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CHAPTER 10

λ-STATISTICAL CONVERGENCE OF FRACTIONAL DIFFERENCE SEQUENCES ON NEUTROSOPHIC NORMED SPACES

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Introduction

Since sequence convergence is very necessary in basic mathematical theory, many different kinds of convergence concepts have been tried to be carried over into summability, approximation theory, and probability theory.The concept of difference sequence space was defined by Kizmaz. These sequence spaces is modification by Et and Colak in 1995. Long before this work Fast, introduced statistical convergence briefly in 1951, and then Schoenberg examined statistical convergence in detail with the help of summation method in 1959. The concept of statistical convergence is worked under different name. (Mursaleen, 2000), introduced the λ statistical convergence and Hazarika and Savas carried these concept on n-normed spaces in 2013. λ-statistically convergent difference sequences using fuzzy numbers is studied in (Altinok et al, 2012).

Some important definitions and features of neutrophic set theory will now be given, which are necessary for the development of this work. Now, let's recall λ_v −statistically convergent.

Let $\lambda = (\lambda_v)$ be a non-decreasing, $\lambda_1 = 1,\ \lambda_{v+1} \leq \lambda_v + 1$ such that $\lim_{v\to\infty} \lambda_v = \infty$. Also, let $\mathcal{J}_v = [v + 1 - \lambda_v,\ v]$ and $t_v(x) = \frac{1}{\lambda_v}\sum_{r\in\mathcal{J}_v} x_r$. In this case, (x_v) is named λ- statistically convergent to ℓ if for all $\varepsilon > 0$,

$$\lim_{v\to\infty} \frac{1}{\lambda_v} |\{r \in \mathcal{J}_v : |x_v - \ell| \geq \varepsilon\}| = 0. \qquad (1)$$

In this situation, it is denoted by $x_v \to \ell(st)_\lambda$.

On the other hand let's examine the general structure of sequence spaces obtained with the help of the fractional difference operator, which is one of the areas where statistical convergence is applied. Let, $\Gamma(u)$ demonstrate the Euler Gamma function where u is real number and $u \notin \{0, -1, -2, \dots\}$. Here the notations in (Kadak and Baliarsingh, 2015) and (Ercan, 2018) will be used. Then a generalization of fractional difference operator is given in (Baliarsingh and Dutta, 2015) as

$$\Delta^u(x_v) = \sum_{j=0}^{\infty} (-1)^j \frac{\Gamma(u+1)}{j!\,\Gamma(u-j+1)} x_{j+v}\,. \qquad (2)$$

After then, $\Delta^u -$ statistical convergence and graded state of this concept is defined in (Ercan, 2018) and (Mursaleen and Baliarsingh, 2022), respectively.

The concepts of fuzzy, intuitinistic fuzzy and neutrosophic sets has revolutionized many areas such as mathematics, science, engineering, medicine. This concepts is given by (Zadeh,1965), (Atanassov,1986) and (Smarandache,1999). The second of these theories deals with an imprecise and uncertain situation by corresponding degree of membership and degree of non-membership to particular object. Neutrosophic logic is a powerful tool for modeling instability and uncertainty in a variety of problems that arise in science and engineering. It has very important and facilitating applications in many fields.

The neutrosophic set is investigated by (Smarandache,1999) and he determined the Neutrosophic set using degree of truth, inaccuracy and uncertainty. Neutrosophic normed spaces is given in (Kirisci and Simsek, 2020) and is examined statistical convergence this spaces. In (Kisi, 2021) lacunary statistical convergent in NNS is studied. Many studies have been done with different types of convergence in this space. e.g(Gonul Bilgin,2022).

The concept of convergence in the NNS space is defined as follows. (Kirisci and Simsek, 2020), (Khan et. al, 2021).

Let Ų be a neutrosophic normed space (NNS), $ꬵ > 0$. For (x_v) in Ų, let $0 < \varepsilon < 1$. So, the sequence (x_v) is called convergence to ℓ if there exists $k \in \mathbb{N}$, such that $Ɠ(x_v - \ell, ꬵ) > 1 - \varepsilon, ß(x_v - \ell, ꬵ) < \varepsilon, ƴ(x_v - \ell, ꬵ) < \varepsilon$. That is $\lim_{v\to\infty} Ɠ(x_v - \ell, ꬵ) = 1$, $\lim_{v\to\infty} ß(x_v - \ell, ꬵ) = 0$, $\lim_{v\to\infty} ƴ(x_v - \ell, ꬵ) = 0$. Then, the sequence (x_v) is named a convergent sequence in Ų. The convergent in NNS is showed by $N - \lim x_v = \ell$.

Many investigations have been made with different types of convergence in different spaces.e.g(Gonul Bilgin and Bozma, 2021)

Now let's recall the definition of statistical convergence is given in (Kirisci and Simsek, 2020). Take a NNS Ų. A sequence (x_k) is named statistical convergence, if there exist ℓ so that the set

$$T_\varepsilon = \{k \le n: Ɠ(x_k - \ell, ꬵ) \le 1 - \varepsilon \; or \; ß(x_k - \ell, ꬵ) \ge \varepsilon, ƴ(x_k - \ell, ꬵ) \ge \varepsilon\}$$

has natural density is zero, for every $\varepsilon > 0$ and $ꬵ > 0$. That is $d(T_\varepsilon) = 0$ or equivalently,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n: G(x_k - \ell, ꬵ) \le 1 - \varepsilon \; or \; B(x_k - \ell, ꬵ) \ge \varepsilon, Y(x_k - \ell, ꬵ) \ge \varepsilon\}| = 0.$$

Then, we show $st - \lim x_k = \ell(N)$ or $x_k \to \ell\big(st(N)\big)$. The set of statistical convergence segences in NN will be denoted by $St(N)$.

