alpha\},\; U_T^{\in}(A;\alpha)^* = \{x \in X \mid A_T(x) > \alpha\},$$

$$U_I^{\in}(A;\beta) = \{x \in X \mid A_I(x) \geq \beta\},\; U_I^{\in}(A;\beta)^* = \{x \in X \mid A_I(x) > \beta\},$$

$$U_F^{\in}(A;\gamma) = \{x \in X \mid A_F(x) \leq \gamma\},\; U_F^{\in}(A;\gamma)^* = \{x \in X \mid A_F(x) < \gamma\},$$

$$L_T^{\in}(A;\alpha) = \{x \in X \mid A_T(x) \leq \alpha\},\; L_T^{\in}(A;\alpha)^* = \{x \in X \mid A_T(x) < \alpha\},$$

$$L_I^{\in}(A;\beta) = \{x \in X \mid A_I(x) \leq \beta\},\; L_I^{\in}(A;\beta)^* = \{x \in X \mid A_I(x) < \beta\},$$

$$L_F^{\in}(A;\gamma) = \{x \in X \mid A_F(x) \geq \gamma\},\; L_F^{\in}(A;\gamma)^* = \{x \in X \mid A_F(x) > \gamma\}.$$

We say $U_T^{\in}(A;\alpha)$, $U_I^{\in}(A;\beta)$ and $U_F^{\in}(A;\gamma)$ are *upper neutrosophic $\in_\Phi$-subsets* of X, and $L_T^{\in}(A;\alpha)$, $L_I^{\in}(A;\beta)$ and $L_F^{\in}(A;\gamma)$ are *lower neutrosophic $\in_\Phi$-subsets* of X, where $\Phi \in \{T, I, F\}$. We say $U_T^{\in}(A;\alpha)^*$, $U_I^{\in}(A;\beta)^*$ and $U_F^{\in}(A;\gamma)^*$ are *strong upper neutrosophic $\in_\Phi$-subsets* of X, and $L_T^{\in}(A;\alpha)^*$, $L_I^{\in}(A;\beta)^*$ and $L_F^{\in}(A;\gamma)^*$ are *strong lower neutrosophic $\in_\Phi$-subsets* of X, where $\Phi \in \{T, I, F\}$.

Definition 3.1 ([3]). A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$-*neutrosophic subalgebra* of X if the following assertions are valid.

$$\begin{aligned} &x \in U_T^{\in}(A; \alpha_x),\ y \in U_T^{\in}(A; \alpha_y) \ \Rightarrow\ x * y \in U_T^{\in}(A; \alpha_x \wedge \alpha_y),\\ &x \in U_I^{\in}(A; \beta_x),\ y \in U_I^{\in}(A; \beta_y) \ \Rightarrow\ x * y \in U_I^{\in}(A; \beta_x \wedge \beta_y),\\ &x \in U_F^{\in}(A; \gamma_x),\ y \in U_F^{\in}(A; \gamma_y) \ \Rightarrow\ x * y \in U_F^{\in}(A; \gamma_x \vee \gamma_y) \end{aligned} \tag{3.1}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Lemma 3.2 ([3]). *A neutrosophic set* $A = (A_T, A_I, A_F)$ *in a* BCK/BCI*-algebra* X *is an* $(\in, \in)$*-neutrosophic subalgebra of* X *if and only if* $A = (A_T, A_I, A_F)$ *satisfies:*

$$(\forall x, y \in X) \begin{pmatrix} A_T(x * y) \geq A_T(x) \wedge A_T(y) \\ A_I(x * y) \geq A_I(x) \wedge A_I(y) \\ A_F(x * y) \leq A_F(x) \vee A_F(y) \end{pmatrix}. \tag{3.2}$$

Proposition 3.3. *Every* $(\in, \in)$*-neutrosophic subalgebra* $A = (A_T, A_I, A_F)$ *of a* BCK/BCI*-algebra* X *satisfies:*

$$(\forall x \in X)\, (A_T(0) \geq A_T(x),\ A_I(0) \geq A_I(x),\ A_F(0) \leq A_F(x))\,. \tag{3.3}$$

Proof. Straightforward. □

Theorem 3.4. *If* $A = (A_T, A_I, A_F)$ *is an* $(\in, \in)$*-neutrosophic subalgebra of a* BCK/BCI*-algebra* X*, then the lower neutrosophic* $\in_\Phi$*-subsets of* X *are* S*-energetic subsets of* X *where* $\Phi \in \{T, I, F\}$.

Proof. Let $x, y \in X$ and $\alpha \in (0, 1]$ be such that $x * y \in L_T^{\in}(A; \alpha)$. Then

$$\alpha \geq A_T(x * y) \geq A_T(x) \wedge A_T(y),$$

and so $A_T(x) \leq \alpha$ or $A_T(y) \leq \alpha$, that is, $x \in L_T^{\in}(A; \alpha)$ or $y \in L_T^{\in}(A; \alpha)$. Thus $\{x, y\} \cap L_T^{\in}(A; \alpha) \neq \emptyset$. Therefore $L_T^{\in}(A; \alpha)$ is an S-energetic subset of X. Similarly, we can verify that $L_I^{\in}(A; \beta)$ is an S-energetic subset of X. Let $x, y \in X$ and $\gamma \in [0, 1)$ be such that $x * y \in L_F^{\in}(A; \gamma)$. Then

$$\gamma \leq A_F(x * y) \leq A_F(x) \vee A_F(y).$$

It follows that $A_F(x) \geq \gamma$ or $A_F(y) \geq \gamma$, i.e., $x \in L_F^{\in}(A; \gamma)$ or $y \in L_F^{\in}(A; \gamma)$. Hence $\{x, y\} \cap L_F^{\in}(A; \gamma) \neq \emptyset$, and therefore $L_F^{\in}(A; \gamma)$ is an S-energetic subset of X. □

Corollary 3.5. *If* $A = (A_T, A_I, A_F)$ *is an* $(\in, \in)$*-neutrosophic subalgebra of a* BCK/BCI*-algebra* X*, then the strong lower neutrosophic* $\in_\Phi$*-subsets of* X *are* S*-energetic subsets of* X *where* $\Phi \in \{T, I, F\}$.

