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A Novel Similarity Measure for Interval-Valued Intuitionistic Fuzzy Sets and Its Applications

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Abstract: In this paper, a novel similarity measure for interval-valued intuitionistic fuzzy sets is introduced, which is based on the transformed interval-valued intuitionistic triangle fuzzy numbers. Its superiority is shown by comparing the proposed similarity measure with some existing similarity measures by some numerical examples. Furthermore, the proposed similarity measure is applied to deal with pattern recognition and medical diagnosis problems.

Keywords: interval-valued intuitionistic fuzzy set; similarity measure; pattern recognition

1. Introduction

As a generalization concept of fuzzy set (FS) introduced by Zadeh [1], the definition of intuitionistic fuzzy set (IFS) was initiated by Atanassov [2] for dealing with vague and uncertain information, which elaborately describe uncertain information by membership degree, non-membership degree and hesitancy degree. In [3], Gau and Buehrer presented the definition of vague set. In [4], Bustince and Burillo have showed that the notion of IFSs and vague sets coincide with each other. In order to deal with indeterminate and inconsistent information, Smarandache [5] proposed a neutrosophic set (NS). In the NS, indeterminacy-membership $I_A(x)$ is independent, thus making the NS more flexible and the most suitable for solving some decision-making problems related to the use of incomplete and imprecise information, uncertainties, predictions and so on. Zhang [6,7] studied algebraic and lattice structure for neutrosophic sets.

The conception of similarity measure for IFSs is one of the most important subjects for degree of similarity between objects in IFS theory. Chen [8] proposed the similarity measure based on a vague set for the first time. Hong [9] introduced a new similarity measure based on vague set and overcame some drawbacks of Chen's similarity measure. Szmidt and Kacprzyk [10] extend Hamming distance and Euclidean distance to construct intuitionistic fuzzy similarity measure. However, Wang and Xin [11] implied that Szmidt and Kacprzyk's distance measure [10] were ineffective in some situations. Grzegorzewski [12] extended some novel similarity measures for IFSs based on Hausdorff distance. Chen [13] pointed out some defects of Grzegorzewski's similarity measure and show some counter examples. On the other hand, some studies defined new similarity measures for IFSs, rather than extending the well-known distance measures. Li and Cheng [14] presented a new similarity measure between IFSs and applied it to pattern recognition. Mitchell [15] indicated that similarity measure of Li and Cheng [14] had some counter-intuitive cases and modified that similarity measure based on a statistical perspective. Furthermore, Liang and Shi [16] presented some counter instances to indicate that the similarity measure of Li and Cheng [14] was not suitable for some situations, and proposed several new similarity measures for IFS. Ye [17] conducted a similarity comparative study of existing similarity measures for IFSs and proposed a cosine similarity measure and weighted cosine similarity measure. Xu [18] acquainted a sequence of similarity measures for IFSs and applied to solve multiple attribute decision-making problems. Boran et al. [19] proposed a new general type of similarity measures for IFSs with two parameters, expressing L_p-norm and give its relation with existing similarity measures. Zhang and Yu [20] presented a new distance measure based on interval comparison, where the IFSs were respectively transformed into the symmetric triangular fuzzy numbers. Comparison with the widely used methods indicated that the proposed method contained more information, with much less loss of information. Luo and Zhao [21] proposed a new distance measure for IFSs, which is based on a matrix norm and a strictly increasing (or decreasing) binary function, and applied it to solve pattern recognition problems.

As the development of IFSs, Atanassov introduced interval-valued intuitionistic fuzzy set (IVIFS) [22], which the membership degree, non-membership degree and hesitancy degree are represented by subinterval of $[0,1]$. It therefore can represent the dynamic character of features accurately. Due to the advantages of IVIFSs in practical application, various similarity measures based on IVIFSs were studied extensively by many researchers from different angles and applied to many areas such as medical diagnosis, pattern recognition problem and so on. Liu [23] proposed a set of axiomatic definitions for entropy measures between IVIFSs, which extends Szmidt and Kacprzyk's axioms formulated for entropy between IFSs. Xu [24] generalized some formulas of similarity measures of IFSs to IVIFSs. Wei [25] proposed an new similarity measure for IVIFSs, and also applied to solve problems on pattern recognitions, multi-criteria fuzzy decision-making and medical diagnosis. Singh [26] introduced a new cosine similarity measure for IVIFSs and applied to pattern recognition. Khalaf [27] advanced a new approach for medical diagnosis by IVIFSs, which is generalized by the application of IFS theory. Dhivya [28] presented a new similarity measure for IVIFSs based on the mid points of transformed triangular fuzzy numbers.

However, there are some drawbacks in some existing similarity measures for IVIFSs, most of which get counterintuitive results in some situations and they cannot get correct classification results for dealing with the pattern recognition problems and medical diagnosis problems. For example, letting $A =< [0.20, 0.30], [0.40, 0.60] >$, $B_1 =< [0.30, 0.40], [0.40, 0.60] >$ and $B_2 =< [0.30, 0.40], [0.30, 0.50] >$ be IVIFSs, we can compute the similarity measures between A and B_i $(i = 1, 2)$ by Formulas (1), (2) and (4) (see Section 3). Obviously, we have the result $B_1 \neq B_2$ because the membership degree of B_1 is identical to that of B_2, and the non-membership degree of B_1 is not identical to that of B_2. Therefore, we should obtain $S_i(A, B_1) \neq S_i(A, B_2) (i = 1, 2)$. However, we can obtain that $S_1(A, B_1) = S_1(A, B_2) = S_2(A, B_1) = S_2(A, B_2) = 0.9$ by the Formulas (1) and (2) (for $p = 1$), which is not reasonable. Meanwhile, we can get $S_D(A, B_1) = 1$ by Formula (4), which does not satisfy the second axiom of the definition for similarity measure. Therefore, we need to develop a new similarity measure to overcome these drawbacks.

The rest of the paper is organized as follows: Section 2 reviews some necessary definitions related to IVIFS. In Section 3, some existing similarity measures are reviewed. In Section 4, a novel similarity measure is introduced. The geometric interpretation of the new similarity measure and the explanation of parameters are briefly given in Section 5. Applications in pattern recognition and medical diagnosis are presented in Section 6. The conclusions for this paper are given in the last section.

2. Preliminary

In this section, we review the basic concepts related to IVIFSs that will be used in this paper.

Definition 1 ([1])**.** *A fuzzy set A in the unverse of discourse $X = \{x_1, x_2, \ldots, x_n\}$ is defined as follows:*

$$A = \{< x, \mu_A(x) > | x \in X\},$$

where $\mu_A(x) : X \to [0,1]$ is the membership degree.

