Annals of **F**uzzy **M**athematics and **I**nformatics
Volume x, No. x, (Month 201y), pp. 1–xx
ISSN: 2093–9310 (print version)
ISSN: 2287–6235 (electronic version)
http://www.afmi.or.kr

Octahedron sets

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Received 19 July 2019; Revised 25 August 2019; Accepted 3 December 2019

Abstract. We define an octahedron set and introduce the notions of internal octahedron sets and external octahedron sets and study some related properties and give some examples. Also, we define Type i-order, Type i-intersection, Type i-union ($i = 1,\ 2,\ 3,\ 4$) and study some of their properties. Moreover, we define an octahedron point and deal with the characterizations of Type i-union (Type i-intersection). Also, we introduce the level set of an octahedron set and obtain one property. Finally, we define the image and preimage of an octahedron set under a mapping and investigate some of their properties.

2010 AMS Classification: 03E72

Keywords: Octahedron set, Internal (external) octahedron set, i-union, i-intersection, Octahedron point, Level set, Image and preimage of octahedron set.

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1. Introduction

To express the real world as it is, numerous mathematicians have been trying to find a mathematical expression of its uncertainty a long time ago. Zadeh [20] (1965) introduced the concept of fuzzy sets as the generalization of ordinary sets. After that time, Zadeh [21] (1975), Pawlak [18] (1982), Atanassov [1] (1983), Atanassov and Gargov [2] (1989), Gau and Buchrer [6] (1993), Coker [3] (1996), Smarandache [19] (1998) and Molodtsov [16] (1999) introduced the concept of interval-valued fuzzy sets, rough sets, intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, vague sets, intuitionistic sets, neutrosophic sets and soft sets, in turn in order to solve various real-life problems. We can see from the literatures that these concepts have been studied in various fields of mathematics, engineering, medicine, and social sciences, etc. Recently, Jun et al. [9] defined a cubic set as a pair of an interval-valued fuzzy set and a fuzzy set, and investigated some of its properties. Smarandache et al. [10] extended the concept of cubic sets to neutrosophic sets and studied some of its properties. Also, Jun et al. [11] introduced the concept of cubic interval-valued intuitionistic fuzzy sets and applied it to BCK/BCI-algebras. Moreover, Jun [8] introduce the notion of cubic intuitionistic set (called an interval-valued intuitionistic cubic set by Kim et al. [14]) and applied it to the cubic intuitionistic set. After then, Kaur and Garg [12, 13] applied it to decision-making and studied cubic intuitionistic fuzzy aggregation operators.

In order to reduce the loss of information in solving the problem of uncertainty, it is considered that the concept of a study should consider the interval-valued fuzzy set, the intuitionistic fuzzy set, and the fuzzy set simultaneously. Then in this paper, by using an interval-valued fuzzy set, an intuitionistic fuzzy set and a fuzzy set, we introduce a new notion, called an (internal, external) octahedron set, and study some properties and give some examples. Also, we define Type i-order, Type i-intersection, Type i-union ($i = 1,\ 2,\ 3,\ 4$) and investigate some of their properties. Moreover, we define an octahedron point and deal with the characterizations of Type i-union (Type i-intersection). Also, we introduce the level set of an octahedron set and obtain one property. Finally, we define the image and preimage of an octahedron set under a mapping and study some of their properties.

2. Preliminaries

In this section, we list some basic definitions needed in the next sections.

Let $I \oplus I = \{\bar{a} = (a^{\in}, a^{\notin}) \in I \times I : a^{\in} + a^{\notin} \leq 1\}$, where $I = [0, 1]$. Then each member $\bar{a}$ of $I \oplus I$ is called an intuitionistic point or intuitionistic number. In particuar, we denote $(0, 1)$ and $(1, 0)$ as $\bar{0}$ and $\bar{1}$, respectively. We define relations $\leq$ and $=$ on $I \oplus I$ as follows (See [5]):

$$(\forall \bar{a},\ \bar{b} \in I \oplus I)(\bar{a} \leq \bar{b} \Longleftrightarrow a^{\in} \leq b^{\in} \text{ and } a^{\notin} \geq b^{\notin}),$$
$$(\forall \bar{a},\ \bar{b} \in I \oplus I)(\bar{a} = \bar{b} \Longleftrightarrow a^{\in} = b^{\in} \text{ and } a^{\notin} = b^{\notin}).$$

For each $\bar{a} \in I \oplus I$, the complement of $\bar{a}$, denoted by $\bar{a}^c$, is defined as follows:

$$\bar{a}^c = (a^{\notin}, a^{\in}).$$

For any $(\bar{a}_j)_{j \in J} \subset I \oplus I$, its infimum $\bigwedge_{j \in J} \bar{a}_j$ and supremum $\bigvee_{j \in J} \bar{a}_j$ are defined as follows:

$$\bigwedge_{j \in J} \bar{a}_j = (\bigwedge_{j \in J} a_j^{\in}, \bigvee_{j \in J} a_j^{\notin}),$$
$$\bigvee_{j \in J} \bar{a}_j = (\bigvee_{j \in J} a_j^{\in}, \bigwedge_{j \in J} a_j^{\notin}).$$

From Theorem 2.1 in [5], we can see that $(I \oplus I, \leq)$ is a complete distributive lattice with the greatest element $\bar{1}$ and the least element $\bar{0}$ satisfying De Morgan's laws.

Definition 2.1 ([1])**.** For a nonempty set X, a mapping $A : X \to I \oplus I$ is called an intuitionistic fuzzy set (briefly, IF set) in X, where for each $x \in X$, $A(x) = (A^{\in}(x), A^{\notin}(x))$, and $A^{\in}(x)$ and $A^{\notin}(x)$ represent the degree of membership and the degree of nonmembership of an element x to A, respectively. Let $(I \oplus I)^X$ denote the set of all IF sets in X and for each $A \in [I]^X$, we write $A = (A^{\in}, A^{\notin})$. In particular,

$\bar{\mathbf{0}}$ and $\bar{\mathbf{1}}$ denote the IF empty set and the IF whole set in X defined by, respectively: for each $x \in X$,

$$\bar{\mathbf{0}}(x) = \bar{0} \text{ and } \bar{\mathbf{1}}(x) = \bar{1}.$$

We define relations $\subset$ and $=$ on $(I \oplus I)^X$ as follows:

$$(\forall\ A,\ B \in (I \oplus I)^X)(A \subset B \Longleftrightarrow (x \in X)(A(x) \leq B(x)),$$

$$(\forall\ A,\ B \in (I \oplus I)^X)(A = B \Longleftrightarrow (x \in X)(A(x) = B(x)).$$

For each $A \in (I \oplus I)^X$, the complement of A, denoted by A^c, is defined as follows: For each $x \in X$,

$$A^c(x) = (A^{\notin}(x), A^{\in}(x)).$$

For each $A \in (I \oplus I)^X$, $[\ ]A$ and $\diamond A$ are IF sets in X defined as follows: For each $x \in X$,

$$[\ ]A(x) = (A^{\in}(x), 1 - A^{\in}(x)) \text{ and } \diamond A(x) = (1 - A^{\notin}(x), A^{\notin}(x)).$$

For any $(A_j)_{j \in J} \subset (I \oplus I)^X$, its intersection $\bigcap_{j \in J} A_j$ and union $\bigcup_{j \in J} A_j$ are defined, respectively as follows: For each $x \in X$,

$$(\bigcap_{j \in J} A_j)(x) = ((\bigcap_{j \in J} A_j^{\in})(x), (\bigcap_{j \in J} A_j^{\notin})(x)) = (\bigwedge_{j \in J} A_j^{\in}(x), \bigvee_{j \in J} A_j^{\notin}(x)),$$

$$(\bigcup_{j \in J} A_j)(x) = ((\bigcup_{j \in J} A_j^{\in})(x), (\bigcup_{j \in J} A_j^{\notin})(x)) = (\bigvee_{j \in J} A_j^{\in}(x), \bigwedge_{j \in J} A_j^{\notin}(x)).$$

The set of all closed subintervals of I is denoted by $[I]$, and members of $[I]$ are called interval numbers and are denoted by $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$, etc., where $\widetilde{a} = [a^-, a^+]$ and $0 \leq a^- \leq a^+ \leq 1$. In particular, if $a^- = a^+$, then we write as $\widetilde{a} = \mathbf{a}$.

We define relations $\leq$ and $=$ on $[I]$ as follows:

$$(\forall\ \widetilde{a},\ \widetilde{b} \in [I])(\widetilde{a} \leq \widetilde{b} \Longleftrightarrow a^- \leq b^- \text{ and } a^+ \leq b^+),$$

$$(\forall\ \widetilde{a},\ \widetilde{b} \in [I])(\widetilde{a} = \widetilde{b} \Longleftrightarrow a^- = b^- \text{ and } a^+ = b^+).$$

For any $\widetilde{a},\ \widetilde{b} \in [I]$, their minimum and maximum, denoted by $\widetilde{a} \wedge \widetilde{b}$ and $\widetilde{a} \vee \widetilde{b}$, are defined as follows:

$$\widetilde{a} \wedge \widetilde{b} = [a^- \wedge b^-, a^+ \wedge b^+],$$

$$\widetilde{a} \vee \widetilde{b} = [a^- \vee b^-, a^+ \vee b^+].$$

Let $(\widetilde{a_j})_{j \in J} \subset [I]$. Then its inf and sup, denoted by $\bigwedge_{j \in J} \widetilde{a_j}$ and $\bigvee_{j \in J} \widetilde{a_j}$ are defined as follows:

$$\bigwedge_{j \in J} \widetilde{a_j} = [\bigwedge_{j \in J} a_j^-, \bigwedge_{j \in J} a_j^+],$$

$$\bigvee_{j \in J} \widetilde{a_j} = [\bigvee_{j \in J} a_j^-, \bigvee_{j \in J} a_j^+].$$

For each $\widetilde{a} \in [I]$, its complement, denoted by $\widetilde{a}^c$, is defined as follows:

$$\widetilde{a}^c = [1 - a^+, 1 - a^-].$$

Definition 2.2 ([7, 21])**.** For a nonempty set X, a mapping $A : X \to [I]$ is called an interval-valued fuzzy set (briefly, an IVF set) in X. Let $[I]^X$ denote the set of all IVF sets in X. For each $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the degree of membership of an element x to A, where $A^-, A^+ \in I^X$ are called a lower fuzzy set and an upper fuzzy set in X, respectively. For each $A \in [I]^X$, we write $A = [A^-, A^+]$. In particular, $\widetilde{0}$ and $\widetilde{1}$ denote the interval-valued fuzzy empty set and the interval-valued fuzzy empty whole set in X, respectively. We define relations $\subset$ and $=$ on $[I]^X$ as follows:

$$(\forall\ A,\ B \in [I]^X)(A \subset B \Longleftrightarrow (x \in X)(A(x) \leq B(x)),$$
$$(\forall\ A,\ B \in [I]^X)(A = B \Longleftrightarrow (x \in X)(A(x) = B(x)).$$

For each $A \in [I]^X$, the complement of A, denoted by A^c, is defined as follows: For each $x \in X$,

$$A^c(x) = [1 - A^-(x), 1 + A^+(x)].$$

For any $(A_j)_{j \in J} \subset [I]^X$, its intersection $\bigcap_{j \in J} A_j$ and union $\bigcup_{j \in J} A_j$ are defined, respectively as follows:
For each $x \in X$,

$$(\bigcap_{j \in J} A_j)(x) = \bigwedge_{j \in J} A_j(x),$$
$$(\bigcup_{j \in J} A_j)(x) = \bigvee_{j \in J} A_j(x).$$

3. Octahedron sets

We will denote members of $[I] \times (I \oplus I) \times I$ as $\widetilde{\widetilde{a}} =< \widetilde{a}, \bar{a}, a >=< [a^-, a^-], (a^{\in}, a^{\notin}), a >$, $\widetilde{\widetilde{b}} =< \widetilde{b}, \bar{b}, b >=< [b^-, b^-], (b^{\in}, b^{\notin}), b >$, etc. and they will be called octahedron numbers. Furthermore, we will define the following order relations between $\widetilde{\widetilde{a}}$ and $\widetilde{\widetilde{b}}$:

(i) (Equality) $\widetilde{\widetilde{a}} = \widetilde{\widetilde{b}} \Leftrightarrow \widetilde{a} = \widetilde{b},\ \bar{a} = \bar{b},\ a = b$,

(ii) (Type 1-order) $\widetilde{\widetilde{a}} \leq_1 \widetilde{\widetilde{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \leq b^+,\ a^{\in} \leq b^{\in},\ a^{\notin} \geq b^{\notin},\ a \leq b$,

(iii) (Type 2-order) $\widetilde{\widetilde{a}} \leq_2 \widetilde{\widetilde{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \leq b^+,\ a^{\in} \leq b^{\in},\ a^{\notin} \geq b^{\notin},\ a \geq b$,

(iv) (Type 3-order) $\widetilde{\widetilde{a}} \leq_3 \widetilde{\widetilde{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \geq b^+,\ a^{\in} \geq b^{\in},\ a^{\notin} \leq b^{\notin},\ a \leq b$,

(v) (Type 4-order) $\widetilde{\widetilde{a}} \leq_4 \widetilde{\widetilde{b}} \Leftrightarrow a^- \leq b^-,\ a^+ \leq b^+,\ a^{\in} \geq b^{\in},\ a^{\notin} \leq b^{\notin},\ a \geq b$.

Definition 3.1. Let X be a nonempty set and let $\mathbf{A} = [A^-, A^+] \in [I]^X,\ A = (A^{\in}, A^{\notin}) \in (I \oplus I)^X,\ \lambda \in I^X$. Then the triple $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ is called an octahedron set in X. In fact, $\mathcal{A} : X \to [I] \times (I \oplus I) \times I$ is a mapping.

We can consider following special octahedron sets in X:

$$\left\langle \widetilde{0}, \bar{\mathbf{0}}, 0 \right\rangle = \ddot{0},$$
$$\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle, \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle,$$
$$\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle,$$
$$\left\langle \widetilde{1}, \bar{\mathbf{1}}, 1 \right\rangle = \ddot{1}.$$

In this case, $\ddot{0}$ (resp. $\ddot{1}$) will be called an octahedron empty set (resp. octahedron whole set) in X. We will denote the set of all octahedron sets as $O(X)$.

It is obvious that for each $A \in 2^X$, $A = \langle [\chi_A, \chi_A], (\chi_A, \chi_{A^c}), \chi_A \rangle \in O(X)$ and then $2^X \subset O(X)$, where 2^X denotes the set of all subsets of X and χ_A denotes the characteristic function of A.

Example 3.2. (1) Let $X = \{a, b, c\}$ be a set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle : X \to [I] \times (I \oplus I) \times I$ be the mapping given by:

$$\mathcal{A}(a) = \langle [0.3, 0.6], (0.7, 0.2), 0.5 \rangle,$$
$$\mathcal{A}(b) = \langle [0.2, 0.4], (0.6, 0.3), 0.7 \rangle,$$
$$\mathcal{A}(c) = \langle [0.4, 0.7], (0.5, 0.4), 0.3 \rangle.$$

Then we can easily see that $\mathcal{A}$ is an octahedron set in X.