Based on these studies, the concepts of $\Delta_\lambda^u -$ statistical convergence and the concept of $\left(\Delta_\lambda^{u,Þ}\right)$ −Cesaro summable, for fractional difference sequences will be introduced in our work. Also, the concept of $(\Delta_\lambda^u)^\alpha$-statistically convergent is given and important coverage relations are given. It is accepted throughout the article that (2) is a convergent series and the number of terms in the sum symbol changes up to u.

2. Matherial and Methods

Now, $\Delta_\lambda^u -$ statistical convergence, $\left(\Delta_\lambda^{u,Þ}\right)$ −Cesaro summable for fractional difference sequences will be introduced in Neutrosophic normed spaces. The connection between these concepts will be defined.

Definition 2.1 Let $\lambda = (\lambda_v)$ be a non-decreasing, $\lambda_1 = 1,\ \lambda_{v+1} \leq \lambda_v + 1$ such that $\lim_{v\to\infty} \lambda_v = \infty$, $\mathcal{J}_v = [v + 1 - \lambda_v,\ v]$ and also u be a convenient fraction. In this case, if there is a ℓ number such that

$$\lim_{v\to\infty} \frac{1}{\lambda_v} |\{t \in \mathcal{J}_v : |\Delta_\lambda^u(x_v) - \ell| \geq \varepsilon\}|$$

for every $\varepsilon > 0$, $x = (x_v)$ sequence is called Δ_λ^u- statistically convergent and the set of Δ_λ^u- statistically convergent sequences is denoted by $S(\Delta_\lambda^u)$. As a result, $x \to \ell\left((st)_{\Delta_\lambda^u}\right)$ can be written.

Theorem 2.1 $x = (x_k), z = (z_k)$ be sequences of real numbers.

i) If $S(\Delta_\lambda^u) - \lim x_k = \breve{x}$ and α belongs to real numbers, then $S(\Delta_\lambda^u) - \lim \alpha x_k = \alpha\breve{x}$.

ii) If $S(\Delta_\lambda^u) - \lim x_k = \breve{x}$, $S(\Delta_\lambda^u) - \lim z_k = \breve{z}$, then $S(\Delta_\lambda^u) - \lim(x_k + z_k) = \breve{x} + \breve{z}$.

Proof.

i) If $\alpha = 0$, it is seen easily. Let assume $\alpha \neq 0$, then we have desired result from

$$\frac{1}{\lambda_v}|\{k \leq n : |\Delta_\lambda^u \alpha x_k - \alpha\breve{x}| \geq \varepsilon\}| \leq \frac{1}{\lambda_v}\left|\left\{k \leq n : |\Delta_\lambda^u x_k - \breve{x}| \geq \frac{\varepsilon}{|\alpha|}\right\}\right|.$$

ii) It is seen from following inequality;

$$\frac{1}{\lambda_v}|\{k \leq n : |\Delta_\lambda^u(x_k + z_k) - (\breve{x} + \breve{z})| \geq \varepsilon\}| \leq \frac{1}{\lambda_v}\left|\left\{k \leq n : |\Delta_\lambda^u x_k - \breve{x}| \geq \frac{\varepsilon}{2}\right\}\right|$$

$$+\frac{1}{\lambda_v}\left|\left\{k \leq n : |\Delta_\lambda^u z_k - \breve{z}| \geq \frac{\varepsilon}{2}\right\}\right|.$$

Definition2.2 Let $\lambda = (\lambda_v)$ be sequence as given in Definition 2.1 and let u be a convenient fraction. Then, if there is a $v^\circ = v^\circ(\varepsilon)$ number such that for every $\varepsilon > 0$

$$\lim_{v\to\infty}\frac{1}{\lambda_v}|\{t\in\mathcal{J}_v: |\Delta_\lambda^u(x_v)-\Delta_\lambda^u(x_{v^\circ})|\geq\varepsilon\}|=0,$$

the sequence $x=(x_v)$ is called the Δ_v^α- statistical Cauchy sequence.

Theorem2.2 If the sequence $x=(x_v)$ is Δ_λ^u- statistically convergent, then the sequence $x=(x_v)$ is the Δ_λ^u- statistical Cauchy sequence.

Proof.

Suppose $x=(x_v)\in S(\Delta_\lambda^u)$ and $\varepsilon>0$. In this case, for almost all v

$$|\Delta_\lambda^u(x_v)-\ell|<\frac{\varepsilon}{2}.$$

For a selected number v°,

$$|\Delta_\lambda^u(x_{v^\circ})-\ell|<\frac{\varepsilon}{2}$$

can be written. From here it becomes

$$|\Delta_\lambda^u(x_v)-\Delta_\lambda^u(x_{v^\circ})|<|\Delta_\lambda^u(x_v)-\ell|+|\Delta_\lambda^u(x_{v^\circ})-\ell|<\varepsilon.$$

Thus x is a Δ_λ^u-statistical Cauchy sequence.

Definition 2.3 Let $Þ>0$. (x_v) is called to be strongly $\left(\Delta_\lambda^{u,Þ}\right)$ –Cesaro summable, if there is a $\ell\in\mathbb{R}$ such that

$$\lim_{v\to\infty}\frac{1}{\lambda_v}\sum_{j=1}^{v}|\Delta_\lambda^u x_v-\ell|^Þ=0.$$

Then, (x_v) is strongly $\left(\Delta_\lambda^{u,Þ}\right)$ –Cesaro summable to ℓ. The set of these sequences is shown with $Ł\left(\Delta_\lambda^{u,Þ}\right)$. If $\ell=0$, then this spaces is shown with $\varpi\left(\Delta_\lambda^{u,Þ}\right)$.