Definition 2 ([2])**.** *An intuitionistic fuzzy set A in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ is defined as follows:*

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\},$$

where $\mu_A(x) : X \to [0,1]$ and $\nu_A(x) : X \to [0,1]$ are membership and non-membership degree, respectively, such that: $0 \leq \mu_A(x) + \nu_A(x) \leq 1$.

The third parameter of intuitionistic fuzzy set A is: $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, which is known as the intuitionistic fuzzy index or the hesitation degree of whether x belongs to A or not. It is obviously seen that $0 \leq \pi_A(x) \leq 1$. If $\pi_A(x)$ is small; then, knowledge about x is more certain; if $\pi_A(x)$ is great, then knowledge about x is more uncertain.

Definition 3 ([22])**.** *An interval-valued intuitionistic fuzzy set A in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ is defined as follows:*

$$A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\} = \{< x, [\mu_A^-(x), \mu_A^+(x)], [\nu_A^-(x), \nu_A^+(x)] > | x \in X\},$$

where $\mu_A(x) \subseteq [0,1]$, $\nu_A(x) \subseteq [0,1]$, which satisfies $0 \leq \mu_A^+(x) + \nu_A^+(x) \leq 1$.

The intervals $\mu_A(x)$ and $\nu_A(x)$ denote the membership degree and non-membership degree, respectively. Furthermore, for each $x \in X$, we can compute the hesitance degree $\pi_A(x) = [\pi_A^-(x_i), \pi_A^+(x_i)] = [1 - \mu_A^+(x) - \nu_A^+(x), 1 - \mu_A^-(x) - \nu_A^-(x)]$.

Definition 4 ([29])**.** *For every two IVIFSs A and B in the universe of discourse X, we have the following relations:*

(1): $A \subseteq B$ *iff* $(\forall x \in X)\mu_A^-(x) \leq \mu_B^-(x)$ *and* $\mu_A^+(x) \leq \mu_B^+(x)$ *and* $\nu_A^-(x) \geq \nu_B^-(x)$ *and* $\nu_A^+(x) \geq \nu_B^+(x)$.
(2): $A \cup B = \langle x, [\max(\mu_A^-(x), \mu_B^-(x)), \max(\mu_A^+(x), \mu_B^+(x))], [\min(\nu_A^-(x), \nu_B^-(x)), \min(\nu_A^+(x), \nu_B^+(x))] \rangle$.
(3): $A \cap B = \langle x, [\min(\mu_A^-(x), \mu_B^-(x)), \min(\mu_A^+(x), \mu_B^+(x))], [\max(\nu_A^-(x), \nu_B^-(x)), \max(\nu_A^+(x), \nu_B^+(x))] \rangle$.
(4): $A = B$ *iff* $(\forall x \in X)\mu_A^-(x) = \mu_B^-(x)$ *and* $\mu_A^+(x) = \mu_B^+(x)$ *and* $\nu_A^-(x) = \nu_B^-(x)$ *and* $\nu_A^+(x) = \nu_B^+(x)$.
(5): $A^c = \langle x, [\nu_A^-(x), \nu_A^+(x)], [\mu_A^-(x), \mu_A^+(x)] \rangle$

Definition 5 ([18])**.** *Let A and B be interval-valued intuitionistic fuzzy sets in the unverse of discourse $X = \{x_1, x_2, \ldots, x_n\}$, a mapping $S : IVIFS(X) \times IVIFS(X) \to [0,1]$, $S(A,B)$ is called to be a similarity measure between A and B, if $S(A,B)$ satisfies the following properties:*

(S1): $0 \leq S(A,B) \leq 1$,
(S2): $S(A,B) = 1$ *if and only if* $A = B$,
(S3): $S(A,B) = S(B,A)$,
(S4): *If* $A \subseteq B \subseteq C$, *then* $S(A,C) \leq S(A,B)$, *and* $S(A,C) \leq S(B,C)$.

3. Some Existing Similarity Measures

In this section, we review some existing similarity measures.

Let $A = \{< x_i, [\mu_A^-(x_i), \mu_A^+(x_i)], [\nu_A^-(x_i), \nu_A^+(x_i)] > | x_i \in X\}$, $B = \{< x_i, [\mu_B^-(x_i), \mu_B^+(x_i)], [\nu_B^-(x_i), \nu_B^+(x_i)] > | x_i \in X\}$ be IVIFSs defined on a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$. The following Formulas (1)–(4) are similarity measures based on IVIFSs:

Xu's similarity measure([24]):

$$S_1(A,B) = 1 - \sqrt[p]{\frac{1}{4n}\sum_{i=1}^{n}(|\mu_A^-(x_i) - \mu_B^-(x_i)|^p + |\mu_A^+(x_i) - \mu_B^+(x_i)|^p + |\nu_A^-(x_i) - \nu_B^-(x_i)|^p + |\nu_A^+(x_i) - \nu_B^+(x_i)|^p)}, \quad (1)$$

$$S_2(A,B) = 1 - \sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}\max(|\mu_A^-(x_i) - \mu_B^-(x_i)|^p, |\mu_A^+(x_i) - \mu_B^+(x_i)|^p, |\nu_A^-(x_i) - \nu_B^-(x_i)|^p, |\nu_A^+(x_i) - \nu_B^+(x_i)|^p)}. \quad (2)$$

Wei's similarity measure ([25]):

$$S_W(A,B) = \frac{1}{n}\sum_{i=1}^{n}\frac{2-\min(\mu_i^-,\nu_i^-)-\min(\mu_i^+,\nu_i^+)}{2+\max(\mu_i^-,\nu_i^-)+\max(\mu_i^+,\nu_i^+)}, \tag{3}$$

where

$$\mu_i^- = \left|\mu_A^-(x_i)-\mu_B^-(x_i)\right|,\ \mu_i^+ = \left|\mu_A^+(x_i)-\mu_B^+(x_i)\right|,$$
$$\nu_i^- = \left|\nu_A^-(x_i)-\nu_B^-(x_i)\right|,\ \nu_i^+ = \left|\nu_A^+(x_i)-\nu_B^+(x_i)\right|.$$

Dhivya's similarity measure ([28]):

$$\begin{aligned} S_D(A,B) &= 1-\tfrac{1}{n}\textstyle\sum_{i=1}^{n}(\tfrac{1}{2}(|\psi_A^-(x_i)-\psi_B^-(x_i)|+|\psi_A^+(x_i)-\psi_B^+(x_i)|)\cdot(1-\tfrac{\sigma_A(x_i)+\sigma_B(x_i)}{2})+ \\ &\quad |\sigma_A(x_i)-\sigma_B(x_i)|\cdot(\tfrac{\sigma_A(x_i)+\sigma_B(x_i)}{2})), \end{aligned} \tag{4}$$

where

$$\begin{aligned} \psi_A^- &= \frac{\mu_A^-(x_i)+1-\nu_A^-(x_i)}{2},\ \psi_A^+ = \frac{\mu_A^+(x_i)+1-\nu_A^+(x_i)}{2}, \\ \psi_B^- &= \frac{\mu_B^-(x_i)+1-\nu_B^-(x_i)}{2},\ \psi_B^+ = \frac{\mu_B^+ v+1-\nu_B^+(x_i)}{2}, \\ \sigma_A(x_i) &= 1-\frac{1}{2}(\mu_A^-(x_i)+\mu_A^+(x_i)+\nu_A^-(x_i)+\nu_A^+(x_i)), \\ \sigma_B(x_i) &= 1-\frac{1}{2}(\mu_B^-(x_i)+\mu_B^+(x_i)+\nu_B^-(x_i)+\nu_B^+(x_i)). \end{aligned}$$

