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Neutrosophic equivalence relation applied on incline algebra

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Abstract

Researchers have great interest in working in the fields of fuzzy algebra and its substructure. Nowadays, the work on fuzzy algebra has attained the greatest height. Considering the incline algebra in this study, the concept of the neutrosophic equivalence relation in incline algebra and its related properties are introduced. Furthermore, the characteristic function and chain conditions are also analyzed, with some results.

Keywords: *Incline algebra, neutrosophic set, neutrosophic subincline, equivalence relation and chain condition.*

MSC (2020): *16Y60, 13E05, 03G25.*

Introduction

Zadeh [28] focused on the thoughts of fuzzy sets and their operations. Fuzzy sets were derived by generalizing the concept of set theory. Fuzzy sets can be thought of as an extension of classical sets. In a classical set (or crisp set), the objects in the set are called elements or members of the set. Researchers are evoked with a great welcome to work in the field of fuzzy set theory, which is developed in different branches of mathematics. Fuzzy sets are generalized by allowing elements to belong to a given set to a certain degree. Instead of taking characteristic functions in $\{0, 1\}$, now by take a new function valued in [0,1]. Atanassov [2] reveals the idea of intuitionistic fuzzy sets by including a component called the non-membership function with a membership function, and Smarandache [19] inserted one more new function named the degree of indeterminacy or neutrality, which is defined on three components, namely, $(t, i, f) = (truth, indeterminacy, falsity)$ and is referred to as the neutrosophic set. This theory, developed by Smarandache [19] in 1999, was extended to the concepts of the classic set and fuzzy sets in 2005.

Incline algebra was originated by Chinese cybernetics expert Cao Zhiqiang [3]. Inclines are generalized from the Boolean and Fuzzy algebras; they found a way to fuse algebras with ordered structures to express the degree of intensity of binary relations. An incline $\mathcal{X}$ is a structure with associative, commutative, and distributive multiplication such that $l+l = l,\ l+lm = m$ $\forall\, l, m \in \mathcal{X}$. It has both a semiring structure and a poset structure. Inclines are formed from the ideals in a ring or semigroup, as do the topologizing filters in a ring. Incline theory is established from semiring and lattice theory. Incline algebra in Monograph was introduced by Cao, Z. Q, Kim, and Roush[3]. In nervous system and automation theory, as well as in many other fields, incline theories are exposed. Ahn [1] et al. investigated the structure of the quotient incline. The inquiry into fuzzy equivalence relations was brought by Chakraborty and Das. Murali has reviewed fuzzy equivalence, congruence relations, and algebraic closure systems in fuzzy set theory. The formal modeling, reasoning, and computing are crisp, accurate, and deterministic in nature with the new mathematical methods. But in reality, crisp data is not always part and parcel of the problems encountered in different fields. As a consequence, various theories, including probability, fuzzy sets, and intuitionistic fuzzy sets, have evolved in the process.

The merging of fuzzy with different algebraic structures was started by Rosenfeld[16], which is evoked with huge interest by many authors, and correlated algebra with fuzzy sets. Jun[9] was the first to apply the theory of fuzzy sets to incline algebra and came up with the concepts of fuzzy subincline algebra and fuzzy ideals of incline algebra, et. al. As a motivation, here in this paper, the fusion of equivalence relations on the neutrosophic set with incline algebra is considered and provided with the related results. Also, the chain conditions and characteristic functions are discussed in an elegant manner.

The work is segmented into the following sections: Section 1 introduces the work, and **Section 2** deals with the fundamental definitions of incline algebra and fuzzy sets. **Section 3** speaks about the idea of the neutrosophic equivalence relation in incline algebra. Whereas, **Section 4** presents the neutrosophic characteristic ideal and chain condition. Finally, **Section 5** concludes about the work.

1. Section 1: Preliminaries

This section reviews some basic facts about incline algebra and fuzzy concepts.

Definition 1.1. *[28] A mapping $\Phi : \mathcal{H} \to [0,1]$ is called a fuzzy set in a universal set $\mathcal{H}$ where $\Phi(l)$ is the membership value of l, $\forall\, l \in \mathcal{H}$.*

Definition 1.2. *[2] An intuitionistic fuzzy set in $\mathcal{H}$ is represented as $\mathcal{B} = \{l, \Phi_{\mathcal{B}}(l), \Omega_{\mathcal{B}}(l)/l \in \mathcal{H}\}$ where $\Phi_{\mathcal{B}} : \mathcal{H} \to [0,1]$ is a membership function and $\Omega_{\mathcal{B}} : \mathcal{H} \to [0,1]$ is a non - membership function satisfying $0 \leq \Phi_{\mathcal{B}}(l) + \Omega_{\mathcal{B}}(r) \leq 1$, $\forall\, l \in \mathcal{H}$.*

Definition 1.3. *[19] An neutrosophic set $\mathcal{R}$ is a structure of the form $\{l, \Phi_{\mathcal{R}}(l), \Psi_{\mathcal{R}}(l), \Omega_{\mathcal{R}}(l)/l \in \mathcal{H}\}$ on $\mathcal{H}$ with $\Phi_{\mathcal{R}}$ is truth membership function, $\Psi_{\mathcal{R}}$ as indeterminate membership function and $\Omega_{\mathcal{R}}$ as an false membership function where $\Phi_{\mathcal{R}}, \Psi_{\mathcal{R}}, \Omega_{\mathcal{R}} : \mathcal{H} \to [0,1]$.*

Definition 1.4. *[3, 9] A non - empty set $(\mathcal{H}, +, *)$ is an incline algebra if $\forall\, l, m \in \mathcal{H}$ the following holds,*
(i) $+$ is commutative and associative
(ii) $*$ is associative and distributive (both left and right) under $+$
(iii) $l + l = l$ (idempotent)

(iv) $l + (l * m) = l$
(v) $m + (l * m) = m$.

Remarks:
1. Every distributive lattice is an incline algebra. But the converse is not true in general.
2. An incline is a distributive lattice if and only if $l * l = l \;\; \forall\, l \in \mathcal{H}$.
3. In an incline, the partial order is defined as $l \leq m \leftrightarrow l + m = m$ and $l \wedge m = min(l, m) \quad l \vee m = max(l, m) \; \forall\, l, m \in \mathcal{H}$.

Definition 1.5. *[3, 10] A subincline of an incline algebra $\mathcal{H}$ is a non - empty subset $\mathcal{I}$ of $\mathcal{H}$ which is closed under addition and multiplication.*

Definition 1.6. *[1] A subincline $\mathcal{I}$ of $\mathcal{H}$ is said to be an ideal if $r \in \mathcal{I}, s \in \mathcal{H}$ and $s \leq r$ then $s \in \mathcal{I}$.*

Definition 1.7. *[14] An non-empty subset $D \in M(\mathcal{H} \times \mathcal{H})$ is called a fuzzy equivalence relation on $\mathcal{H}$, where $M(\mathcal{H} \times \mathcal{H})$ denotes the set of all fuzzy subsets of $\mathcal{H} \times \mathcal{H}$ if*
(i) $D(l, l) = sup\{D(m, n) | m, n \in \mathcal{H}\}$,
(ii) $D(l, m) = D(m, l)$,
(iii) $D(l, n) \wedge D(n, m) \leq D(l, m), \forall\, l, m, n \in \mathcal{H}$
And if moreover, it satisfies
(iv) $D(l_1 + l_2, m_1 + m_2) \wedge D(l_1 * l_2, m_1 * m_2) \geq D(l_1, m_1) \wedge D(l_2, m_2)$,
$\forall\, l_1, l_2, m_1, m_2 \in \mathcal{H}$, D *is a fuzzy congruence relation on* $\mathcal{H}$.

Definition 1.8. *[14, 15] A fuzzy subincline(ideal) $\mathcal{I}$ of $\mathcal{H}$ is said to be fuzzy characteristic if $\mathcal{I}(\Phi(l)) = \mathcal{I}(l) \forall\, l \in \mathcal{H}$ and $\Phi \in Aut(\mathcal{H})$*

2. Section 2: Neutrosophic Equivalence Relation On Incline Algebra

This segregation is based on the study of equivalence relation on incline algebra.