Proof. Straightforward. □

The converse of Theorem 3.4 is not true as seen in the following example.

Example 3.6. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1 (see [10]).
Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X which is given in Table 2.
If $\alpha \in [0.4, 0.6)$, $\beta \in [0.5, 0.8)$ and $\gamma \in (0.2, 0.5]$, then $L_T^{\in}(A; \alpha) = \{1, 2, 3\}$, $L_I^{\in}(A; \beta) = \{1, 2, 3\}$ and $L_F^{\in}(A; \gamma) = \{1, 2, 3\}$ are S-energetic subsets of X. Since

$$A_T(4 * 4) = A_T(0) = 0.6 \ngeq 0.7 = A_T(4) \wedge A_T(4)$$

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	1
3	3	2	1	0	2
4	4	1	1	1	0

Table 2: Tabulation representation of $A = (A_T,\, A_I,\, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.6	0.8	0.2
1	0.4	0.5	0.7
2	0.4	0.5	0.6
3	0.4	0.5	0.5
4	0.7	0.8	0.2

and/or

$$A_F(3 * 2) = A_F(1) = 0.7 \not\leq 0.6 = A_F(3) \vee A_F(2),$$

it follows from Lemma 3.2 that $A = (A_T,\, A_I,\, A_F)$ is not an $(\in, \in)$-neutrosophic subalgebra of X.

Definition 3.7. Let $A = (A_T,\, A_I,\, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0, 1]$. Then (α, β, γ) is called a *neutrosophic permeable S-value* for $A = (A_T,\, A_I,\, A_F)$ if the following assertion is valid.

$$(\forall x, y \in X) \left(\begin{array}{lcl} x * y \in U_T^{\in}(A; \alpha) & \Rightarrow & A_T(x) \vee A_T(y) \geq \alpha, \\ x * y \in U_I^{\in}(A; \beta) & \Rightarrow & A_I(x) \vee A_I(y) \geq \beta, \\ x * y \in U_F^{\in}(A; \gamma) & \Rightarrow & A_F(x) \wedge A_F(y) \leq \gamma \end{array} \right). \tag{3.4}$$

Example 3.8. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $*$ which is given in Table 3.

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	1	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Then $(X, *, 0)$ is a BCK-algebra (see [10]). Let $A = (A_T,\, A_I,\, A_F)$ be a neutrosophic set in X which is given in Table 4

Table 4: Tabulation representation of $A = (A_T,\, A_I,\, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.2	0.3	0.7
1	0.6	0.4	0.6
2	0.5	0.3	0.4
3	0.4	0.8	0.5
4	0.7	0.6	0.2

It is routine to verify that $(\alpha, \beta, \gamma) \in (0, 2, 1] \times (0.3, 1] \times [0, 0.7)$ is a neutrosophic permeable S-value for $A = (A_T,\, A_I,\, A_F)$.

Theorem 3.9. *Let $A = (A_T,\, A_I,\, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0, 1]$. If $A = (A_T,\, A_I,\, A_F)$ satisfies the following condition*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x * y) \le A_T(x) \vee A_T(y) \\ A_I(x * y) \le A_I(x) \vee A_I(y) \\ A_F(x * y) \ge A_F(x) \wedge A_F(y) \end{array} \right), \tag{3.5}$$

then (α, β, γ) is a neutrosophic permeable S-value for $A = (A_T,\, A_I,\, A_F)$.

Proof. Let $x, y \in X$ be such that $x * y \in U_T^{\in}(A; \alpha)$. Then

$$\alpha \le A_T(x * y) \le A_T(x) \vee A_T(y).$$

Similarly, if $x * y \in U_I^{\in}(A; \beta)$ for $x, y \in X$, then $A_I(x) \vee A_I(y) \ge \beta$. Now, let $a, b \in X$ be such that $a * b \in U_F^{\in}(A; \gamma)$. Then

$$\gamma \ge A_F(a * b) \ge A_F(a) \wedge A_F(b).$$

Therefore (α, β, γ) is a neutrosophic permeable S-value for $A = (A_T,\, A_I,\, A_F)$. □

Theorem 3.10. *Let $A = (A_T,\, A_I,\, A_F)$ be a neutrosophic set in a BCK-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0, 1]$. If $A = (A_T,\, A_I,\, A_F)$ satisfies the following conditions*

$$(\forall x \in X)\, (A_T(0) \le A_T(x), A_I(0) \le A_I(x), A_F(0) \ge A_F(x)) \tag{3.6}$$

and

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x) \le A_T(x * y) \vee A_T(y) \\ A_I(x) \le A_I(x * y) \vee A_I(y) \\ A_F(x) \ge A_F(x * y) \wedge A_F(y) \end{array} \right), \tag{3.7}$$

then (α, β, γ) is a neutrosophic permeable S-value for $A = (A_T,\, A_I,\, A_F)$.