4. A New Similarity Measure between Interval-Valued Intuitionistic Fuzzy Sets

Definition 6. *Let A, B be IVIFSs defined in universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, and $A = \{< x_i, [\mu_A^-(x_i), \mu_A^+(x_i)], [\nu_A^-(x_i), \nu_A^+(x_i)] > | x_i \in X\}$, $B = \{< x_i, [\mu_B^-(x_i), \mu_B^+(x_i)], [\nu_B^-(x_i), \nu_B^+(x_i)] > | x_i \in X\}$. We call*

$$S^p(A,B) = 1-\left\{\begin{array}{l} \frac{1}{2n}\sum\limits_{i=1}^{n}\left|\frac{t_1[(\mu_A^-(x_i)-\mu_B^-(x_i))+(\mu_A^+(x_i)-\mu_B^+(x_i))]-[(\nu_A^-(x_i)-\nu_B^-(x_i))+(\nu_A^+(x_i)-\nu_B^+(x_i))]}{2(t_1+1)}\right|^p \\ +\left|\frac{t_2[(\nu_A^-(x_i)-\nu_B^-(x_i))+(\nu_A^+(x_i)-\nu_B^+(x_i))]-[(\mu_A^-(x_i)-\mu_B^-(x_i))+(\mu_A^+(x_i)-\mu_B^+(x_i))]}{2(t_2+1)}\right|^p \end{array}\right\}^{\frac{1}{p}} \tag{5}$$

a similarity measure between A and B. $t_1, t_2, p \in [1, +\infty)$. Here, three parameters: p is the L_p-norm and t_1, t_2 identifies the level of uncertainty.

Theorem 1. *$S^p(A,B)$ is a similarity measure between IVIFSs A and B.*

Proof. Let A, B, C be IVIFSs defined on a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$, and $A = \{< x_i, [\mu_A^-(x_i), \mu_A^+(x_i)], [\nu_A^-(x_i), \nu_A^+(x_i)] > | x_i \in X\}$, $B = \{< x_i, [\mu_B^-(x_i), \mu_B^+(x_i)], [\nu_B^-(x_i), \nu_B^+(x_i)] > | x_i \in X\}$, and $C = \{< x_i, [\mu_C^-(x_i), \mu_C^+(x_i)], [\nu_C^-(x_i), \nu_C^+(x_i)] > | x_i \in X\}$.

(1) Firstly, we know that, for arbitrary $x_i \in X$:

$$\begin{aligned} & t_1[(\mu_A^-(x_i)-\mu_B^-(x_i))+(\mu_A^+(x_i)-\mu_B^+(x_i))]-[(\nu_A^-(x_i)-\nu_B^-(x_i))+(\nu_A^+(x_i)-\nu_B^+(x_i))] \\ = \ & [t_1(\mu_A^-(x_i)-\mu_B^-(x_i))-(\nu_A^-(x_i)-\nu_B^-(x_i))]+[t_1(\mu_A^+(x_i)-\mu_B^+(x_i))-(\nu_A^+(x_i)-\nu_B^+(x_i))]. \end{aligned}$$

For $\mu_A^-(x_i), \mu_B^-(x_i), \nu_A^-(x_i), \nu_B^-(x_i) \in [0,1]$, then we have $-t_1 \leq t_1(\mu_A^-(x_i) - \mu_B^-(x_i)) \leq t_1$, $-1 \leq \nu_A^-(x_i) - \nu_B^-(x_i) \leq 1$. Thus, we obtain that

$$-(t_1+1) \leq t_1(\mu_A^-(x_i) - \mu_B^-(x_i)) - (\nu_A^-(x_i) - \nu_B^-(x_i)) \leq t_1 + 1.$$

Similarly,

$$-(t_1+1) \leq t_1(\mu_A^+(x_i) - \mu_B^+(x_i)) - (\nu_A^+(x_i) - \nu_B^+(x_i)) \leq t_1 + 1.$$

Thus,

$$0 \leq \left| \frac{t_1[(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))] - [(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))]}{2(t_1+1)} \right|^p \leq 1.$$

By the same way, we have

$$0 \leq \left| \frac{t_2[(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))] - [(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))]}{2(t_2+1)} \right|^p \leq 1.$$

Therefore,

$$0 \leq \left\{ \begin{array}{l} \frac{1}{2n} \sum\limits_{i=1}^{n} \left| \frac{t_1[(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))] - [(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))]}{2(t_1+1)} \right|^p \\ + \left| \frac{t_2[(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))] - [(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))]}{2(t_2+1)} \right|^p \end{array} \right\} \leq 1.$$

That is, $0 \leq S^p(A,B) \leq 1$.

(2) $A = B$, if and only if for arbitrary $x_i \in X$, we have $\mu_A^-(x_i) = \mu_B^-(x_i)$, $\mu_A^+(x_i) = \mu_B^+(x_i)$, $\nu_A^-(x_i) = \nu_B^-(x_i)$, $\nu_A^+(x_i) = \nu_B^+(x_i)$. It is obvious that $S^p(A,B) = 1$.

(3) For $S^p(A,B)$, we have

$$\begin{aligned} & \left| t_1[(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))] - [(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))] \right|^p \\ = \; & \left| -t_1[(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))] + [(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+ v)] \right|^p \\ = \; & \left| t_1[(\mu_B^-(x_i) - \mu_A^-(x_i)) + (\mu_B^+(x_i) - \mu_A^+(x_i))] - [(\nu_B^-(x_i) - \nu_A^-(x_i)) - (\nu_B^+(x_i) - \nu_A^+(x_i))] \right|^p. \end{aligned}$$

Similarly,

$$\begin{aligned} & \left| t_2[(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))] - [(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))] \right|^p \\ = \; & \left| -t_2[(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))] + [(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))] \right|^p \\ = \; & \left| t_2[(\nu_B^-(x_i) - \nu_A^-(x_i)) - (\nu_B^+(x_i) - \nu_A^+(x_i))] - [(\mu_B^-(x_i) - \mu_A^-(x_i)) - (\mu_B^+(x_i) - \mu_A^+(x_i))] \right|^p. \end{aligned}$$

Thus, $S^p(A,B) = S^p(B,A)$.