Definition 2.1. *Let ß$= \{\Phi_{\mathcal{E}}, \Psi_{\mathcal{E}}, \Omega_{\mathcal{E}}\} \in M(\mathcal{H} \times \mathcal{H})$ ($M(\mathcal{H} \times \mathcal{H})$ is the set of all neutrosophic subsets of $\mathcal{H} \times \mathcal{H}$) is said to be a neutrosophic equivalence relation on $\mathcal{H}$ if*

(i) $\Phi_{\mathcal{E}}(l, l) = sup\{\Phi_{\mathcal{E}}(m, n)/m, n \in \mathcal{H}\}$,
$\Psi_{\mathcal{E}}(l, l) = sup\{\Psi_{\mathcal{E}}(m, n)/m, n \in \mathcal{H}\}$,

$\Omega_{\mathcal{E}}(l,l) = inf\{\Omega_{\mathcal{E}}(m,n)/m,n \in \mathcal{H}\}$,
(ii) $\Phi_{\mathcal{E}}(l,m) = \Phi_{\mathcal{E}}(m,l)$,
$\Psi_{\mathcal{E}}(l,m) = \Psi_K(m,l)$,
$\Omega_{\mathcal{E}}(l,m) = \Omega_{\mathcal{E}}(m,l)$,
(iii) $\Phi_{\mathcal{E}}(l,n) \wedge \Phi_{\mathcal{E}}(n,m) \leq \Phi_{\mathcal{E}}(l,m)$,
$\Psi_{\mathcal{E}}(l,n) \wedge \Psi_{\mathcal{E}}(n,m) \leq \Psi_{\mathcal{E}}(l,m)$,
$\Omega_{\mathcal{E}}(l,n) \wedge \Omega_{\mathcal{E}}(n,m) \leq \Omega_{\mathcal{E}}(l,m)\ \forall\ l,m,n \in \mathcal{H}$

Also, if it satisifies,
(iv) $\Phi_{\mathcal{E}}(l_1+l_2,m_1+m_2) \wedge \Phi_{\mathcal{E}}(l_1 * l_2,m_1 * m_2) \geq \Phi_{\mathcal{E}}(l_1,m_1) \wedge \Phi_{\mathcal{E}}(l_2,m_2)$
$\Psi_{\mathcal{E}}(l_1+l_2,m_1+m_2) \wedge \Psi_{\mathcal{E}}(l_1 * l_2,m_1 * m_2) \geq \Psi_{\mathcal{E}}(l_1,m_1) \wedge \Psi_{\mathcal{E}}(l_2,m_2)$
$\Omega_{\mathcal{E}}(l_1+l_2,m_1+m_2) \vee \Omega_{\mathcal{E}}(l_1 * l_2,m_1 * m_2) \leq \Omega_{\mathcal{E}}(l_1,m_1) \vee \Omega_{\mathcal{E}}(l_2,m_2)\ \forall$ $l_1,l_2,m_1,m_2 \in \mathcal{H}$ *implies that* $\mathcal{E}$ *is a neutrosophic congruence relation on* $\mathcal{H}$.

Theorem 2.2. *Let* $\mathcal{R} \in M(\mathcal{H})$,*(*$M(\mathcal{H})$ *is the set of all neutrosophic subsets of* $\mathcal{H}$*) be a neutrosophic ideal, define a neutrosophic subset* $\mathcal{A}_{\mathcal{R}} \in M(\mathcal{H}\times\mathcal{H})$ *by* $\mathcal{A}_{\Phi_{\mathcal{R}}}(l,m) = sup\{\Phi_{\mathcal{R}}(c)/l+c = m+c, c \in \mathcal{H}\}$, $\mathcal{A}_{\Psi_{\mathcal{R}}}(l,m) = sup\{\Psi_{\mathcal{R}}(c)/l+c = m+c, c \in \mathcal{H}\}$; $\mathcal{A}_{\Omega_{\mathcal{R}}}(l,m) = inf\{\Omega_{\mathcal{R}}(c)/l+c = m+c, c \in \mathcal{H}\}\ \forall\ l,m\ \in \mathcal{H}$ *then* $\mathcal{A}_{\mathcal{R}}$ *is a neutrosophic congruence relation on* $\mathcal{H}$, $\mathcal{A}_{\mathcal{R}}$ *is named as neutrosophic relation induced by* $\mathcal{R}$.

Proof:
For $l,m,n \in \mathcal{H}$

$$\begin{aligned}
\mathcal{A}_{\Phi_{\mathcal{R}}}(l,l) &= sup\{\Phi_{\mathcal{R}}(c)/l+c = l+c, c \in \mathcal{H}\}\\
&= sup\{\Phi_{\mathcal{R}}(c)/c \in \mathcal{H}\}\\
\mathcal{A}_{\Phi_{\mathcal{R}}}(m,n) &= sup\{\Phi_{\mathcal{R}}(d)/m+d = n+d, d \in \mathcal{H}\}\\
&= sup\{\Phi_{\mathcal{R}}(d)/d \in \mathcal{H}\} = \mathcal{A}_{\Phi_{\mathcal{R}}}(l,l)
\end{aligned}$$

Hence, $\mathcal{A}_{\Phi_{\mathcal{R}}}(l,l) = sup\{\Phi_{\mathcal{R}}(m,n)/m,n \in \mathcal{H}\}$
Also, $\mathcal{A}_{\Phi_{\mathcal{R}}}(l,m) = \mathcal{A}_{\Phi_{\mathcal{R}}}(m,l)\ \forall\ l,m \in \mathcal{H}$
If $l+c = n+c$ & $n+d = m+d$ then $l+e = m+e$ where $e = c+d$, by using the Proposition (3.3) in [7]

$$\begin{aligned}
\mathcal{A}_{\Phi_{\mathcal{R}}}(l,m) \wedge \mathcal{A}_{\Phi_{\mathcal{R}}}(m,n) &= sup\{\Phi_{\mathcal{R}}(c)/l+c = m+c, c \in \mathcal{H}\}\wedge\\
&\quad\ sup\{\Phi_{\mathcal{R}}(d)/m+d = n+d, d \in \mathcal{R}\}\\
&= sup\{\Phi_{\mathcal{R}}(c+d)/l+c = m+c, m+d\\
&= n+d; c,d \in \mathcal{H}\}\\
&\leq sup\{\Phi_{\mathcal{R}}(e)/l+e = m+e, e \in \mathcal{H}\}\\
&= \mathcal{A}_{\Phi_{\mathcal{R}}}(l,m)
\end{aligned}$$

Let $l_1,l_2,m_1,m_2\ \in \mathcal{H}$, then

$$\begin{aligned}\mathcal{A}_{\Phi_{\mathcal{R}}}(l_1,m_1)\wedge\mathcal{A}_{\Phi_{\mathcal{R}}}(l_2,m_2) &= sup\{\Phi_{\mathcal{R}}(c)/l_1+c=m_1+c,c\in\mathcal{H}\}\wedge\\ &\quad sup\{\Phi_{\mathcal{R}}(d)/l_1+d=m_2+d,d\in L\}\\ &= sup\{\Phi_{\mathcal{R}}(l+m)/l_1+c=m_1+c,l_2+d\\ &= m_2+d;c,d\in\mathcal{H}\}\\ &\leq sup\{\Phi_{\mathcal{R}}(e)/(l_1+l_2)+e\\ &= (m_1+m_2)+e,e\in\mathcal{H}\}\\ &= \mathcal{A}_{\Phi_{\mathcal{R}}}(l_1+l_2,m_1+m_2)\end{aligned}$$

Morever, if $l_1+c=m_1+c$ and $l_2+b=m_2+d$ then
$(l_1+c)*(l_2+d)=(m_1+c)*(m_2+d)$ and so
$(l_1*l_2)+\{(c*l_2)+(l_1*d)+(c,d)\}=(m_1*m_2)+\{(c+m_2)+(m_1+d)+(c*d)\}$
Since $(l_1+c)*d=(m_1+c)*d$ and $c*(l_2+d)=c*(m_2+d)$, we get
$(c*l_2)+(l_1*d)+(c*d)=(c*m_2)+(m_1*d)+(c*d)$
Since $\mathcal{R}$ is a neutrosophic ideal of $\mathcal{H}$,
$\Phi_{\mathcal{R}}((c*l_2)+(l_1*d)+(c*d))=\Phi_{\mathcal{R}}(c*l_2)\wedge\Phi_{\mathcal{R}}(l_1*d)\wedge\Phi_{\mathcal{R}}(c*d)$
$\geq\Phi_{\mathcal{R}}(c)\wedge\Phi_{\mathcal{R}}(d)$