(2) Let $X = I$ and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle : X \to [I] \times (I \oplus I) \times I$ be the mapping defined as follows: for each $x \in X$,

$$\mathcal{A}(x) = \left\langle [\frac{x}{4}, \frac{1+x}{2}], (\frac{x}{3}, \frac{1+x}{5}), x \right\rangle.$$

Then we can easily calculate that $\mathcal{A}$ is an octahedron set in X.

(3) Let $\mathbf{A} = [A^-, A^+] \in [I]^X$. Then clearly,

$$\langle \mathbf{A}, \bar{\mathbf{0}}, 0 \rangle \text{ (resp. } \langle \mathbf{A}, \bar{\mathbf{1}}, 0 \rangle, \langle \mathbf{A}, \bar{\mathbf{0}}, 1 \rangle, \langle \mathbf{A}, \bar{\mathbf{1}}, 1 \rangle)$$

is an octahedron set in X. In this case, we will denote

$$\langle \mathbf{A}, \bar{\mathbf{0}}, 0 \rangle \text{ (resp. } \langle \mathbf{A}, \bar{\mathbf{1}}, 0 \rangle, \langle \mathbf{A}, \bar{\mathbf{0}}, 1 \rangle, \langle \mathbf{A}, \bar{\mathbf{1}}, 1 \rangle)$$

as $\mathcal{O}_{\bar{\mathbf{0}},0}$ (resp. $\mathcal{O}_{\bar{\mathbf{1}},0}$, $\mathcal{O}_{\bar{\mathbf{0}},1}$, $\mathcal{O}_{\bar{\mathbf{1}},1}$).

Now let us $A : X \to I \oplus I$ and $\lambda : X \to I$ be the mappings defined as follows, respectively:
For each $x \in X$,

$$A(x) = (A^{\in}(x), A^{\notin}(x)) = (A^-(x), 1 - A^+(x)),$$
$$\lambda(x) = \frac{A^-(x) + A^+(x)}{2}.$$

Then we can easily see that $\langle \mathbf{A}, A, \lambda \rangle$ is an octahedron set in X. In this case, $< \mathbf{A}, A, \lambda >$ will be called the octahedron set in X induced by $\mathbf{A}$ and will be denoted by $\mathcal{O}_{\mathbf{A}}$.

(4) Let $A = (A^{\in}, A^{\notin}) \in (I \oplus I)^X$. Then clearly

$$\left\langle \widetilde{0}, A, 0 \right\rangle \text{ (resp. } \left\langle \widetilde{1}, A, 0 \right\rangle, \left\langle \widetilde{0}, A, 1 \right\rangle, \left\langle \widetilde{1}, A, 1 \right\rangle)$$

is an octahedron set in X. In this case,

$$\left\langle \widetilde{0}, A, 0 \right\rangle \text{ (resp. } \left\langle \widetilde{1}, A, 0 \right\rangle, \left\langle \widetilde{0}, A, 1 \right\rangle, \left\langle \widetilde{1}, A, 1 \right\rangle)$$

will be denoted by $\mathcal{O}_{\widetilde{0},0}$ (resp. $\mathcal{O}_{\widetilde{1},0}$, $\mathcal{O}_{\widetilde{0},1}$, $\mathcal{O}_{\widetilde{1},1}$).

Now let us $\mathbf{A} : X \to [I]$ and $\lambda : X \to I$ be the mappings defined as follows, respectively:
For each $x \in X$,

$$\mathbf{A}(x) = [A^{\in}(x), 1 - A^{\notin}(x)],$$
$$\lambda(x) = \frac{A^{\in}(x) + 1 - A^{\notin}(x)}{2}.$$

Then clearly $\langle \mathbf{A}, A, \lambda\rangle$ is an octahedron set in X. In this case, $\langle \mathbf{A}, A, \lambda\rangle$ will be called the octahedron set in X induced by A and will be denoted by $\mathcal{O}_A$.

(5) Let $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle$ be an octahedron set in X. Then clearly $\langle \mathbf{A}, [\,]A, \lambda\rangle$ and $\langle \mathbf{A}, \diamond A, \lambda\rangle$ are octahedron sets in X.

Definition 3.3. Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu\rangle \in O(X)$. Then we can define following order relations between $\mathcal{A}$ and $\mathcal{B}$:

(i) (Equality) $\mathcal{A} = \mathcal{B} \Leftrightarrow \mathbf{A} = \mathbf{B},\ A = B,\ \lambda = \mu$,
(ii) (Type 1-order) $\mathcal{A} \subset_1 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \subset B,\ \lambda \leq \mu$,
(iii) (Type 2-order) $\mathcal{A} \subset_2 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \subset B,\ \lambda \geq \mu$,
(iv) (Type 3-order) $\mathcal{A} \subset_3 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \supset B,\ \lambda \leq \mu$,
(v) (Type 4-order) $\mathcal{A} \subset_4 \mathcal{B} \Leftrightarrow \mathbf{A} \subset \mathbf{B},\ A \supset B,\ \lambda \geq \mu$.

Definition 3.4. Let X be a nonempty set and let $(\mathcal{A}_j)_{j\in J} = (\langle \mathbf{A}_j, A_j, \lambda_j\rangle)_{j\in J}$ be a family of octahedron sets in X. Then the Type i-union $\cup^i$ and Type i-intersection $\cap^i$ of $(\mathcal{A}_j)_{j\in J}$, $(i = 1,\ 2,\ ,3, 4)$, are defined as follows, respectively:

(i) (Type i-union) $\bigcup^1_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \mathbf{A}_j, \bigcup_{j\in J} A_j, \bigcup_{j\in J} \lambda_j \right\rangle$,

$$\bigcup^2_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \mathbf{A}_j, \bigcup_{j\in J} A_j, \bigcap_{j\in J} \lambda_j \right\rangle,$$
$$\bigcup^3_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \mathbf{A}_j, \bigcap_{j\in J} A_j, \bigcup_{j\in J} \lambda_j \right\rangle,$$
$$\bigcup^4_{j\in J} \mathcal{A}_j = \left\langle \bigcup_{j\in J} \mathbf{A}_j, \bigcap_{j\in J} A_j, \bigcap_{j\in J} \lambda_j \right\rangle,$$

(ii) (Type i-intersection) $\bigcap^1_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \mathbf{A}_j, \bigcap_{j\in J} A_j, \bigcap_{j\in J} \lambda_j \right\rangle$,

$$\bigcap^2_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \mathbf{A}_j, \bigcap_{j\in J} A_j, \bigcup_{j\in J} \lambda_j \right\rangle,$$
$$\bigcap^3_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \mathbf{A}_j, \bigcup_{j\in J} A_j, \bigcap_{j\in J} \lambda_j \right\rangle,$$
$$\bigcap^4_{j\in J} \mathcal{A}_j = \left\langle \bigcap_{j\in J} \mathbf{A}_j, \bigcup_{j\in J} A_j, \bigcup_{j\in J} \lambda_j \right\rangle.$$

The followings are the immediate results of Definitions 3.3 and 3.4.

Proposition 3.5. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda_{\mathcal{A}}\rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \lambda_{\mathcal{B}}\rangle$, $\mathcal{C} = \langle \mathbf{C}, C, \lambda_{\mathcal{C}}\rangle$ and $\mathcal{D} = \langle \mathbf{D}, D, \lambda_{\mathcal{D}}\rangle$ be octahedron sets in X. Then for each $i = 1,\ 2,\ 3,\ 4$,*

(1) *if $\mathcal{A} \subset_i \mathcal{B}$ and $\mathcal{B} \subset_i \mathcal{C}$, then $\mathcal{A} \subset_i \mathcal{C}$,*
(2) *if $\mathcal{A} \subset_i \mathcal{B}$ and $\mathcal{A} \subset_i \mathcal{C}$, then $\mathcal{A} \subset_i \mathcal{B} \cap_i \mathcal{C}$,*
(3) *if $\mathcal{A} \subset_i \mathcal{B}$ and $\mathcal{C} \subset_i \mathcal{B}$, then $\mathcal{A} \cup_i \mathcal{C} \subset_i \mathcal{B}$, .*
(4) *if $\mathcal{A} \subset_i \mathcal{B}$ and $\mathcal{C} \subset_i \mathcal{D}$, then $\mathcal{A} \cup_i \mathcal{C} \subset_i \mathcal{B} \cup_i \mathcal{D}$ and $\mathcal{A} \cap_i \mathcal{C} \subset_i \mathcal{B} \cap_i \mathcal{D}$.*

Definition 3.6. Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle$ be an octahedron set in X. Then the complement $\mathcal{A}^c$, operators $[\,]$ and $\diamond$ of $\mathcal{A}$ are defined as follows, respectively: for each $x \in X$,

(i) $\mathcal{A}^c = \langle \mathbf{A}^c, A^c, \lambda^c\rangle$,
(ii) $[\,]\mathcal{A} = \langle \mathbf{A}, [\,]A, \lambda\rangle$,
(iii) $\diamond\mathcal{A} = \langle \mathbf{A}, \diamond A, \lambda\rangle$.

From Definition 3.6 (i), we can easily see that the followings hold:

$\ddot{0}^c = \ddot{1},\ \ddot{1}^c = \ddot{0}$,

$\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle^c = \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle,\ \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle^c = \left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle$,

$$\left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle^c = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle, \ \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle^c = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle,$$
$$\left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle^c = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle, \ \left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle^c = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle,$$
$$\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle^c = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle, \ \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle^c = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle,$$
$$\left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle^c = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle, \ \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle^c = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle,$$
$$\left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle^c = \left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle, \ \left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle^c = \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle.$$

The followings are the immediate results of Definitions 3.3 and 3.6 (i).

Proposition 3.7. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ and $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle$ be octahedron sets in X. If $\mathcal{A} \subset_i \mathcal{B}$, then $\mathcal{B}^c \subset_i \mathcal{A}^c$, for each $i = 1,\ 2,\ 3,\ 4$.*

Proposition 3.8. *Let $\mathcal{A} \in O(X)$ and let $(\mathcal{A}_j)_{j \in J} \subset O(X)$. Then*
(1) $(\mathcal{A}^c)^c = \mathcal{A}$,
(2) *for each* $i = 1,\ 2,\ 3,\ 4$,

$$(\bigcup_{j\in J}^{i} \mathcal{A}_j)^c = \bigcap_{j\in J}^{i} \mathcal{A}_j^c, \ (\bigcap_{j\in J}^{i} \mathcal{A}_j)^c = \bigcup_{j\in J}^{i} \mathcal{A}_j^c.$$

Remark 3.9. For any $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in O(X)$ and each $i = \{1,\ 2,\ 3,\ 4\}$, the followings do not hold, in general:

$$\mathcal{A} \cup^i \mathcal{A}^c = \ddot{1} \text{ and } \mathcal{A} \cap^i \mathcal{A}^c = \ddot{0}.$$

Example 3.10. Consider the IVF set $\mathbf{A}$, the IF set A and the fuzzy set λ in a nonempty set X given by respectively: for each $x \in X$,

$$\mathbf{A}(x) = [0.5, 0.5],\ A(x) = (0.5, 0.5) \text{ and } \lambda = 0.5.$$

Then, clearly, $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ is an octahedron set in X. Moreover,

$$(\mathbf{A} \cup^i \mathbf{A}^c)(x) = \langle [0.5, 0.5], (0.5, 0.5), 0.5 \rangle \neq \ddot{1}(x)$$

and

$$(\mathbf{A} \cap^i \mathbf{A}^c)(x) = \langle [0.5, 0.5], (0.5, 0.5), 0.5 \rangle \neq \ddot{0}(x).$$

Thus $\mathcal{A} \cup^i \mathcal{A}^c \neq \ddot{1}$ and $\mathcal{A} \cap^i \mathcal{A}^c \neq \ddot{0}$.

The followings are the immediate results of Definition 3.6.

Proposition 3.11. *Let X be a nonempty set, $\mathcal{A} = \langle \mathbf{A}, A, \lambda_{\mathcal{A}} \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \lambda_{\mathcal{B}} \rangle$, $\mathcal{C} = \langle \mathbf{C}, C, \lambda_{\mathcal{C}} \rangle \in O(X)$ and $(\mathcal{A}_j)_{j \in J} = (\langle \mathbf{A}_j, A_j, \lambda_j \rangle)_{j \in J} \subset O(X)$. Then for each $i = 1,\ 2,\ 3,\ 4$;*
(1) $\mathcal{A} \cup^i \mathcal{A} = \mathcal{A}$, $\ \mathcal{A} \cap^i \mathcal{A} = \mathcal{A}$,
(2) $\mathcal{A} \cup^i \mathcal{B} = \mathcal{B} \cup^i \mathcal{A}$, $\ \mathcal{A} \cap^i \mathcal{B} = \mathcal{B} \cap^i \mathcal{A}$,
(3) $\mathcal{A} \cup^i (\mathcal{B} \cup^i \mathcal{C}) = (\mathcal{A} \cup^i \mathcal{B}) \cup^i \mathcal{C}$, $\ \mathcal{A} \cap^i (\mathcal{B} \cap^i \mathcal{C}) = (\mathcal{A} \cap^i \mathcal{B}) \cap^i \mathcal{C}$,
(4) $\mathcal{A} \cup^i (\mathcal{B} \cap^i \mathcal{C}) = (\mathcal{A} \cup^i \mathcal{B}) \cap^i (\mathcal{A} \cup^i \mathcal{C})$, $\ \mathcal{A} \cap^i (\mathcal{B} \cup^i \mathcal{C}) = (\mathcal{A} \cap^i \mathcal{B}) \cup^i (\mathcal{A} \cap^i \mathcal{C})$,
(4)$'$ $\mathcal{A} \cup^i (\bigcap^i_{j \in J} \mathcal{A}_j) = \bigcap^i_{j \in J} (\mathcal{A} \cup^i \mathcal{A}_j)$, $\ \mathcal{A} \cap^i (\bigcup^i_{j \in J} \mathcal{A}_j) = \bigcup^i_{j \in J} (\mathcal{A} \cap^i \mathcal{A}_j)$.

From the above Propositions 3.8 and 3.11, we can see that $(O(X), \cup^i, \cap^i, \ddot{0}, \ddot{1})$ forms a Boolean algebra except the property of Remark 3.9.

From Definition 3.6, we have the similar results to Theorem 2 in [1].

Proposition 3.12. *Let $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle$ be an octahedron set in a nonempty set X. Then*
(1) $([\,]\mathcal{A}^c)^c = \diamond\mathcal{A}$, $(\diamond\mathcal{A}^c)^c = [\,]\mathcal{A}$,
(2) $[\,]\mathcal{A} \subset_i \mathcal{A} \subset_i \diamond\mathcal{A}$ *for each* $i = 1,\ 2,\ 3,\ 4$,
(3) $[\,][\,]\mathcal{A} = [\,]\mathcal{A}$,
(4) $[\,]\diamond\mathcal{A} = \diamond\mathcal{A}$,
(5) $\diamond[\,]\mathcal{A} = [\,]\mathcal{A}$,
(6) $\diamond\diamond\mathcal{A} = \diamond\mathcal{A}$.

Also, we have the similar results to Theorems 3 and 4 in [1].