(4) For A, B, C be IVIFSs, the similarity measure A and B, and A and C are the following:

$$S^p(A,B) = 1 - \left\{ \begin{array}{l} \frac{1}{2n} \sum\limits_{i=1}^{n} \left| \frac{t_1[(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))] - [(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))]}{2(t_1+1)} \right|^p \\ + \left| \frac{t_2[(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))] - [(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))]}{2(t_2+1)} \right|^p \end{array} \right\}^{\frac{1}{p}},$$

$$S^p(A,C) = 1 - \left\{ \begin{array}{l} \frac{1}{2n} \sum\limits_{i=1}^{n} \left| \frac{t_1[(\mu_A^-(x_i) - \mu_C^-(x_i)) + (\mu_A^+(x_i) - \mu_C^+(x_i))] - [(\nu_A^-(x_i) - \nu_C^-(x_i)) + (\nu_A^+(x_i) - \nu_C^+(x_i))]}{2(t_1+1)} \right|^p \\ + \left| \frac{t_2[(\nu_A^-(x_i) - \nu_C^-(x_i)) + (\nu_A^+(x_i) - \nu_C^+(x_i))] - [(\mu_A^-(x_i) - \mu_C^-(x_i)) + (\mu_A^+(x_i) - \mu_C^+(x_i))]}{2(t_2+1)} \right|^p \end{array} \right\}^{\frac{1}{p}}.$$

If $A \subseteq B \subseteq C$, then $\mu_A^-(x_i) \leq \mu_B^-(x_i) \leq \mu_C^-(x_i)$, $\mu_A^+(x_i) \leq \mu_B^+(x_i) \leq \mu_C^+(x_i)$, $\nu_C^-(x_i) \leq \nu_B^-(x_i) \leq \nu_A^-(x_i)$, and $\nu_C^+(x_i) \leq \nu_B^+(x_i) \leq \nu_A^+(x_i)$. Then, we have

$$\begin{aligned}
& \left|t_1[(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))] - [(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))]\right| \\
= \; & t_1[(\mu_B^-(x_i) - \mu_A^-(x_i)) + (\mu_B^+(x_i) - \mu_A^+(x_i))] + [(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))] \\
\leq \; & t_1[(\mu_C^-(x_i) - \mu_A^-(x_i)) + (\mu_C^+(x_i) - \mu_A^+(x_i))] + [(\nu_A^-(x_i) - \nu_C^-(x_i)) + (\nu_A^+(x_i) - \nu_C^+(x_i))] \\
= \; & \left|t_1[(\mu_A^-(x_i) - \mu_C^-(x_i)) + (\mu_A^+(x_i) - \mu_C^+(x_i))] - [(\nu_A^-(x_i) - \nu_C^-(x_i)) + (\nu_A^+(x_i) - \nu_C^+(x_i))]\right|.
\end{aligned}$$

By the same reason, we have

$$\begin{aligned}
& \left|t_2[(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))] - [(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))]\right| \\
\leq \; & \left|t_2[(\nu_A^-(x_i) - \nu_C^-(x_i)) + (\nu_A^+(x_i) - \nu_C^+(x_i))] - [(\mu_A^-(x_i) - \mu_C^-(x_i)) + (\mu_A^+(x_i) - \mu_C^+(x_i))]\right|.
\end{aligned}$$

Therefore, $S^p(A,B) \geq S^p(A,C)$, and $S^p(B,C) \geq S^p(A,C)$.

In conclusion, $S^p(A,B)$ is a similarity measure between IVIFSs A and B. □

Remark 1. *If interval-valued intuitionistic fuzzy sets A and B degenerates to intuitionistic fuzzy set, i.e., $\mu_A^- = \mu_A^+$, $\nu_A^- = \nu_A^+$, and $\mu_B^- = \mu_B^+$, $\nu_B^- = \nu_B^+$, then*

$$S^p(A,B) = 1 - \left\{\frac{1}{n}\sum_{i=1}^{n}\left|\frac{t_1(\mu_A - \mu_B) - (\nu_A - \nu_B)}{2(t_1+1)}\right|^p + \left|\frac{t_2(\nu_A - \nu_B) - (\mu_A - \mu_B)}{2(t_2+1)}\right|^p\right\}^{\frac{1}{p}} \tag{6}$$

is a new similarity measure between intuitionistic fuzzy sets A and B.

Remark 2. *In the environment of IFSs, and when $t_1 = t_2 = t$, the proposed similarity measure*

$$S^p(A,B) = 1 - \left\{\frac{1}{2n(t+1)^p}\sum_{i=1}^{n}(|t(\mu_A - \mu_B) - (\nu_A - \nu_B)|^p + |t(\nu_A - \nu_B) - (\mu_A - \mu_B)|^p)\right\}^{\frac{1}{p}} \tag{7}$$

is the similarity measure between intuitionistic fuzzy sets A and B in the literature ([19]).

Example 1. *Supposing that A_i and B_i are two IVIFSs, we can compute the similarity measures between A_i and B_i by different similarity measures listed in Table 1.*

Table 1. Comparison of similarity measures in the environment of IVIFSs (interval-valued intuitionistic fuzzy set) (counter-intuitive cases are in bold type; $p = 1$ in S_1 and S_2; $p = 1, t_1 = 2, t_2 = 3$ in S^p).

	1	**2**	**3**	**4**
A_i	$<[0.20, 0.30], [0.40, 0.60]>$	$<[0.20, 0.30], [0.40, 0.60]>$	$<[0.20, 0.30], [0.30, 0.50]>$	$<[0.20, 0.30], [0.30, 0.50]>$
B_i	$<[0.30, 0.40], [0.40, 0.60]>$	$<[0.30, 0.40], [0.30, 0.50]>$	$<[0.30, 0.40], [0.40, 0.60]>$	$<[0.30, 0.40], [0.30, 0.50]>$
S_1 [24]	**0.90**	**0.90**	0.90	0.95
S_2 [24]	**0.90**	**0.90**	**0.90**	**0.90**
S_D [28]	**1.00**	0.98	0.95	0.94
S^p	0.95	0.90	0.80	0.94

In Table 1, by comparing the first column and the second column, we can find that $S_i(A_1, B_1) = S_i(A_2, B_2) (i = 1, 2)$ when $A_1 = A_2$, $B_1 \neq B_2$. Similarly, by comparing the third column and the fourth column, we can find $S_2(A_3, B_3) = S_2(A_4, B_4)$ when $A_3 = A_4$, $B_3 \neq B_4$. Therefore, we can determine that the similarity measure S_1 and S_2 is not reasonable. Meanwhile, we find that $S_D(A_1, B_1) = 1$ when $A_1 \neq B_1$, which is not satisfy the second axiom of the definition for similarity measure. Most importantly, we can observe that the proposed similarity measure S^p can overcome these drawbacks. Therefore, our novel similarity measure for IVIFSs is more reasonable than others.