Hence,

$$\begin{aligned}\mathcal{A}_{\Phi_{\mathcal{R}}}(l_1,m_1)\wedge\mathcal{A}_{\Phi_{\mathcal{R}}}(l_2,m_2) &= sup\{\Phi_{\mathcal{R}}(c)/l_1+c=m_1+c,c\in\mathcal{H}\}\wedge\\ &\quad sup\{\Phi_{\mathcal{R}}(d)/m_2+d=m_2+d,d\in\mathcal{H}\}\\ &= sup\{\Phi_{\mathcal{R}}(c)\wedge\Phi_{\mathcal{R}}(d)/l_1+c\\ &= m_1+c,l_2+d=m_2+d,c,d\in\mathcal{H}\}\\ &\leq sup\{\Phi_{\mathcal{R}}((c*l_2)+(l_1*d)+(c*d))/l_1+c\\ &= m_1+c,l_2+d=m_2+d,c,d\in\mathcal{H}\}\\ &\leq sup\{\Phi_{\mathcal{R}}((c*l_2)+(l_1*d)+(c*d))/\\ &\quad (l_1*l_2)+(c*l_2)+(l_1*d)+(c,d)\\ &\quad = (m_1*m_2)+\{(c+m_2)+(m_1+d)\\ &+(c*d)\}\\ &\leq sup\{\Phi_{\mathcal{R}}(e)/(l_1*l_2)+e\\ &= (m_1*m_2)+e,\ e\in\mathcal{H}\}\\ &= \mathcal{A}_{\Phi_{\mathcal{R}}}(l_1*l_2,m_1*m_2)\end{aligned}$$

On combining,
$\mathcal{A}_{\Phi_{\mathcal{R}}}(l_1+l_2,m_1+m_2)\wedge\mathcal{A}_{\Phi_{\mathcal{R}}}(l_1*l_2,m_1*m_2)\geq\mathcal{A}_{\Phi_{\mathcal{R}}}(l_1,m_1)\wedge\mathcal{A}_{\Phi_{\mathcal{R}}}(l_2,m_2)$
Similarly, for the intermediate and falsity function.

Theorem 2.3. *Considering an idempotent incline $\mathcal{H}$ and if $\mathcal{E}\in M(\mathcal{H}\times\mathcal{H})$ is a neutrosophic congruence relation on $\mathcal{H}$, then the neutrosophic set $\mathcal{K}_{\mathcal{E}}=(\Phi_{\mathcal{E}},\Psi_{\mathcal{E}},\Omega_{\mathcal{E}})\in M(\mathcal{H})$ is defined as $\mathcal{K}_{\Phi_{\mathcal{E}}}(l)=inf\{\Phi_{\mathcal{E}}(l*m,l)/m\in\mathcal{H}\},\mathcal{K}_{\Psi_{\mathcal{E}}}(l)=inf\{\Psi_{\mathcal{E}}(l*m,l)/m\in\mathcal{H}\};\mathcal{K}_{\Omega_{\mathcal{E}}}(l)=sup\{\Omega_{\mathcal{E}}(l*m,l)/m\in\mathcal{H}\}$, $\forall\ l\ \in\mathcal{H}$ is said to be a neutrosophic ideal of $\mathcal{H}$.*

Proof:

For $l, m \in \mathcal{H}$ and if $l \leq m$ then, $\mathcal{E}$ is a neutrosophic congruence relation.

$$\begin{aligned} \mathcal{K}_{\Phi_{\mathcal{E}}}(l) &= inf\{\Phi_{\mathcal{E}}(l * n, m)/n \in \mathcal{H}\} \\ &= inf\{\Phi_{\mathcal{E}}(l * (m * n), l * m)/n \in \mathcal{H}\} \end{aligned}$$

Since by a lemma 3,17 in [7]

If $\mathcal{H}$ is an idempotent incline and if $l \leq m$ in $\mathcal{H}$ then $l * m = m$

$$\begin{aligned} &\geq inf\{\Phi_{\mathcal{E}}(l, l) \wedge \Phi_{\mathcal{E}}(m * n), m)/n \in \mathcal{H}\} \\ &= inf\{\Phi_{\mathcal{E}}(m * n), m)/n \in \mathcal{H}\} \\ &= \mathcal{K}_{\Phi_{\mathcal{E}}}(m) \\ \mathcal{K}_{\Phi_{\mathcal{E}}}(l) &\geq \mathcal{K}_{\Phi_{\mathcal{E}}}(m) \end{aligned}$$

Thus, $\mathcal{K}_{\mathcal{E}}$ is a order reversing.

Since, $l + (l * m) = l$ and $m + (l * m) = m$, implies

$\mathcal{K}_{\Phi_{\mathcal{E}}}(l * m) \geq \mathcal{K}_{\Phi_{\mathcal{E}}}(l)$ & $\mathcal{K}_{\Phi_{\mathcal{E}}}(l * m) \geq \mathcal{K}_{\Phi_{\mathcal{E}}}(m)$

$\mathcal{K}_{\Phi_{\mathcal{E}}}(l * m) \geq \mathcal{K}_{\Phi_{\mathcal{E}}}(l) \wedge \mathcal{K}_{\Phi_{\mathcal{E}}}(m)$

And

$$\begin{aligned} \mathcal{K}_{\Phi_{\mathcal{E}}}(l + m) &= inf\{\Phi_{\mathcal{E}}((l + m) * n, l + m)/n \in \mathcal{H}\} \\ &= inf\{\Phi_{\mathcal{E}}((l * n) + (m * n), l + m)/n \in \mathcal{H}\} \\ &\geq inf\{\Phi_{\mathcal{E}}((l * n, l) \wedge \Phi_{\mathcal{E}}(m * n), m)/n \in \mathcal{H}\} \\ &= inf\{\Phi_{\mathcal{E}}((l * n, l)/n \in \mathcal{H}\} \wedge inf\{\Phi_{\mathcal{E}}(m * n), m)/n \in \mathcal{H}\} \\ &= \mathcal{K}_{\Phi_{\mathcal{E}}}(l) \wedge \mathcal{K}_{\Phi_{\mathcal{E}}}(m) \end{aligned}$$

where $\mathcal{E}$ is a neutrosophic congruence relation

$\mathcal{K}_{\Phi_{\mathcal{E}}}(l + m) \wedge \mathcal{K}_{\Phi_{\mathcal{E}}}(l * m) \geq \mathcal{K}_{\Phi_{\mathcal{E}}}(l) \wedge \mathcal{K}_{\Phi_{\mathcal{E}}}(m)$

Similarly, $\mathcal{K}_{\Psi_{\mathcal{E}}}(l + m) \wedge \mathcal{K}_{\Psi_{\mathcal{E}}}(l * m) \geq \mathcal{K}_{\Psi_{\mathcal{E}}}(l) \wedge \mathcal{K}_{\Psi_{\mathcal{E}}}(m)$

And

$$\begin{aligned} \mathcal{K}_{\Omega_{\mathcal{E}}}(l) &= sup\{\Omega_{\mathcal{E}}(l * n, l)/n \in \mathcal{H}\} \\ &= sup\{\Omega_{\mathcal{E}}(l * (m * n), l * m)/n \in \mathcal{H}\} \\ &\leq sup\{\Omega_{\mathcal{E}}(l, l) \wedge \Omega_{\mathcal{E}}(m * n), m)/n \in \mathcal{H}\} \\ &= sup\{\Omega_{\mathcal{E}}(m * n), m)/n \in \mathcal{H}\} \\ &= \mathcal{K}_{\Omega_{\mathcal{E}}}(m) \\ \mathcal{K}_{\Omega_{\mathcal{E}}}(l) &\leq \mathcal{K}_{\Omega_{\mathcal{E}}}(m) \end{aligned}$$

$\mathcal{K}_{\Omega_{\mathcal{E}}}(l * m) \leq \mathcal{K}_{\Omega_{\mathcal{E}}}(l)$ or $\mathcal{K}_{\Omega_{\mathcal{E}}}(l * m) \leq \mathcal{K}_{\Omega_{\mathcal{E}}}(m)$, we get