Proof. Let $x, y, a, b, u, v \in X$ be such that $x * y \in U_T^{\in}(A;\alpha)$, $a * b \in U_I^{\in}(A;\beta)$ and $u * v \in U_F^{\in}(A;\gamma)$. Then

$$\begin{aligned}\alpha &\leq A_T(x*y) \leq A_T((x*y)*x) \vee A_T(x)\\ &= A_T((x*x)*y) \vee A_T(x) = A_T(0*y) \vee A_T(x)\\ &= A_T(0) \vee A_T(x) = A_T(x),\end{aligned}$$

$$\begin{aligned}\beta &\leq A_I(a*b) \leq A_I((a*b)*a) \vee A_I(a)\\ &= A_I((a*a)*b) \vee A_I(a) = A_I(0*b) \vee A_I(a)\\ &= A_I(0) \vee A_I(a) = A_I(a)\end{aligned}$$

and

$$\begin{aligned}\gamma &\geq A_F(u*v) \geq A_F((u*v)*u) \wedge A_F(u)\\ &= A_F((u*u)*v) \wedge A_F(u) = A_F(0*v) \wedge A_F(v)\\ &= A_F(0) \wedge A_F(v) = A_F(v)\end{aligned}$$

by (2.3), (V), (3.6) and (3.7). It follows that

$$\begin{aligned}&A_T(x) \vee A_T(y) \geq A_T(x) \geq \alpha,\\ &A_I(a) \vee A_I(b) \geq A_I(a) \geq \beta,\\ &A_F(u) \wedge A_F(v) \leq A_F(u) \leq \gamma.\end{aligned}$$

Therefore (α, β, γ) is a neutrosophic permeable S-value for $A = (A_T, A_I, A_F)$. □

Theorem 3.11. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0, 1]$. If (α, β, γ) is a neutrosophic permeable S-value for $A = (A_T, A_I, A_F)$, then upper neutrosophic $\in_\Phi$-subsets of X are S-energetic where $\Phi \in \{T, I, F\}$.*

Proof. Let $x, y, a, b, u, v \in X$ be such that $x * y \in U_T^{\in}(A;\alpha)$, $a * b \in U_I^{\in}(A;\beta)$ and $u * v \in U_F^{\in}(A;\gamma)$. Using (3.4), we have $A_T(x) \vee A_T(y) \geq \alpha$, $A_I(a) \vee A_I(b) \geq \beta$, and $A_F(u) \wedge A_F(v) \leq \gamma$. It follows that

$$A_T(x) \geq \alpha \text{ or } A_T(y) \geq \alpha, \text{ that is, } x \in U_T^{\in}(A;\alpha) \text{ or } y \in U_T^{\in}(A;\alpha),$$

$$A_I(a) \geq \beta \text{ or } A_I(b) \geq \beta, \text{ that is, } a \in U_I^{\in}(A;\beta) \text{ or } b \in U_I^{\in}(A;\beta),$$

and

$$A_F(u) \leq \gamma \text{ or } A_F(v) \leq \gamma, \text{ that is, } u \in U_F^{\in}(A;\gamma) \text{ or } v \in U_F^{\in}(A;\gamma).$$

Hence $\{x, y\} \cap U_T^{\in}(A;\alpha) \neq \emptyset$, $\{a, b\} \cap U_I^{\in}(A;\beta) \neq \emptyset$, and $\{u, v\} \cap U_F^{\in}(A;\gamma) \neq \emptyset$. Therefore $U_T^{\in}(A;\alpha)$, $U_I^{\in}(A;\beta)$ and $U_F^{\in}(A;\gamma)$ are S-energetic subsets of X. □

Definition 3.12. Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0, 1]$. Then (α, β, γ) is called a *neutrosophic anti-permeable S-value* for $A = (A_T, A_I, A_F)$ if the following assertion is valid.

$$(\forall x, y \in X)\left(\begin{array}{l} x * y \in L_T^{\in}(A;\alpha) \;\Rightarrow\; A_T(x) \wedge A_T(y) \leq \alpha,\\ x * y \in L_I^{\in}(A;\beta) \;\Rightarrow\; A_I(x) \wedge A_I(y) \leq \beta,\\ x * y \in L_F^{\in}(A;\gamma) \;\Rightarrow\; A_F(x) \vee A_F(y) \geq \gamma \end{array}\right). \tag{3.8}$$

Table 5: Cayley table for the binary operation “$*$”

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	1	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Table 6: Tabulation representation of $A = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.7	0.6	0.4
1	0.4	0.5	0.6
2	0.4	0.5	0.6
3	0.5	0.2	0.7
4	0.3	0.3	0.9

Example 3.13. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $*$ which is given in Table 5. Then $(X, *, 0)$ is a BCK-algebra (see [10]). Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X which is given in Table 6
It is routine to verify that $(\alpha, \beta, \gamma) \in (0.3, 1] \times (0.2, 1] \times [0, 0.9)$ is a neutrosophic anti-permeable S-value for $A = (A_T, A_I, A_F)$.

Theorem 3.14. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0, 1]$. If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X, then (α, β, γ) is a neutrosophic anti-permeable S-value for $A = (A_T, A_I, A_F)$.*

Proof. Let $x, y, a, b, u, v \in X$ be such that $x * y \in L_T^{\Subset}(A; \alpha)$, $a * b \in L_I^{\Subset}(A; \beta)$ and $u * v \in L_F^{\Subset}(A; \gamma)$. Using Lemma 3.2, we have

$$\begin{aligned} A_T(x) \wedge A_T(y) &\leq A_T(x * y) \leq \alpha, \\ A_I(a) \wedge A_I(b) &\leq A_I(a * b) \leq \beta, \\ A_F(u) \vee A_F(v) &\geq A_F(u * v) \geq \gamma, \end{aligned}$$

and so (α, β, γ) is a neutrosophic anti-permeable S-value for $A = (A_T, A_I, A_F)$. □

Theorem 3.15. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0, 1]$. If (α, β, γ) is a neutrosophic anti-permeable S-value for $A = (A_T, A_I, A_F)$, then lower neutrosophic $\in_\Phi$-subsets of X are S-energetic where $\Phi \in \{T, I, F\}$.*

Proof. Let $x, y, a, b, u, v \in X$ be such that $x * y \in L_T^{\Subset}(A; \alpha)$, $a * b \in L_I^{\Subset}(A; \beta)$ and $u * v \in L_F^{\Subset}(A; \gamma)$. Using (3.8), we have $A_T(x) \wedge A_T(y) \leq \alpha$, $A_I(a) \wedge A_I(b) \leq \beta$, and $A_F(u) \vee A_F(v) \geq \gamma$, which imply that

$$A_T(x) \leq \alpha \text{ or } A_T(y) \leq \alpha, \text{ that is, } x \in L_T^{\Subset}(A; \alpha) \text{ or } y \in L_T^{\Subset}(A; \alpha),$$

$$A_I(a) \leq \beta \text{ or } A_I(b) \leq \beta, \text{ that is, } a \in L_I^{\in}(A;\beta) \text{ or } b \in L_I^{\in}(A;\beta),$$

and

$$A_F(u) \geq \gamma \text{ or } A_F(v) \geq \gamma, \text{ that is, } u \in L_F^{\in}(A;\gamma) \text{ or } v \in L_F^{\in}(A;\gamma).$$