5. Geometric Interpretation of the Novel Similarity Measure

In this section, we briefly interpret the proposed similarity measure and explain the functionality of parameters t_1, t_2 and p defined in the proposed similarity measure.

Let $A = <[\mu_A^-, \mu_A^+], [\nu_A^-, \nu_A^+]>$, $B = <[\mu_B^-, \mu_B^+], [\nu_B^-, \nu_B^+]>$ be interval-valued intuitionistic fuzzy numbers. We can split A into two intuitionistic fuzzy numbers, i.e., $A^- = <\mu_A^-, \nu_A^->$ and $A^+ = <\mu_A^+, \nu_A^+>$. For intuitionistic fuzzy set A^-, μ_A^- can be equal to any value in $[\mu_A^-, \mu_A^- + \pi_A^+]$ and ν_A^- can be equal to any value in $[\nu_A^-, \nu_A^- + \pi_A^+]$, where $\pi_A^+ = 1 - \mu_A^- - \nu_A^-$. Similarly, μ_A^+ can be equal to any value in $[\mu_A^+, \mu_A^+ + \pi_A^-]$ and ν_A^+ can be equal to any value in $[\nu_A^+, \nu_A^+ + \pi_A^-]$ for intuitionistic fuzzy set A^+, where $\pi_A^- = 1 - \mu_A^+ - \nu_A^+$. Then, the possible values for A^- and A^+ illustrated in Figure 1 as the two triangles. As the center of gravity, D^- and D^+ are the most informative points in the triangle A^- and A^+, respectively.

However, $<\mu_A^- + \frac{\pi_A^+}{t_1+1}, \nu_A^- + \frac{\pi_A^+}{t_2+1}>$ $(t_1, t_2 \in [1, +\infty))$ can represent any point in the triangle A^-. Especially when $t_1 = t_2 = t$, $<\mu_A^- + \frac{\pi_A^+}{t+1}, \nu_A^- + \frac{\pi_A^+}{t+1}>$ can denote the point of middle line of the triangle bevel. In the same way, $<\mu_A^+ + \frac{\pi_A^-}{t_1+1}, \nu_A^+ + \frac{\pi_A^-}{t_2+1}>$ $(t_1, t_2 \in [1, +\infty))$ represents any point in the triangle A^+.

The following is the calculation process:

Firstly, $A'^- = \left\langle \mu_A^- + \frac{\pi_A^+}{t_1+1}, \nu_A^- + \frac{\pi_A^+}{t_2+1} \right\rangle$ denotes possible points of triangle A^-. By the same token, $A'^+ = \left\langle \mu_A^+ + \frac{\pi_A^-}{t_1+1}, \nu_A^+ + \frac{\pi_A^-}{t_2+1} \right\rangle$ denotes possible points of triangle A^+. Similarly, we can obtain that $B'^- = \left\langle \mu_B^- + \frac{\pi_B^+}{t_1+1}, \nu_B^- + \frac{\pi_B^+}{t_1+1} \right\rangle$ and $B'^+ = \left\langle \mu_B^+ + \frac{\pi_B^-}{t_1+1}, \nu_B^+ + \frac{\pi_B^-}{t_1+1} \right\rangle$ denote any points in triangles B^- and B^+, respectively.

Secondly, the average of A'^- and A'^+ can be computed as follows:

$$A'' = <\mu_A'', \nu_A''> = \left\langle \frac{2 + t_1(\mu_A^- + \mu_A^+) - (\nu_A^- + \nu_A^+)}{2(t_1+1)}, \frac{2 + t_2(\mu_A^- + \mu_A^+) - (\nu_A^- + \nu_A^+)}{2(t_2+1)} \right\rangle.$$

We can also get the mean value of B'^- and B'^+:

$$B'' = <\mu_B'', \nu_B''> = \left\langle \frac{2 + t_1(\mu_B^- + \mu_B^+) - (\nu_B^- + \nu_B^+)}{2(t_1+1)}, \frac{2 + t_2(\mu_B^- + \mu_B^+) - (\nu_B^- + \nu_B^+)}{2(t_2+1)} \right\rangle.$$

The absolute difference between A'' and B'' is calculated as follows:

$$\left|\mu_A'' - \mu_B''\right| = \left|\frac{t_1[(\mu_A^- - \mu_B^-) + (\mu_A^+ - \mu_B^+)] - [(\nu_A^- - \nu_B^-) + (\nu_A^+ - \nu_B^+)]}{2(t_1+1)}\right|,$$

$$\left|\nu_A'' - \nu_B''\right| = \left|\frac{t_2[(\nu_A^- - \nu_B^-) + (\nu_A^+ - \nu_B^+)] - [(\mu_A^- - \mu_B^-) + (\mu_A^+ - \mu_B^+)]}{2(t_2+1)}\right|.$$

$\left|\mu_A'' - \mu_B''\right|$ and $\left|\nu_A'' - \nu_B''\right|$ to the power of p is equal to the following:

$$\left|\mu_A'' - \mu_B''\right|^p = \frac{\left|t_1[(\mu_A^- - \mu_B^-) + (\mu_A^+ - \mu_B^+)] - [(\nu_A^- - \nu_B^-) + (\nu_A^+ - \nu_B^+)]\right|^p}{2^p(t_1+1)^p},$$

$$\left|\nu_A'' - \nu_B''\right|^p = \frac{\left|t_2[(\nu_A^- - \nu_B^-) + (\nu_A^+ - \nu_B^+)] - [(\mu_A^- - \mu_B^-) + (\mu_A^+ - \mu_B^+)]\right|^p}{2^p(t_2+1)^p}.$$

The average value of $\left|\mu_A'' - \mu_B''\right|^p$ and $\left|\nu_A'' - \nu_A''\right|^p$ is calculated as follows:

$$\begin{aligned}
&\frac{1}{2}\left(\left|\mu_A'' - \mu_B''\right|^p + \left|\nu_A'' - \nu_A''\right|^p\right) \\
= \;& \frac{1}{2}\left|\frac{t_1[(\mu_A^- - \mu_B^-) + (\mu_A^+ - \mu_B^+)] - [(\nu_A^- - \nu_B^-) + (\nu_A^+ - \nu_B^+)]}{2(t_1+1)}\right|^p \\
&+\frac{1}{2}\left|\frac{t_2[(\nu_A^- - \nu_B^-) + (\nu_A^+ - \nu_B^+)] - [(\mu_A^- - \mu_B^-) + (\mu_A^+ - \mu_B^+)]}{2(t_2+1)}\right|^p.
\end{aligned}$$