$\mathcal{K}_{\Omega_{\mathcal{E}}}(l * m) \leq \mathcal{K}_{\Omega_{\mathcal{E}}}(l) \vee \mathcal{K}_{\Omega_{\mathcal{E}}}(m)$

$$\begin{aligned} \mathcal{K}_{\Omega_{\mathcal{E}}}(l + m) &= sup\{\Omega_{\mathcal{E}}((l + m) * n, l + m)/n \in \mathcal{H}\} \\ &= sup\{\Omega_{\mathcal{E}}((l * n) + (m * n), l + m)/n \in \mathcal{H}\} \\ &\leq sup\{\Omega_{\mathcal{E}}((l * n, l) \vee \Omega_{\mathcal{E}}(m * n), m)/n \in \mathcal{H}\} \\ &= sup\{\Omega_{\mathcal{E}}((l * n, l)/n \in \mathcal{H}\} \vee inf\{\Omega_{\mathcal{E}}(m * n), m)/n \in \mathcal{H}\} \\ &= \mathcal{K}_{\Omega_{\mathcal{E}}}(l) \vee \mathcal{K}_{\Omega_{\mathcal{E}}}(m) \end{aligned}$$

Theorem 2.4. *If $\mathcal{R}$ in $M(\mathcal{H})$ is a neutrosophic ideal, then $\mathcal{R} \subseteq \mathcal{K}_{\mathcal{A}_{\mathcal{R}}}$*

Proof:
Let $l \in \mathcal{H}$, then $\mathcal{K}_{\mathcal{A}_{\mathcal{R}}}(l) = (\mathcal{A}_{\Phi_{\mathcal{R}}}, \mathcal{A}_{\Psi_{\mathcal{R}}}, \mathcal{A}_{\Omega_{\mathcal{R}}})(l)$
Now,
$\mathcal{K}_{\mathcal{A}_{\mathcal{R}}}(l) = (inf\{\mathcal{A}_{\Phi_{\mathcal{R}}}(l*m, l)/m \in \mathcal{H}\}, inf\{\mathcal{A}_{\Psi_{\mathcal{R}}}(l*m, l)/m \in \mathcal{H}\}; sup\{\mathcal{A}_{\Omega_{\mathcal{R}}}(l* m, l)/m \in \mathcal{H}\})$
since $l + (l * m) = l = l + l$

$$\begin{aligned}
\mathcal{A}_{\Phi_{\mathcal{R}}}(l*m, l) &= sup\{\Phi_{\mathcal{R}}(c)/(l*m) + c = l + c, c \in \mathcal{H}\} \\
&\geq \Phi_{\mathcal{R}}(l) \\
\mathcal{A}_{\Psi_{\mathcal{R}}}(l*m, l) &= sup\{\Psi_{\mathcal{R}}(c)/(l*m) + c = l + c, c \in \mathcal{H}\} \\
&\geq \Psi_{\mathcal{R}}(l) \\
\mathcal{A}_{\Omega_{\mathcal{R}}}(l*m, l) &= inf\{\Phi_{\mathcal{R}}(c/(l*m) + c = l + c, c \in \mathcal{H}\} \\
&\leq \Omega_{\mathcal{R}}(l) \\
\mathcal{K}_{\mathcal{A}_{\Phi_{\mathcal{R}}}}(l) &\geq inf\{\Phi_{\mathcal{R}}(l)/m \in \mathcal{H}\} \\
\mathcal{K}_{\mathcal{A}_{\Psi_{\mathcal{R}}}}(l) &\geq inf\{\Psi_{\mathcal{R}}(l)/m \in \mathcal{H}\} \\
\mathcal{K}_{\mathcal{A}_{\Omega_{\mathcal{R}}}}(l) &\leq sup\{\Omega_{\mathcal{R}}(l)/m \in \mathcal{H}\} \\
\mathcal{K}_{\mathcal{A}_{\mathcal{R}}} &= \mathcal{R}(l) \\
i.e., \mathcal{R} &\subseteq \mathcal{K}_{\mathcal{A}_{\mathcal{R}}}
\end{aligned}$$

Definition 2.5. [7] *An element 0 in an incline $\mathcal{H}$ is said to be zero if for any $l \in \mathcal{H}$, $l + 0 = l = 0 + l$ and $l * 0 = 0 * l = 0$*

Theorem 2.6. *If $\mathcal{H}$ satisfies $l + 0 = l$ & $l * 0 = 0 * l = 0 \ \forall \ l \in \mathcal{H}$ then $\mathcal{K}_{\mathcal{A}_{\mathcal{R}}} \subseteq \mathcal{R}$ for every neutrosophic ideal $\mathcal{R}$ of $\mathcal{H}$.*

Proof:
It should be noted that $(l*0)+c = 0+c = c$, $\forall \ c, l \in \mathcal{H}$ and $\forall \ l \in \mathcal{H}$. Now,

$$\begin{aligned}
\mathcal{K}_{\mathcal{A}_{\mathcal{R}}}(l) &= \{\mathcal{A}_{\Phi_{\mathcal{R}}}, \mathcal{A}_{\Psi_{\mathcal{R}}}, \mathcal{A}_{\Omega_{\mathcal{R}}}\}(l) \\
\mathcal{K}_{\mathcal{A}_{\mathcal{R}}}(l) &= (inf\{\mathcal{A}_{\Phi_{\mathcal{R}}}(l*m, l)/m \in \mathcal{H}\}, \\
&\quad inf\{\mathcal{A}_{\Psi_{\mathcal{R}}}(l*m, l)/m \in \mathcal{H}\}; \\
&\quad sup\{\mathcal{A}_{\Omega_{\mathcal{R}}}(l*m, l)/m \in \mathcal{H}\}) \\
&\leq (\mathcal{A}_{\Phi_{\mathcal{R}}}(l*0, l)), \mathcal{A}_{\Psi_{\mathcal{R}}}(l*0, l), \mathcal{A}_{\Omega_{\mathcal{R}}}(l*0, l)) \\
\mathcal{A}_{\Phi_{\mathcal{R}}}(l*0, l) &= sup\{\Phi_{\mathcal{R}}(c)/(l*0) + c = l + c, c \in \mathcal{H} \ \} \\
&= sup\{\Phi_{\mathcal{R}}(c)/c = l + c, c \in \mathcal{H}\} \\
&= sup\{\Phi_{\mathcal{R}}(c)/l \leq c, c \in \mathcal{H}\} \\
&\leq \Phi_{\mathcal{R}}(l)
\end{aligned}$$

Similarly, $\mathcal{A}_{\Psi_\mathcal{R}}(l * 0, l) \leq \Psi_\mathcal{R}(l)$ *and* $\mathcal{A}_{\Omega_\mathcal{R}}(l * 0, l) \leq \Omega_\mathcal{R}(l)$
i.e., $\mathcal{R}$ is order reversing.
$\mathcal{K}_{\mathcal{A}_\mathcal{R}} \subseteq \mathcal{R}$.

Definition 2.7. *Let $\mathcal{R}$ be the family of all neutrosophic ideals of an incline algebra $\mathcal{H}$ and let $u, v, w \in [0,1]$. Define a binary relation $\mathcal{U}^u, \mathcal{V}^v$ & $\mathcal{W}^w$ on $\mathcal{R}$ such that $\mathcal{R}_1 = (\Phi_{\mathcal{R}_1}, \Psi_{\mathcal{R}_1}, \Omega_{\mathcal{R}_1})$ & $\mathcal{R}_2 = (\Phi_{\mathcal{R}_2}, \Psi_{\mathcal{R}_2}, \Omega_{\mathcal{R}_2})$ in $\mathcal{R}$.*

$$(\mathcal{R}_1, \mathcal{R}_2) \in \mathcal{U}^u \quad \Leftrightarrow \mathcal{U}(\Phi_{\mathcal{R}_1}, u) = \mathcal{U}(\Phi_{\mathcal{R}_2}, u)$$
$$(\mathcal{R}_1, \mathcal{R}_2) \in \mathcal{V}^v \quad \Leftrightarrow \mathcal{V}(\Psi_{\mathcal{R}_1}, v) = \mathcal{V}(\Phi_{\mathcal{R}_2}, v)$$
$$(\mathcal{R}_1, \mathcal{R}_2) \in \mathcal{W}^w \quad \Leftrightarrow \mathcal{W}(\Omega_{\mathcal{R}_1}, w) = \mathcal{W}(\Omega_{\mathcal{R}_2}, w)$$