Hence $\{x,y\} \cap L_T^{\in}(A;\alpha) \neq \emptyset$, $\{a,b\} \cap L_I^{\in}(A;\beta) \neq \emptyset$, and $\{u,v\} \cap L_F^{\in}(A;\gamma) \neq \emptyset$. Therefore $L_T^{\in}(A;\alpha)$, $L_I^{\in}(A;\beta)$ and $L_F^{\in}(A;\gamma)$ are S-energetic subsets of X. □

Definition 3.16. A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called an $(\in, \in)$-*neutrosophic ideal* of X if the following assertions are valid.

$$(\forall x \in X) \begin{pmatrix} x \in U_T^{\in}(A;\alpha) \;\Rightarrow\; 0 \in U_T^{\in}(A;\alpha) \\ x \in U_I^{\in}(A;\beta) \;\Rightarrow\; 0 \in U_I^{\in}(A;\beta) \\ x \in U_F^{\in}(A;\gamma) \;\Rightarrow\; 0 \in U_F^{\in}(A;\gamma) \end{pmatrix}, \tag{3.9}$$

$$(\forall x,y \in X) \begin{pmatrix} x*y \in U_T^{\in}(A;\alpha_x),\, y \in U_T^{\in}(A;\alpha_y) \;\Rightarrow\; x \in U_T^{\in}(A;\alpha_x \wedge \alpha_y) \\ x*y \in U_I^{\in}(A;\beta_x),\, y \in U_I^{\in}(A;\beta_y) \;\Rightarrow\; x \in U_I^{\in}(A;\beta_x \wedge \beta_y) \\ x*y \in U_F^{\in}(A;\gamma_x),\, y \in U_F^{\in}(A;\gamma_y) \;\Rightarrow\; x \in U_F^{\in}(A;\gamma_x \vee \gamma_y) \end{pmatrix} \tag{3.10}$$

for all $\alpha, \beta, \alpha_x, \alpha_y, \beta_x, \beta_y \in (0,1]$ and $\gamma, \gamma_x, \gamma_y \in [0,1)$.

Theorem 3.17. *A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in)$-neutrosophic ideal of X if and only if $A = (A_T, A_I, A_F)$ satisfies:*

$$(\forall x,y \in X) \begin{pmatrix} A_T(0) \geq A_T(x) \geq A_T(x*y) \wedge A_T(y) \\ A_I(0) \geq A_I(x) \geq A_I(x*y) \wedge A_I(y) \\ A_F(0) \leq A_F(x) \leq A_F(x*y) \vee A_F(y) \end{pmatrix}. \tag{3.11}$$

Proof. Assume that (3.11) is valid and let $x \in U_T^{\in}(A;\alpha)$, $a \in U_I^{\in}(A;\beta)$ and $u \in U_F^{\in}(A;\gamma)$ for any $x, a, u \in X$, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$. Then $A_T(0) \geq A_T(x) \geq \alpha$, $A_I(0) \geq A_I(a) \geq \beta$, and $A_F(0) \leq A_F(u) \leq \gamma$. Hence $0 \in U_T^{\in}(A;\alpha)$, $0 \in U_I^{\in}(A;\beta)$ and $0 \in U_F^{\in}(A;\gamma)$, and so (3.9) is valid. Let $x, y, a, b, u, v \in X$ be such that $x*y \in U_T^{\in}(A;\alpha_x)$, $y \in U_T^{\in}(A;\alpha_y)$, $a*b \in U_I^{\in}(A;\beta_a)$, $b \in U_I^{\in}(A;\beta_b)$, $u*v \in U_F^{\in}(A;\gamma_u)$ and $v \in U_F^{\in}(A;\gamma_v)$ for all $\alpha_x, \alpha_y, \beta_a, \beta_b \in (0,1]$ and $\gamma_u, \gamma_v \in [0,1)$. Then $A_T(x*y) \geq \alpha_x$, $A_T(y) \geq \alpha_y$, $A_I(a*b) \geq \beta_a$, $A_I(b) \geq \beta_b$, $A_F(u*v) \leq \gamma_u$ and $A_F(v) \leq \gamma_v$. It follows from (3.11) that

$$\begin{aligned} &A_T(x) \geq A_T(x*y) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y, \\ &A_I(a) \geq A_I(a*b) \wedge A_I(b) \geq \beta_a \wedge \beta_b, \\ &A_F(u) \leq A_F(u*v) \vee A_F(v) \leq \gamma_u \vee \gamma_v. \end{aligned}$$

Hence $x \in U_T^{\in}(A;\alpha_x \wedge \alpha_y)$, $a \in U_I^{\in}(A;\beta_a \wedge \beta_b)$ and $u \in U_F^{\in}(A;\gamma_u \vee \gamma_v)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of X.