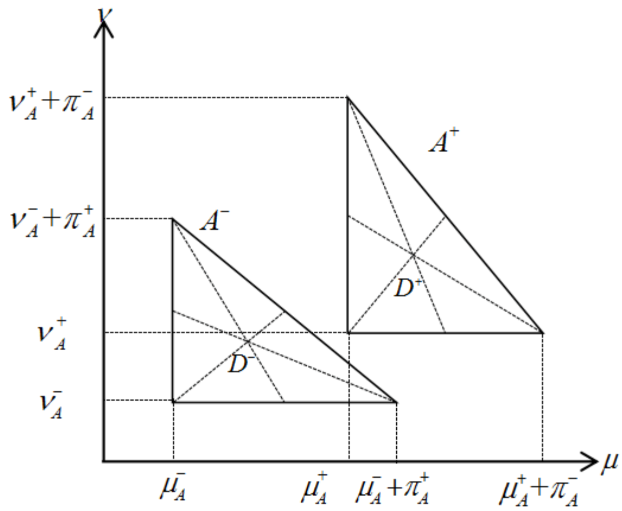

Figure 1. Possible value for A^- and A^+.

The p root of the average value of $\left|\mu_A'' - \mu_B''\right|^p$ and $\left|\nu_A'' - \nu_A''\right|^p$ is calculated as:

$$\left\{\frac{1}{2}\left(\left|\mu_A'' - \mu_B''\right|^p + \left|\nu_A'' - \nu_B''\right|^p\right)\right\}^{\frac{1}{p}} = \left\{\begin{array}{l} \left|\frac{t_1[(\mu_A^- - \mu_C^-) + (\mu_A^+ - \mu_C^+)] - [(\nu_A^- - \nu_C^-) + (\nu_A^+ - \nu_C^+)]}{2(t_1+1)}\right|^p \\ + \left|\frac{t_2[(\nu_A^- - \nu_C^-) + (\nu_A^+ - \nu_C^+)] - [(\mu_A^- - \mu_C^-) + (\mu_A^+ - \mu_C^+)]}{2(t_2+1)}\right|^p \end{array}\right\}^{\frac{1}{p}}.$$

For an interval-valued intuitionistic fuzzy set instead of interval-valued intuitionistic fuzzy number, i.e., there is more than one feature in the discourse of universe, such as $X = \{x_1, x_2, \ldots, x_n\}$:

$$S^p(A,B) = 1 - \left\{\begin{array}{l} \frac{1}{2n}\sum\limits_{i=1}^{n}\left|\frac{t_1[(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))] - [(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))]}{2(t_1+1)}\right|^p \\ + \left|\frac{t_2[(\nu_A^-(x_i) - \nu_B^-(x_i)) + (\nu_A^+(x_i) - \nu_B^+(x_i))] - [(\mu_A^-(x_i) - \mu_B^-(x_i)) + (\mu_A^+(x_i) - \mu_B^+(x_i))]}{2(t_2+1)}\right|^p \end{array}\right\}^{\frac{1}{p}}.$$

In particular, $A'^- = D^- = \left\langle \mu_A^- + \frac{1-\mu_A^- - \nu_A^-}{3}, \nu_A^- + \frac{1-\mu_A^- - \nu_A^-}{3}\right\rangle$ and $A'^+ = D^+ = \left\langle \mu_A^+ + \frac{1-\mu_A^+ - \nu_A^+}{3}, \nu_A^+ + \frac{1-\mu_A^+ - \nu_A^+}{3}\right\rangle$ when $t_1 = t_2 = 2$. Without a doubt, D^- and D^+ are the most concentrated points of information in triangle A^- and A^+, respectively; therefore, they are also the most significant points in all possible meaningful points.

6. Applications

In this section, the proposed similarity measure is used to solve the real life problems under the IVIFSs environment and obtained results have been compared with some existing similarity measures.

6.1. Pattern Recognition

6.1.1. Algorithms for Pattern Recognition

Letting $X = \{x_1, x_2, \ldots, x_n\}$ be a finite universe of discourse, there exists m patterns which are denoted by IVIFSs $A_j = \{< x_1, [\mu^-_{A_j}(x_1), \mu^+_{A_j}(x_1)], [\nu^-_{A_j}(x_1), \nu^+_{A_j}(x_1)] >, \ldots, < x_1, [\mu^-_{A_j}(x_n), \mu^+_{A_j}(x_n)], [\nu^-_{A_j}(x_n), \nu^+_{A_j}(x_n)] > | x_1, \ldots, x_n \in X\}$ $(j = 1, 2, \ldots, m)$ and there is a test sample to be classified which is denoted by an IVIFS $B = \{< x_1, [\mu^-_B(x_1), \mu^+_B(x_1)], [\nu^-_B(x_1), \nu^+_B(x_1)] >, \ldots, < x_1, [\mu^-_B(x_n), \mu^+_B(x_n)], [\nu^-_B(x_n), \nu^+_B(x_n)] > | x_1, \ldots, x_n \in X\}$. The recognition process is as follows:

Step 1. Calculate the similarity measure $S(B, A_j)$ between B and $A_j (j = 1, \ldots, m)$.

Step 2. Choose the maximum one $S(B, A_{j_0})$ from $S(B, A_j)$ $(j = 1, 2, \ldots, m)$, i.e., $S(B, A_{j_0}) = \max\limits_{1 \le j \le m} S(B, A_j)$. Then, the test sample B is classified the pattern A_{j_0}.

6.1.2. Applications for Pattern Recognition

Example 2. *Assume that there are four classes of ores $A_i (i = 1, 2, 3, 4)$ in the area developed by a coal mine company, for which the related feature information are expressed by IVIFSs, and $A_i = \{< x_1, \left[\mu^-_{A_i}(x_1), \mu^+_{A_i}(x_1)\right], \left[\nu^-_{A_i}(x_1), \nu^+_{A_i}(x_1)\right] >, \ldots, < x_4, \left[\mu^-_{A_i}(x_4), \mu^+_{A_i}(x_4)\right], \left[\nu^-_{A_i}(x_4), \nu^+_{A_i}(x_4)\right] > | x_1, x_2, x_3, x_4 \in X\}$, which are presented in Table 2. Now, there is an unknown ore B and our aim is to classify B into the four kinds of ores above.*

Table 2. Feature matrix of A_1, A_2, A_3, A_4 and B.