Clearly, $\mathcal{U}^u, \mathcal{V}^v, \mathcal{W}^w$ are equivalence relations on $\mathcal{R}$

(i) For $\mathcal{R}_1 = (\Phi_{\mathcal{R}_1}, \Psi_{\mathcal{R}_1}, \Omega_{\mathcal{R}_1}) \in \mathcal{R}$, let $[\Phi_{\mathcal{R}_1}]_{\mathcal{U}^u}$ [respectively $[\Psi_{\mathcal{R}_1}]_{\mathcal{V}^v}$, $[\Omega_{\mathcal{R}_1}]_{\mathcal{W}^w}$] denotes the equivalence class of $\Phi_{\mathcal{R}_1}$ modulo $\mathcal{U}^u$ [$\Psi_{\mathcal{R}_1}$ of $\mathcal{V}^v$, $\Omega_{\mathcal{R}_1}$ of $\mathcal{W}^w$]

(ii) $\Phi_{\mathcal{R}_1}/\mathcal{U}^u$ [$\Psi_{\mathcal{R}_1}/\mathcal{V}^v, \Omega_{\mathcal{R}_1}/\mathcal{W}^w$] denote the system of all equivalence classes modulo $\mathcal{U}^u$ [$\mathcal{V}^v, \mathcal{W}^w$], where $\Phi_{\mathcal{R}_1}/\mathcal{U}^u = \{[\Phi_{\mathcal{R}_1}]_{\mathcal{U}^u}/\Phi_{\mathcal{R}_1} \in \mathcal{R}\}$, respectively, $\Psi_{\mathcal{R}_1}/\mathcal{V}^v = \{[\Psi_{\mathcal{R}_1}]_{\mathcal{V}^v}/\Psi_{\mathcal{R}_1} \in \mathcal{R}\}$ and $\Omega_{\mathcal{R}_1}/\mathcal{W}^w = \{[\Omega_{\mathcal{R}_1}]_{\mathcal{W}^w}/\Omega_{\mathcal{R}_1} \in \mathcal{R}\}$.

Let $\mathcal{Z}$ be the family of all ideals of $\mathcal{R}$ and let $u, v, w \in [0,1]$, now define a mapping f_u, g_v, h_w from neutrosophic ideal of $\mathcal{R}$ to $\mathcal{Z} \cup \{\phi\}$ as

$$f_u[\Phi_{\mathcal{R}_1}] = \mathcal{U}[\Phi_{\mathcal{R}_1}, u];$$
$$g_v[\Psi_{\mathcal{R}_1}] = \mathcal{V}[\Psi_{\mathcal{R}_1}, v];$$
$$h_w[\Omega_{\mathcal{R}_1}] = \mathcal{W}[\Omega_{\mathcal{R}_1}, w] \ \ \forall \ \mathcal{R}_1 = (\Phi_{\mathcal{R}_1}, \Psi_{\mathcal{R}_1}, \Omega_{\mathcal{R}_1}) \in \mathcal{R}$$

Hence clearly, f_u, g_v, h_w are well defined.

Theorem 2.8. *For any $u, v, w \in [0,1]$ the mapping f_u, g_v, h_w are surjective from $\mathcal{R}$ to $\mathcal{Z} \cup \{\phi\}$.*

Proof:
Let $u, v, w \in [0,1]$, since $0^* = (0,1)$ in neutrosophic ideal and is defined as $0(l) = 0$ & $1(l) = 1 \ \forall \ l \in \mathcal{H}$, now obviously

$$f_u(0^*) = \mathcal{U}(0, u) = \phi,$$
$$g_v(0^*) = \mathcal{V}(0, v) = \phi,$$
$$h_w(0^*) = \mathcal{W}(0, w) = \phi$$

Let $\mathcal{Z}_1(\neq \phi)$ in $\mathcal{Z}$ and let $\mathcal{Z}_1^* = (\Phi_{\mathcal{Z}_1}, \Psi_{\mathcal{Z}_1}, \Omega_{\mathcal{Z}_1})$ in neutrosophic ideal $\mathcal{R}$

$f_u(\Phi_{\mathcal{Z}_1}) = \mathcal{U}(\Phi_{\mathcal{Z}_1}, u) = \mathcal{Z},$
$g_v(\Psi^*_{\mathcal{Z}_1}) = \mathcal{V}(\Psi_{\mathcal{Z}_1}, v) = \mathcal{Z},$
$h_w(\Omega^*_{\mathcal{Z}_1}) = \mathcal{W}(\Omega_{\mathcal{Z}_1}, w) = \mathcal{Z}$

Thus f_u, g_v, h_w are surjective.

Theorem 2.9. *Let $\Phi_{\mathcal{R}}/\mathcal{U}^u$ $[\Psi_{\mathcal{R}}/\mathcal{V}^v, \Omega_{\mathcal{R}}/\mathcal{W}^w]$ are the quotient sets and are equipotent to $\mathcal{Z} \cup \{\phi\}$, $\forall\ u, v, w\ \in (0,1)$.*

Proof:
For $u, v, w\ \in (0,1)$, let $f_u^{\sim}, g_v^{\sim}, h_w^{\sim} : \Phi_{\mathcal{R}_1}/\mathcal{U}^u, \Psi_{\mathcal{R}_1}/\mathcal{V}^v, \Omega_{\mathcal{R}_1}/\mathcal{W}^w$ to $\mathcal{Z} \cup \{\phi\}$ is defined by

$f_u^{\sim}[[\Phi_{\mathcal{R}_1}]_{\mathcal{U}^u}] = f_u[\Phi_{\mathcal{R}_1}]$
$g_v^{\sim}[[\Psi_{\mathcal{R}_1}]_{\mathcal{V}^v}] = g_v[\Psi_{\mathcal{R}_1}]$
$h_w^{\sim}[[\Omega_{\mathcal{R}_1}]_{\mathcal{W}^w}] = h_w[\Omega_{\mathcal{R}_1}]\ \forall\ \mathcal{R}_1 = (\Phi_{\mathcal{R}_1}, \Psi_{\mathcal{R}_1}, \Omega_{\mathcal{R}_1}) \in \mathcal{R}$

Now, $\mathcal{U}[\Phi_{\mathcal{R}_1}, u] = \mathcal{U}[\Phi_{\mathcal{R}_2}, u]$
$\mathcal{V}[\Phi_{\mathcal{R}_1}, v] = \mathcal{V}[\Phi_{\mathcal{R}_2}, v]$
$\mathcal{W}[\Omega_{\mathcal{R}_1}, w] = \mathcal{W}[\Omega_{\mathcal{R}_2}, w]$, for $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{R}$

then $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{U}^u$, $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{V}^v$ and $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{W}^w$,

$[\Phi_{\mathcal{R}_1}]_{\mathcal{U}^u} = [\Phi_{\mathcal{R}_2}]_{\mathcal{U}^u}, [\Psi_{\mathcal{R}_1}]_{\mathcal{V}^v} = [\Psi_{\mathcal{R}_2}]_{\mathcal{V}^v}$ *and* $[\Omega_{\mathcal{R}_1}]_{\mathcal{W}^w} = [\Omega_{\mathcal{R}_2}]_{\mathcal{W}^w}$

Thus, $f_u^{\sim}, g_v^{\sim}, h_w^{\sim}$ are injective.

Let $\mathcal{Z}_1(\neq \phi)$ in $\mathcal{Z}$ and $\mathcal{Z}_1^* = (\Phi_{\mathcal{Z}_1}, \Psi_{\mathcal{Z}_1}, \Omega_{\mathcal{Z}_1})$ in neutrosophic ideal $\mathcal{R}$ of $\mathcal{H}$,

$f_u^{\sim}[[\mathcal{Z}_1^*]_{\mathcal{U}^u}] = f_u[\mathcal{Z}_1^*] = \mathcal{U}[\Phi_{\mathcal{Z}_1}, u] = \mathcal{Z}$
$g_v^{\sim}[[\mathcal{Z}_1^*]_{\mathcal{V}^v}] = g_w[\mathcal{Z}_1^*] = \mathcal{V}[\Phi_{\mathcal{Z}_1}, v] = \mathcal{Z}$
$h_w^{\sim}[[\mathcal{Z}_1^*]_{\mathcal{W}^w}] = h_w[\mathcal{Z}_1^*] = \mathcal{W}[\Omega_{\mathcal{Z}_1}, w] = \mathcal{Z}$

And, $0^* \in (0,1)\ \in \mathcal{R}$

$f_u^{\sim}[[0^*]_{\mathcal{U}^u}] = f_u[0^*] = \mathcal{U}[0, u] = \mathcal{Z}$
$g_v^{\sim}[[0^*]_{\mathcal{V}^v}] = g_v[0^*] = \mathcal{V}[0, v] = \mathcal{Z}$
$h_w^{\sim}[[0^*]_{\mathcal{W}^w}] = h_w[0^*] = \mathcal{W}[0, w] = \mathcal{Z}$

$f_u^{\sim}, g_v^{\sim}, h_w^{\sim}$ are surjective.

