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ARTICLE

# A New Method to Evaluate Linear Programming Problem in Bipolar Single-Valued Neutrosophic Environment

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**ABSTRACT**

A bipolar single-valued neutrosophic set can deal with the hesitation relevant to the information of any decision making problem in real life scenarios, where bipolar fuzzy sets may fail to handle those hesitation problems. In this study, we first develop a new method for solving linear programming problems based on bipolar single-valued neutrosophic sets. Further, we apply the score function to transform bipolar single-valued neutrosophic problems into crisp linear programming problems. Moreover, we apply the proposed technique to solve fully bipolar single-valued neutrosophic linear programming problems with non-negative triangular bipolar single-valued neutrosophic numbers ($TBS_vNNs$) and non-negative trapezoidal bipolar single-valued neutrosophic numbers ($T_rBS_vNNs$).

**KEYWORDS**

Bipolar single-valued neutrosophic numbers; score function; trapezoidal numbers; linear programming

## 1 Introduction

The origin of linear programming is the 1940s (World War II). Linear programming (LP) has a linear objective function and a group of linear equalities and inequalities. LP was first used in petroleum manufacturing. Well elaborated data with much information is used for LP problems. On the other hand in real life problems the accuracy of data is often deceitful and affects the optimal solution of LP problems. Probability distribution cannot transact with inaccurate and unclear information.

In 1965, Zadeh [1] introduced fuzzy sets to handle imprecise information. Atanassove [2] gave the concept of intuitionistic fuzzy sets. The intuitionistic fuzzy sets consider both truth-membership and falsity-membership. Intuitionistic fuzzy sets can only handle incomplete information and not the indeterminate information and inconsistent information which exist commonly in the belief system. In 1998, Smarandache [3] presented the notion of neutrosophic set theory. Smarandache [3] and Wang et al. [4] defined a single-valued neutrosophic set which takes the value from the subset of [0, 1]. Deli et al. [5] introduced the concept of bipolar single-valued neutrosophic sets as an extension of bipolar fuzzy sets [6].

Bellman et al. [7] first introduced the concept of decision making in a fuzzy environment. Zimmermann [8] proposed the fuzzy programming technique to solve the multiobjective linear programming problem under a fuzzy environment. Tanaka et al. [9] studied fuzzy-mathematical programming. Lotfi et al. [10] discussed full fuzzy linear programming (FFLP) problems in which all parameters and variables are triangular fuzzy numbers. They used the concept of the symmetric triangular fuzzy number and introduced an approach to defuzzify a general fuzzy quantity. Allahviranloo et al. [11] suggested a method to solve FFLP problems by using a ranking function. Veeramani et al. [12] suggested a method to deal with a kind of fuzzy linear programming (FLP) problem involving symmetric trapezoidal fuzzy numbers. Kumar et al. [13–15] worked on fuzzy linear programming by using non-negative and unrestricted variables to find optimal solutions of FFLP problems. By using trapezoidal fuzzy numbers Behera et al. [16,17] presented a new method to solve linear programming (LP) problems. Najafi et al. [18,19] proposed an efficient technique for solving FFLP by using unrestricted parameters and variables. Moloudzadeh et al. [20] suggested a simple method to solve an arbitrary fully fuzzy linear system. Akram et al. [21] proposed a method to solve LR-bipolar fuzzy linear systems. Mehmood et al. [22] suggested a method to solve fully bipolar fuzzy linear programming (FBFLP) problems by using non-negative bipolar fuzzy numbers and unrestricted bipolar fuzzy numbers with equality constraints. They transformed FBFLP problem into a crisp linear programming problem and achieved the exact bipolar fuzzy optimal solution.

Intuitionistic optimization is an extension of fuzzy optimization. Garg et al. [23] used an Intuitionistic fuzzy optimization method for solving reliability optimization problems in interval environment. Angelov [24] worked in an intuitionistic fuzzy environment. Many researchers solved intuitionistic fuzzy linear programming (IFLP) problems by using triangular intuitionistic fuzzy numbers [25–29]. In an interval-valued intuitionistic fuzzy environment, Bharati et al. [30] gave the solution of multiobjective linear programming problems. Parvathi et al. [31] proposed linear regression analysis in an intuitionistic fuzzy environment and also worked on intuitionistic fuzzy linear programming [32].