Definition 2.4 Let $\lambda=(\lambda_v)$ be a sequences with the properties given in Definition 2.1, $\mathcal{J}_v=[v+1-\lambda_v,\ v]$ and u be a convenient fraction. For $0<\alpha\leq1$ let ${\lambda_v}^\alpha=({\lambda_1}^\alpha,{\lambda_2}^\alpha,\dots,{\lambda_v}^\alpha,\dots)$. Then, if there is a ℓ number such that

$$\lim_{v\to\infty}\frac{1}{{\lambda_v}^\alpha}|\{t\in\mathcal{J}_v: |\Delta_\lambda^u(x_v)-\ell|\geq\varepsilon\}|$$

for every $\varepsilon>0$, $x=(x_v)$ sequence is called $(\Delta_\lambda^u)^\alpha$- statistically convergent to ℓ and the set of $(\Delta_\lambda^u)^\alpha$- statistically convergent sequences is denoted by $S((\Delta_\lambda^u)^\alpha)$. As a result, $x\to\ell\left((st)_{\Delta_\lambda^u}\right)$ can be written.

Clearly, for all $0 < \alpha \leq 1$, $(\Delta_\lambda^u)^\alpha$- statistically convergent is well defined, however for $\alpha > 1$ this property does not always have to be true.

In the lemma below, a density comparison will be made for the type of convergence we defined using the density concept of a well-known set in the literature.

Lemma 2.1 Let $\mathcal{I}_v = [v + 1 - \lambda_v,\ v]$, u be a convenient fraction and $\aleph \subseteq \mathbb{N}$. If for all $0 < \text{ɋ} \leq \text{ꝓ} \leq 1$, then

$$(\delta_\lambda^u)^{\text{ꝓ}}(\aleph) \leq (\delta_\lambda^u)^{\text{ɋ}}(\aleph).$$

Proof.

Let $0 < \text{ɋ} \leq \text{ꝓ} \leq 1$ and ${\lambda_v}^{\text{ɋ}} \leq {\lambda_v}^{\text{ꝓ}}$. In this case, $\frac{1}{{\lambda_v}^{\text{ꝓ}}} \leq \frac{1}{{\lambda_v}^{\text{ɋ}}}$. Hence,

$$\frac{1}{{\lambda_v}^{\text{ꝓ}}} |\{t \in \mathcal{I}_v : |\Delta_\lambda^u(x_v) - \ell| \geq \varepsilon\}| \leq \frac{1}{{\lambda_v}^{\text{ꝓ}}} |\{t \in \mathcal{I}_v : |\Delta_\lambda^u(x_v) - \ell| \geq \varepsilon\}|.$$

So,

$$(\delta_\lambda^u)^{\text{ꝓ}}(\aleph) \leq (\delta_\lambda^u)^{\text{ɋ}}(\aleph).$$

Lemma2.2 Let $\lambda = (\lambda_v)$ be a sequences with the properties given in Definition 2.1, $\mathcal{I}_v = [v + 1 - \lambda_v,\ v]$ and u be a convenient fraction. For $0 < \alpha \leq 1$, if (x_v) is Δ_λ^u- statistically convergent then these sequence is $(\Delta_\lambda^u)^\alpha$- statistically convergent.

Proof.

The proof is easily obtained from the definition.

Definition2.5 Let $\lambda = (\lambda_v)$ be sequence as given in Definition 2.1 and let u be a convenient fraction. Then, if there is a $v^\circ = v^\circ(\varepsilon)$ number such that for every $\varepsilon > 0$

$$\lim_{v \to \infty} \frac{1}{{\lambda_v}^\alpha} |\{t \in \mathcal{I}_v : |(\Delta_\lambda^u)^\alpha(x_v) - (\Delta_\lambda^u)^\alpha(x_{v^\circ})| \geq \varepsilon\}| = 0,$$

the sequence $x = (x_v)$ is called the $(\Delta_\lambda^u)^\alpha$- statistical Cauchy sequence.

Proposition2.1 Let $\lambda = (\lambda_v)$ be a sequences with the properties given in Definition 2.1, $\mathcal{I}_v = [v + 1 - \lambda_v,\ v]$ and u be a convenient fraction. For $0 < \alpha \leq 1$, if $\liminf_v \frac{\lambda_v}{v} > 0$ then $S(\Delta^u) \subset S(\Delta_\lambda^u)$.

Proof.

Let $\liminf_v \frac{\lambda_v}{v} > 0$ and (x_v) is Δ^u- statistical convergent sequences. Then,

$$\frac{1}{v}|\{t \leq v : |\Delta^u(x_v) - \ell| \geq \varepsilon\}| \geq \frac{1}{v}|\{t \in \mathcal{I}_v : |\Delta^u(x_v) - \ell| \geq \varepsilon\}|$$
$$= \frac{\lambda_v}{v}\frac{1}{\lambda_v}|\{t \in \mathcal{I}_v : |\Delta^u(x_v) - \ell| \geq \varepsilon\}|.$$

So, (x_v) is (Δ^u_λ)- statistical convergent sequences. Then, $S(\Delta^u) \subset S(\Delta^u_\lambda)$.