Conversely, let $A = (A_T, A_I, A_F)$ be an $(\in, \in)$-neutrosophic ideal of X. If there exists $x_0 \in X$ such that $A_T(0) < A_T(x_0)$, then $x_0 \in U_T^{\in}(A;\alpha)$ and $0 \notin U_T^{\in}(A;\alpha)$ where $\alpha = A_T(x_0)$. This is a contradiction, and so $A_T(0) \geq A_T(x)$ for all $x \in X$. Assume that $A_T(x_0) < A_T(x_0*y_0) \wedge A_T(y_0)$ for some $x_0, y_0 \in X$. Taking $\alpha := A_T(x_0*y_0) \wedge A_T(y_0)$ implies that $x_0*y_0 \in U_T^{\in}(A;\alpha)$ and $y_0 \in U_T^{\in}(A;\alpha)$, but $x_0 \notin U_T^{\in}(A;\alpha)$. This is a contradiction, and thus $A_T(x) \geq A_T(x*y) \wedge A_T(y)$ for all $x, y \in X$. Similarly, we can verify that $A_I(0) \geq A_I(x) \geq A_I(x*y) \wedge A_I(y)$ for all $x, y \in X$. Now, suppose that $A_F(0) > A_F(a)$ for some $a \in X$. Then $a \in U_F^{\in}(A;\gamma)$ and $0 \notin U_F^{\in}(A;\gamma)$ by taking $\gamma = A_F(a)$. This is impossible, and thus

$A_F(0) \leq A_F(x)$ for all $x \in X$. Suppose there exist $a_0, b_0 \in X$ such that $A_F(a_0) > A_F(a_0 * b_0) \vee A_F(b_0)$ and take $\gamma := A_F(a_0 * b_0) \vee A_F(b_0)$. Then $a_0 * b_0 \in U_F^{\in}(A;\gamma)$, $b_0 \in U_F^{\in}(A;\gamma)$ and $a_0 \notin U_F^{\in}(A;\gamma)$, which is a contradiction. Thus $A_F(x) \leq A_F(x * y) \vee A_F(y)$ for all $x, y \in X$. Therefore $A = (A_T,\, A_I,\, A_F)$ satisfies (3.11). □

Lemma 3.18. *Every* $(\in,\, \in)$*-neutrosophic ideal* $A = (A_T,\, A_I,\, A_F)$ *of a* BCK/BCI*-algebra* X *satisfies:*

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; A_T(x) \geq A_T(y),\, A_I(x) \geq A_I(y),\, A_F(x) \leq A_F(y))\,. \tag{3.12}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x * y = 0$, and so

$$\begin{aligned} A_T(x) &\geq A_T(x * y) \wedge A_T(y) = A_T(0) \wedge A_T(y) = A_T(y), \\ A_I(x) &\geq A_I(x * y) \wedge A_I(y) = A_I(0) \wedge A_I(y) = A_I(y), \\ A_F(x) &\leq A_F(x * y) \vee A_F(y) = A_F(0) \vee A_F(y) = A_F(y) \end{aligned}$$

by (3.11). This completes the proof. □

Theorem 3.19. *A neutrosophic set* $A = (A_T,\, A_I,\, A_F)$ *in a* BCK*-algebra* X *is an* $(\in,\, \in)$*-neutrosophic ideal of* X *if and only if* $A = (A_T,\, A_I,\, A_F)$ *satisfies:*

$$(\forall x, y, z \in X) \left(x * y \leq z \;\Rightarrow \left\{ \begin{array}{l} A_T(x) \geq A_T(y) \wedge A_T(z) \\ A_I(x) \geq A_I(y) \wedge A_I(z) \\ A_F(x) \leq A_F(y) \vee A_F(z) \end{array} \right. \right). \tag{3.13}$$

Proof. Let $A = (A_T,\, A_I,\, A_F)$ be an $(\in,\, \in)$-neutrosophic ideal of X and let $x, y, z \in X$ be such that $x * y \leq z$. Using Theorem 3.17 and Lemma 3.18, we have

$$\begin{aligned} A_T(x) &\geq A_T(x * y) \wedge A_T(y) \geq A_T(y) \wedge A_T(z), \\ A_I(x) &\geq A_I(x * y) \wedge A_I(y) \geq A_I(y) \wedge A_I(z), \\ A_F(x) &\leq A_F(x * y) \vee A_F(y) \leq A_F(y) \vee A_F(z). \end{aligned}$$

Conversely, assume that $A = (A_T,\, A_I,\, A_F)$ satisfies (3.13). Since $0 * x \leq x$ for all $x \in X$, it follows from (3.13) that

$$\begin{aligned} A_T(0) &\geq A_T(x) \wedge A_T(x) = A_T(x), \\ A_I(0) &\geq A_I(x) \wedge A_I(x) = A_I(x), \\ A_F(0) &\leq A_F(x) \vee A_F(x) = A_F(x) \end{aligned}$$

for all $x \in X$. Since $x * (x * y) \leq y$ for all $x, y \in X$, we have

$$\begin{aligned} A_T(x) &\geq A_T(x * y) \wedge A_T(y), \\ A_I(x) &\geq A_I(x * y) \wedge A_I(y), \\ A_F(x) &\leq A_F(x * y) \vee A_F(y) \end{aligned}$$

for all $x, y \in X$ by (3.13). It follows from Theorem 3.17 that $A = (A_T,\, A_I,\, A_F)$ is an $(\in,\, \in)$-neutrosophic ideal of X. □

Theorem 3.20. *If* $A = (A_T,\, A_I,\, A_F)$ *is an* $(\in,\, \in)$*-neutrosophic ideal of a* BCK/BCI*-algebra* X*, then the lower neutrosophic* $\in_\Phi$*-subsets of* X *are* I*-energetic subsets of* X *where* $\Phi \in \{T, I, F\}$.