The first contribution of neutrosophic linear programming theory was studied by Abdel-Basset et al. [33]. They introduced the neutrosophic linear programming (NLP) models in which the parameters are presented with trapezoidal neutrosophic numbers ($T_rNNs$) and presented a technique for solving them. Das et al. [34] worked to solve NLP problems by using mixed constraints. Bera et al. [35] proposed the Big-M simplex technique to solve NLP problems. Edalatpanah [36] proposed a new direct algorithm for solving the LP problems including neutrosophic variables. Hussian et al. [37] proposed LP problems based on neutrosophic environment. Khalifa et al. [38] suggested a method to solve NLP by using single-valued neutrosophic numbers ($S_vNNs$). Recently, Akram et al. [39,40] have presented new methods to solve Pythagorean fuzzy linear programming problems.

In this paper, we are going to extend the NLP problems into bipolar single-valued neutrosophic linear programming ($BS_vNLP$) problems in which all the coefficients, variables and right hand side are represented by bipolar single-valued neutrosophic numbers ($BS_vNNs$). The fully bipolar single-valued neutrosophic linear programming ($FBS_vNLP$) problems are superior to crisp linear programming (CLP) problems and up to our knowledge, there is no work in literature on $FBS_vNLP$. The $BS_vNLP$ problems are more appropriate to avert unrealistic modeling. This is unique research that deals with LP problems in a $BS_vN$ environment with $TBS_vNNs$ and $T_rBS_vNNs$. Score function is used to convert $BS_vNNs$ to CLP problems.

This paper has been categorized as follows: In Section 2, basic concepts of $BS_vNs$, $TBS_vNNs$, $T_rBS_vNNs$ and their arithmetic operations are discussed. In Section 3, methodology for solving $FBS_vNLP$ problems are explained. In Section 4, some examples and practical models are solved. In Section 5, comparative analysis is depicted and conclusion is given in Section 6. The list of acronyms used in the research paper is given below:

| | |
|---|---|
| $BS_vNN$ | Bipolar single-valued neutrosophic number |
| $TBS_vNN$ | Triangular Bipolar single-valued neutrosophic number |
| $T_rBS_vNN$ | Trapezoidal Bipolar single-valued neutrosophic number |
| LPP | Linear programming Problem |
| FFLP | Fully fuzzy linear programming |

## 2 Preliminaries

**Definition 2.1.** [5] Let $X$ be a non-empty set. A bipolar single-valued Neutrosophic set $\tilde{B}$ in $X$ is an object having the form

$$\tilde{B} = \{\prec x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x) \succ : x \in X\}, \tag{1}$$

where $T^+(x), I^+(x), F^+(x) : X \to [0,1]$ and $T^-(x), I^-(x), F^-(x) : X \to [-1,0]$. The positive membership degree $T^+(x), I^+(x), F^+(x)$ denotes the truth membership, indeterminate membership and falsity membership of an element $x \in X$ corresponding to a bipolar neutrosophic set $\tilde{B}$ similarly negative membership degree $T^-(x), I^-(x), F^-(x)$ denotes the truth membership, indeterminate membership and falsity membership of an element $x \in X$ to some implicit counter-property corresponding to a bipolar neutrosophic set $\tilde{B}$.

**Definition 2.2.** Based on [15], we define a $TBS_vNN$ defined on $\mathbb{R}$

$$\tilde{B} = \prec \tilde{P}, \tilde{N} \succ = \prec ([a_i, b_i, c_i]; \chi_p, \beta_p, \zeta_p), ([e_i, f_i, g_i]; \alpha_n, \varphi_n, \nu_n) \succ \tag{2}$$

is said to be non-negative $TBS_vNN$ if and only if $a_i \geq 0$ and $e_i \geq 0$. where $i = 1, 2, 3$ such that $a_i \leq b_i \leq c_i$ similarly $e_i \leq f_i \leq g_i$ also $\chi_p, \beta_p, \zeta_p \in [0,1]$ and $\alpha_n, \varphi_n, \nu_n \in [-1,0] \subset \mathbb{R}$.

**Definition 2.3.** Based on [15], we define a $T_rBS_vNN$ defined on $\mathbb{R}$

$$\tilde{B} = \prec \tilde{P}, \tilde{N} \succ = \prec \left([a_i, b_i, c_i, d_i]; \chi_p, \beta_p, \zeta_p\right), ([e_i, f_i, g_i, h_i]; \alpha_n, \varphi_n, \nu_n) \succ \tag{3}$$

is said to be non-negative $T_rBS_vNN$ if and only if $a_i \geq 0$ and $e_i \geq 0$. where $i = 1, 2, 3$ and $a_i \leq b_i \leq c_i \leq d_i$ similarly $e_i \leq f_i \leq g_i \leq h_i$ also $\chi_p, \beta_p, \zeta_p \in [0,1]$ and $\alpha_n, \varphi_n, \nu_n \in [-1,0] \subset \mathbb{R}$.