Theorem 2.10. *For $u, v, w \in (0,1)$, a mapping $\sigma_{u,v,w} : \mathcal{R} \to \mathcal{Z} \cup \{\phi\}$ is defined as $\sigma_{u,v,w}(\mathcal{R}) = f_u(\Phi_{\mathcal{R}}) \cap g_v(\Psi_{\mathcal{R}}) \cap h_w(\Omega_{\mathcal{R}})$ for each $\mathcal{R}_1 = (\Phi_{\mathcal{R}_1}, \Psi_{\mathcal{R}_1}, \Omega_{\mathcal{R}_1}) \in \mathcal{R}$ is surjective.*

Proof:
Let $u, v, w \in (0,1)$ & $0^* \in (0,1)\ \in \mathcal{R}$

$$\sigma_{u,v,w}(0^*) = f_u(0^*) \cap g_v(0^*) \cap h_w(0^*) = \mathcal{U}(0, u) \cap \mathcal{V}(0, v) \cap \mathcal{W}(1, w) = \phi$$

For any $\mathcal{R}^* \in \mathcal{R}$, there exists $\mathcal{R}^{'^*} = (\Phi_{\mathcal{R}'^*}, \Psi_{\mathcal{R}'^*}, \Omega_{\mathcal{R}'^*})$ in $\mathcal{R}$ such that

$$\begin{aligned}\sigma_{u,v,w}(\mathcal{R}^{'^*}) &= f_u(\Phi_{\mathcal{R}'^*}) \cap g_v(\Psi_{\mathcal{R}'^*}) \cap h_w(\Omega_{\mathcal{R}'^*}) \\ &= \mathcal{U}(\Phi_{\mathcal{R}'^*}, u) \cap \mathcal{V}(\Psi_{\mathcal{R}'^*}, v) \cap \mathcal{W}(\Omega_{\mathcal{R}'^*}, w) \\ &= \mathcal{R}^*\end{aligned}$$

Remark:
Let us define another relation $G^{u,v,w}$ on R such that $u, v, w \in [0, 1]$

$$\begin{aligned}(\mathcal{R}_1, \mathcal{R}_2) \in G^{u,v,w} &\Leftrightarrow \mathcal{U}(\Phi_{\mathcal{R}_1}, u) \cap \mathcal{V}(\Phi_{R_1}, v) \cap \mathcal{W}(\Omega_{R_1}, w) \\ &= \mathcal{U}(\Phi_{\mathcal{R}_2}, u) \cap \mathcal{V}(\Phi_{\mathcal{R}_2}, v) \cap \mathcal{W}(\Omega_{\mathcal{R}_2}, w)\end{aligned}$$

where $\mathcal{R}_1, \mathcal{R}_2 \in \mathcal{R}$ thus, the relation $G^{u,v,w}$ is also an equivalence relation on $\mathcal{R}$.

Theorem 2.11. *For $u, v, w \in (0, 1)$, the quotient set $\mathcal{R}/G^{u,v,w}$ is equipotent to $\mathcal{Z} \cup \{\phi\}$.*

3. Section 3: Neutrosophic Characteristic Ideal and Chain Condition

This part deals with the neutrosophic characteristic ideal and the chain conditions.

Definition 3.1. *If $\mathcal{R}$ be a neutrosophic ideal of $\mathcal{H}$, τ is a mapping from $\mathcal{H}$ to $\mathcal{H}$ and let us define a mapping $\mathcal{R}^\tau; \mathcal{H} \to [0,1]$ by $\mathcal{R}^\tau(l) = \{\Phi^\tau_{\mathcal{R}}(l), \Psi^\tau_{\mathcal{R}}(l), \Omega^\tau_{\mathcal{R}}(l)\} = \{\Phi_{\mathcal{R}}(\tau(l)), \Psi_{\mathcal{R}}(\tau(l)), \Omega_{\mathcal{R}}(\tau(l))/l \in \mathcal{H}\}$.*

Theorem 3.2. *If $\mathcal{R}$ is a neutrosophic ideal of $\mathcal{H}$ and τ is an endomorphism of $\mathcal{H}$, then $\mathcal{R}^\tau$ is a neutrosophic ideal of $\mathcal{H}$.*

Proof:
For $l, m \in \mathcal{H}$ and $\mathcal{R} = (\Phi_{\mathcal{R}}, \Psi_{\mathcal{R}}, \Omega_{\mathcal{R}})$

$$\begin{aligned}\Phi^\tau_{\mathcal{R}}(l+m) \wedge \Phi^\tau_{\mathcal{R}}(l*m) &= \Phi_{\mathcal{R}}(\tau(l+m)) \wedge \Phi_{\mathcal{R}}(\tau(l*m)) \\ &= \Phi_{\mathcal{R}}(\tau(l)+\tau(m)) \wedge \Phi_{\mathcal{R}}(\tau(l)*\tau(m)) \\ &\geq \Phi_{\mathcal{R}}(\tau(l)) \wedge \Phi_{\mathcal{R}}(\tau(m)) \\ &= \Phi^\tau_{\mathcal{R}}(l) \wedge \Phi^\tau_{\mathcal{R}}(m)\end{aligned}$$

Let $l \leq m \,\&\, l+m = m, so \tau(m) = \tau(l+m) = \tau(l)+\tau(m) therefore \tau(l) \leq \tau(m) \Phi^\tau_{\mathcal{R}}(l) = \Phi_{\mathcal{R}}(\tau(l)) \geq \Phi_{\mathcal{R}}(\tau(m)) \geq \Phi^\tau_{\mathcal{R}}(m)$ *Similarly,* $\Psi^\tau_{\mathcal{R}}(l+m) \wedge \Psi^\tau_{\mathcal{R}}(l*m) \geq \Psi^\tau_{\mathcal{R}}(l) \wedge \Psi^\tau_{\mathcal{R}}(m) \,\& \Psi^\tau_{\mathcal{R}}(l) \geq \Psi^\tau_{\mathcal{R}}(m) \Omega^\tau_{\mathcal{R}}(l+m) \vee \Omega^\tau_{\mathcal{R}}(l*m) \leq \Omega^\tau_{\mathcal{R}}(l) \vee \Omega^\tau_{\mathcal{R}}(m) \,\& \Omega^\tau_{\mathcal{R}}(l) \leq \Omega^\tau_{\mathcal{R}}(m)$

$\mathcal{R}^\tau$ *is a neutrosophic ideal of* $\mathcal{H}$.