Proposition 3.13. *Let $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle$ and $\mathcal{B} = \langle \mathbf{B}, B, \mu\rangle$ be octahedron sets in a nonempty set X and let $i = 1,\ 2,\ 3,\ 4$. Then*
(1) $[\,](\mathcal{A} \cup^i \mathcal{B}) = [\,]\mathcal{A} \cup^i [\,]\mathcal{B}$, $[\,](\mathcal{A} \cap^i \mathcal{B}) = [\,]\mathcal{A} \cap^i [\,]\mathcal{B}$,
(2) $\diamond(\mathcal{A} \cup^i \mathcal{B}) = \diamond\mathcal{A} \cup^i \diamond\mathcal{B}$, $\diamond(\mathcal{A} \cap^i \mathcal{B}) = \diamond\mathcal{A} \cap^i \diamond\mathcal{B}$.

Definition 3.14. Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle \in O(X)$. Then $\mathcal{A}$ is called:
(i) an $\in$-internal octahedron set (briefly, $\in$-IOS) in X, if for each $x \in X$,

$$A^{\in}(x),\ \lambda(x) \in \mathbf{A}(x) = [A^-(x), A^+(x)],$$

(ii) a $\notin$-internal octahedron set (briefly, $\notin$-IOS) in X, if for each $x \in X$,

$$1 - A^{\notin}(x),\ \lambda(x) \in \mathbf{A}(x) = [A^-(x), A^+(x)],$$

(iii) an internal octahedron set (briefly, IOS) in X, if it is both an $\in$-IOS and a $\notin$-IOS in X,
(iv) an $\in$-external octahedron set (briefly, $\in$-EOS) in X, if for each $x \in X$,

$$A^{\in}(x),\ \lambda(x) \notin (A^-(x), A^+(x)),$$

(v) a $\notin$-external octahedron set (briefly, $\notin$-EOS) in X, if for each $x \in X$,

$$1 - A^{\notin}(x),\ \lambda(x) \notin (A^-(x), A^+(x)),$$

(iv) an external octahedron set (briefly, EOS) in X, if it is both an $\in$-EOS and a $\notin$-EOS in X.

Example 3.15. (1) Let $\mathcal{A}_1 = \langle \mathbf{A}_1, A_1, \lambda_1\rangle$ be the octahedron set in I given by: for each $x \in I$,

$$\mathcal{A}_1(x) = \left\langle [\frac{x}{4}, \frac{1+x}{2}], (\frac{x}{3}, \frac{1+x}{5}), \frac{x}{2} \right\rangle.$$

Then we can easily calculate that $A_1^{\in}(x),\ \lambda_1(x) \in \mathbf{A}_1(x)$, for each $x \in I$ but $A_1^{\notin}(x) \notin (A_1^-(x), A_1^+(x))$, for each $x \in I$ such that $x > \frac{3}{7}$. Thus $\mathcal{A}_1$ is an $\in$-IOS but not a $\notin$-IOS in X.
(2) Let $\mathcal{A}_2 = \langle \mathbf{A}_2, A_2, \lambda_2\rangle$ be the octahedron set in I given by: for each $x \in I$,

$$\mathcal{A}_2(x) = \left\langle ([\frac{x}{4}, \frac{x}{2}], (\frac{1+x}{5}, 1 - \frac{x}{3}), \frac{x}{3} \right\rangle.$$

Then we can easily see that $\mathcal{A}_2$ is a $\notin$-IOS in X.
(3) Let $\mathcal{A}_3 = \langle \mathbf{A}_3, A_3, \lambda_3\rangle$ be the octahedron set in I given by: for each $x \in I$,

$$\mathcal{A}_3(x) = \left\langle [\frac{x}{4}, \frac{x}{2}], (\frac{x}{4}, 1 - \frac{x}{3}), \frac{x}{3} \right\rangle.$$

Then we can easily calculate that $\mathcal{A}_3$ is an IOS in X.

(4) Let $\mathcal{A}_4 = \langle \mathbf{A}_4, A_4, \lambda_4 \rangle$ be the octahedron set in I given by: for each $x \in I$,

$$\mathcal{A}_4(x) = \left\langle [\frac{1+x}{4}, \frac{1+x}{2}], (\frac{x}{4}, \frac{1+x}{3}), \frac{2+x}{3} \right\rangle.$$

Then we can easily see that $\mathcal{A}_4$ is an $\in$-EOS in X.

(5) Let $\mathcal{A}_5 = \langle \mathbf{A}_5, A_5, \lambda_5 \rangle$ be the octahedron set in I given by: for each $x \in I$,

$$\mathcal{A}_5(x) = \left\langle [\frac{1+x}{4}, \frac{1+x}{2}], (\frac{1+x}{3}, 1 - \frac{\frac{1}{2}+x}{4}), \frac{2+x}{3} \right\rangle.$$

Then we can easily calculate that $\mathcal{A}_5$ is a $\notin$-EOS in X.

(6) Let $\mathcal{A}_6 = \langle \mathbf{A}_6, A_6, \lambda_6 \rangle$ be the octahedron set in I given by: for each $x \in I$,

$$\mathcal{A}_6(x) = \left\langle [\frac{1+x}{4}, \frac{1+x}{2}], (\frac{1+x}{5}, 1 - \frac{\frac{1}{2}+x}{4}), \frac{2+x}{3} \right\rangle.$$

Then we can easily see that $\mathcal{A}_6$ is an EOS in X.

The followings are the immediate results of Definition 3.14.

Proposition 3.16. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in O(X)$. If $\mathcal{A}$ is not external, then there is $x \in X$ such that $A^{\in}(x) \in \mathbf{A}(x)$ or $1 - A^{\notin}(x) \in \mathbf{A}(x)$ and $\lambda(x) \in \mathbf{A}(x)$.*

Proposition 3.17. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in O(X)$. If $\mathcal{A}$ is both internal and external, then for each $x \in X$,*

$$A^{\in}(x),\ 1 - A^{\notin}(x),\ \lambda(x) \in U(\mathbf{A}) \cup L(\mathbf{A}),$$

where $U(\mathbf{A}) = \{A^+(x) : x \in X\}$ and $L(\mathbf{A}) = \{A^-(x) : x \in X\}$.

The following is the immediate result of Definitions 3.6 (i) and 3.14.

Proposition 3.18. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in O(X)$. If $\mathcal{A}$ is internal (resp. external), then $\mathcal{A}^c$ is external (resp. internal).*

Proposition 3.19. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in O(X)$. If $\mathcal{A}$ is internal, then $[\,]\mathcal{A}$ and $\diamond\mathcal{A}$ are internal.*

Proof. The proofs are straightforward. □

Proposition 3.20. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in O(X)$.*

(1) If $\mathcal{A}$ is $\in$-external, then $[\,]\mathcal{A}$ is $\in$-external.

(2) If $\mathcal{A}$ is $\notin$-external, then $\diamond\mathcal{A}$ is $\notin$-external.

Proof. The proofs are straightforward. □

For any $\notin$-external (resp. $\in$-external) octahedron set $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ in a nonempty set X, $[\,]\mathcal{A}$ (resp. $\diamond\mathcal{A}$) need not be $\notin$-external (resp. $\in$-external) as shown in following example.

Example 3.21. (1) In Example 3.6 (5), consider the $\notin$-EOS $\mathcal{A}_5$ in I given by: for each $x \in I$,

$$\mathcal{A}_5(x) = \left\langle [\frac{1+x}{4}, \frac{1+x}{2}], (\frac{1+x}{3}, 1 - \frac{\frac{1}{2}+x}{4}), \frac{2+x}{3} \right\rangle.$$

Since $[\,]\mathcal{A}_5 = \langle \mathbf{A}_5, [\,]A_5, \lambda_5 \rangle = \langle \mathbf{A}_5, (A_5^{\in}, 1 - A_5^{\in}), \lambda_5 \rangle$, $([\,]A_5)^{\notin}(x) = 1 - A_5^{\in}(x)$. Then $([\,]A_5)^{\notin}(x) = 1 - \frac{1+x}{3}$. Thus $1 - [\,]A_5)^{\notin}(x) = \frac{1+x}{3} \in \mathbf{A}_5$. So $[\,]\mathcal{A}_5$ is not $\notin$-external.

(2) In Example 3.6 (4), consider the $\in$-EOS $\mathcal{A}_4$ in I given by: for each $x \in I$,

$$\mathcal{A}_4(x) = \left\langle [\frac{1+x}{4}, \frac{1+x}{2}], (\frac{x}{4}, \frac{1+x}{3}), \frac{2+x}{3} \right\rangle.$$

Since $\diamond\mathcal{A}_4 = \langle \mathbf{A}_4, \diamond A_4, \lambda_4 \rangle = \left\langle \mathbf{A}_5, (1 - A_4^{\notin}, A_4^{\notin}), \lambda_5 \right\rangle$, $(\diamond A_4)^{\in}(x) = 1 - A_4^{\notin}(x)$. Then $(\diamond A_4)^{\in}(x) = 1 - \frac{1+x}{3}(x) = \frac{2-x}{3}$ and $(\diamond A_4)^{\in}(x) \in \mathbf{A}_4(x)$, for each $x \in [\frac{5}{17}, \frac{1}{7}]$. Thus $\diamond\mathcal{A}_4$ is not $\in$-external.

Proposition 3.22. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in O(X)$.*

(1) *If $\mathcal{A}$ is $\in$-internal (resp. $\in$-external), then $[\,]\mathcal{A}$ is $\in$-internal (resp. $\in$-external).*

(2) *If $\mathcal{A}$ is $\notin$-internal (resp. $\notin$-external), then $\diamond\mathcal{A}$ is $\notin$-internal (resp. $\notin$-external).*

Proof. The proofs are straightforward. □

For any $\notin$-internal (resp. $\notin$-external) octahedron set $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ in a nonempty set X, $[\,]\mathcal{A}$ need not be $\notin$-internal (resp. $\notin$-external). Also for any $\in$-internal (resp. $\in$-external) octahedron set $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ in a nonempty set X, $\diamond\mathcal{A}$ need not be $\in$-internal (resp. $\in$-external) as shown in following examples.

Example 3.23. (1) In Example 3.6 (2), consider the $\notin$-IOS $\mathcal{A}_2$ in I given by: for each $x \in I$,

$$\mathcal{A}_2(x) = \left\langle [\frac{x}{4}, \frac{x}{2}], (\frac{1+x}{5}, 1 - \frac{x}{3}), \frac{x}{3} \right\rangle.$$

Then $1 - ([\,]A_2)^{\notin}(x) = 1 - (1 - \frac{1+x}{5}) = \frac{1+x}{5} \notin [\frac{x}{4}, \frac{x}{2}]$, for each $x \in [0, \frac{2}{3})$. Thus $[\,]\mathcal{A}_2$ is not $\notin$-internal.

(2) In Example 3.6 (5), consider the $\notin$-EOS $\mathcal{A}_5$ in I given by: for each $x \in I$,

$$\mathcal{A}_5(x) = \left\langle [\frac{1+x}{4}, \frac{1+x}{2}], (\frac{1+x}{3}, 1 - \frac{\frac{1}{2}+x}{4}), \frac{2+x}{3} \right\rangle.$$

Then $1 - ([\,]A_5)^{\notin}(x) = 1 - (1 - \frac{1+x}{3}) = \frac{1+x}{3} \in [\frac{1+x}{4}, \frac{1+x}{2}]$, for each $x \in I$. Thus $[\,]\mathcal{A}_2$ is not $\notin$-external.

(3) In Example 3.6 (1), consider the $\in$-IOS $\mathcal{A}_1$ in I given by: for each $x \in I$,

$$\mathcal{A}_1(x) = \left\langle [\frac{x}{4}, \frac{1+x}{2}], (\frac{x}{3}, \frac{1+x}{5}), \frac{x}{2} \right\rangle.$$

Then $(\diamond A_1)^{\in}(x) = 1 - \frac{1+x}{5} = \frac{4-x}{5} \notin [\frac{x}{4}, \frac{1+x}{2}]$, for each $x \in [0, \frac{3}{7})$. Thus $\diamond\mathcal{A}_2$ is not $\in$-internal.

(4) In Example 3.6 (4), consider the $\in$-IOS $\mathcal{A}_4$ in I given by: for each $x \in I$,

$$\mathcal{A}_4(x) = \left\langle [\frac{1+x}{4}, \frac{1+x}{2}], (\frac{x}{4}, \frac{1+x}{3}), \frac{2+x}{3} \right\rangle.$$

Then $(\diamond A_4)^{\in}(x) = 1 - \frac{1+x}{3} = \frac{2-x}{3} \notin [\frac{1+x}{4}, \frac{1+x}{2}]$, for each $x \in (\frac{1}{5}, \frac{5}{7})$. Thus $\diamond \mathcal{A}_4$ is not $\in$-internal.

The followings are the immediate results of Propositions 3.19, 3.20 and 3.22

Corollary 3.24. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in O(X)$.*

(1) If $\mathcal{A}$ is $\in$-internal (resp. $\in$-external), then $[\ ]\mathcal{A}$ is $\in$-internal (resp. $\in$-external).

(2) If $\mathcal{A}$ is $\notin$-external, then $\diamond \mathcal{A}$ is $\notin$-external.

Proposition 3.25. *Let X be a nonempty set and let $(\mathcal{A}_j)_{j \in J} = (\langle \mathbf{A}_j, A_j, \lambda_j \rangle)_{j \in J}$ be a family octahedron sets in X. If $\mathcal{A}_j$ is internal for each $j \in J$, then $\bigcup^1_{j \in J} \mathcal{A}_j$ and $\bigcap^1_{j \in J} \mathcal{A}_j$ are internal.*

Proof. The proof is straightforward. □

We can see that Type i-union and Type i-intersection ($i = 2,\ 3,\ 4$) of internal octahedron sets may not be internal octahedron sets as shown in the following examples.

Example 3.26. Consider two octahedron sets $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ and $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle$ in I defined as follows: for each $x \in I$,

$$\mathcal{A}(x) = \left\langle [\frac{1+x}{4}, \frac{1+x}{2}], (\frac{1+x}{3}, \frac{1-x}{3}), \frac{2+x}{5} \right\rangle$$

and

$$\mathcal{B} = \left\langle [\frac{x}{5}, \frac{2+x}{3}], (\frac{x}{3}, \frac{2-x}{5}), \frac{x}{4} \right\rangle.$$

Then we have the followings:

$$(\mathcal{A} \cup^2 \mathcal{B})(x) = \left\langle [\frac{1+x}{4}, \frac{2+x}{3}], (\frac{1+x}{3}, \frac{1-x}{3}), \frac{x}{4} \right\rangle, \tag{3.1}$$

$$(\mathcal{A} \cup^3 \mathcal{B})(x) = \left\langle [\frac{1+x}{4}, \frac{2+x}{3}], (\frac{x}{3}, \frac{2-x}{5}), \frac{2+x}{5} \right\rangle, \tag{3.2}$$

$$(\mathcal{A} \cup^4 \mathcal{B})(x) = \left\langle [\frac{1+x}{4}, \frac{2+x}{3}], (\frac{x}{3}, \frac{2-x}{5}), \frac{x}{4} \right\rangle, \tag{3.3}$$

$$(\mathcal{A} \cap^2 \mathcal{B})(x) = \left\langle [\frac{x}{5}, \frac{1+x}{2}], (\frac{x}{3}, \frac{2-x}{5}), \frac{2+x}{5} \right\rangle, \tag{3.4}$$

$$(\mathcal{A} \cap^3 \mathcal{B})(x) = \left\langle [\frac{x}{5}, \frac{1+x}{2}], (\frac{1+x}{3}, \frac{1-x}{3}), \frac{x}{4} \right\rangle, \tag{3.5}$$

$$(\mathcal{A} \cap^4 \mathcal{B})(x) = \left\langle [\frac{x}{5}, \frac{1+x}{2}], (\frac{1+x}{3}, \frac{1-x}{3}), \frac{2+x}{5} \right\rangle. \tag{3.6}$$

Thus we can see the followings.