Definition 2.6 Let $Þ > 0$. (x_v) is called to be strongly $\left(\Delta^{u,Þ}_\lambda\right)^\alpha$ −Cesaro summable, if there is a $\ell \in \mathbb{R}$ such that

$$\lim_{v\to\infty}\frac{1}{\lambda_v{}^\alpha}\sum_{j=1}^{v}|\Delta^u_\lambda x_v - \ell|^{Þ} = 0.$$

Then, (x_v) is strongly $\left(\Delta^{u,Þ}_\lambda\right)^\alpha$ −Cesaro summable to ℓ. The set of these sequences is shown with $Ł\left(\left(\Delta^{u,Þ}_\lambda\right)^\alpha\right)$. If $\ell = 0$, then this spaces is shown with $\varpi\left(\left(\Delta^{u,Þ}_\lambda\right)^\alpha\right)$.

Theorem2.3 Let $0 < \inf Þ_v \leq Þ_v \leq \sup Þ_v < \infty$ and u be a convenient fraction. In this case $Ł\left(\left(\Delta^{u,Þ}_\lambda\right)\right) \subset S(\Delta^u_\lambda)$.

Proof.

Let $(x_v) \in Ł\left(\left(\Delta^{u,Þ}_\lambda\right)\right)$, $t \in \mathcal{I}_v$. In this case,

$$\frac{1}{\lambda_v}\sum_{t\in\mathcal{I}_v}|\Delta^u_\lambda(x_t) - \ell|^{Þ_v} \geq \frac{1}{\lambda_v}\sum_{\substack{|\Delta^u_\lambda(x_t)-\ell|\geq\varepsilon \\ t\in\mathcal{I}_v}}|\Delta^\alpha(x_t) - \ell|^{Þ_v}$$
$$\geq \frac{1}{\lambda_v}\sum_{\substack{|\Delta^u_\lambda(x_t)-\ell|\geq\varepsilon \\ t\in\mathcal{I}_v}}\varepsilon^{\inf Þ_v}$$
$$\geq \frac{1}{\lambda_v}|\{t \in \mathcal{I}_v : |\Delta^\alpha(x_t) - \ell| \geq \varepsilon\}|\varepsilon^{\inf Þ_v}.$$

For $v \to \infty$, we get

$$\lim_{v\to\infty}\frac{1}{\lambda_v}|\{t \in \mathcal{I}_v : |\Delta^u_\lambda(x_t) - \ell| \geq \varepsilon\}|$$
$$\leq \frac{1}{\varepsilon^{\inf Þ_v}}\left(\lim_{v\to\infty}\frac{1}{\lambda_v}\sum_{t\in\mathcal{I}_v}|\Delta^u_\lambda(x_t) - \ell|^{Þ_v}\right) = 0,$$

so, $x \in S(\Delta^u_\lambda)$ then, $Ł\left(\left(\Delta^{u,Þ}_\lambda\right)\right) \subset S(\Delta^u_\lambda)$.

Theorem2.4 Let $0 < \text{Þ} < \infty$. If $x = (x_v)$ is strongly $\left(\Delta_\lambda^{u,\text{Þ}}\right)$-Cesaro summable to ℓ, then it is (Δ_λ^u)-statistically convergent to ℓ.

Proof. For any sequence $x = (x_v)$ and $\varepsilon > 0$,

$$\sum_{v=1}^{m} |\Delta_\lambda^u x_v - \ell|^{\text{Þ}} = \sum_{\substack{v=1 \\ |x_v - \ell| \geq \varepsilon}}^{m} |\Delta_\lambda^u x_v - \ell|^{\text{Þ}} + \sum_{\substack{v=1 \\ |x_v - \ell| < \varepsilon}}^{m} |\Delta_\lambda^u x_v - \ell|^{\text{Þ}}$$

$$\geq \sum_{v=1}^{m} |\Delta_\lambda^u x_v - \ell|^{\text{Þ}}$$

$$\geq |\{k \leq n : |\Delta^u x_v - \ell| \geq \varepsilon\}| \varepsilon^{\text{Þ}}$$

and so

$$\frac{1}{\lambda_v} \sum_{v=1}^{m} |\Delta_\lambda^u x_v - \ell|^{\text{Þ}} \geq \frac{1}{\lambda_v} |\{k \leq n : |\Delta_\lambda^u x_v - \ell| \geq \varepsilon\}| \varepsilon^{\text{Þ}}.$$

Therefore, if $x = (x_v)$ is $\left(\Delta_\lambda^{u,\text{Þ}}\right)$-Cesaro summable to ℓ, then, this sequence is (Δ_λ^u)-statistically convergent to ℓ.

3. Main Results

Now, the concept of (Δ_λ^u)-statistical convergence in Neutrosophic normed spaces will be defined and important properties of convergence in these spaces will be introduced.