Proof. Let $x, a, u \in X$, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ be such that $x \in L_T^{\in}(A;\alpha)$, $a \in L_I^{\in}(A;\beta)$ and $u \in L_F^{\in}(A;\gamma)$. Using Theorem 3.17, we have

$$\begin{aligned}&\alpha \geq A_T(x) \geq A_T(x*y) \wedge A_T(y),\\&\beta \geq A_I(a) \geq A_I(a*b) \wedge A_I(b),\\&\gamma \leq A_F(u) \leq A_F(u*v) \vee A_F(v),\end{aligned}$$

for all $y, b, v \in X$. It follows that

$$A_T(x*y) \leq \alpha \text{ or } A_T(y) \leq \alpha, \text{ that is, } x*y \in L_T^{\in}(A;\alpha) \text{ or } y \in L_T^{\in}(A;\alpha),$$

$$A_I(a*b) \leq \beta \text{ or } A_I(b) \leq \beta, \text{ that is, } a*b \in L_T^{\in}(A;\beta) \text{ or } b \in L_T^{\in}(A;\beta),$$

and

$$A_F(u*v) \geq \gamma \text{ or } A_F(v) \geq \gamma, \text{ that is, } u*v \in L_T^{\in}(A;\gamma) \text{ or } v \in L_T^{\in}(A;\gamma).$$

Hence $\{y, x*y\} \cap L_T^{\in}(A;\alpha)$, $\{b, a*b\} \cap L_I^{\in}(A;\beta)$ and $\{v, u*v\} \cap L_F^{\in}(A;\gamma)$ are nonempty, and therefore $L_T^{\in}(A;\alpha)$, $L_I^{\in}(A;\beta)$ and $L_F^{\in}(A;\gamma)$ are I-energetic subsets of X. □

Corollary 3.21. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of a BCK/BCI-algebra X, then the strong lower neutrosophic $\in_\Phi$-subsets of X are I-energetic subsets of X where $\Phi \in \{T, I, F\}$.*

Proof. Straightforward. □

Theorem 3.22. *Let $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0,1]$. If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic ideal of a BCK-algebra X, then*

(1) *The (strong) upper neutrosophic $\in_\Phi$-subsets of X are right stable where $\Phi \in \{T, I, F\}$.*

(2) *The (strong) lower neutrosophic $\in_\Phi$-subsets of X are right vanished where $\Phi \in \{T, I, F\}$.*

Proof. (1) Let $x \in X$, $a \in U_T^{\in}(A;\alpha)$, $b \in U_I^{\in}(A;\beta)$ and $c \in U_F^{\in}(A;\gamma)$. Then $A_T(a) \geq \alpha$, $A_I(b) \geq \beta$ and $A_F(c) \leq \gamma$. Since $a*x \leq a$, $b*x \leq b$ and $c*x \leq c$, it follows from Lemma 3.18 that $A_T(a*x) \geq A_T(a) \geq \alpha$, $A_I(b*x) \geq A_I(b) \geq \beta$ and $A_F(c*x) \leq A_F(c) \leq \gamma$, that is, $a*x \in U_T^{\in}(A;\alpha)$, $b*x \in U_I^{\in}(A;\beta)$ and $c*x \in U_F^{\in}(A;\gamma)$. Hence the upper neutrosophic $\in_\Phi$-subsets of X are right stable where $\Phi \in \{T, I, F\}$. Similarly, the strong upper neutrosophic $\in_\Phi$-subsets of X are right stable where $\Phi \in \{T, I, F\}$.

(2) Assume that $x*y \in L_T^{\in}(A;\alpha)$, $a*b \in L_I^{\in}(A;\beta)$ and $c*d \in L_F^{\in}(A;\gamma)$ for any $x, y, a, b, c, d \in X$. Then $A_T(x*y) \leq \alpha$, $A_I(a*b) \leq \beta$ and $A_F(c*d) \geq \gamma$. Since $x*y \leq x$, $a*b \leq a$ and $c*d \leq c$, it follows from Lemma 3.18 that $\alpha \geq A_T(x*y) \geq A_T(x)$, $\beta \geq A_I(a*b) \geq A_I(a)$, and $\gamma \leq A_F(c*d) \leq A_F(c)$, that is, $x \in L_T^{\in}(A;\alpha)$, $a \in L_I^{\in}(A;\beta)$ and $c \in L_F^{\in}(A;\gamma)$. Therefore the lower neutrosophic $\in_\Phi$-subsets of X are right vanished where $\Phi \in \{T, I, F\}$. By the similar way, we know that the strong lower neutrosophic $\in_\Phi$-subsets of X are right vanished where $\Phi \in \{T, I, F\}$. □

Definition 3.23. Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0,1]$. Then (α, β, γ) is called a *neutrosophic permeable I-value* for $A = (A_T, A_I, A_F)$ if the following assertion is valid.

$$(\forall x, y \in X)\left(\begin{array}{lcl} x \in U_T^{\in}(A;\alpha) & \Rightarrow & A_T(x*y) \vee A_T(y) \geq \alpha,\\ x \in U_I^{\in}(A;\beta) & \Rightarrow & A_I(x*y) \vee A_I(y) \geq \beta,\\ x \in U_F^{\in}(A;\gamma) & \Rightarrow & A_F(x*y) \wedge A_F(y) \leq \gamma \end{array}\right). \tag{3.14}$$

Table 7: Cayley table for the binary operation “$*$”

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

Example 3.24. (1) In Example 3.8, (α, β, γ) is a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$.

(2) Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the binary operation $*$ which is given in Table 7 (see [10]).

Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X which is given in Table 8

Table 8: Tabulation representation of $A = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.33	0.38	0.77
1	0.44	0.48	0.66
a	0.55	0.68	0.44
b	0.66	0.58	0.44
c	0.66	0.68	0.55

It is routine to check that $(\alpha, \beta, \gamma) \in (0.33, 1] \times (0.38, 1] \times [0, 0.77)$ is a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$.