In (3.1), $(\lambda \wedge \mu)(1) = \frac{1}{4} \notin [\frac{1}{2}, 1] = (\mathbf{A} \cup \mathbf{B})(1)$.
In (3.2), $(A \cap B)^{\in}(1) = \frac{1}{3} \notin [\frac{1}{2}, 1] = (\mathbf{A} \cup \mathbf{B})(1)$.
In (3.3), also $(A \cap B)^{\in}(1) = \frac{1}{3} \notin [\frac{1}{2}, 1] = (\mathbf{A} \cup \mathbf{B})(1)$.
In (3.4), $1 - (A \cap B)^{\notin}(0) = \frac{3}{5} \notin [0, \frac{1}{2}] = (\mathbf{A} \cap \mathbf{B})(0)$.
In (3.5), $1 - (A \cup B)^{\notin}(0) = \frac{2}{3} \notin [0, \frac{1}{2}] = (\mathbf{A} \cap \mathbf{B})(0)$.
In (3.6), also $1 - (A \cup B)^{\notin}(0) = \frac{2}{3} \notin [0, \frac{1}{2}] = (\mathbf{A} \cap \mathbf{B})(0)$.
So $\mathcal{A} \cup^i \mathcal{B}$ and $\mathcal{A} \cap^i \mathcal{B}$ are not internal octahedron sets in I, for $i = 2,\ 3,\ 4$.

We provide a condition for the Type i-union ($i = 2,\ 3,\ 4$) of two IOSs to be an IOS.

Proposition 3.27. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in O(X)$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are internal such that for each $x \in X$,*

$$A^-(x) \vee B^-(x) \leq (\lambda \wedge \mu)(x). \tag{3.7}$$

Then $\mathcal{A} \cup^2 \mathcal{B}$ is internal.

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ be IOSs in X satisfying the condition (3.7) and let $x \in X$. Then clearly,

$$A^-(x) \leq A^{\in}(x) \leq A^+(x),\ A^-(x) \leq 1 - A^{\notin}(x) \leq A^+(x),\ A^-(x) \leq \lambda(x) \leq A^+(x),$$

$$B^-(x) \leq B^{\in}(x) \leq B^+(x),\ B^-(x) \leq 1 - B^{\notin}(x) \leq B^+(x),\ A^-(x) \leq \mu(x) \leq A^+(x).$$

Thus

$$(\mathbf{A} \cup \mathbf{B})^-(x) \leq (A \cup B)^{\in}(x) \leq (\mathbf{A} \cup \mathbf{B})^+(x)$$

and

$$(\mathbf{A} \cup \mathbf{B})^-(x) \leq 1 - (A \cup B)^{\notin}(x) \leq (\mathbf{A} \cup \mathbf{B})^+(x).$$

By the condition (3.7), we have

$$(\mathbf{A} \cup \mathbf{B})^-(x) = A^-(x) \vee B^-(x) \leq (\lambda \wedge \mu)(x) \leq (\mathbf{A} \cup \mathbf{B})^+(x).$$

So $\mathcal{A} \cup^2 \mathcal{B}$ is internal. □

Proposition 3.28. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in O(X)$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are internal such that for each $x \in X$,*

$$A^-(x) \vee B^-(x) \leq (A \cap B)^{\in}(x),\ A^-(x) \vee B^-(x) \leq 1 - (A \cap B)^{\notin}(x). \tag{3.8}$$

Then $\mathcal{A} \cup^3 \mathcal{B}$ is internal.

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ be IOSs in X satisfying the condition (3.7) and let $x \in X$. Then clearly,

$$A^-(x) \leq A^{\in}(x) \leq A^+(x),\ A^-(x) \leq 1 - A^{\notin}(x) \leq A^+(x),\ A^-(x) \leq \lambda(x) \leq A^+(x),$$

$$B^-(x) \leq B^{\in}(x) \leq B^+(x),\ B^-(x) \leq 1 - B^{\notin}(x) \leq B^+(x),\ A^-(x) \leq \mu(x) \leq A^+(x).$$

Thus $(\mathbf{A} \cup \mathbf{B})^-(x) \leq (\lambda \vee \mu)(x) \leq (\mathbf{A} \cup \mathbf{B})^+(x)$. By the condition (3.8), we have Thus

$$(\mathbf{A} \cup \mathbf{B})^-(x) = A^-(x) \vee B^-(x) \leq (A \cap B)^{\in}(x) \leq (\mathbf{A} \cup \mathbf{B})^+(x)$$

and

$$(\mathbf{A} \cup \mathbf{B})^-(x) = A^-(x) \vee B^-(x) \leq 1 - (A \cap B)^{\notin}(x) \leq (\mathbf{A} \cup \mathbf{B})^+(x).$$

So $\mathcal{A} \cup^3 \mathcal{B}$ is internal. □

The following is the immediate result of Propositions 3.27 and 3.28.

Corollary 3.29. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in O(X)$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are IOSs satisfying the conditions (3.7) and 3.8). Then $\mathcal{A} \cup^4 \mathcal{B}$ is internal.*

We provide a condition for the Type i-intersection ($i = 2,\ 3,\ 4$) of two IOSs to be an IOS.

Proposition 3.30. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in O(X)$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are internal such that for each $x \in X$,*

$$A^+(x) \wedge B^+(x) \geq (\lambda \vee \mu)(x). \tag{3.9}$$

Then $\mathcal{A} \cap^2 \mathcal{B}$ is internal.

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ be IOSs in X satisfying the condition (3.9) and let $x \in X$. Then clearly,

$$A^-(x) \leq A^{\in}(x) \leq A^+(x),\ A^-(x) \leq 1 - A^{\notin}(x) \leq A^+(x),\ A^-(x) \leq \lambda(x) \leq A^+(x),$$

$$B^-(x) \leq B^{\in}(x) \leq B^+(x),\ B^-(x) \leq 1 - B^{\notin}(x) \leq B^+(x),\ A^-(x) \leq \mu(x) \leq A^+(x).$$

Thus

$$(\mathbf{A} \cap \mathbf{B})^-(x) \leq (A \cap B)^{\in}(x) \leq (\mathbf{A} \cap \mathbf{B})^+(x)$$

and

$$(\mathbf{A} \cap \mathbf{B})^-(x) \leq 1 - (A \cap B)^{\notin}(x) \leq (\mathbf{A} \cap \mathbf{B})^+(x).$$

By the condition (3.9), we have

$$(\mathbf{A} \cap \mathbf{B})^-(x) \leq (\lambda \vee \mu)(x) \leq A^+(x) \wedge B^+(x) = (\mathbf{A} \cap \mathbf{B})^+(x).$$

So $\mathcal{A} \cap^2 \mathcal{B}$ is internal. □

Proposition 3.31. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in O(X)$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are internal such that for each $x \in X$,*

$$A^+(x) \wedge B^+(x) \geq (A \cup B)^{\in}(x),\ A^+(x) \wedge B^+(x) \geq 1 - (A \cup B)^{\notin}(x). \tag{3.10}$$

Then $\mathcal{A} \cap^3 \mathcal{B}$ is internal.

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ be IOSs in X satisfying the condition (3.10) and let $x \in X$. Then clearly,

$$A^-(x) \leq A^{\in}(x) \leq A^+(x),\ A^-(x) \leq 1 - A^{\notin}(x) \leq A^+(x),\ A^-(x) \leq \lambda(x) \leq A^+(x),$$

$$B^-(x) \leq B^{\in}(x) \leq B^+(x),\ B^-(x) \leq 1 - B^{\notin}(x) \leq B^+(x),\ A^-(x) \leq \mu(x) \leq A^+(x).$$

Thus $(\mathbf{A} \cap \mathbf{B})^-(x) \leq (\lambda \wedge \mu)(x) \leq (\mathbf{A} \cap \mathbf{B})^+(x)$. By the condition (3.10), we have Thus

$$(\mathbf{A} \cap \mathbf{B})^-(x) = A^-(x) \leq (A \cup B)^{\in}(x) \leq A^+(x) \wedge B^+(x) = (\mathbf{A} \cup \mathbf{B})^+(x)$$

and

$$(\mathbf{A} \cap \mathbf{B})^-(x) \leq 1 - (A \cup B)^{\notin}(x) \leq A^+(x) \wedge B^+(x) = (\mathbf{A} \cap \mathbf{B})^+(x).$$

So $\mathcal{A} \cap^3 \mathcal{B}$ is internal. □

The following is the immediate result of Propositions 3.30 and 3.31.

Corollary 3.32. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in O(X)$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are IOSs satisfying the conditions (3.9) and (3.10). Then $\mathcal{A} \cap^4 \mathcal{B}$ is internal.*

Remark 3.33. Type i-union and Type i-intersection ($i = 1,\ 2,\ 3,\ 4$) of two external octahedron sets may not be external, in general.

Example 3.34. In Example 3.6 (6), consider the EOS $\mathcal{A}_6$ in I given by: for each $x \in I$,

$$\mathcal{A}_6(x) = \left\langle [\frac{1+x}{4}, \frac{1+x}{2}], (\frac{1+x}{5}, 1 - \frac{\frac{1}{2}+x}{4}), \frac{2+x}{3} \right\rangle.$$

Let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ be the octahedron set in I defined as follows: for each $x \in I$,

$$\mathcal{A}(x) = \left\langle [\frac{1+x}{3}, \frac{x}{2}], (\frac{1+x}{4}, \frac{1-x}{2}), \frac{1+x}{2} \right\rangle.$$

Then we can easily see that $\mathcal{A}$ is an EOS in I.

(Case 1) Type 1-union and Type 1-intersection: for each $x \in I$,

$$(\mathcal{A} \cup^1 \mathcal{A}_6)(x) = \left\langle [\frac{1+x}{3}, \frac{1+x}{2}], (\frac{1+x}{4}, \frac{1-x}{2}), \frac{2+x}{3} \right\rangle$$

and

$$(\mathcal{A} \cap^1 \mathcal{A}_6)(x) = \left\langle [\frac{1+x}{4}, \frac{x}{2}], (\frac{1+x}{5}, 1 - \frac{\frac{1}{2}+x}{4}), \frac{1+x}{2} \right\rangle.$$

Then $(A \cup A_6)^{\in}(x)$, $(\lambda \vee \lambda_6)(x) \notin (\frac{1+x}{3}, \frac{1+x}{2})$ but $1-(A \cup A_6)^{\notin}(x) = \frac{1+x}{2} \in (\mathbf{A} \cup \mathbf{B})(x)$, for each $x \in I$ and $(A \cap A_6)^{\in}(x)$, $(\lambda \wedge \lambda_6)(x) \notin (\frac{1+x}{4}, \frac{x}{2})$ but $1 - (A \cap A_6)^{\notin}(x) = \frac{\frac{1}{2}-x}{4} \in (\mathbf{A} \cap \mathbf{B})(x)$, for each $x \in [\frac{1}{2}, 1]$. Thus $\mathcal{A} \cup^1 \mathcal{A}_6$ and $\mathcal{A} \cap^1 \mathcal{A}_6$ are not EOS in I.

(Case 2) Type 2-union and Type 2-intersection: for each $x \in I$,

$$(\mathcal{A} \cup^2 \mathcal{A}_6)(x) = \left\langle [\frac{1+x}{3}, \frac{1+x}{2}], (\frac{1+x}{4}, \frac{1-x}{2}), \frac{1+x}{2} \right\rangle$$

and

$$(\mathcal{A} \cap^2 \mathcal{A}_6)(x) = \left\langle [\frac{1+x}{4}, \frac{x}{2}], (\frac{1+x}{5}, 1 - \frac{\frac{1}{2}+x}{4}), \frac{2+x}{3} \right\rangle.$$

Then $(A \cup A_6)^{\in}(x)$ but $(\lambda \wedge \lambda_6)(x) = \frac{1+x}{2} \in (\mathbf{A} \cup \mathbf{B})(x)$ and $(A \cap A_6)^{\in}(x)$, $(\lambda \vee \lambda_6)(x) \notin (\frac{1+x}{4}, \frac{x}{2})$ but $1 - (A \cap A_6)^{\notin}(x) = \frac{\frac{1}{2}-x}{4} \in (\mathbf{A} \cap \mathbf{B})(x)$, for each $x \in [\frac{1}{2}, 1]$. Thus $\mathcal{A} \cup^2 \mathcal{A}_6$ and $\mathcal{A} \cap^2 \mathcal{A}_6$ are not EOS in I.

Similarly, we can easily calculate that $\mathcal{A} \cup^3 \mathcal{A}_6$, $\mathcal{A} \cap^3 \mathcal{A}_6$, $\mathcal{A} \cup^4 \mathcal{A}_6$ and $\mathcal{A} \cap^4 \mathcal{A}_6$ are not EOS in I.

We give a condition for the Type i-intersection of two EOSs to be an EOS.

Proposition 3.35. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in O(X)$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are external satisfying the following conditions: for each $x \in X$,*

$$(3.11) \quad (A^+(x) \vee B^-(x)) \wedge (A^-(x) \vee B^+(x)) \geq (\lambda \wedge \mu)(x) > (A^+(x) \wedge B^-(x)) \vee (A^-(x) \wedge B^+(x)),$$

(3.12) $(A^+(x) \vee B^-(x)) \wedge (A^-(x) \vee B^+(x))$
$\geq (A \cap B)^{\in}(x) > (A^+(x) \wedge B^-(x)) \vee (A^-(x) \wedge B^+(x)),$

(3.13) $(A^+(x) \vee B^-(x)) \wedge (A^-(x) \vee B^+(x))$
$\geq 1 - (A \cap B)^{\notin}(x) > (A^+(x) \wedge B^-(x)) \vee (A^-(x) \wedge B^+(x)).$

Then $\mathcal{A} \cap^1 \mathcal{B}$ is external.

Proof. Suppose the condition (3.12) holds and for each $x \in X$, take

$$\alpha_x := (A^+(x) \vee B^-(x)) \wedge (A^-(x) \vee B^+(x))$$

and

$$\beta_x := ((A^+(x) \wedge B^-(x)) \vee (A^-(x) \wedge B^+(x)).$$

Then clearly, $\alpha_x = A^-(x)$ or $\alpha_x = B^-(x)$ or $\alpha_x = A^+(x)$ or $\alpha_x = B^+(x)$.