Definition 3.3. *A neutrosophic ideal* $\mathcal{R} = (\Phi_{\mathcal{R}}, \Psi_{\mathcal{R}}, \Omega_{\mathcal{R}})$ *of* $\mathcal{H}$ *is said to be neutrosophic characteristic if* $\Phi_{\mathcal{R}}(\tau(l)) = \Phi_{\mathcal{R}}(l), \Psi_{\mathcal{R}}(\tau(l)) = \Psi_{\mathcal{R}}(l), \Omega_{\mathcal{R}}(\tau(l)) = \Omega_{\mathcal{R}}(l)\ \forall\ l \in \mathcal{H}$ *and* $\tau \in Aut(\mathcal{H})$*, where* $Aut(\mathcal{H})$ *is the set of all automorphism of* $\mathcal{H}$*.*

Theorem 3.4. $\mathcal{R} \in M(\mathcal{H})$ *be a neutrosophic characteristic ideal of* $\mathcal{H}$*. Then each level ideal of* $\mathcal{R}$ *is a characteristic ideal of* $\mathcal{H}$*.*

Proof:
Let $\mathcal{R} \in M(\mathcal{H})$ *be a neutrosophic characteristic ideal of* $\mathcal{H}$*,* $\mathcal{R}_{u,v,w,}$ *where* $u, v, w \in Im(\mathcal{R})$ *is a ideal of* $\mathcal{H}$
It suffices to show that $\tau(\Phi_{\mathcal{R}_u}) = \Phi_{\mathcal{R}_u}, \tau(\Psi_{\mathcal{R}_v}) = \Psi_{\mathcal{R}_v}, \tau(\Omega_{\mathcal{R}_w}) = \Omega_{\mathcal{R}_w},\ u \in Im(\Phi_{\mathcal{R}}),\ v \in Im(\Psi_{\mathcal{R}}),\ w \in Im(\Omega_{\mathcal{R}})$
Let $\tau \in Aut(\mathcal{H})$ & $l \in \mathcal{R}_{u,v,w}$ *since* $\mathcal{R}$ *is a neutrosophic characteristic,*
$\Phi_{\mathcal{R}}(\tau(l)) = \Phi_{\mathcal{R}}(l) \geq u$
It follows that $\tau(l) \in \Phi_{\mathcal{R}_u}$ *and hence* $\tau(\Phi_{\mathcal{R}_u}) \subseteq \Phi_{\mathcal{R}_u}$
Similarly, for $\Psi_{\mathcal{R}_u}, \Omega_{\mathcal{R}_u}$.
Conversely, let $m \in \Phi_{\mathcal{R}_u}$ & $l \in \mathcal{H}$ *such that*
$\tau(l) = m$ *then* $\Phi_{\mathcal{R}}(l) = \Phi_{\mathcal{R}}(\tau(l)) = \Phi_{\mathcal{R}}(m) \geq u,\ l \in \Phi_{\mathcal{R}_u}$
$m = \tau(l) \in \tau(\Phi_{\mathcal{R}_u})$ *so that* $\Phi_{\mathcal{R}_u} \subseteq \tau(\Phi_{\mathcal{R}_u})$
Similarly, for $\Psi_{\mathcal{R}}, \Omega_{\mathcal{R}}$.
$\mathcal{R}_{u,v,w}$ *is a characteristic ideal of* $\mathcal{H}$*.*

The converse of theorem 4.4 is given in lemma 4.5

Let $\mathcal{R}$ *be a neutrosophic ideal of* $\mathcal{H}$ *and let* $l \in \mathcal{H}$ *then* $\mathcal{R}(l) = (\Phi_{\mathcal{R}}(l), \Psi_{\mathcal{R}}(l), \Omega_{\mathcal{R}}(l)) = (u, v, w) \Leftrightarrow l \in \Phi_{\mathcal{R}_u}\ l \notin \Phi_{\mathcal{R}_{u_1}}, l \in \Psi_{\mathcal{R}_v}, l \notin \Psi_{\mathcal{R}_{v_1}}\ ;\ l \in \Omega_{\mathcal{R}_w}, l \notin \Omega_{\mathcal{R}_{w_1}}\ \forall\ u_1 \geq u, v_1 \geq v$ *and* $w_1 \geq w$.
It's obvious.

Theorem 3.5. $\mathcal{R}$ *be a neutrosophic ideal of* $\mathcal{H}$ *and if each level ideal of* $\mathcal{R}$ *is characteristic, then* $\mathcal{R}$ *is a neutrosophic characteristic ideal of* $\mathcal{H}$*.*

Proof:
Let $\mathcal{R} \in M(\mathcal{R})$ *be a neutrosophic ideal and let* $l \in \mathcal{H}, \tau \in Aut(\mathcal{H})$ *and* $\Phi_{\mathcal{R}}(l) = u, \Psi_{\mathcal{R}}(l) = v$ & $\Omega_{\mathcal{R}}(l) = w$
By lemma 4.5, $l \in \Phi_{\mathcal{R}_u}\ l \notin \Phi_{\mathcal{R}_u}\ \forall u_1 \geq u$.
It follow $\tau(\Phi_{\mathcal{R}_u}) = \Phi_{\mathcal{R}_u}$.
Thus $\tau(l) \in \tau(\Phi_{\mathcal{R}_u}) = \Phi_{\mathcal{R}_u}$ *and so* $\Phi_{\mathcal{R}}(\tau(l)) \geq u$.
Let $\Phi_{\mathcal{R}}(\tau(l)) = u_1$ *and assume that* $u_1 \geq u$*, then* $\tau(l) \in \Phi_{\mathcal{R}_{u_1}} = \tau(\Phi_{\mathcal{R}_{u_1}})$

Since τ *is* $1-1$ $l \in \Phi_{\mathcal{R}_{u_1}}$ *which arries at a contradiction.*
$\Phi_{\mathcal{R}}(\tau(l)) = u = \Phi_{\mathcal{R}}(l)$ *also similarly for,* $\Psi_{\mathcal{R}}(\tau(l)) = v = \Psi_{\mathcal{R}}(l)$ *and* $\Omega_{\mathcal{R}}(\tau(l)) = w = \Omega_{\mathcal{R}}(l)$
This proves that $\mathcal{R}$ *is a neutrosophic characteristic ideals of* $\mathcal{H}$*.*

Definition 3.6. *[16] An incline algebra* $\mathcal{H}$ *is said to have ascending chain condition with respect to ideal if* $\mathcal{H}$ *contains no infinite proper ascending chain of ideals*
$\mathcal{S}_1\mathcal{S}_2\mathcal{S}_3$....... *Similarly, for the descending chain.*

Theorem 3.7. $\mathcal{H}$ *satisifies the descending chain condition with respect to ideal and let* $\mathcal{R}$ *is a neutrosophic ideal of* $M(\mathcal{H})$ *then there is no finite ascending sequence of elements of* $Im(\mathcal{R})$

Proof:
Suppose $|Im(\Phi_{\mathcal{R}})| = \infty$ *and let* $\{u_n\}$ *be a strictly increasing sequence of elements of* $Im(\Phi_{\mathcal{R}})$ *then* $0 \leq u_1 < u_2 .... < 1$
Now define $\Phi_{\mathcal{R}_u} = \{l \in \mathcal{H}/\Phi_{\mathcal{R}}(l) \geq u_k\},\ k = 2, 3, ...$
then $\Phi_{\mathcal{R}_k}$ *is an ideal of* $\mathcal{H}$*.*
Let $l \in \Phi_{\mathcal{R}_k}$, $\Phi_k(l) \geq u_k > u_{k-1}$ *implies* $l \in \Phi_{R_{k-1}}$
Since $u_{k-1} \in Im(\mathcal{R})$, *there exists* $l_{k-1} \in \mathcal{H}$ *such that* $\Phi_R(l_{k-1}) = u_{k-1}$.
Hence $\Phi_{\mathcal{R}_k} \subseteq \Phi_{\mathcal{R}_{k-1}}$
Thus, $\Phi_{\mathcal{R}_k}\Phi_{\mathcal{R}_{k-1}}$ *and so on.*
Will get as, $\Phi_{\mathcal{R}_1}\Phi_{\mathcal{R}_2}\Phi_{\mathcal{R}_3}$..... *of ideal of* $\mathcal{H}$ *which goes on.*

This is a contradiciton and the same procedure is carried out for other two components.