Definition3.1 Let $\left(X, \mu^T, \gamma^{\text{Ŧ}}, \eta^{\text{Ï}}, \boxtimes, \otimes\right)$ be a Neutrosophic normed spaces and let $\lambda = (\lambda_v)$ be sequence as given in Definition 2.1 and let u be a convenient fraction. $x = (x_v)$ is called (Δ_λ^u)-statistical convergence according to neutrosophic normed where for every $\varepsilon > 0$ and $\text{ƅ} > 0$, there exist a ℓ:

$$\delta_\lambda^u\left\{k : \mu^T(\Delta_\lambda^u x_v - \ell, \text{ƅ}) \leq 1 - \varepsilon \; or \; \eta^{\text{Ï}}(\Delta_\lambda^u x_v - \ell, \text{ƅ}) \geq \varepsilon, \gamma^{\text{Ŧ}}(\Delta_\lambda^u x_v - \ell, \text{ƅ}) \geq \varepsilon\right\} = 0.$$

Equivalent to this information, we will say that

$$\lim_{v} \frac{\left|\left\{k \in \mathcal{I}_v : \mu^T(\Delta_\lambda^u x_v - \ell, \text{ƅ}) \leq 1 - \varepsilon \; or \; \eta^{\text{Ï}}(\Delta_\lambda^u x_v - \ell, \text{ƅ}) \geq \varepsilon, \gamma^{\text{Ŧ}}(\Delta_\lambda^u x_v - \ell, \text{ƅ}) \geq \varepsilon\right\}\right|}{\lambda_v} = 0,$$

is also (Δ_λ^u)-statistical convergence according to neutrosophic normed. In this case, we use $st_{\Delta_\lambda^u}^{\mathcal{N}} - lim\, x = \ell$. The set of all this sequences will demonstrated with $St_{\Delta_\lambda^u}^{\mathcal{N}}$.

Lemma 3.1 Let $\left(X,\mu^{T},\gamma^{Ғ},\eta^{Ï},\boxtimes,\otimes\right)$ be a Neutrosophic normed spaces, $\lambda=(\lambda_v)$ be sequence as given in Definition 2.1 and let u be a convenient fraction. Let $x=(x_v)$ is (Δ_λ^u)-statistical convergence according to neutrosophic normed where for every $\varepsilon>0$ and $ƅ>0$, then the next situations are equivalent in $(X,\mu_T,\nu_F,\eta_I,\circledcirc,\circledast)$:

i. $st_{\Delta_\lambda^u}^{\mathcal{N}}-lim\,x=\ell,$

ii.
$$\lim_v \frac{\left|\left\{k\in\mathcal{I}_v:\mu^{T}(\Delta_\lambda^u x_v-\ell,ƅ)>1-\varepsilon\ or\ \eta^{Ï}(\Delta_\lambda^u x_v-\ell,ƅ)\le\varepsilon,\gamma^{Ғ}(\Delta_\lambda^u x_v-\ell,ƅ)\le\varepsilon\right\}\right|}{\lambda_v}$$
$$=0,$$

iii.
$$\lim_v \frac{|\{k\in\mathcal{I}_v:\mu^{T}(\Delta_\lambda^u x_v-\ell,ƅ)\le 1-\varepsilon\}|}{\lambda_v}=0,\ \lim_v \frac{\left|\left\{k\in\mathcal{I}_v:\eta^{Ï}(\Delta_\lambda^u x_v-\ell,ƅ)\ge 1-\varepsilon\right\}\right|}{\lambda_v}=0,$$
$$\lim_v \frac{|\{k\in\mathcal{I}_v:\gamma^{Ғ}(\Delta_\lambda^u x_v-\ell,ƅ)\ge 1-\varepsilon\}|}{\lambda_v}=0,$$

iv.
$$\lim_v \frac{|\{k\in\mathcal{I}_v:\mu^{T}(\Delta_\lambda^u x_v-\ell,ƅ)>1-\varepsilon\}|}{\lambda_v}=0,\ \lim_v \frac{\left|\left\{k\in\mathcal{I}_v:\eta^{Ï}(\Delta_\lambda^u x_v-\ell,ƅ)\le 1-\varepsilon\right\}\right|}{\lambda_v}=0,$$
$$\lim_v \frac{|\{k\in\mathcal{I}_v:\gamma^{Ғ}(\Delta_\lambda^u x_v-\ell,ƅ)\le 1-\varepsilon\}|}{\lambda_v}=0,$$

The proof is easily obtained from Definition3.1.

Theorem 3.1 Let $\left(X,\mu^{T},\gamma^{Ғ},\eta^{Ï},\boxtimes,\otimes\right)$ be a NNS. If, $x=(x_v)$ is (Δ_λ^u)-statistical convergence, then this limit is unique.

Proof Let $st_{\Delta_\lambda^u}^{\mathcal{N}}-lim\,x=\ell_1$ and $st_{\Delta_\lambda^u}^{\mathcal{N}}-lim\,x=\ell_2$ such that $\ell_1\ne\ell_2$. For given $\varepsilon>0$ and $ƅ>0$,

$(1-\varepsilon)\boxtimes(1-\varepsilon)>1-ꟼ$ and $\varepsilon\otimes\varepsilon<ꟼ$. For any $ƅ>0$ is defined the next sets:

$\wp_{\mu^T}(\varepsilon,\text{ƅ}) = \{k \in \mathcal{I}_v : \mu^T(\Delta_\lambda^u x_v - \ell_1,\text{ƅ}) \le 1-\varepsilon\}, \wp_{\breve{\mu}^T}(\varepsilon,\text{ƅ})$
$= \{k \in \mathcal{I}_v : \breve{\mu}^T(\Delta_\lambda^u x_v - \ell_2,\text{ƅ}) \le 1-\varepsilon\}$