Case 1: Suppose $\alpha_x = A^-(x)$. Then $B^-(x) \leq B^+(x) \leq A^-(x) \leq A^+(x)$. Thus $\beta_x = B^+(x)$. So

$$B^-(x) = (\mathbf{A} \cap \mathbf{B})^-(x) \leq (\mathbf{A} \cap \mathbf{B})^+(x) = B^+(x) = \beta_x < (A \cap B)^{\in}(x).$$

Hence $(A \cap B)^{\in}(x) \notin ((\mathbf{A} \cap \mathbf{B})^-(x), (\mathbf{A} \cap \mathbf{B})^+(x))$.

Case 2: Suppose $\alpha_x = A^+(x)$. Then $B^-(x) \leq A^+(x) \leq B^+(x)$. Thus

$$\beta_x = A^-(x) \vee B^+(x).$$

If $\beta_x = A^-(x)$, then

$$B^-(x) \leq A^-(x) < (A \cap B)^{\in}(x) \leq A^+(x) \leq B^+(x). \tag{3.14}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are external, the inequality

$$B^-(x) \leq A^-(x) < (A \cap B)^{\in}(x) < A^+(x) \leq B^+(x)$$

does not hold. Thus by the inequality (3.14), we have

$$B^-(x) \leq A^-(x) < (A \cap B)^{\in}(x) = A^+(x) \leq B^+(x).$$

So $(A \cap B)^{\in}(x) = A^+(x) = (\mathbf{A} \cap \mathbf{B})^+(x)$. Hence

$$(A \cap B)^{\in}(x) \notin ((\mathbf{A} \cap \mathbf{B})^-(x), (\mathbf{A} \cap \mathbf{B})^+(x)).$$

If $\beta_x = B^-(x)$, then

$$A^-(x) \leq B^-(x) < (A \cap B)^{\in}(x) \leq A^+(x) \leq B^+(x). \tag{3.15}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are external, the inequality

$$A^-(x) \leq B^-(x) < (A \cap B)^{\in}(x) < A^+(x) \leq B^+(x)$$

does not hold. Thus by the inequality (3.15), we have

$$A^-(x) \leq B^-(x) < (A \cap B)^{\in}(x) = A^+(x) \leq B^+(x).$$

So $(A \cap B)^{\in}(x) = A^+(x) = (\mathbf{A} \cap \mathbf{B})^+(x)$. Hence

$$(A \cap B)^{\in}(x) \notin ((\mathbf{A} \cap \mathbf{B})^-(x), (\mathbf{A} \cap \mathbf{B})^+(x)).$$

Case 3: Suppose $\alpha_x = B^-(x)$ or $\alpha_x = B^+(x)$. Then the proof is similar to Case 1 and Case 2.

When the conditions (3.11) and (3.12) hold, we can also prove similarly that

$$(\lambda \wedge \mu)(x) \notin ((\mathbf{A} \cap \mathbf{B})^-(x), (\mathbf{A} \cap \mathbf{B})^+(x))$$

and

$$1-(A\cap B)^{\notin}(x)\notin((\mathbf{A}\cap\mathbf{B})^-(x),(\mathbf{A}\cap\mathbf{B})^+(x)).$$

Therefore $\mathcal{A}\cap^1\mathcal{B}$ is an EOS in X. □

Proposition 3.36. *Let X be a nonempty set and let $\mathcal{A}=\langle\mathbf{A},A,\lambda\rangle$, $\mathcal{B}=\langle\mathbf{B},B,\mu\rangle\in O(X)$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are external satisfying the conditions (3.12) and (3.13), and the following condition: for each $x\in X$,*

$$(A^+(x)\vee B^-(x))\wedge(A^-(x)\vee B^+(x)) \geq(\lambda\vee\mu)(x)>(A^+(x)\wedge B^-(x))\vee(A^-(x)\wedge B^+(x)). \tag{3.16}$$

Then $\mathcal{A}\cap^2\mathcal{B}$ is external.

Proof. From the proof of Proposition 3.35, it is obvious that

$$(A\cap B)^{\in}(x)\notin((\mathbf{A}\cap\mathbf{B})^-(x),(\mathbf{A}\cap\mathbf{B})^+(x)),$$

$$1-(A\cap B)^{\notin}(x)\notin((\mathbf{A}\cap\mathbf{B})^-(x),(\mathbf{A}\cap\mathbf{B})^+(x)).$$

Suppose the condition (3.16) holds and for each $x\in X$, take

$$\alpha_x:=(A^+(x)\vee B^-(x))\wedge(A^-(x)\vee B^+(x))$$

and

$$\beta_x:=((A^+(x)\wedge B^-(x))\vee(A^-(x)\wedge B^+(x)).$$

Then by the proof process of Proposition 3.35,

$$(\lambda\vee\mu)(x)\notin((\mathbf{A}\cap\mathbf{B})^-(x),(\mathbf{A}\cap\mathbf{B})^+(x)).$$

Thus $\mathcal{A}\cap^2\mathcal{B}$ is external. □

Proposition 3.37. *Let X be a nonempty set and let $\mathcal{A}=\langle\mathbf{A},A,\lambda\rangle$, $\mathcal{B}=\langle\mathbf{B},B,\mu\rangle\in O(X)$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are external satisfying the condition (3.11), and the following conditions: for each $x\in X$,*

$$(A^+(x)\vee B^-(x))\wedge(A^-(x)\vee B^+(x)) \geq(A\cup B)^{\in}(x)>(A^+(x)\wedge B^-(x))\vee(A^-(x)\wedge B^+(x)), \tag{3.17}$$

$$(A^+(x)\vee B^-(x))\wedge(A^-(x)\vee B^+(x)) \geq 1-(A\cup B)^{\notin}(x)>(A^+(x)\wedge B^-(x))\vee(A^-(x)\wedge B^+(x)). \tag{3.18}$$

Then $\mathcal{A}\cap^3\mathcal{B}$ is external.

Proof. The proof is clear from the proofs of Propositions 3.35 and 3.36. □

The followings are immediate results of Propositions 3.36 and 3.37.

Corollary 3.38. *Let X be a nonempty set and let $\mathcal{A}=\langle\mathbf{A},A,\lambda\rangle$, $\mathcal{B}=\langle\mathbf{B},B,\mu\rangle\in O(X)$.*

If $\mathcal{A}$ and $\mathcal{B}$ are external satisfying the conditions (3.16), (3.17) and (3.18), then $\mathcal{A}\cap^4\mathcal{B}$ is external.

We give a condition for the Type i-union ($i=1,\ 2,\ 3,\ 4$) of two EOSs to be an EOS.

Proposition 3.39. *Let X be a nonempty set and let $\mathcal{A}=\langle\mathbf{A},A,\lambda\rangle$, $\mathcal{B}=\langle\mathbf{B},B,\mu\rangle\in O(X)$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are external satisfying the following conditions: for each $x\in X$,*

(3.19) $(A^+(x)\vee B^-(x))\wedge(A^-(x)\vee B^+(x))$

$$> (\lambda \vee \mu)(x) \geq (A^+(x) \wedge B^-(x)) \vee (A^-(x) \wedge B^+(x)),$$

(3.20) $$(A^+(x) \vee B^-(x)) \wedge (A^-(x) \vee B^+(x)) > (A \cup B)^{\in}(x) \geq (A^+(x) \wedge B^-(x)) \vee (A^-(x) \wedge B^+(x)),$$

(3.21) $$(A^+(x) \vee B^-(x)) \wedge (A^-(x) \vee B^+(x)) > 1 - (A \cup B)^{\notin}(x) \geq (A^+(x) \wedge B^-(x)) \vee (A^-(x) \wedge B^+(x)).$$

Then $\mathcal{A} \cup^1 \mathcal{B}$ *is external.*

Proof. We prove only when the condition (3.20) holds. Suppose (3.20) holds and for each $x \in X$, take

$$\alpha_x := (A^+(x) \vee B^-(x)) \wedge (A^-(x) \vee B^+(x))$$

and

$$\beta_x := ((A^+(x) \wedge B^-(x)) \vee (A^-(x) \wedge B^+(x)).$$

Then clearly, $\alpha_x = A^-(x)$ or $\alpha_x = B^-(x)$ or $\alpha_x = A^+(x)$ or $\alpha_x = B^+(x)$. We consider only $\alpha_x = A^-(x)$ or $\alpha_x = A^+(x)$, since the proofs of the remainder are similar.

Case 1: Suppose $\alpha_x = A^-(x)$. Then $B^-(x) \leq B^+(x) \leq A^-(x) \leq A^+(x)$. Thus $\beta_x = B^+(x)$. So

$$(\mathbf{A} \cup \mathbf{B})^-(x) = A^-(x) = \alpha_x > (A \cup B)^{\in}(x).$$

Hence $(A \cup B)^{\in}(x) \notin ((\mathbf{A} \cup \mathbf{B})^-(x), (\mathbf{A} \cup \mathbf{B})^+(x))$.

Case 2: Suppose $\alpha_x = A^+(x)$. Then $B^-(x) \leq A^+(x) \leq B^+(x)$. Thus

$$\beta_x = A^-(x) \vee B^-(x).$$

If $\beta_x = A^-(x)$, then

(3.22) $$B^-(x) \leq A^-(x) \leq (A \cup B)^{\in}(x) < A^+(x) \leq B^+(x).$$

Since $\mathcal{A}$ and $\mathcal{B}$ are external, the inequality

$$B^-(x) \leq A^-(x) < (A \cup B)^{\in}(x) < A^+(x) \leq B^+(x)$$

does not hold. Thus by the inequality (3.22), we have

$$B^-(x) \leq A^-(x) = (A \cup B)^{\in}(x) = A^+(x) \leq B^+(x).$$

So $(A \cup B)^{\in}(x) = A^-(x) = (\mathbf{A} \cup \mathbf{B})^+(x)$. Hence

$$(A \cup B)^{\in}(x) \notin ((\mathbf{A} \cup \mathbf{B})^-(x), (\mathbf{A} \cup \mathbf{B})^+(x)).$$

If $\beta_x = B^-(x)$, then

(3.23) $$A^-(x) \leq B^-(x) \leq (A \cup B)^{\in}(x) < A^+(x) \leq B^+(x).$$

Since $\mathcal{A}$ and $\mathcal{B}$ are external, the inequality

$$A^-(x) \leq B^-(x) < (A \cup B)^{\in}(x) < A^+(x) \leq B^+(x)$$

does not hold. Thus by the inequality (3.23), we have

$$A^-(x) \leq B^-(x) = (A \cup B)^{\in}(x) < A^+(x) \leq B^+(x).$$

So $(A \cup B)^{\in}(x) = B^-(x) = (\mathbf{A} \cup \mathbf{B})^-(x)$. Hence

$$(A \cup B)^{\in}(x) \notin ((\mathbf{A} \cup \mathbf{B})^-(x), (\mathbf{A} \cup \mathbf{B})^+(x)).$$

When the conditions (3.19) and (3.21), we can prove similarly that

$$(\lambda \vee \mu)(x) \notin ((\mathbf{A} \cup \mathbf{B})^-(x), (\mathbf{A} \cup \mathbf{B})^+(x))$$

and

$$1 - (A \cup B)^{\notin}(x) \notin ((\mathbf{A} \cup \mathbf{B})^-(x), (\mathbf{A} \cup \mathbf{B})^+(x)).$$

Therefore $\mathcal{A} \cup^1 \mathcal{B}$ is an EOS in X. □

Proposition 3.40. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in O(X)$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are external satisfying the conditions (3.20) and (3.21), and the following condition: for each $x \in X$,*

(3.24) $$(A^+(x) \vee B^-(x)) \wedge (A^-(x) \vee B^+(x)) > (\lambda \wedge \mu)(x) \geq (A^+(x) \wedge B^-(x)) \vee (A^-(x) \wedge B^+(x)).$$

Then $\mathcal{A} \cup^2 \mathcal{B}$ is external.

Proof. From the proof of Proposition 3.39, it is obvious that for each $x \in X$,

$$(A \cap B)^{\in}(x) \notin ((\mathbf{A} \cup \mathbf{B})^-(x), (\mathbf{A} \cup \mathbf{B})^+(x))$$

and

$$1 - (A \cap B)^{\notin}(x) \notin ((\mathbf{A} \cup \mathbf{B})^-(x), (\mathbf{A} \cup \mathbf{B})^+(x)).$$

Suppose (3.24) holds and for each $x \in X$, take

$$\alpha_x := (A^+(x) \vee B^-(x)) \wedge (A^-(x) \vee B^+(x))$$

and

$$\beta_x := ((A^+(x) \wedge B^-(x)) \vee (A^-(x) \wedge B^+(x)).$$

Then clearly, $\alpha_x = A^-(x)$ or $\alpha_x = B^-(x)$ or $\alpha_x = A^+(x)$ or $\alpha_x = B^+(x)$. We consider only $\alpha_x = B^-(x)$ or $\alpha_x = B^+(x)$, since the proofs of the remainder are similar.

Case 1: Suppose $\alpha_x = B^-(x)$. Then $A^-(x) \leq A^+(x) \leq B^-(x) \leq B^+(x)$. Thus $\beta_x = A^+(x)$. So by the inequality (3.24),

$$(\mathbf{A} \cup \mathbf{B})^-(x) = B^-(x) = \alpha_x > (\lambda \wedge \mu)(x).$$

Hence $(\lambda \wedge \mu)(x) \notin ((\mathbf{A} \cup \mathbf{B})^-(x), (\mathbf{A} \cup \mathbf{B})^+(x))$.

Case 2: Suppose $\alpha_x = B^+(x)$. Then $A^-(x) \leq B^+(x) \leq A^+(x)$. Thus

$$\beta_x = A^-(x) \vee B^-(x).$$

If $\beta_x = A^-(x)$, then

(3.25) $$B^-(x) \leq A^-(x) \leq (\lambda \wedge \mu)(x) < B^+(x) \leq A^+(x).$$

Since $\mathcal{A}$ and $\mathcal{B}$ are external, the inequality

$$B^-(x) \leq A^-(x) < (\lambda \wedge \mu)(x) < B^+(x) \leq A^+(x)$$

does not hold. Thus by the inequality (3.25), we have

$$B^-(x) \leq A^-(x) = (\lambda \wedge \mu)(x) < B^+(x) \leq A^+(x).$$

So $(\lambda \wedge \mu)(x) = A^-(x) = (\mathbf{A} \cup \mathbf{B})^-(x)$. Hence

$$(\lambda \wedge \mu)(x) \notin ((\mathbf{A} \cup \mathbf{B})^-(x), (\mathbf{A} \cup \mathbf{B})^+(x)).$$

If $\beta_x = B^-(x)$, then

(3.26) $$A^-(x) \leq B^-(x) \leq (\lambda \wedge \mu)(x) < B^+(x) \leq A^+(x).$$

Since $\mathcal{A}$ and $\mathcal{B}$ are external, the inequality

$$A^-(x) \leq B^-(x) < (\lambda \wedge \mu)(x) < B^+(x) \leq A^+(x)$$

does not hold. Thus by the inequality (3.26), we have

$$A^-(x) \leq B^-(x) = (\lambda \wedge \mu)(x) < B^+(x) \leq A^+(x).$$

So $(\lambda \wedge \mu)(x) = B^-(x) = (\mathbf{A} \cup \mathbf{B})^-(x)$. Hence

$$(\lambda \wedge \mu) \notin ((\mathbf{A} \cup \mathbf{B})^-(x), (\mathbf{A} \cup \mathbf{B})^+(x)).$$

Hence $\mathcal{A} \cup^2 \mathcal{B}$ is an EOS in X. □

Proposition 3.41. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in O(X)$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are external satisfying the condition (3.19), the following conditions: for each $x \in X$,*

(3.27) $$(A^+(x) \vee B^-(x)) \wedge (A^-(x) \vee B^+(x)) > (A \cap B)^{\in}(x) \geq (A^+(x) \wedge B^-(x)) \vee (A^-(x) \wedge B^+(x)),$$

(3.28) $$(A^+(x) \vee B^-(x)) \wedge (A^-(x) \vee B^+(x)) > 1 - (A \cap B)^{\notin}(x) \geq (A^+(x) \wedge B^-(x)) \vee (A^-(x) \wedge B^+(x)).$$

Then $\mathcal{A} \cup^3 \mathcal{B}$ is external.