Theorem 3.8. *If* $|Im(_1)| < \infty$ *for every neutrosophic ideal* $_1$ *of then satisfies the descending chain condition with respect to ideal.*

Proof:
Assuming in a contradiction, that does not satisfies the descending chain condition with respect to ideal, then there exists a strict infinite descending chain $\mathcal{S}_0\mathcal{S}_1\mathcal{S}_2$..... *of ideals of* $\mathcal{H}$*. Now define* $\mathcal{R} \in' ()$ *by*

$$\chi_{\mathcal{M}}(l) = \begin{cases} \frac{r}{r+1} : & if\ l \in \frac{\mathcal{S}_r}{\mathcal{S}_{r+1}},\ r = 0, 1... \\ 1 : & if\ l \in \cap_{r=0}^{\infty}\mathcal{S}_r \end{cases}$$

$$\Psi_{\mathcal{R}}(l) = \begin{cases} \frac{r}{r+1} : & if\ l \in \frac{\mathcal{S}_j}{\mathcal{S}_{r+1}},\ r = 0, 1... \\ 1 : & if\ l \in \cap_{r=0}^{\infty} \mathcal{S}_r \end{cases}$$

$$\Omega_{\mathcal{R}}(l) = \begin{cases} \frac{r}{r+1} : & if\ l \in \frac{\mathcal{S}_r}{\mathcal{S}_{r+1}},\ r = 0, 1... \\ 1 : & if\ l \in \cap_{r=0}^{\infty} \mathcal{S}_r \end{cases}$$

where $\mathcal{S}_0$ *stands for*
Assume $l \in \frac{\mathcal{S}_r}{\mathcal{S}_{r+1}}$ $m \in \frac{\mathcal{S}_r}{\mathcal{S}_{r+1}}$, *for* $r = 0, 1, ..$ *and* $r_0 = 0, 1, 2....$
without loss of generality, assume that $r \leq r_0$ *then* $l + m \in \mathcal{S}_r$ & $l * m \in \mathcal{S}_r$

$$\begin{aligned} \Phi_{\mathcal{R}}(l+m) \wedge \Phi_{\mathcal{R}}(l*m) &\geq \tfrac{r}{r+1} \\ &= \Phi_{\mathcal{R}}(l) \wedge \Phi_{\mathcal{R}}(m) \end{aligned}$$

If $l, m \in \cap_{r=0}^{\infty} \mathcal{S}_j$ *then* $l + m, l * m \in \cap_{r=0}^{\infty} \mathcal{S}_r$ *implies* $\Phi_{\mathcal{R}}(l+m) \wedge \Phi_{\mathcal{R}}(l * m) = \Phi_{\mathcal{R}}(l) \wedge \Phi_{\mathcal{R}}(m)$
If $l \notin \cap_{r=0}^{\infty} \mathcal{S}_r$ $m \in \cap_{r=0}^{\infty} \mathcal{S}_r$, *then there exists* r_0 *such that* $l \in \frac{\mathcal{S}_r}{\mathcal{S}_{r+1}}$.

$$\begin{aligned} \Phi_{\mathcal{R}}(l+m) \wedge \Phi_{\mathcal{R}}(l*m) &\geq \tfrac{r_0}{r_0+1} \\ &= \Phi_{\mathcal{R}}(l) \wedge \Phi_{\mathcal{R}}(m) \end{aligned}$$

Similarly, for the other two components and if $l \in \frac{\mathcal{S}_r}{\mathcal{S}_{r+1}}, m \leq l$ *then* $m \in \frac{\mathcal{S}_r}{\mathcal{S}_{r+1}}$
$\Phi_{\mathcal{R}}(l) = \frac{r}{r+1} \geq \Phi_{\mathcal{R}}(m)$. *If* $l \in \cap_{r=0}^{\infty} \mathcal{S}_r$ $m \leq l$ *then* $\Phi_{\mathcal{R}}(l) = 1 \geq \Phi_{\mathcal{R}}(m)$

$\mathcal{R}$ *is a neutrosophic ideal of having an infinite number of different values, which is a contradiction.*

4. Section 4: Conclusion

In this article, the concept of the neutrosophic equivalence relation in incline algebra is discussed, along with its properties and some results. Also, chain conditions and characteristic functions are discussed. This also helps to promote the concept of merging fuzzy and incline theories. Also, this can be preferred to some other algebraic structures.

References

[1] Ahn S. S., Jun Y. B., Kim H. S., Ideals and Quotients of Incline Algebras, *Comm. Koren Math. Soc.*, Vol. 16 (4), pp. 573-583, 2001.

[2] Atanassov K. T., Intuitionistic fuzzy sets, *Fuzzy sets and systems*, Vol. 20 (1), pp. 87-96, 1986.

[3] Cao Z. Q., Kim K. H., Roush F. W., Incline Algebras and applications, *Ellis Horwood, Chicheater, England and Wiley, New York*, 1984.

[4] Cao Z., A new algebraic system-slope, B usefal, Vol. 6 (1), pp. 22-27, 1981.

[5] Chakraborty M. K., Das M., On fuzzy equivalences I, Fuzzy sets and systems, Vol. 11, pp. 185-193, 1983.

[6] Chakraborty M. K., Das M., On fuzzy equivalences II, Fuzzy sets and systems, Vol. 12, pp. 299-307, 1983.

[7] Hemavathi P., P. Muralikrishna and K. Palanivel, i-v-f beta-ideals of beta-Algebras, Materials Science and Engineering Conference Series, Vol. 263 (4), pp. 1-10, 2017.

[8] Hasankhani A., Talaie M., Zahedi M. M., Fuzzy congruence relations on BCK - algebras, J. Fuzzy Math, Vol. 6, 1998.

[9] Jun Y. B., Ahn S. S., Kim, H. S., Fuzzy Subincline(Ideals) of Incline Algebras, *Fuzzy Sets and Systems*, Vol. 123, pp. 217-255, 2001.

[10] Kim K. H., Roush F. W., Inclines of algebraic structures, *Fuzzy Sets and Systems*, Vol. 72, pp. 189-196, 1995.

[11] Kim K. H., Roush F. W., Markowsky G., Representation of Incline Algebras, *Algebra Coll*, Vol. 4, pp. 461-470, 1997.

[12] Liu L. L., Fu W. Q., and Li S. G., The order aspects of fuzzy subinclines, fuzzy equivalence relations and fuzzy congruence relations, Fuzzy systems and Mathemations, Vol. 25 (5), pp. 43-49, 2011.

[13] Liu L. L., Fu, W. Q., and Li, S. G., Characterizations of fuzzy subinclines, fuzzy ideals, fuzzy filters and fuzzy congruence reltions, Journal of Shandong University, Vol. 46 (11), pp. 48-52, 2011.

[14] Murali V., Fuzzy equivalence relations, Fuzzy sets and systems, Vol. 30, pp. 155-163, 1989.

[15] Murali V., Fuzzy Congruence relations, Fuzzy sets and systems, Vol. 41, pp. 359-369, 1991.

[16] Rosenfeld A., Fuzzy groups, Journal of Mathematical Anakysis & Applications, Vol. 35, pp. 512-217, 1971.

[17] P. Hemavathi P. Muralikrishna, K. Palanivel, Study on i-v fuzzy transltion and multiplication on i-v fuzzy β - subalgebra, *International Journal of Pure and Applied Mathematics,* Vol. 100 (2), pp. 245-256, 2018.

[18] P. Muralikrishna, Surya M., A Crystal Study on Neutrosophic Incline, *South East Asian J. of Mathematics and Mathematical Sciences*, Vol. 19 (2), pp. 65-76, 2023.

[19] Smarandache F., Neutrosophic Set, A generalization of Intuitionistic fuzzy sets, *International Journal of Pure and Applied Mathematics*, Vol. 24 (5), pp. 287-297, 2005.

[20] Surya M., P. Muralikrishna, Application of MBJ Neutrosophic Set on Filters of Incline Algebra, *International Journal of Neutrosophic Science*, Vol. 19 (1), pp. 60-67, 2022.

[21] Surya M., P. Muralikrishna, Some Interpretations on Neutrosophic Filter on Incline Algebra, *Missouri Journal of Mathematical Sciences*, Vol. 35 (1), pp. 123-128, 2013.

[22] P. Muralikrishna, Surya M., Study of MBJ Neutrosophic Level Sets in β - Substructures, *AIP Conference Proceedings* 2829, 020003, 2023.

[23] Surya Manokaran, Muralikrishna Prakasam, Study of MBJ Neutrosophic Level Sets in β - Ideal, *Scopus indexed Springer Book Series Trends in Mathematics*, 177 - 184.

[24] Muralikrishna Prakasam, Surya M, Study of Neutrosophic Magnified Translation on Incline Algebra, *Journal of the Indonesian Mathematical Society*, Vol. 29 (1), pp. 125-134, 2023.

[25] Venkata Arvinda Raju, A Note on Incline Algebras, *International Journal of Mathematical Archive*, Vol. 8 (9), pp. 154-157, 2017.

[26] Volety V. S. Ramachandram, On some properties of Inclines, *Journal of Science & Arts*, Vol. 1 (18), pp. 13-16, 2012.

[27] Yuan B., Wu W., Fuzzy ideals on a distributive lattice, Fuzzy sets and systems, Vol. 35, pp. 231-240, 1990.

[28] Zadeh L. A., Fuzzy Sets, *Information and Control.* Vol. 8 (3), pp. 338-353, 1965.

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