Lemma 3.25. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X satisfies the condition (3.5), then*

$$(\forall x \in X)\, (A_T(0) \leq A_T(x),\, A_I(0) \leq A_I(x),\, A_F(0) \geq A_F(x))\,. \tag{3.15}$$

Proof. Straightforward. □

Theorem 3.26. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK-algebra X satisfies the condition (3.5), then every neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$ is a neutrosophic permeable S-value for $A = (A_T, A_I, A_F)$.*

Proof. Let (α, β, γ) be a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$. Let $x, y, a, b, u, v \in X$ be such that $x * y \in U_T^{\in}(A; \alpha)$, $a * b \in U_I^{\in}(A; \beta)$ and $u * v \in U_F^{\in}(A; \gamma)$. It follows from (3.14), (2.3), (III), (V) and Lemma 3.25 that

$$\begin{aligned} \alpha &\leq A_T((x * y) * x) \vee A_T(x) = A_T((x * x) * y) \vee A_T(x) \\ &= A_T(0 * y) \vee A_T(x) = A_T(0) \vee A_T(x) = A_T(x), \end{aligned}$$

$$\begin{aligned} \beta &\leq A_I((a * b) * a) \vee A_I(a) = A_I((a * a) * b) \vee A_I(a) \\ &= A_I(0 * b) \vee A_I(a) = A_I(0) \vee A_I(a) = A_I(a), \end{aligned}$$

and

$$\gamma \geq A_F((u * v) * u) \wedge A_F(u) = A_F((u * u) * v) \wedge A_F(u)$$
$$= A_F(0 * v) \wedge A_F(u) = A_F(0) \wedge A_F(u) = A_F(u).$$

Hence $A_T(x) \vee A_T(y) \geq A_T(x) \geq \alpha$, $A_I(a) \vee A_I(b) \geq A_I(a) \geq \beta$, and $A_F(u) \wedge A_F(v) \leq A_F(u) \leq \gamma$. Therefore (α, β, γ) is a neutrosophic permeable S-value for $A = (A_T, A_I, A_F)$. □

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, any upper neutrosophic $\in_\Phi$-subsets of X may not be I-energetic where $\Phi \in \{T, I, F\}$ as seen in the following example.

Example 3.27. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 9 (see [10]).

Table 9: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	1	1	0	0
4	4	2	1	2	0

Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X which is given in Table 10

Table 10: Tabulation representation of $A = (A_T, A_I, A_F)$

X	$A_T(x)$	$A_I(x)$	$A_F(x)$
0	0.75	0.73	0.34
1	0.53	0.45	0.58
2	0.67	0.86	0.34
3	0.53	0.56	0.58
4	0.46	0.56	0.66

Then $U_T^{\in}(A; 0.6) = \{0, 2\}$, $U_I^{\in}(A; 0.7) = \{0, 2\}$ and $U_F^{\in}(A; 0.4) = \{0, 2\}$. Since $2 \in \{0, 2\}$ and $\{1, 2 * 1\} \cap \{0, 2\} = \emptyset$, we know that $\{0, 2\}$ is not an I-energetic subset of X.

We now provide conditions for the upper neutrosophic $\in_\Phi$-subsets to be I-energetic where $\Phi \in \{T, I, F\}$.

Theorem 3.28. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0, 1]$. If (α, β, γ) is a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$, then the upper neutrosophic $\in_\Phi$-subsets of X are I-energetic subsets of X where $\Phi \in \{T, I, F\}$.*

Proof. Let $x, a, u \in X$ and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0, 1]$ such that $x \in U_T^{\in}(A; \alpha)$, $a \in U_I^{\in}(A; \beta)$ and $u \in U_F^{\in}(A; \gamma)$. Since (α, β, γ) is a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$, it follows from (3.14) that

$$A_T(x * y) \vee A_T(y) \geq \alpha,\ A_I(a * b) \vee A_I(b) \geq \beta \text{ and } A_F(u * v) \wedge A_F(v) \leq \gamma$$

for all $y, b, v \in X$. Hence

$$A_T(x * y) \geq \alpha \text{ or } A_T(y) \geq \alpha, \text{ that is, } x * y \in U_T^{\in}(A; \alpha) \text{ or } y \in U_T^{\in}(A; \alpha),$$

$$A_I(a * b) \geq \beta \text{ or } A_I(b) \geq \beta, \text{ that is, } a * b \in U_I^{\in}(A; \beta) \text{ or } b \in U_I^{\in}(A; \beta),$$

and

$$A_F(u * v) \leq \gamma \text{ or } A_F(v) \leq \gamma, \text{ that is, } u * v \in U_F^{\in}(A; \gamma) \text{ or } v \in U_F^{\in}(A; \gamma).$$

Hence $\{y, x * y\} \cap U_T^{\in}(A; \alpha)$, $\{b, a * b\} \cap U_I^{\in}(A; \beta)$ and $\{v, u * v\} \cap U_F^{\in}(A; \gamma)$ are nonempty, and therefore the upper neutrosophic $\in_\Phi$-subsets of X are I-energetic subsets of X where $\Phi \in \{T, I, F\}$. □

Theorem 3.29. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0, 1]$. If $A = (A_T, A_I, A_F)$ satisfies the following condition*

$$(\forall x, y \in X) \begin{pmatrix} A_T(x) \leq A_T(x * y) \vee A_T(y) \\ A_I(x) \leq A_I(x * y) \vee A_I(y) \\ A_F(x) \geq A_F(x * y) \wedge A_F(y) \end{pmatrix}, \tag{3.16}$$

then (α, β, γ) is a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$.

Proof. Let $x, a, u \in X$ and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0, 1]$ such that $x \in U_T^{\in}(A; \alpha)$, $a \in U_I^{\in}(A; \beta)$ and $u \in U_F^{\in}(A; \gamma)$. Using (3.16), we get

$$\begin{aligned} &\alpha \leq A_T(x) \leq A_T(x * y) \vee A_T(y), \\ &\beta \leq A_I(a) \leq A_I(a * b) \vee A_I(b), \\ &\gamma \geq A_F(u) \geq A_F(u * v) \wedge A_F(v) \end{aligned}$$

for all $y, b, v \in X$. Therefore (α, β, γ) is a neutrosophic permeable I-value for $A = (A_T, A_I, A_F)$. □

Combining Theorems 3.28 and 3.29, we have the following corollary.