Proof. The proof is similar to Proposition 3.40. □

The following is the immediate result of Propositions 3.40 and 3.41.

Corollary 3.42. *Let X be a nonempty set and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in O(X)$. If $\mathcal{A}$ and $\mathcal{B}$ are external satisfying the conditions (3.24), (3.27) and (3.28), then $\mathcal{A} \cup^4 \mathcal{B}$ is external.*

4. Octahedron points and level sets

Definition 4.1 ([15])**.** $A \in (I \oplus I)^X$ is called an intuitionistic fuzzy point (briefly, an IF point) with the support $x \in X$ and the value $\bar{a} \in I \oplus I$ with $\bar{a} \neq \bar{0}$, denoted by $A = x_{\bar{a}}$, if for each $y \in X$,

$$x_{\bar{a}}(y) = \begin{cases} \bar{a} & \text{if } y = x \\ \bar{0} & \text{otherwise.} \end{cases}$$

The set of all IFF points in X is denoted by $IF_P(X)$.

For each $x_{\bar{a}} \in IF_P(X)$ and $A \in (I \oplus I)^X$, $x_{\bar{a}}$ is said to belong to A, denoted by $x_{\bar{a}} \in A$, if $a^{\in} \leq A^{\in}(x)$ and $a^{\notin} \geq A^{\notin}(x)$.

It is well-known (See Theorem 2.4 in [15]) that $A = \bigcup_{x_{\bar{a}} \in A} x_{\bar{a}}$, for each $A \in (I \oplus I)^X$.

Definition 4.2 ([17])**.** $A \in [I]^X$ is called an interval-valued fuzzy point (briefly, an IVF point) with the support $x \in X$ and the value $\widetilde{a} \in [I]$ with $a^+ > 0$, denoted by $A = x_{\widetilde{a}}$, if for each $y \in X$,

$$x_{\widetilde{a}}(y) = \begin{cases} \widetilde{a} & \text{if } y = x \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

The set of all IVF points in X is denoted by $IVF_P(X)$.

For each $x_{\widetilde{a}} \in IVF_P(X)$ and $A \in [I]^X$, $x_{\widetilde{a}}$ is said to belong to A, denoted by $x_{\widetilde{a}} \in A$, if $a^- \leq A^-(x)$ and $a^+ \leq A^+(x)$.

It is clear that $A = \bigcup_{x_{\tilde{a}} \in A} x_{\tilde{a}}$, for each $A \in [I]^X$.

Definition 4.3. Let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in O(X)$, let $\tilde{a} \in [I]$ with $a^+ > 0$, $\bar{b} \in I \oplus I$ with $\bar{b} \neq \bar{0}$, $\alpha \in I$ with $\alpha \neq 0$. Then A is called an octahedron point with the support $x \in X$ and the value $\langle \tilde{a}, \bar{a}, \alpha \rangle$, denoted by $A = x_{\langle \tilde{a}, \bar{a}, \alpha \rangle}$, if for each $y \in X$,

$$x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle}(y) = \begin{cases} \left\langle \tilde{a}, \bar{b}, \alpha \right\rangle & \text{if } y = x \\ \left\langle \tilde{0}, \bar{0}, 0 \right\rangle & \text{otherwise.} \end{cases}$$

The set of all octahedron points in X is denoted by $O_P(X)$.

Definition 4.4. Let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in O(X)$ and let $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in O_P(X)$. Then $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle}$ is said to:

(i) belong to $\mathcal{A}$ with respect to Type 1-order, denoted by $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_1 A$, if
$\tilde{a} \leq \mathbf{A}(x)$, $\bar{b} \leq A(x)$ and $\alpha \leq \lambda(x)$, i.e., $x_{\tilde{a}} \in \mathbf{A}$, $x_{\bar{b}} \in A$ and $x_\alpha \in \lambda$,

(ii) belong to $\mathcal{A}$ with respect to Type 2-order, denoted by $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_2 A$, if
$\tilde{a} \leq \mathbf{A}(x)$, $\bar{b} \leq A(x)$ and $\alpha \geq \lambda(x)$,

(iii) belong to $\mathcal{A}$ with respect to Type 3-order, denoted by $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_3 A$, if
$\tilde{a} \leq \mathbf{A}(x)$, $\bar{b} \geq A(x)$ and $\alpha \leq \lambda(x)$,

(iv) belong to $\mathcal{A}$ with respect to Type 4-order, denoted by $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_4 A$, if
$\tilde{a} \leq \mathbf{A}(x)$, $\bar{b} \geq A(x)$ and $\alpha \geq \lambda(x)$.

It is clear that $A = \bigcup_{x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in \mathcal{A}} x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle}$, for each $\mathcal{A} \in O^X$.

Theorem 4.5. *Let $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in O_P(X)$, $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$ and $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in O(X)$ and let $i = \{1, 2, 3, 4\}$. Then*

$$\mathcal{A} \subset_i \mathcal{B} \text{ if and only if } x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_i \mathcal{B}, \text{ for each } x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_i \mathcal{A}.$$

Proof. Suppose $\mathcal{A} \subset_1 \mathcal{B}$ and let $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_1 \mathcal{A}$. Then
$\tilde{a} = [a^-, a^+] \leq [A^-(x), A^+(x)] = \mathbf{A}(x)$,
$\bar{b} = (b^\in, b^\notin) \leq (A^\in(x), A^\notin(x)) = A(x)$, $\alpha \leq \lambda(x)$.
Since $\mathcal{A} \subset_1 \mathcal{B}$, $\mathbf{A}(x) \leq \mathbf{B}(x)$, $A(x) \leq B(x)$, $\lambda(x) \leq \mu(x)$. Thus

$$\tilde{a} \leq \mathbf{B}(x), \ \bar{b} \leq B(x), \ \alpha \leq \mu(x).$$

So $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_1 \mathcal{B}$.

Conversely, suppose the necessary condition holds and Assume that $\mathcal{A} \not\subset_1 \mathcal{B}$. Then there is $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in O_P(X)$ such that $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_1 \mathcal{A}$, but $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \notin_1 \mathcal{B}$. Thus $x_{\tilde{a}} \in \mathbf{A}$, $x_{\bar{b}} \in A$, $x_\alpha \in \lambda$ but $x_{\tilde{a}} \notin \mathbf{B}$ or $x_{\bar{b}} \notin B$ or $x_\alpha \notin \lambda$. This is a contradiction. So $\mathcal{A} \subset_1 \mathcal{B}$.

The remainders can be proved similarly. □

Proposition 4.6. *Let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu \rangle \in O(X)$, let $(\mathcal{A}_j)_{j \in J} = (\langle \mathbf{A}_j, A_j, \lambda_j \rangle)_{j \in J} \subset O(X)$, $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in O_P(X)$ and let $i = \{1, 2, 3, 4\}$.*

(1) *If $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_i \mathcal{A}$ or $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_i \mathcal{B}$, then $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_P \mathcal{A} \cup^i \mathcal{B}$.*

(2) *If there is $j \in J$ such that $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_i \mathcal{A}_j$, then $x_{\left\langle \tilde{a}, \bar{b}, \alpha \right\rangle} \in_i \bigcup^i_{j \in J} \mathcal{A}_j$.*

Proof. The proofs are straightforward. □

The converse of Proposition 4.6 need not to be true in general as shown in the following example.

Example 4.7. Let $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle$ and $\mathcal{B} = \langle \mathbf{B}, B, \mu\rangle$ be two octahedron sets in I given in Example 3.27 as follows: for each $x \in I$,

$$\mathcal{A}(x) = \left\langle [\frac{1+x}{4}, \frac{1+x}{2}], (\frac{1+x}{3}, \frac{1-x}{3}), \frac{2+x}{5} \right\rangle$$

and

$$\mathcal{B} = \left\langle [\frac{x}{5}, \frac{2+x}{3}], (\frac{x}{3}, \frac{2-x}{5}), \frac{x}{4} \right\rangle.$$

Then clearly, $\mathcal{A} \cup^1 \mathcal{B} = \left\langle [\frac{1+x}{4}, \frac{2+x}{3}], (\frac{1+x}{3}, \frac{1-x}{3}), \frac{2+x}{5} \right\rangle$. Let $\widetilde{a} = [\frac{1}{6}, \frac{2}{3}]$, $\bar{b} = (\frac{3}{5}, \frac{1}{6})$, $\alpha = \frac{2}{5}$ and Consider octahedron point $1_{\langle \widetilde{a}, \bar{b}, \alpha\rangle}$. Since $(\mathcal{A} \cup^1 \mathcal{B})(1) = \left\langle [\frac{1}{2}, 1], (\frac{2}{3}, 0), \frac{3}{5} \right\rangle$, $\mathcal{A}(1) = \left\langle [\frac{1}{2}, 1], (\frac{2}{3}, 0), \frac{3}{5} \right\rangle$ and $\mathcal{B}(1) = \left\langle [\frac{1}{5}, 1], (\frac{1}{3}, \frac{1}{5}), \frac{1}{4} \right\rangle$,
$\left\langle \widetilde{a}, \bar{b}, \alpha \right\rangle \leq (\mathcal{A} \cup^1 \mathcal{B})(1)$ but $\frac{3}{5} > \frac{1}{3}$ and $\frac{2}{5} > \frac{1}{4}$. Thus $1_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_1 \mathcal{A} \cup^1 \mathcal{B}$ but $1_{\bar{b}} \notin B$ and $1_\alpha \notin \lambda$. So $1_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_1 \mathcal{A} \cup^1 \mathcal{B}$ but $1_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \notin_1 \mathcal{B}$.

On the other hand, by (3.1) of Example 3.27, $(\mathcal{A} \cup^2 \mathcal{B})(1) = \left\langle [\frac{1}{2}, 1], (\frac{2}{3}, 0), \frac{1}{4} \right\rangle$. Then $1_{\widetilde{a}} \in \mathbf{A} \cup \mathbf{B}$, $1_{\bar{b}} \in A \cup B$, $\alpha > \frac{1}{4} = \lambda(1)$ but $\bar{b} \not\leq B(x)$, i.e., $1_{\bar{b}} \notin B$. Thus $1_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_2 \mathcal{A} \cup^2 \mathcal{B}$ but $1_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \notin_2 \mathcal{B}$.

Similarly, we can see that for $i = 2,\ 3$, $1_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_i \mathcal{A} \cup^i \mathcal{B}$ but $1_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \notin_i \mathcal{B}$.

Theorem 4.8. *Let* $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle$, $\mathcal{B} = \langle \mathbf{B}, B, \mu\rangle \in O(X)$, *let* $(\mathcal{A}_j)_{j\in J} = (\langle \mathbf{A}_j, A_j, \lambda_j\rangle)_{j\in J} \subset O(X)$, $x_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in O_P(X)$*and let* $i = \{1,\ 2,\ 3,\ 4\}$. *Then*

(1) $x_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_i \mathcal{A} \cap^i \mathcal{B}$ *if and only if* $x_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_i \mathcal{A}$ *and* $x_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_i \mathcal{B}$,

(2) $x_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_i \cap^i_{j\in J} \mathcal{A}_j$ *if and only if* $x_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_i \mathcal{A}_j$, *for each* $j \in J$.

Proof. (1) Suppose $x_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_1 \mathcal{A} \cap^1 \mathcal{B}$. Then $x_{\widetilde{a}} \in \mathbf{A} \cap \mathbf{B}$, $x_{\bar{b}} \in A \cap B$ and $x_\alpha \in \lambda \wedge \mu$.

Thus $\widetilde{a} = [a^-, a^+] \leq (\mathbf{A} \cap \mathbf{B})(x) = [A^-(x)] \wedge B^-(x), A^+(x)] \wedge B^+(x)]$,
$\bar{b} = (b^\in, b^\notin) \leq (A \cap B)(x) = (A^\in(x) \wedge B^\in(x), A^\notin(x) \vee B^\notin(x)$,
$\alpha \leq (\lambda \wedge \mu)(x) = \lambda(x) \wedge \mu(x)$.

So $a^- \leq A^-(x),\ a^+ \leq A^+(x)$ and $a^- \leq B^-(x),\ a^+ \leq B^+(x)$,
$b^\in \leq A^\in(x),\ b^\notin \geq A^\notin(x)$ and $b^\in \leq B^\in(x),\ b^\notin \geq B^\notin(x)$,
$\alpha \leq \lambda(x)$ and $\alpha \leq \mu(x)$.

Hence $x_{\widetilde{a}} \in \mathbf{A},\ x_{\bar{b}} \in A,\ x_\alpha \in \lambda$ and $x_{\widetilde{a}} \in \mathbf{B},\ x_{\bar{b}} \in B,\ x_\alpha \in \mu$. Therefore $x_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_1 \mathcal{A}$ and $x_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_1 \mathcal{B}$.

Conversely, suppose $x_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_1 \mathcal{A}$ and $x_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \in_1 \mathcal{B}$. Then

$$a^- \leq A^-(x),\ a^+ \leq A^+(x),\ b^\in \leq A^\in(x),\ b^\notin \geq A^\notin(x),\ \alpha \leq \lambda(x)$$

and

$$a^- \leq B^-(x),\ a^+ \leq B^+(x),\ b^\in \geq B^\in(x),\ b^\notin \geq B^\notin(x),\ \alpha \leq \mu(x).$$
$$b^\in \leq B^\in(x),\ b^\notin \geq B^\notin(x).$$

Thus

$$a^- \leq A^-(x) \wedge B^-(x) = (\mathbf{A} \cap \mathbf{B})^-(x),\ a^+ \leq A^+(x) \wedge B^+(x) = (\mathbf{A} \cap \mathbf{B})^+(x),$$

$$b^{\in} \leq A^{\in}(x) \wedge B^{\in}(x) = (A \cap B)^{\in}(x),\ b^{\notin} \geq A^{\notin}(x) \vee B^{\notin}(x) = (A \cap B)^{\notin}(x),$$

$$\alpha \leq \lambda(x) \wedge \mu(x) = (\lambda \wedge \mu)(x).$$

So $x_{\widetilde{a}} \in \mathbf{A} \cap \mathbf{B}$, $x_{\bar{b}} \in A \cap B$ and $x_\alpha \in \lambda \wedge \mu$. Hence $x_{\langle \widetilde{a}, \bar{b}, \alpha \rangle} \in_i \mathcal{A} \cap^i \mathcal{B}$.