**Definition 2.4.** [30] Let $M = \prec (T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)) \succ$ be a $BS_vNN$ then the score function is presented by:

$$(M) = \frac{1}{6}(T^+(x) + 1 - I^+(x) + 1 - F^+(x) + 1 + T^-(x) - I^-(x) - F^-(x)). \tag{4}$$

**Definition 2.5.** Based on [41], we define a $BS_vNN$ on $\mathbb{R}$ is a $BS_vN$ set such that:

$$\tilde{B} = \prec \tilde{P}, \tilde{N} \succ = \prec ([a_1, b_1, c_1, d_1]; \chi_p), ([a_2, b_2, c_2, d_2]; \beta_p), ([a_3, b_3, c_3, d_3]; \zeta_p), ([e_1, f_1, g_1, h_1]; \alpha_n), \tag{5}$$

$$([e_2, f_2, g_2, h_2]; \varphi_n), ([e_3, f_3, g_3, h_3]; \nu_n) \succ \tag{6}$$

where $\chi_p, \beta_p, \zeta_p \in [0,1]$ and $\alpha_n, \varphi_n, \nu_n \in [-1,0] \subset \mathbb{R}$ whose true membership values are given as:

$$T_p^+(x) = \begin{cases} S_T^l(x), & a_1 \le x \le b_1, \\ \chi_p, & b_1 \le x \le c_1, \\ S_T^r(x), & c_1 \le x \le d_1, \\ 0, & \text{otherwise.} \end{cases} \quad T_n^-(x) = \begin{cases} U_T^l(x), & e_1 \le x \le f_1, \\ \alpha_n, & f_1 \le x \le g_1, \\ U_T^r(x), & g_1 \le x \le h_1, \\ 0, & \text{otherwise.} \end{cases}$$

$S_T^l(x)$ and $U_T^r(x)$ are continuous and non-decreasing functions satisfying the following conditions: $S_T^l(a_1) = 0, S_T^l(b_1) = \chi_p$, and $U_T^r(g_1) = \alpha_n, U_T^r(h_1) = 0$, while $S_T^r(x)$ and $U_T^l(x)$ are continuous and non-increasing functions and satisfying the following conditions:

$S_T^r(c_1) = \chi_p, S_T^r(d_1) = 0, U_T^l(e_1) = 0, U_T^l(f_1) = \alpha_n$, where $\chi_p \in [0,1], \alpha_n \in [-1,0]$. The indeterminacy membership functions are given as:

$$I_p^+(x) = \begin{cases} S_I^l(x), & a_2 \le x \le b_2, \\ \beta_p, & b_2 \le x \le c_2, \\ S_I^r(x), & c_2 \le x \le d_2, \\ 1, & \text{otherwise.} \end{cases} \quad I_n^-(x) = \begin{cases} U_I^l(x), & e_2 \le x \le f_2, \\ \varphi_n, & f_2 \le x \le g_2, \\ U_I^r(x), & g_2 \le x \le h_2, \\ -1, & \text{otherwise.} \end{cases}$$

$S_I^r(x)$ and $U_I^l(x)$ are continuous and non-decreasing functions satisfying the following conditions: $S_I^r(c_2) = \beta_p, S_I^r(d_2) = 1, U_I^l(e_2) = -1, U_I^l(f_2) = \varphi_n$,
while $S_I^l(x)$ and $U_I^r(x)$ are continuous and non-increasing functions and are satisfying the following conditions: $S_I^l(a_2) = 1, S_I^l(b_2) = \beta_p, U_I^r(g_2) = \varphi_n, U_I^r(h_2) = -1$, where $\beta_p \in [0,1], \varphi_n \in [-1,0]$. The falsity membership function are given as:

$$F_p^+(x) = \begin{cases} S_F^l(x), & a_3 \le x \le b_3, \\ \zeta_p, & b_3 \le x \le c_3, \\ S_F^r(x), & c_3 \le x \le d_3, \\ 1, & \text{otherwise.} \end{cases} \quad F_n^-(x) = \begin{cases} U_F^l(x), & e_3 \le x \le f_3, \\ \nu_n, & f_3 \le x \le g_3, \\ U_F^r(x), & g_3 \le x \le h_3, \\ -1, & \text{otherwise.} \end{cases}$$