$\wp_{\eta^{\text{Ï}}}(\varepsilon,\text{ƅ}) = \{k \in \mathcal{I}_v : \eta^{\text{Ï}}(\Delta_\lambda^u x_v - \ell_1,\text{ƅ}) \le 1-\varepsilon\}$, $\wp_{\breve{\eta}^{\text{Ï}}}(\varepsilon,\text{ƅ}) = \{k \in \mathcal{I}_v : \breve{\eta}^{\text{Ï}}(\Delta_\lambda^u x_v - \ell_2,\text{ƅ}) \le 1-\varepsilon\}$

and

$\wp_{\gamma^{\text{ꟻ}}}(\varepsilon,\text{ƅ}) = \{k \in \mathcal{I}_v : \gamma^{\text{ꟻ}}(\Delta_\lambda^u x_v - \ell_1,\text{ƅ}) \le 1-\varepsilon\}, \wp_{\breve{\gamma}^{\text{ꟻ}}}(\varepsilon,\text{ƅ})$
$= \{k \in \mathcal{I}_v : \breve{\gamma}^{\text{ꟻ}}(\Delta_\lambda^u x_v - \ell_2,\text{ƅ}) \le 1-\varepsilon\}$

Using $st_{\Delta_\lambda^u}^{\mathcal{N}} - lim\, x = \ell_1$ and Lemma3.1, for each $\text{ƅ} > 0$,

$$\delta_\lambda^u\left(\wp_{\mu^T}(\varepsilon,\text{ƅ})\right) = \delta_\lambda^u\left(\wp_{\eta^{\text{Ï}}}(\varepsilon,\text{ƅ})\right) = \delta_\lambda^u\left(\wp_{\gamma^{\text{ꟻ}}}(\varepsilon,\text{ƅ})\right) = 0$$

can be written. Also, with the help of $st_{\Delta_\lambda^u}^{\mathcal{N}} - lim\, x = \ell_2$ for each $\text{ƅ} > 0$

is obtained. Now;

$$\wp(\varepsilon,\text{ƅ}) = \left(\left(\wp_{\mu^T}(\varepsilon,\text{ƅ})\right)\cup\left(\wp_{\breve{\mu}^T}(\varepsilon,\text{ƅ})\right)\right)\cap\left(\left(\wp_{\eta^{\text{Ï}}}(\varepsilon,\text{ƅ})\right)\cup\left(\wp_{\breve{\eta}^{\text{Ï}}}(\varepsilon,\text{ƅ})\right)\right)\cap\left(\left(\wp_{\gamma^{\text{ꟻ}}}(\varepsilon,\text{ƅ})\right)\cup\left(\wp_{\breve{\gamma}^{\text{ꟻ}}}(\varepsilon,\text{ƅ})\right)\right)$$

is defined. Then, $\delta_\lambda^u\left(\wp(\varepsilon,\text{ƅ})\right) = 0$, $\delta_\lambda^u\left(\mathbb{N} / \wp(\varepsilon,\text{ƅ})\right) = 1$, can be written. So; three possible situations is written, if $k \in \left(\mathbb{N} / \wp(\varepsilon,\text{ƅ})\right)$ is taken;

$i.)\, k \in \left(\mathbb{N} / \left(\left(\wp_{\mu^T}(\varepsilon,\text{ƅ})\right)\cup\left(\wp_{\breve{\mu}^T}(\varepsilon,\text{ƅ})\right)\right)\right)$,

$ii.)\, k \in \left(\mathbb{N} / \left(\left(\wp_{\eta^{\text{Ï}}}(\varepsilon,\text{ƅ})\right)\cup\left(\wp_{\breve{\eta}^{\text{Ï}}}(\varepsilon,\text{ƅ})\right)\right)\right)$ and

$iii.)\, k \in \left(\mathbb{N} / \left(\left(\wp_{\gamma^{\text{ꟻ}}}(\varepsilon,\text{ƅ})\right)\cup\left(\wp_{\breve{\gamma}^{\text{ꟻ}}}(\varepsilon,\text{ƅ})\right)\right)\right)$

From i.) $\mu^T(\ell_1 - \ell_2,\text{ƅ}) \ge \mu^T\left(\Delta_\lambda^u x_v - \ell_1, \frac{\text{ƅ}}{2}\right) \boxtimes \mu^T\left(\Delta_\lambda^u x_v - \ell_2, \frac{\text{ƅ}}{2}\right) > (1-\varepsilon) \boxtimes (1-\varepsilon) > 1 - \beta$.

is obtained. Then for all $\text{ƅ} > 0$, and $\varepsilon > 0$ is arbitrary, $\mu^T(\ell_1 - \ell_2,\text{ƅ}) = 1$. So, $\ell_1 = \ell_2$.

Now considering ii.) and so using $k \in \left(\mathbb{N}/\left(\left(\text{Ꝑ}_{\eta^{\ddot{I}}}(\varepsilon, \text{ƅ})\right) \cup \left(\text{Ꝑ}_{\breve{\eta}^{\ddot{I}}}(\varepsilon, \text{ƅ})\right)\right)\right)$,

$$\eta^{\ddot{I}}(\ell_1 - \ell_2, \text{ƅ}) \geq \eta^{\ddot{I}}\left(\Delta_\lambda^u x_v - \ell_1, \frac{\text{ƅ}}{2}\right) \otimes \eta^{\ddot{I}}\left(\Delta_\lambda^u x_v - \ell_2, \frac{\text{ƅ}}{2}\right) < \varepsilon \otimes \varepsilon < \text{ꟻ}$$

is getting. So, for $\varepsilon > 0$ is arbitrary, $\eta^{\ddot{I}}(\ell_1 - \ell_2, \text{ƅ}) = 0$ Then, $\ell_1 = \ell_2$. Furthermore, using iii.) since

$$k \in \left(\mathbb{N} / \left(\left(\text{Ꝑ}_{\gamma^{\text{ғ}}}(\varepsilon, \text{ƅ})\right) \cup \left(\text{Ꝑ}_{\breve{\gamma}^{\text{ғ}}}(\varepsilon, \text{ƅ})\right)\right)\right)$$

$$\gamma^{\text{ғ}}(\ell_1 - \ell_2, \text{ƅ}) \geq \gamma^{\text{ғ}}\left(\Delta_\lambda^u x_v - \ell_1, \frac{\text{ƅ}}{2}\right) \otimes \gamma^{\text{ғ}}\left(\Delta_\lambda^u x_v - \ell_2, \frac{\text{ƅ}}{2}\right) < \varepsilon \otimes \varepsilon < \text{ꟻ}$$

can be written. Then, for $\varepsilon > 0$ is arbitrary, $\gamma^{\text{ғ}}(\ell_1 - \ell_2, \text{ƅ}) = 0$. So, $\ell_1 = \ell_2$.