For $i = 2,\ 3,\ 4$, the proofs is similar. □

Let $\langle \widetilde{a}, \bar{b}, \alpha \rangle,\ \left\langle \widetilde{a'}, \bar{b'}, \beta \right\rangle \in [I] \times (I \oplus I) \times I$. Then

$$\langle \widetilde{a}, \bar{b}, \alpha \rangle \leq < \widetilde{a'}, \bar{b'}, \beta > \text{ if and only if } \widetilde{a} \leq \widetilde{a'},\ \bar{b} \leq \bar{b'},\ \alpha \leq \beta.$$

It is clear that $\langle \widetilde{a}, \bar{b}, \alpha \rangle \leq < \widetilde{a'}, \bar{b'}, \beta >$ if and only if
$a^- \leq (a')^-$ and $a^+ \leq (a')^+,\ b^{\in} \leq (b')^{\in}$ and $b^{\notin} \geq (b')^{\notin},\ \alpha \leq \beta.$

Definition 4.9. Let X be a nonempty set, let $\langle \widetilde{a}, \bar{b}, \alpha \rangle \in [I] \times (I \oplus I) \times I$ and let $\mathcal{A} = \langle \mathbf{A}, A, \lambda \rangle \in O(X)$. Then two subsets $[\mathcal{A}]_{\langle \widetilde{a}, \bar{b}, \alpha \rangle}$ and $[\mathcal{A}]^*_{\langle \widetilde{a}, \bar{b}, \alpha \rangle}$ of X are defined as follows:

$$[\mathcal{A}]_{\langle \widetilde{a}, \bar{b}, \alpha \rangle} = \{x \in X : \mathbf{A}(x) \geq \widetilde{a},\ A(x) \geq \bar{b},\ \lambda(x) \geq \alpha\},$$

$$[\mathcal{A}]^*_{\langle \widetilde{a}, \bar{b}, \alpha \rangle} = \{x \in X : \mathbf{A}(x) > \widetilde{a},\ A(x) > \bar{b},\ \lambda(x) > \alpha\}.$$

In this case, $[\mathcal{A}]_{\langle \widetilde{a}, \bar{b}, \alpha \rangle}$ is called an $\langle \widetilde{a}, \bar{b}, \alpha \rangle$-level set of $\mathcal{A}$ and $[\mathcal{A}]^*_{\langle \widetilde{a}, \bar{b}, \alpha \rangle}$ is called a strong $\langle \widetilde{a}, \bar{b}, \alpha \rangle$-level set of $\mathcal{A}$.

Example 4.10. Consider the octahedron set in I given by: for each $x \in I$,

$$\mathcal{A} = \left\langle [\frac{1+x}{6}, \frac{2+x}{3}], (\frac{1+x}{2}, \frac{x}{5}), \frac{1+x}{4} \right\rangle.$$

Let $\widetilde{a} = [\frac{1}{2}, \frac{2}{3}],\ \bar{b} = (\frac{2}{3}, \frac{1}{4}),\ \alpha = \frac{1}{2}$. Then

$$\begin{aligned}
[\mathcal{A}]_{\langle \mathbf{0}, \bar{0}, 0 \rangle} &= \{x \in I : [\tfrac{1+x}{6}, \tfrac{2+x}{3}] \geq \mathbf{0},\ (\tfrac{1+x}{2}, \tfrac{x}{5}) \geq \bar{0},\ \tfrac{1+x}{4} \geq 0\} \\
&= \{x \in I : \tfrac{1+x}{6} \geq 0,\ \tfrac{2+x}{3} \geq 0,\ \tfrac{1+x}{2} \geq 0,\ \tfrac{2+x}{3} \leq 1,\ \tfrac{1+x}{4} \geq 0\} \\
&= I, \\
[\mathcal{A}]^*_{\langle \mathbf{0}, \bar{0}, 0 \rangle} &= \{x \in I : [\tfrac{1+x}{6}, \tfrac{2+x}{3}] > \mathbf{0},\ (\tfrac{1+x}{2}, \tfrac{x}{5}) > \bar{0},\ \tfrac{1+x}{4} > 0\} \\
&= \{x \in I : \tfrac{1+x}{6} > 0,\ \tfrac{2+x}{3} > 0,\ \tfrac{1+x}{2} > 0,\ \tfrac{2+x}{3} < 1,\ \tfrac{1+x}{4} > 0\} \\
&= I \setminus \{1\}, \\
[\mathcal{A}]_{\langle \mathbf{1}, \bar{1}, 1 \rangle} &= \{x \in I : [\tfrac{1+x}{6}, \tfrac{2+x}{3}] \geq \mathbf{1},\ (\tfrac{1+x}{2}, \tfrac{x}{5}) \geq \bar{1},\ \tfrac{1+x}{4} \geq 1\} \\
&= \{x \in I : \tfrac{1+x}{6} \geq 1,\ \tfrac{2+x}{3} \geq 1,\ \tfrac{1+x}{2} \geq 1,\ \tfrac{2+x}{3} \leq 0,\ \tfrac{1+x}{4} \geq 1\} \\
&= \phi = [\mathcal{A}]^*_{<\mathbf{1}, \bar{1}, 1>}, \\
[\mathcal{A}]_{\langle \widetilde{a}, \bar{b}, \alpha \rangle} &= \{x \in I : [\tfrac{1+x}{6}, \tfrac{2+x}{3}] \geq \widetilde{a},\ (\tfrac{1+x}{2}, \tfrac{x}{5}) \geq \bar{b},\ \tfrac{1+x}{4} \geq \alpha\} \\
&= \{x \in I : \tfrac{1+x}{6} \geq \tfrac{1}{2},\ \tfrac{2+x}{3} \geq \tfrac{2}{3},\ \tfrac{1+x}{2} \geq \tfrac{2}{3},\ \tfrac{2+x}{3} \leq \tfrac{1}{4},\ \tfrac{1+x}{4} \geq \tfrac{1}{2}\} \\
&= [\tfrac{1}{3}, 1], \\
[\mathcal{A}]^*_{\langle \widetilde{a}, \bar{b}, \alpha \rangle} &= \{x \in I : [\tfrac{1+x}{6}, \tfrac{2+x}{3}] > \widetilde{a},\ (\tfrac{1+x}{2}, \tfrac{x}{5}) > \bar{b},\ \tfrac{1+x}{4} > \alpha\} \\
&= \{x \in I : \tfrac{1+x}{6} > \tfrac{1}{2},\ \tfrac{2+x}{3} > \tfrac{2}{3},\ \tfrac{1+x}{2} > \tfrac{2}{3},\ \tfrac{2+x}{3} < \tfrac{1}{4},\ \tfrac{1+x}{4} > \tfrac{1}{2}\} \\
&= (\tfrac{1}{3}, 1].
\end{aligned}$$

It is obvious that that for each $\langle \widetilde{a}, \bar{b}, \alpha\rangle \in [I] \times (I \oplus I) \times I$ and each $\mathcal{A} \in O^X$, $[\mathcal{A}]^*_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \subset [\mathcal{A}]_{\langle \widetilde{a}, \bar{b}, \alpha\rangle}$.

Proposition 4.11. *Let $\mathcal{A} \in O(X)$ and let $\langle \widetilde{a}, \bar{b}, \alpha\rangle$, $\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle \in [I] \times (I \oplus I) \times I$. Then have the following properties:*

(1) *if $\langle \widetilde{a}, \bar{b}, \alpha\rangle \leq \left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle$, then $[\mathcal{A}]_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle} \subset [\mathcal{A}]_{\langle \widetilde{a}, \bar{b}, \alpha\rangle}$,*

(2) $[\mathcal{A}]_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} = \bigcap_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle < \langle \widetilde{a}, \bar{b}, \alpha\rangle} [\mathcal{A}]_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle}$, *where* $\widetilde{a} \neq \mathbf{0}$, $\bar{b} \neq \bar{0}$, $\alpha \neq 0$,

$(1)'$ *if $\langle \widetilde{a}, \bar{b}, \alpha\rangle \leq \left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle$, then $[\mathcal{A}]^*_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle} \subset [\mathcal{A}]^*_{\langle \widetilde{a}, \bar{b}, \alpha\rangle}$,*

$(2)'$ $[\mathcal{A}]^*_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} = \bigcup_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle > \langle \widetilde{a}, \bar{b}, \alpha\rangle} [\mathcal{A}]^*_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle}$, *where* $\widetilde{a} \neq \mathbf{1}$, $\bar{b} \neq \bar{0}$, $\alpha \neq 0$.

Proof. (1) The proof is obvious from Definition 4.9.

(2) From (1), it is obvious that $([\mathcal{A}]_{\langle \widetilde{a}, \bar{b}, \alpha\rangle})_{\langle \widetilde{a}, \bar{b}, \alpha\rangle \in ([I]\times(I\oplus I)\times I)\setminus\{\langle \mathbf{0}, \bar{0}, 0\rangle\}}$ is a descending family of subsets of X. Then clearly,

$$[\mathcal{A}]_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} \subset \bigcap_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle < \langle \widetilde{a}, \bar{b}, \alpha\rangle} [\mathcal{A}]_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle},$$

for each $\langle \widetilde{a}, \bar{b}, \alpha\rangle \in ([I] \times (I \oplus I) \times I) \setminus \{\langle \mathbf{0}, \bar{0}, 0\rangle\}$.

Suppose $x \notin [\mathcal{A}]_{\langle \widetilde{a}, \bar{b}, \alpha\rangle}$. Then $\mathcal{A}(x) < \langle \widetilde{a}, \bar{b}, \alpha\rangle$. Thus

$$\exists \left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle \in ([I] \times (I \oplus I) \times I) \setminus \{\langle \mathbf{0}, \bar{0}, 0\rangle\}$$

such that $\mathcal{A}(x) < \left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle < \langle \widetilde{a}, \bar{b}, \alpha\rangle$. So $x \notin [\mathcal{A}]_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle}$, for some $\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle \in ([I] \times (I \oplus I) \times I) \setminus \{\langle \mathbf{0}, \bar{0}, 0\rangle\}$ such that $\left\langle \widetilde{a'}, \bar{b'}, \beta\beta\right\rangle < \langle \widetilde{a}, \bar{b}, \alpha\rangle$, i.e., $x \notin \bigcap_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle < \langle \widetilde{a}, \bar{b}, \alpha\rangle} [\mathcal{A}]_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle}$. Hence $\bigcap_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle < \langle \widetilde{a}, \bar{b}, \alpha\rangle} [\mathcal{A}]_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle} \subset [\mathcal{A}]_{\langle \widetilde{a}, \bar{b}, \alpha\rangle}$. Therefore $[\mathcal{A}]_{\langle \widetilde{a}, \bar{b}, \alpha\rangle} = \bigcap_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle < \langle \widetilde{a}, \bar{b}, \alpha\rangle} [\mathcal{A}]_{\left\langle \widetilde{a'}, \bar{b'}, \beta\right\rangle}$.

$(2)'$ The proof is similar to (2). □

5. The image and the preimage of an octahedron set under a mapping

Definition 5.1 ([4])**.** Let X, Y be two sets, let $f : X \to Y$ be a mapping and let $A \in (I \oplus I)^X$, $B \in (I \oplus I)^Y$.

(i) The preimage of B under f, denoted by $f^{-1}(B)$, is the IF set in X defined as follows: For each $x \in X$,

$$f^{-1}(B)(x) = (B^{\in}(f(x)), B^{\notin}(f(x))) = ((B^{\in} \circ f)(x), (B^{\notin} \circ f)(x)).$$

(ii) The image of A under f, denoted by $f(A)=(f(A^{\in}),f_{-}(A^{\notin}))$, is the IF set in Y defined as follows: For each $y\in Y$,

$$f(A^{\in})(y)=\begin{cases}\bigvee_{x\in f^{-1}(y)}A^{\in}(x) & \text{if } f^{-1}(y)\neq\phi\\ \bar{0} & \text{otherwise,}\end{cases}$$

$$f_{-}(A^{\notin})(y)=(1-f(1-A^{\notin}))(y)=\begin{cases}\bigwedge_{x\in f^{-1}(y)}A^{\notin,-}(x) & \text{if } f^{-1}(y)\neq\phi\\ \bar{1} & \text{otherwise.}\end{cases}$$

Definition 5.2 ([17]). Let $X,\ Y$ be two sets, let $f:X\to Y$ be a mapping and let $A\in[I]^X,\ B\in[I]^Y$.

(i) The preimage of B under f, denoted by $f^{-1}(B)$, is the IVF set in X defined as follows: for each $x\in X$,

$$f^{-1}(B)(x)=[(B^{-}(f(x)),B^{+}(f(x))]=[(B^{-}\circ f)(x)),(B^{+}\circ f)(x)].$$

(ii) The image of A under f, denoted by $f(A)=[f(A^{-}),f(A^{+})]$, is the IVF set in Y defined as follows: for each $y\in Y$,

$$f(A)(y)=\begin{cases}[\bigvee_{x\in f^{-1}(y)}A^{-}(x),\bigvee_{x\in f^{-1}(y)}A^{+}(x)] & \text{if } f^{-1}(y)\neq\phi\\ \mathbf{0} & \text{otherwise.}\end{cases}$$

Result 5.3 (Theorem 2 in [17]). *Let $f:X\to Y$ be a mapping.*

(1) *$f^{-1}(B^c)=[f^{-1}(B)]^c$, for each $B\in[I]^Y$.*
(2) *$[f(A)]^c\subset f(A^c)$, for each $A\in[I]^X$.*
(3) *If $B_1\subset B_2$, then $f^{-1}(B_1)\subset f^{-1}(B_2)$, for any $B_1,\ B_2\in[I]^Y$.*
(4) *If $A_1\subset A_2$, then $f(A_1)\subset f(A_2)$, for any $A_1,\ A_2\in[I]^X$.*
(5) *$f(f^{-1}(B))\subset B$, for each $B\in[I]^Y$.*
(6) *$A\subset f^{-1}(f(A))$, for each $A\in[I]^X$.*
(7) *$f(\bigcup_{j\in J}A_j)=\bigcup_{j\in J}f(A_j)$.*
(8) *$f^{-1}(\bigcup_{j\in J}B_j)=\bigcup_{j\in J}f^{-1}(B_j)$, for each $(B_j)_{j\in J}\subset[I]^Y$.*
(9) *$f^{-1}(\bigcap_{j\in J}B_j)=\bigcap_{j\in J}f^{-1}(B_j)$, for each $(B_j)_{j\in J}\subset[I]^Y$.*
(10) *If $g:Y\to Z$ is a mapping, then $(g\circ f)^{-1}(C)=f^{-1}(g^{-1}(C))$, for each $C\in[I]^X$, where $g\circ f$ is the composition of f and g.*

Definition 5.4. Let $X,\ Y$ be two sets, let $f:X\to Y$ be a mapping and let $\mathcal{A}=\langle\mathbf{A},A,\lambda\rangle\in O(X)$, $\mathcal{B}=\langle\mathbf{B},B,\mu\rangle\in O(Y)$.