$S_F^r(x)$ and $U_F^l(x)$ are continuous and non-decreasing functions satisfying the following conditions: $S_F^r(c_3) = \zeta_p, S_F^r(d_3) = 1, U_F^l(e_3) = -1, U_F^l(f_3) = \nu_n$,

$S_F^l(x)$ and $U_F^r(x)$ are continuous and non-increasing functions and are satisfying the following conditions: $S_F^l(a_3) = 1, S_F^l(b_3) = \zeta_p, U_F^r(g_3) = \nu_n, U_F^r(h_3) = -1$, where $\zeta_p \in [0,1], \nu_n \in [-1,0]$.

Some useful information are given in Eqs. (1)–(6).

**Definition 2.6.** Based on [41], we define: If $[a_1,b_1,c_1,d_1]=[a_2,b_2,c_2,d_2]=[a_3,b_3,c_3,d_3]$ and $[e_1,f_1,g_1,h_1]=[e_2,f_2,g_2,h_2]=[e_3,f_3,g_3,h_3]$, then the $BS_vNN$ is reduced to a $T_rBS_vNN$ as:

(1) $\tilde{B}=\prec \tilde{P},\tilde{N}\succ=\prec \left([a_1,b_1,c_1,d_1];\chi_p,\beta_p,\zeta_p\right),([e_1,f_1,g_1,h_1];\alpha_n,\varphi_n,\nu_n)\succ$ .

(2) $\tilde{B}=\prec \tilde{P},\tilde{N}\succ=\prec \left([a_1,b_1,d_1];\chi_p,\beta_p,\zeta_p\right),([e_1,f_1,h_1];\alpha_n,\varphi_n,\nu_n)\succ$ is called a $TBS_vNN$ if and only if $b_1=c_1$ and $f_1=g_1$.

**Definition 2.7.** Based on [42], we define a $TBS_vNN$ defined on $\mathbb{R}$ denoted by:

$\tilde{B}=\prec \tilde{P},\tilde{N}\succ=\prec ([a_1,b_1,c_1];\chi_p),([a_2,b_2,c_2];\beta_p),([a_3,b_3,c_3];\zeta_p),([e_1,f_1,g_1];\alpha_n),$

$([e_2,f_2,g_2];\varphi_n),([e_3,f_3,g_3];\nu_n)\succ$

whose truth, indeterminacy and falsity membership functions are presented by:

$$T_p^+(x)=\begin{cases}\dfrac{x-a_1}{b_1-a_1}\chi_p, & a_1\le x\le b_1,\\[2mm] \dfrac{c_1-x}{c_1-b_1}\chi_p, & b_1\le x\le c_1,\\[2mm] 0, & \text{otherwise.}\end{cases}\qquad T_n^-(x)=\begin{cases}\dfrac{x-e_1}{f_1-e_1}\alpha_n, & e_1\le x\le f_1,\\[2mm] \dfrac{g_1-x}{g_1-f_1}\alpha_n, & f_1\le x\le g_1,\\[2mm] 0, & \text{otherwise.}\end{cases}$$

$$I_p^+(x)=\begin{cases}\dfrac{(b_2-x)+\beta_p(x-a_2)}{b_2-a_2}, & a_2\le x\le b_2,\\[2mm] \dfrac{(x-b_2)+\beta_p(c_2-x)}{c_2-b_2}, & b_2\le x\le c_2,\\[2mm] 1, & \text{otherwise.}\end{cases}\qquad I_n^-(x)=\begin{cases}\dfrac{(f_2-x)+\varphi_n(x-e_2)}{f_2-e_2}, & e_2\le x\le f_2,\\[2mm] \dfrac{(x-f_2)+\varphi_n(g_2-x)}{g_2-f_2}, & f_2\le x\le g_2,\\[2mm] -1, & \text{otherwise.}\end{cases}$$

$$F_p^+(x)=\begin{cases}\dfrac{(b_3-x)+\zeta_p(x-a_3)}{b_3-a_3}, & a_3\le x\le b_3,\\[2mm] \dfrac{(x-b_3)+\zeta_p(c_3-x)}{c_3-b_3}, & b_3\le x\le c_3,\\[2mm] 1, & \text{otherwise.}\end{cases}\qquad F_n^-(x)=\begin{cases}\dfrac{(f_3-x)+\nu_p(x-e_3)}{f_3-e_3}, & e_3\le x\le f_3,\\[2mm] \dfrac{(x-f_3)+\nu_n(g_3-x)}{g_3-f_3}, & f_3\le x\le g_3,\\[2mm] -1, & \text{otherwise.}\end{cases}$$

where $\chi_p,\beta_p,\zeta_p\in[0,1]$ and $\alpha_n,\varphi_n,\nu_n\in[-1,0]\subset\mathbb{R}$.