	Feature1	Feature2	Feature3	Feature4
A_1	$<[0.10, 0.50], [0.20, 0.30]>$	$<[0.10, 0.30], [0.00, 0.20]>$	$<[0.30, 0.50], [0.20, 0.40]>$	$<[0.20, 0.50], [0.10, 0.30]>$
A_2	$<[0.20, 0.40], [0.15, 0.35]>$	$<[0.20, 0.20], [0.05, 0.15]>$	$<[0.20, 0.60], [0.30, 0.30]>$	$<[0.30, 0.40], [0.15, 0.25]>$
A_3	$<[0.15, 0.30], [0.30, 0.40]>$	$<[0.20, 0.40], [0.50, 0.60]>$	$<[0.50, 0.60], [0.15, 0.35]>$	$<[0.25, 0.45], [0.30, 0.40]>$
A_4	$<[0.20, 0.35], [0.10, 0.65]>$	$<[0.35, 0.60], [0.05, 0.30]>$	$<[0.15, 0.30], [0.40, 0.55]>$	$<[0.15, 0.25], [0.45, 0.55]>$
B	$<[0.30, 0.40], [0.10, 0.50]>$	$<[0.10, 0.40], [0.25, 0.40]>$	$<[0.20, 0.30], [0.10, 0.35]>$	$<[0.15, 0.40], [0.20, 0.50]>$

Compute the similarity measures $S(A_i, B)$ between B and A_i. By analyzing the computed results in Table 3, we can easily see that, if S_1 is used for pattern recognition, we can obtain that $S_1(A_1, B) = S_1(A_2, B) = S_1(A_4, B) > S_1(A_3, B)$. In this way, we can not classify the sample B into a certain pattern accurately. If S_W is used for pattern recognition, we can obtain that $S_W(A_2, B) = S_W(A_4, B) > S_W(A_1, B) = S_W(A_3, B)$. In this way, we can not make sure if the sample B belongs to one of A_2 and A_4. If we use S_D for pattern recognition, we can get $S(A_3, B) = S(A_4, B) > S(A_2, B) > S(A_1, B)$. In this way, we can not classify the sample B into one of A_3 and A_4. If we use S^p for pattern recognition, we can get $S(A_1, B) > S(A_2, B) > S(A_3, B) > S(A_4, B)$. According to the principle of recognition, S_2 and S^p can get the same recognition result, i.e., the sample B can be classified into the pattern A_3. However, we can not distinguish which one is bigger between A_2 and A_4 when using S_2 to calculate the similarity measure. Therefore, we can assign the sample B to the pattern A_3.

Table 3. Pattern recognition result under different similarity measures (counter-intuitive cases are in bold type; $p = 1$ in S_1 and S_2; $p = 1, t_1 = 2, t_2 = 3$ in S^p; **N.A.** means method is not applicable).

	$S(A_1, B)$	$S(A_2, B)$	$S(A_3, B)$	$S(A_4, B)$	Classification Results
S_1 [24]	**0.87**	**0.87**	0.86	**0.87**	**N.A.**
S_2 [24]	0.75	**0.76**	0.79	**0.76**	A_3
S_W [25]	0.78	**0.79**	0.78	**0.79**	**N.A.**
S_D [28]	0.82	0.86	**0.88**	**0.88**	**N.A.**
S^p	0.82	0.81	0.88	0.75	A_3

Example 3 ([30]). *In this example, a pattern recognition example about classification of building materials is used to illustrate the proposed similarity measure. Suppose that there are four classes of building material, which are denoted by the IVIFSs* $A_j = \{< x_1, [\mu^-_{A_j}(x_1), \mu^+_{A_j}(x_1)], [\nu^-_{A_j}(x_1), \nu^+_{A_j}(x_1)] >, \ldots, < x_{12}, [\mu^-_{A_j}(x_{12}), \mu^+_{A_j}(x_{12})], [\nu^-_{A_j}(x_{12}), \nu^+_{A_j}(x_{12})] > | x_1, \ldots, x_{12} \in X\}$ $(j = 1, \ldots, 4)$ *in the feature space* $X = \{x_1, x_2, \ldots, x_{12}\}$*, and there is an unknown pattern B:*

$$\begin{aligned} A_1 = \{ & < x_1, [0.1, 0.2], [0.5, 0.6] >, < x_2, [0.1, 0.2], [0.7, 0.8] >, < x_3, [0.5, 0.6], [0.3, 0.4] >, \\ & < x_4, [0.8, 0.9], [0.0, 0.1] >, < x_5, [0.4, 0.5], [0.3, 0.4] >, < x_6, [0.0, 0.1], [0.8, 0.9] >, \\ & < x_7, [0.3, 0.4], [0.5, 0.6] >, < x_8, [1.0, 1.0], [0.0, 0.0] >, < x_9, [0.2, 0.3], [0.6, 0.7] >, \\ & < x_{10}, [0.4, 0.5], [0.4, 0.5] >, < x_{11}, [0.7, 0.8], [0.1, 0.2] >, < x_{12}, [0.4, 0.5], [0.4, 0.5] > \}, \end{aligned}$$

$$\begin{aligned} A_2 = \{ & < x_1, [0.5, 0.6], [0.3, 0.4] >, < x_2, [0.6, 0.7], [0.1, 0.2] >, < x_3, [1.0, 1.0], [0.0, 0.0] >, \\ & < x_4, [0.1, 0.2], [0.6, 0.7] >, < x_5, [0.0, 0.1], [0.8, 0.9] >, < x_6, [0.7, 0.8], [0.1, 0.2] >, \\ & < x_7, [0.5, 0.6], [0.3, 0.4] >, < x_8, [0.6, 0.7], [0.2, 0.3] >, < x_9, [1.0, 1.0], [0.0, 0.0] >, \\ & < x_{10}, [0.1, 0.2], [0.7, 0.8] >, < x_{11}, [0.0, 0.1], [0.8, 0.9] >, < x_{12}, [0.7, 0.8], [0.1, 0.2] > \}, \end{aligned}$$

$$\begin{aligned} A_3 = \{ & < x_1, [0.4, 0.5], [0.3, 0.4] >, < x_2, [0.6, 0.7], [0.2, 0.3] >, < x_3, [0.9, 1.0], [0.0, 0.0] >, \\ & < x_4, [0.0, 0.1], [0.8, 0.9] >, < x_5, [0.0, 0.1], [0.8, 0.9] >, < x_6, [0.6, 0.7], [0.2, 0.3] >, \\ & < x_7, [0.1, 0.2], [0.7, 0.8] >, < x_8, [0.2, 0.3], [0.6, 0.7] >, < x_9, [0.5, 0.6], [0.2, 0.4] >, \\ & < x_{10}, [1.0, 1.0], [0.0, 0.0] >, < x_{11}, [0.3, 0.4], [0.4, 0.5] >, < x_{12}, [0.0, 0.1], [0.8, 0.9] > \}, \end{aligned}$$