(i) The preimage of $\mathcal{B}$ under f, denoted by $f^{-1}(\mathcal{B})=\left\langle f^{-1}(\mathbf{B}),f^{-1}(B),f^{-1}(\mu)\right\rangle$, is the octahedron set in X defined as follows: for each $x\in X$,

$$f^{-1}(\mathcal{B})(x)=\left\langle[(B^{-}\circ f)(x),(B^{+}\circ f)(x))],((B^{\in}\circ f)(x),(B^{\notin}\circ f)(x)),(\mu\circ f)(x)\right\rangle.$$

(ii) The image of $\mathcal{A}$ under f, denoted by $f(\mathcal{A})=\langle f(\mathbf{A}),f(A),f(\lambda)\rangle$, is the octahedron set in Y defined as follows: for each $y\in Y$,

$$f(\mathbf{A})(y)=\begin{cases}[\bigvee_{x\in f^{-1}(y)}A^{-}(x),\bigvee_{x\in f^{-1}(y)}A^{+}(x)] & \text{if } f^{-1}(y)\neq\phi\\ \mathbf{0} & \text{otherwise,}\end{cases}$$

$$f(A)(y)=\begin{cases}(\bigvee_{x\in f^{-1}(y)}A^{\in}(x),\bigwedge_{x\in f^{-1}(y)}A^{\notin}(x)) & \text{if } f^{-1}(y)\neq\phi\\ \bar{0} & \text{otherwise,}\end{cases}$$

$$f(\lambda)(y)=\begin{cases}\bigvee_{x\in f^{-1}(y)}\lambda(x) & \text{if } f^{-1}(y)\neq\phi\\ 0 & \text{otherwise.}\end{cases}$$

It is obvious that $f(x_{\langle\widetilde{a},\bar{b},\alpha\rangle}) = [f(x)]_{\langle\widetilde{a},\bar{b},\alpha\rangle}$, for each $x_{\langle\widetilde{a},\bar{b},\alpha\rangle} \in O_P(X)$.

Example 5.5. Let $X = \{x, y, z\}$, $Y = \{a, b, c, d\}$ and let $f : X \to Y$ be the mapping defined by: $f(x) = f(y) = a$, $f(z) = c$. Let $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle$ be the octahedron set in X and let $\mathcal{B} = \langle \mathbf{B}, B, \mu\rangle$ be the octahedron set in Y defined by Table 5.1:

X	$\mathbf{A}(t)$	$A(t)$	$\lambda(t)$
x	([0.2, 0.6]	(0.6, 0.3)	0.7
y	([0.3, 0.5]	(0.5, 0.2)	0.6
z	([0.4, 0.7]	(0.7, 0.2)	0.8

Table 5.1

Then we have easily the following Table 5.2 for $f(\mathcal{A})$:

Y	$f(\mathbf{A})(x)$	$f(A)(x)$	$f(\lambda)(x)$
a	[0.3, 0.6]	(0.6, 0.2)	0.7
b	$\mathbf{0}$	$\widetilde{0}$	0
c	[0.4, 0.7]	(0.7, 0.2)	0.8
d	$\mathbf{0}$	$\widetilde{0}$	0

Table 5.2

Now let $\mathcal{B} = \langle \mathbf{B}, B, \mu\rangle$ be the octahedron set in Y defined by the following Table:

Y	$\mathbf{B}(x)$	$B(x)$	$\mu(x)$
a	([0.3, 0.5]	(0.5, 0.4)	0.6
b	([0.2, 0.6]	(0.7, 0.2)	0.8
c	([0.4, 0.7]	(0.6, 0.3)	0.7
d	([0.2, 0.5]	(0.4, 0.5)	0.5

Table 5.3

Then we have easily the following Table 5.4 for $f^{-1}(\mathcal{B})$:

X	$f^{-1}(\mathbf{B})(t)$	$f^{-1}(B)(t)$	$f^{-1}(\mu)(t)$
x	[0.3, 0.5]	(0.5, 0.4)	0.6
y	[0.3, 0.5]	(0.5, 0.4)	0.6
z	[0.4, 0.7]	(0.6, 0.3)	0.7

Table 5.4

Proposition 5.6. *Let* $\mathcal{A} = \langle \mathbf{A}, A, \lambda\rangle$, $\mathcal{A}_1 = \langle \mathbf{A}_1, A_1, \lambda_1\rangle$, $\mathcal{A}_2 = \langle \mathbf{A}_2, A_2, \lambda_2\rangle \in O(X)$, $(\mathcal{A}_j)_{j\in J} = (\langle \mathbf{A}_j, A_j, \lambda_j\rangle)_{j\in J} \subset O(X)$, *let* $\mathcal{B} = \langle \mathbf{B}, B, \mu\rangle$, $\mathcal{B}_1 = \langle \mathbf{B}_1, B_1, \mu_1\rangle$, $\mathcal{B}_2 = \langle \mathbf{B}_2, B_2, \mu_2\rangle \in O(Y)$, $(\mathcal{B}_j)_{j\in J} = (\langle \mathbf{B}_j, B_j, \mu_j\rangle)_{j\in J} \subset O(Y)$ *and let* $f : X \to Y$ *be a mapping. Then for each* $i = 1, 2, 3, 4$,

(1) *if* $\mathcal{A}_1 \subset_i \mathcal{A}_2$, *then* $f(\mathcal{A}_1) \subset_i f(\mathcal{A}_2)$,

(2) *if* $\mathcal{B}_1 \subset_i \mathcal{B}_2$, *then* $f^{-1}(\mathcal{B}_1) \subset_i f^{-1}(\mathcal{B}_2)$,
(3) $\mathcal{A} \subset_1 f^{-1}(f(\mathcal{A}))$ *and if* f *is injective, then* $\mathcal{A} = f^{-1}(f(\mathcal{A}))$,
(4) $f(f^{-1}(\mathcal{B})) \subset_1 \mathcal{B}$ *and if* f *is surjective,* $f(f^{-1}(\mathcal{B})) = \mathcal{B}$,
(5) $f^{-1}(\bigcup^i_{j\in J} \mathcal{B}_j) = \bigcup^i_{j\in J} f^{-1}(\mathcal{B}_j)$,
(6) $f^{-1}(\bigcap^i_{j\in J} \mathcal{B}_j) = \bigcap^i_{j\in J} f^{-1}(\mathcal{B}_j)$,
(7) $f(\bigcup^1_{j\in J} \mathcal{A}_j) = \bigcup^1_{j\in J} f(\mathcal{A}_j)$,
(8) $f(\bigcap^i_{j\in J} \mathcal{A}_j) \subset_i \bigcap^i_{j\in J} f(\mathcal{A}_j)$ *and if* f *is injective, then* $f(\bigcap^i_{j\in J} \mathcal{A}_j) = \bigcap^i_{j\in J} f(\mathcal{A}_j)$,
(9) *if* f *is surjective, then* $f(\mathcal{A})^c \subset_1 f(\mathcal{A}^c)$.
(10) $f^{-1}(\mathcal{B}^c) = f^{-1}(\mathcal{B})^c$.
(11) $f^{-1}(\ddot{0}) = \ddot{0}$, $f^{-1}(\ddot{1}) = \ddot{1}$, $f^{-1}(\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle$,
$f^{-1}(\left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle$, $f^{-1}(\left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle$,
$f^{-1}(\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle$, $f^{-1}(\left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle$,
$f^{-1}(\left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle$.
(12) $f(\ddot{0}) = \ddot{0}$ *and if* f *is surjective, then the following hold:*

$$f(\left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{0}}, 1 \right\rangle, \ f(\left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 0 \right\rangle,$$

$$f(\left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 0 \right\rangle, \ f(\left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle) = \left\langle \widetilde{0}, \bar{\mathbf{1}}, 1 \right\rangle,$$

$$f(\left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{0}}, 1 \right\rangle, \ f(\left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle) = \left\langle \widetilde{1}, \bar{\mathbf{1}}, 0 \right\rangle, \ f(\ddot{1}) = \ddot{1}.$$

Proof. $\geq \bigvee_{x\in f^{-1}(y)} \lambda_2(x)$
Thus
The proofs are straightforward. □

Example 5.7. In Example 5.5, $f^{-1}(f(\mathcal{A}))(x) = \langle [0.3, 0.6], (0.6, 0.2), 0.7\rangle \geq \mathcal{A}(x)$, $f^{-1}(f(\mathcal{A}))(y) = \langle [0.3, 0.6], (0.6, 0.2), 0.7\rangle \geq \mathcal{A}(y)$ and $f^{-1}(f(\mathcal{A}))(z) = \mathcal{A}(z)$. Then $\mathcal{A} \subset_1 f^{-1}(f(\mathcal{A}))$. Moreover, $\mathcal{A} \neq f^{-1}(f(\mathcal{A}))$. On the other hand, we can easily calculate that $f(f^{-1}(\mathcal{B})) = \mathcal{B}$. Thus we can confirm that Proposition 5.6 (3) and (4) hold. Note that f is surjective but not injective.

Remark 5.8. $f(\bigcup^1_{j\in J} \mathcal{A}_j) \neq \bigcup^i_{j\in J} f(\mathcal{A}_j)$ for $i = 2,\ 4,\ 4$, in general.

Example 5.9. Let $X = \{x, y, z\}$, $Y = \{a, b, c\}$, let $\mathcal{A}_1 = \langle \mathbf{A}_1, A_1, \lambda_1\rangle$ and $\mathcal{A}_2 = \langle \mathbf{A}_2, A_2, \lambda_2\rangle$ be two octahedron sets in X given by:
$\mathcal{A}_1(x) = \langle [0.3, 0.6], (0.6, 0.3), 0.6\rangle$, $\mathcal{A}_1(y) = \langle [0.2, 0.7], (0.7, 0.2), 0.7\rangle$,
$\mathcal{A}_1(z) = \langle [0.5, 0.6], (0.7, 0.1), 0.5\rangle$,
$\mathcal{A}_2(x) = \langle [0.4, 0.5], (0.7, 0.2), 0.7\rangle$, $\mathcal{A}_2(y) = \langle [0.3, 0.4], (0.6, 0.3), 0.6\rangle$,
$\mathcal{A}_1(z) = \langle [0.3, 0.8], (0.5, 0.3), 0.7\rangle$.
Let $f : X \to Y$ be the mapping defined by $f(x) = f(y) = a,\ f(z) = c$. Then

$$f(\lambda_1 \wedge \lambda_2)(a) = 0.6 \neq 0.7 = (f(\lambda_1) \wedge f(\lambda_2))(a)$$

and

$$f(A_1 \cap A_2)(a) = (0.6, 0.3) \neq (0.7, 0.2) = (f(A_1) \cap f(A_2)).$$

Thus $f(\mathcal{A}_1 \cup^2 \mathcal{A}_2)(a) \neq (f(\mathcal{A}_1) \cup^2 f(\mathcal{A}_1))(a)$, $f(\mathcal{A}_1 \cup^3 \mathcal{A}_2)(a) \neq (f(\mathcal{A}_1)^4 f(\mathcal{A}_1))(a)$ and $f(\mathcal{A}_1 \cup^4 \mathcal{A}_2)(a) \neq (f(\mathcal{A}_1) \cup^4 f(\mathcal{A}_1))(a)$. So $f(\bigcup^1_{j\in J} \mathcal{A}_j) \neq \bigcup^i_{j\in J} f(\mathcal{A}_j)$ for $i = 2,\ 4,\ 4$.

The following is an immediate result of Definition 5.4 (i).

Proposition 5.10. *If $g : Y \to Z$ is a mapping, then $(g \circ f)^{-1}(\mathcal{C}) = f^{-1}(g^{-1}(\mathcal{C}))$, for each $\mathcal{C} \in O^X$, where $g \circ f$ is the composition of f and g.*

6. Conclusions

We defined an octahedron set and we introduced concepts of internal octahedron sets and external octahedron sets. Moreover, we studied some related properties and gave some examples. Furthermore, we defined an octahedron point and deal with the characterizations of Type i-union (Type i-intersection). Finally, we defined the image and preimage of an octahedron set under a mapping and investigated some of their properties. In the future, we expect that one applies octahedron sets to algebras, topologies, decision-making, measures and entropy measures, etc. Furthermore, In solving real-world problems, we think that it is necessary to extend the concept of octahedron sets as a way of reducing possible information loss, for example, interval-valued intuitionistic octahedron set, soft octahedron set, neutrosophic octahedron set, etc.

Funding: This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2018R1D1A1B07049321)

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[2] K. T. Atanassov and G. Gargov, Interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems 31 (1989) 343–349.
[3] D. Coker A note on intuitionistic sets and intuitionistic points, Tr. J. of Mathematics 20 (1996) 343–351.
[4] D. Coker, An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems 88 (1997)81–89.
[5] Minseok Cheong and K. Hur, Intuitionistic interval-valued fuzzy sets, J. Korean Institute of Intelligent Systems 20 (6) (2010) 864–874.
343–351.
[6] W. L. Gau and D. J. Buchrer, Vague sets, IEEE Trans. Systems Man Cybernet. 23 (2) (1993) 610–614.
[7] M. B. Gorzalczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy sets and Systems 21 (1987) 1–17.
[8] Y. B. Jun, A novel extension of cubic sets and its applications in BCI/BCK-algebras, Ann. Fuzzy Math. Inform. 14 (5) (2017) 475–486.
[9] Y. B. Jun, C. S. Kim and K. O. Yang, Cubic sets, Ann. Fuzzy Math. Inform. 4 (1) (2012) 83–98.
[10] Y. B. Jun, S. Smarandache and C. S. Kim, Neutrosophic cubic sets, New Mathematics and Natural Computation (2015) 1–15.
[11] Y. B. Jun, S. Z. Song and S. J. Kim Cubic interval-valued intuitionistic fuzzy sets and their application in BCK/BCI-algebras, Axioms 2018, 7, 7; doi:10.3390/axioms7010007.

[12] G. Kaur and H. Garg, Multi-attribute decision-making based on bonferroni mean operators under cubic intuitionistic fuzzy environment, Entropy 20 (1) (2018) pp.65 doi:10.3390/e20010065.
[13] G. Kaur and H. Garg, Cubic intuitionistic fuzzy aggregation operators, International Journal for Uncertainty Quantification 8 (5) (2018) 405–428.
[14] J. H. Kim, S. M. Mostafa, P. K. Lim, J. G. Lee, K. Hur, Interval-valued intutionistic cubic sets, To be submitted.
[15] S. J. Lee and E. P. Lee, The category of intuitionistic fuzzy topological spaces, Bull. Korean Math. Soc. 37 (1) (2000) 63–76.
[16] D. Molodtsov, Soft set theory–First results, Computer Math. Applic. 37 (1999) 19–31.
[17] T. K. Mondal and S. K. Samanta, Topology of interval-valued fuzzy sets, Indian J. Pure Appl. Math. 30 (1) (1999) 133–189.
[18] Z. Pawlak, Rough sets, International Journal of Information and Computer Sciences 11 (1982) 341–356.
[19] F. Smarandache, Neutrosophy, Neutrosophic Probability, and Logic, American Research Press, Rehoboth, USA 1998.
[20] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[21] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, Inform. Sci. 8 (1975) 199–249.

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