**Definition 2.8.** Based on [42], we define $T_rBS_vNN$ defined $\mathbb{R}$ denoted by:

$\tilde{B}=\prec \tilde{P},\tilde{N}\succ=\prec ([a_1,b_1,c_1,d_1];\chi_p),([a_2,b_2,c_2,d_2];\beta_p),([a_3,b_3,c_3,d_3];\zeta_p),([e_1,f_1,g_1,h_1];\alpha_n),$

$([e_2,f_2,g_2,h_2];\varphi_n),([e_3,f_3,g_3,h_3];\nu_n)\succ$

whose truth, indeterminacy and falsity membership functions are presented by:

$$T_p^+(x)=\begin{cases}\dfrac{x-a_1}{b_1-a_1}\chi_p, & a_1\le x\le b_1,\\ \chi_p, & b_1\le x\le c_1,\\ \dfrac{d_1-x}{d_1-c_1}\chi_p, & c_1\le x\le d_1,\\ 0, & \text{otherwise.}\end{cases}\qquad T_n^-(x)=\begin{cases}\dfrac{x-e_1}{f_1-e_1}\alpha_n, & e_1\le x\le f_1,\\ \alpha_n, & f_1\le x\le g_1,\\ \dfrac{h_1-x}{h_1-g_1}\alpha_n, & g_1\le x\le h_1,\\ 0, & \text{otherwise.}\end{cases}$$

$$I_p^+(x)=\begin{cases}\dfrac{(b_2-x)+\beta_p(x-a_2)}{b_2-a_2}, & a_2\le x\le b_2,\\ \beta_p, & b_2\le x\le c_2,\\ \dfrac{(x-c_2)+\beta_p(d_2-x)}{d_2-c_2}, & c_2\le x\le d_2,\\ 1, & \text{otherwise.}\end{cases}\qquad I_n^-(x)=\begin{cases}\dfrac{(f_2-x)+\varphi_n(x-e_2)}{f_2-e_2}, & e_2\le x\le f_2,\\ \varphi_n, & f_2\le x\le g_2,\\ \dfrac{(x-g_2)+\varphi_n(h_2-x)}{h_2-g_2}, & g_2\le x\le h_2,\\ -1, & \text{otherwise.}\end{cases}$$

$$F_p^+(x)=\begin{cases}\dfrac{(b_3-x)+\zeta_p(x-a_3)}{b_3-a_3}, & a_3\le x\le b_3,\\ \zeta_p, & b_3\le x\le c_3,\\ \dfrac{(x-c_3)+\zeta_p(d_3-x)}{d_3-c_3}, & c_3\le x\le d_3,\\ 1, & \text{otherwise.}\end{cases}\qquad F_n^-(x)=\begin{cases}\dfrac{(f_3-x)+\nu_p(x-e_3)}{f_3-e_3}, & e_3\le x\le f_3,\\ \nu_n, & f_3\le x\le g_3,\\ \dfrac{(x-g_3)+\nu_n(h_3-x)}{h_3-g_3}, & g_3\le x\le h_3,\\ -1, & \text{otherwise.}\end{cases}$$

where $\chi_p,\beta_p,\zeta_p\in[0,1]$ and $\alpha_n,\varphi_n,\nu_n\in[-1,0]\subset\mathbb{R}$.