$$\begin{aligned} A_4 = \{ & < x_1, [1.0, 1.0], [0.0, 0.0] >, < x_2, [1.0, 1.0], [0.0, 0.0] >, < x_3, [0.8, 0.9], [0.0, 0.1] >, \\ & < x_4, [0.7, 0.8], [0.1, 0.2] >, < x_5, [0.0, 0.1], [0.7, 0.9] >, < x_6, [0.0, 0.1], [0.8, 0.9] >, \\ & < x_7, [0.1, 0.2], [0.7, 0.8] >, < x_8, [0.1, 0.2], [0.7, 0.8] >, < x_9, [0.4, 0.5], [0.3, 0.4] >, \\ & < x_{10}, [1.0, 1.0], [0.0, 0.0] >, < x_{11}, [0.3, 0.4], [0.4, 0.5] >, < x_{12}, [0.0, 0.1], [0.8, 0.9] > \}, \end{aligned}$$

$$\begin{aligned} B = \{ & < x_1, [0.9, 1.0], [0.0, 0.0] >, < x_2, [0.9, 1.0], [0.0, 0.0] >, < x_3, [0.7, 0.8], [0.1, 0.2] >, \\ & < x_4, [0.6, 0.7], [0.1, 0.2] >, < x_5, [0.0, 0.1], [0.8, 0.9] >, < x_6, [0.1, 0.2], [0.7, 0.8] >, \\ & < x_7, [0.1, 0.2], [0.7, 0.8] >, < x_8, [0.1, 0.2], [0.7, 0.8] >, < x_9, [0.4, 0.5], [0.3, 0.4] >, \\ & < x_{10}, [1.0, 1.0], [0.0, 0.0] >, < x_{11}, [0.3, 0.4], [0.4, 0.5] >, < x_{12}, [0.0, 0.1], [0.7, 0.9] > \}. \end{aligned}$$

Calculate the similarity measure $S(A_j, B)$ *between IVIFSs* A_j $(j = 1, 2, 3, 4)$ *and B by use of Formulas (1)–(5). It is obvious that the similarity measure in the literature ([30]) is the special case of* S_1 *and* S_2*, and the computed result is the same as ([30]). According to Table 4 and the recognition principle, the unknown pattern can be classified properly in* A_4 *by the computation of similarity measure. This conclusion coincides with that in [30].*

Table 4. Pattern recognition results under different similarity measures (counter-intuitive cases are in bold type; $p = 1$ in S_1 and S_2, $p = 1, t_1 = 2, t_2 = 3$ in S^p).

	$S(A_1,B)$	$S(A_2,B)$	$S(A_3,B)$	$S(A_4,B)$	Recognition Results
S_1 [24]	0.59	0.58	0.81	0.97	A_4
S_2 [24]	**0.53**	**0.53**	0.79	0.94	A_4
S_W [25]	0.48	0.47	0.74	0.94	A_4
S_D [28]	0.64	0.56	0.83	0.98	A_4
S^p	0.60	0.58	0.85	0.97	A_4

6.2. Applications for Medical Diagnosis

Researchers proposed a lot of methods from different points of view to deal with problems of medical diagnosis. Refs. [27,31–33] presented several ways to deal with the problems of medical diagnosis. In this section, the methods of pattern recognition are used for solving medical diagnosis problems, i.e., patients are unknown test samples, diseases are several patterns, and the symptom set is the set universe of discourse. Our aim is to classify patients in one of the illnesses, respectively.

Example 4. *Let $A = \{A_1$ (Viral fever), A_2 (Typhoid), A_3 (Pneumonia), A_4 (Stomach problem)$\}$ be a set of diagnoses and $X = \{x_1$ (Temperature), x_2 (Cough), x_3 (Headache), x_4 (Stomach pain)$\}$ be a set of symptoms. The disease–symptom matrix that is represented by IVIFSs is listed in Table 5.*

Table 5. Disease–symptom matrix.

	x_1 (Temperature)	x_2 (Cough)	x_3 (Headache)	x_4 (Stomach Pain)
A_1 (Viral fever)	$<[0.8,0.9],[0.0,0.1]>$	$<[0.7,0.8],[0.1,0.2]>$	$<[0.5,0.6],[0.2,0.3]>$	$<[0.6,0.8],[0.1,0.2]>$
A_2 (Typhoid)	$<[0.5,0.6],[0.1,0.3]>$	$<[0.8,0.9],[0.0,0.1]>$	$<[0.6,0.8],[0.1,0.2]>$	$<[0.4,0.6],[0.1,0.2]>$
A_3 (Pneumonia)	$<[0.7,0.8],[0.1,0.2]>$	$<[0.7,0.9],[0.0,0.1]>$	$<[0.4,0.6],[0.2,0.4]>$	$<[0.3,0.5],[0.2,0.4]>$
A_4 (Stomach problem)	$<[0.8,0.9],[0.0,0.1]>$	$<[0.7,0.8],[0.1,0.2]>$	$<[0.7,0.9],[0.0,0.1]>$	$<[0.8,0.9],[0.0,0.1]>$

Suppose the patient B can be represented as:

$B = \{< x_1,[0.4,0.5],[0.1,0.2] >,< x_2,[0.7,0.8],[0.1,0.2] >,< x_3,[0.9,0.9],[0.0,0.1] >,< x_4,[0.3,0.5],[0.2,0.4] >\}$.

Our aim is to classify the patient B in one of the illnesses A_1, A_2, A_3 and A_4. Then, we can have the following results in the environment of IVIFSs, which are listed in Table 6.

Table 6. Computed results under different similarity measures (counter-intuitive cases are in bold type; $p = 1$ in S_1 and S_2; $p = 1, t_1 = 2, t_2 = 3$ in S^p).

	$S(A_1,B)$	$S(A_2,B)$	$S(A_3,B)$	$S(A_4,B)$	Recognition Result
S_1 [24]	0.81	0.89	0.86	0.84	A_2
S_2 [24]	**0.73**	0.80	0.78	**0.73**	A_2
S_W [25]	0.82	0.80	0.79	0.77	A_2
S_D [28]	0.82	0.91	0.86	0.84	A_2
S^p	0.83	0.89	0.87	0.85	A_2

Considering the recognition principle of the maximum similarity degree for the IVIFSs, we can obtain the consequence that the similarity measure between A_2 and B is the largest one. However, the similarity measures S_2 could not distinguish which one is bigger between A_1 and A_4. Thus, we can classify the patient B to illness A_2 due to the recognition principle. Therefore, we can diagnose that the patient's disease is typhoid.

7. Conclusions

In this paper, a novel similarity measure for IVIFSs is proposed, which is obtained by splitting an IVIFS into two IFSs and computing the average value of the p power of any points in two triangles composed of the two intuitionistic fuzzy sets. Its superiority is presented by comparing the developed similarity measure with some existing similarity measures. Thus, we can use the similarity measure to deal with the problems with vagueness and uncertainty. For example, pattern recognition, medical diagnosis, game theory and so on.

In fact, we can choose different values of the three parameters (t_1, t_2 and p in Formula (5)) when facing different problems. However, there are some difficulties when choosing the value of parameters. This is also a problem to be solved in the future.

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