Since all cases are taken into account, $\ell_1 = \ell_2$.

Proposition 3.1 Let $st_{\Delta_\lambda^u}^{\mathcal{N}} - lim\, x = \ell_1$, $st_{\Delta_\lambda^u}^{\mathcal{N}} - lim\, x = \ell_2$ **.** In this case,

$i)\ st_{\Delta_\lambda^u}^{\mathcal{N}} - lim(x_v + \breve{x}_v) = \ell_1 + \ell_2,$

$ii)\ st_{\Delta_\lambda^u}^{\mathcal{N}} - lim\, \check{c}(x_v) = \check{c}\ell_1,$

$iii)\ st_{\Delta_\lambda^u}^{\mathcal{N}} - lim(x_v \breve{x}_v) = \ell_1 \ell_2.$

Definition 3.2 Let $\left(X, \mu^T, \gamma^{\text{ғ}}, \eta^{\ddot{I}}, \boxtimes, \otimes\right)$ be a Neutrosophic normed spaces, $\lambda = (\lambda_v)$ be sequence as given in Definition 2.1 and let u be a convenient fraction. Let $x = (x_v)$ is a sequences in neutrosophic normed where for every $\varepsilon > 0$ and $\text{ƅ} > 0$, if there exists $\ddot{r}$ such that for all $\varepsilon > 0$ and $\text{ƅ} > 0$,

$$\delta_\lambda^u\{k: \mu^T(\Delta_\lambda^u x_v - \ell, \text{ƅ}) \leq 1 - \varepsilon \ or\ \eta^{\ddot{I}}(\Delta_\lambda^u x_v - \ell, \text{ƅ}) \geq \varepsilon, \gamma^{\text{ғ}}(\Delta_\lambda^u x_v - \ell, \text{ƅ}) \geq \varepsilon\} = 0.$$

then $x = (x_v)$ is called (Δ_λ^u)-statistical Cauchy sequences on $\left(X, \mu^T, \gamma^{\text{ғ}}, \eta^{\ddot{I}}, \boxtimes, \otimes\right)$.

Now, similar to the proof in literature , the relationship between (Δ_λ^u) $-$statistical convergence of sequences and being Cauchy sequences will also be given in the following lemmas.

Lemma 3.2 On $\left(X,\mu^{T},\gamma^{\text{ғ}},\eta^{\text{Ї}},\boxtimes,\otimes\right)$ if $x=(x_v)$ is (Δ_λ^u) −statistical convergence then this sequences is (Δ_λ^u) −statistical Cauchy sequences.

Proof Let $x=(x_v)$ is (Δ_λ^u) −statistical convergence sequences on $\left(X,\mu^{T},\gamma^{\text{ғ}},\eta^{\text{Ї}},\boxtimes,\otimes\right)$ then for given $\varepsilon>0$ and choosing $\text{ƅ}>0$ such that $(1-\varepsilon)\boxtimes(1-\varepsilon)>1-\text{Ꝑ}$ and $\varepsilon\otimes\varepsilon<\text{Ꝑ}$. For any $\text{ƅ}>0$, it can be written. Let

$$\delta_\lambda^u(\text{Ɠ}_1)=\delta_\lambda^u\{k\in\mathcal{I}_v:\mu^T(\Delta_\lambda^u x_v-\ell,\text{ƅ})\le 1-\varepsilon\ or\ \eta^{\text{Ї}}(\Delta_\lambda^u x_v-\ell,\text{ƅ})\ge\varepsilon,\gamma^{\text{ғ}}(\Delta_\lambda^u x_v-\ell,\text{ƅ})\ge\varepsilon\}=0,$$

$$\delta_\lambda^u(\text{Ɠ}_2)=\delta_\lambda^u\{k\in\mathcal{I}_v:\mu^T(\Delta_\lambda^u x_v-\ell,\text{ƅ})>1-\varepsilon\ or\ \eta^{\text{Ї}}(\Delta_\lambda^u x_v-\ell,\text{ƅ})<\varepsilon,\gamma^{\text{ғ}}(\Delta_\lambda^u x_v-\ell,\text{ƅ})<\varepsilon\}=1.$$

Let $d\in\text{Ɠ}_2$ then $\mu^T\left(\Delta_\lambda^u x_v-\ell,\frac{\text{ƅ}}{2}\right)>1-\varepsilon$ and $\eta^{\text{Ї}}\left(\Delta_\lambda^u x_v-\ell,\frac{\text{ƅ}}{2}\right)<\varepsilon$, $\gamma^{\text{ғ}}\left(\Delta_\lambda^u x_v-\ell,\frac{\text{ƅ}}{2}\right)<\varepsilon$