Corollary 3.30. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0, 1]$. If $A = (A_T, A_I, A_F)$ satisfies the condition (3.16), then the upper neutrosophic $\in_\Phi$-subsets of X are I-energetic subsets of X where $\Phi \in \{T, I, F\}$.*

Definition 3.31. Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0, 1]$. Then (α, β, γ) is called a *neutrosophic anti-permeable I-value* for $A = (A_T, A_I, A_F)$ if the following assertion is valid.

$$(\forall x, y \in X) \begin{pmatrix} x \in L_T^{\in}(A; \alpha) \ \Rightarrow \ A_T(x * y) \wedge A_T(y) \leq \alpha, \\ x \in L_I^{\in}(A; \beta) \ \Rightarrow \ A_I(x * y) \wedge A_I(y) \leq \beta, \\ x \in L_F^{\in}(A; \gamma) \ \Rightarrow \ A_F(x * y) \vee A_F(y) \geq \gamma \end{pmatrix}. \tag{3.17}$$

Theorem 3.32. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0,1]$. If $A = (A_T, A_I, A_F)$ satisfies the condition (3.10), then (α, β, γ) is a neutrosophic anti-permeable I-value for $A = (A_T, A_I, A_F)$.*

Proof. Let $x, a, u \in X$ be such that $x \in L_T^{\in}(A;\alpha)$, $a \in L_I^{\in}(A;\beta)$ and $u \in L_F^{\in}(A;\gamma)$. Then

$$\begin{aligned}&A_T(x*y) \wedge A_T(y) \leq A_T(x) \leq \alpha,\\&A_I(a*b) \wedge A_I(b) \leq A_I(a) \leq \beta,\\&A_F(u*v) \vee A_F(v) \geq A_F(u) \geq \gamma,\end{aligned}$$

for all $y, b, v \in X$ by (3.11). Hence (α, β, γ) is a neutrosophic anti-permeable I-value for $A = (A_T, A_I, A_F)$. □

Theorem 3.33. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0,1]$. If (α, β, γ) is a neutrosophic anti-permeable I-value for $A = (A_T, A_I, A_F)$, then the lower neutrosophic $\in_\Phi$-subsets of X are I-energetic where $\Phi \in \{T, I, F\}$.*

Proof. Let $x \in L_T^{\in}(A;\alpha)$, $a \in L_I^{\in}(A;\beta)$ and $u \in L_F^{\in}(A;\gamma)$. Then $A_T(x*y) \wedge A_T(y) \leq \alpha$, $A_I(a*b) \wedge A_I(b) \leq \beta$, $A_F(u*v) \vee A_F(v) \geq \gamma$ for all $y, b, v \in X$ by (3.17). It follows that

$$A_T(x*y) \leq \alpha \text{ or } A_T(y) \leq \alpha, \text{ that is, } x*y \in L_T^{\in}(A;\alpha) \text{ or } y \in L_T^{\in}(A;\alpha),$$

$$A_I(a*b) \leq \beta \text{ or } A_I(b) \leq \beta, \text{ that is, } a*b \in L_I^{\in}(A;\beta) \text{ or } b \in L_I^{\in}(A;\beta),$$

and

$$A_F(u*v) \geq \gamma \text{ or } A_F(v) \geq \gamma, \text{ that is, } u*v \in L_F^{\in}(A;\gamma) \text{ or } v \in L_F^{\in}(A;\gamma).$$

Hence $\{y, x*y\} \cap L_T^{\in}(A;\alpha)$, $\{b, a*b\} \cap L_I^{\in}(A;\beta)$ and $\{v, u*v\} \cap L_F^{\in}(A;\gamma)$ are nonempty, and therefore the lower neutrosophic $\in_\Phi$-subsets of X are I-energetic where $\Phi \in \{T, I, F\}$. □

Combining Theorems 3.32 and 3.33, we get the following corollary.

Corollary 3.34. *Let $A = (A_T, A_I, A_F)$ be a neutrosophic set in a BCK/BCI-algebra X and $(\alpha, \beta, \gamma) \in \Lambda_T \times \Lambda_I \times \Lambda_F$ where Λ_T, Λ_I and Λ_F are subsets of $[0,1]$. If $A = (A_T, A_I, A_F)$ satisfies the condition (3.10), then the lower neutrosophic $\in_\Phi$-subsets of X are I-energetic where $\Phi \in \{T, I, F\}$.*

Theorem 3.35. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of a BCK-algebra X, then every neutrosophic anti-permeable I-value for $A = (A_T, A_I, A_F)$ is a neutrosophic anti-permeable S-value for $A = (A_T, A_I, A_F)$.*

Proof. Let (α, β, γ) be a neutrosophic anti-permeable I-value for $A = (A_T, A_I, A_F)$. Let $x, y, a, b, u, v \in X$ be such that $x*y \in L_T^{\in}(A;\alpha)$, $a*b \in L_I^{\in}(A;\beta)$ and $u*v \in L_F^{\in}(A;\gamma)$. It follows from (3.17), (2.3), (III), (V) and Proposition 3.3 that

$$\begin{aligned}\alpha &\geq A_T((x*y)*x) \wedge A_T(x) = A_T((x*x)*y) \wedge A_T(x)\\&= A_T(0*y) \wedge A_T(x) = A_T(0) \wedge A_T(x) = A_T(x),\end{aligned}$$

$$\begin{aligned}\beta &\geq A_I((a*b)*a) \wedge A_I(a) = A_I((a*a)*b) \wedge A_I(a)\\&= A_I(0*b) \wedge A_I(a) = A_I(0) \wedge A_I(a) = A_I(a),\end{aligned}$$

and

$$\gamma \leq A_F((u*v)*u) \vee A_F(u) = A_F((u*u)*v) \vee A_F(u)$$
$$= A_F(0*v) \vee A_F(u) = A_F(0) \vee A_F(u) = A_F(u).$$

Hence $A_T(x) \wedge A_T(y) \leq A_T(x) \leq \alpha$, $A_I(a) \wedge A_I(b) \leq A_I(a) \leq \beta$, and $A_F(u) \vee A_F(v) \geq A_F(u) \geq \gamma$. Therefore (α, β, γ) is a neutrosophic anti-permeable S-value for $A = (A_T, A_I, A_F)$. □

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