Let $\mathfrak{G}=\{k\in\mathcal{I}_v:\mu^T(\Delta_\lambda^u x_v-\ell,\text{ƅ})\le 1-\text{Ꝑ}\ or\ \eta^{\text{Ї}}(\Delta_\lambda^u x_v-\ell,\text{ƅ})\ge\text{Ꝑ},\gamma^{\text{ғ}}(\Delta_\lambda^u x_v-\ell,\text{ƅ})\ge\text{Ꝑ}\}$

It is desired to show that $\mathfrak{G}\subset\text{Ɠ}_1$, so $d\in(\mathfrak{G}/\text{Ɠ}_1)$. Then

$$\mu^T(\Delta_\lambda^u x_v-\ell,\text{ƅ})\le 1-\text{Ꝑ}\ or\ \mu^T\left(\Delta_\lambda^u x_v-\ell,\frac{\text{ƅ}}{2}\right)>1-\text{Ꝑ},\mu^T\left(\Delta_\lambda^u x_v-\ell,\frac{\text{ƅ}}{2}\right)>1-\text{Ꝑ}$$

Here, $\mu^T\left(\Delta_\lambda^u x_v-\ell,\frac{\text{ƅ}}{2}\right)>1-\text{Ꝑ}$. On the other hands,

$$1-\text{Ꝑ}\ge\mu^T(\Delta_\lambda^u x_d-\Delta_\lambda^u x_v,\text{ƅ})\ge\mu^T\left(\Delta_\lambda^u x_d-\ell,\frac{\text{ƅ}}{2}\right)\boxtimes\mu^T\left(\Delta_\lambda^u x_v-\ell,\frac{\text{ƅ}}{2}\right)>(1-\varepsilon)\boxtimes(1-\varepsilon)>1-\text{Ꝑ}.$$

But it is not possible.

On the other hands,

$\eta^{\text{Ї}}(\Delta_\lambda^u x_v-\ell,\text{ƅ})\ge\varepsilon$ and $\eta^{\text{Ї}}\left(\Delta_\lambda^u x_v-\ell,\frac{\text{ƅ}}{2}\right)<\varepsilon$,

with a similar technique, $\eta^{\text{Ї}}\left(\Delta_\lambda^u x_d-\ell,\frac{\text{ƅ}}{2}\right)<\varepsilon$ So,

$$\text{Ꝑ}\le\eta^{\text{Ї}}(\Delta_\lambda^u x_d-\Delta_\lambda^u x_v,\text{ƅ})\le\eta^{\text{Ї}}\left(\Delta_\lambda^u x_d-\ell,\frac{\text{ƅ}}{2}\right)\otimes\eta^{\text{Ї}}\left(\Delta_\lambda^u x_v-\ell,\frac{\text{ƅ}}{2}\right)<\varepsilon\otimes\varepsilon<\text{Ꝑ}.$$

But this is not possible. Again in the same way,

$\gamma^{ғ}(\Delta_\lambda^u x_v - \ell, \mathrm{b}) \geq \varepsilon$ and $\gamma^{ғ}\left(\Delta_\lambda^u x_v - \ell, \frac{\mathrm{b}}{2}\right) < \varepsilon$,

with a similar technique, $\gamma^{ғ}\left(\Delta_\lambda^u x_d - \ell, \frac{\mathrm{b}}{2}\right) < \varepsilon$ So,

$$\text{Ꞔ} \leq \gamma^{ғ}(\Delta_\lambda^u x_d - \Delta_\lambda^u x_v, \mathrm{b}) \leq \gamma^{ғ}\left(\Delta_\lambda^u x_d - \ell, \frac{\mathrm{b}}{2}\right) \otimes \gamma^{ғ}\left(\Delta_\lambda^u x_v - \ell, \frac{\mathrm{b}}{2}\right) < \varepsilon \otimes \varepsilon < \text{Ꞔ}.$$

It is not possible. Hence, $\mathfrak{G} \subset \text{Ɠ}_1$. So, $\delta_\lambda^u(\mathfrak{G}) = 0$. Then, $x = (x_v)$ is (Δ_λ^u) −statistical Cauchy sequences in Neutrosophic normed spaces.

Definition 12 $\left(X, \mu^T, \gamma^{ғ}, \eta^{\ddot{\mathrm{I}}}, \boxtimes, \otimes\right)$ is named (Δ_λ^u)-statistical complete, if all (Δ_λ^u)-statistical Cauchy sequences is (Δ_λ^u)-statistical convergent.

Definition 14 Let $\left(X, \mu^T, \gamma^{ғ}, \eta^{\ddot{\mathrm{I}}}, \boxtimes, \otimes\right)$ be a Neutrosophic normed spaces. $x = (x_v)$ is called (Δ_λ^u)-statistical bounded, if there exists some $\mathrm{b} > 0$ such that

$$\delta_\lambda^u\{k: \mu^T(\Delta_\lambda^u x_v - \ell, \mathrm{b}) > 1 - \varepsilon \; or \; \eta^{\ddot{\mathrm{I}}}(\Delta_\lambda^u x_v - \ell, \mathrm{b}) < \varepsilon, \gamma^{ғ}(\Delta_\lambda^u x_v - \ell, \mathrm{b}) < \varepsilon\} = 0,$$

4. Conclusion

In this paper, we have defined Δ_λ^u − statistical convergence and $\left(\Delta_\lambda^{u,\text{Þ}}\right)$ −Cesaro summable, for fractional difference sequences. Also, Δ_λ^u − statistical convergence with respect to neutrosophic norm is introduced and some fundamental properties are examined. Then, important coverage relations are given for the concept of $(\Delta_\lambda^u)^\alpha$-statistically convergent.

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