**Definition 2.9.** Let

$\tilde{A}_1=\prec\left([a_1,b_1,c_1,d_1];\chi_p^1\right),\left([a_2,b_2,c_2,d_2];\beta_p^1\right),\left([a_3,b_3,c_3,d_3];\zeta_p^1\right),\left([a_4,b_4,c_4,d_4];\alpha_n^1\right),$

$\left([a_5,b_5,c_5,d_5];\varphi_n^1\right),\left([a_6,b_6,c_6,d_6];\nu_n^1\right)\succ$ and

$\tilde{A}_2=\prec\left([e_1,f_1,g_1,h_1];\chi_p^2\right),\left([e_2,f_2,g_2,h_2];\beta_p^2\right),\left([e_3,f_3,g_3,h_3];\zeta_p^2\right),\left([e_4,f_4,g_4,h_4];\alpha_n^2\right),$

$\left([e_5,f_5,g_5,h_5];\varphi_n^2\right),\left([e_6,f_6,g_6,h_6];\nu_n^2\right)\succ.$

be two non-negative $T_rBS_vNNs$, then

(1) $\tilde{A}_1\oplus\tilde{A}_2=\prec\left([a_1+e_1,b_1+f_1,c_1+g_1,d_1+h_1];\chi_p^1\wedge\chi_p^2\right),$
$\left([a_2+e_2,b_2+f_2,c_2+g_2,d_2+h_2];\beta_p^1\vee\beta_p^2\right),$
$\left([a_3+e_3,b_3+f_3,c_3+g_3,d_3+h_3];\zeta_p^1\vee\zeta_p^2\right),\left([a_4+e_4,b_4+f_4,c_4+g_4,d_4+h_4];\alpha_n^1\vee\alpha_n^2\right),$
$\left([a_5+e_5,b_5+f_5,c_5+g_5,d_5+h_5];\varphi_n^1\wedge\varphi_n^2\right),\left([a_6+e_6,b_6+f_6,c_6+g_6,d_6+h_6];\nu_n^1\wedge\nu_n^2\right)\succ.$

(2) $\tilde{A}_1 \ominus \tilde{A}_2 = \prec \left([a_1 - h_1, b_1 - g_1, c_1 - f_1, d_1 - e_1]; \chi_p^1 \wedge \chi_p^2\right),$
$\left([a_2 - h_2, b_2 - g_2, c_2 - f_2, d_2 - e_2]; \beta_p^1 \vee \beta_p^2\right),$
$\left([a_3 - h_3, b_3 - g_3, c_3 - f_3, d_3 - e_3]; \zeta_p^1 \vee \zeta_p^2\right), \left([a_4 - h_4, b_4 - g_4, c_4 - f_4, d_4 - e_4]; \alpha_n^1 \vee \alpha_n^2\right),$
$\left([a_5 - h_5, b_5 - g_5, c_5 - f_5, d_5 - e_5]; \varphi_n^1 \wedge \varphi_n^2\right), \left([a_6 - h_6, b_6 - g_6, c_6 - f_6, d_6 - e_6]; \nu_n^1 \wedge \nu_n^2\right) \succ .$

$$(3)\ \tilde{A}_1 \otimes \tilde{A}_2 = \begin{cases} \prec \left([a_1e_1, b_1f_1, c_1g_1, d_1h_1]; \chi_p^1 \wedge \chi_p^2\right), \left([a_2e_2, b_2f_2, c_2g_2, d_2h_2]; \beta_p^1 \vee \beta_p^2\right), \\ \left([a_3e_3, b_3f_3, c_3g_3, d_3h_3]; \zeta_p^1 \vee \zeta_p^2\right), \left([a_4e_4, b_4f_4, c_4g_4, d_4h_4]; \alpha_n^1 \vee \alpha_n^2\right), \\ \left([a_5e_5, b_5f_5, c_5g_5, d_5h_5]; \varphi_n^1 \wedge \varphi_n^2\right), \left([a_6e_6, b_6f_6, c_6g_6, d_6h_6]; \nu_n^1 \wedge \nu_n^2\right) \succ . \end{cases}$$

For other notations and applications, readers are referred to [43–45].

## 3 Methodology

In this section, a new method is presented to find the non-negative bipolar single-valued neutrosophic optimal solution of $FBS_vNLP$ problems with equality constraints, in which all the parameters are represented by non-negative $BS_vNNs$.

$$\text{Maximize/Minimize } \sum_{j=1}^{n} \tilde{c}_j{}^H \otimes \tilde{x}_j{}^H; \tag{7}$$

subject to

$$\sum_{j=1}^{n} \tilde{a}_{ij}{}^H \otimes \tilde{x}_j{}^H = \tilde{b}_i{}^H, \quad \forall i = 1, 2, 3, \ldots, m.$$

where $\tilde{c}_j^H, \tilde{a}_{ij}^H, \tilde{b}_i^H$ and $\tilde{x}_j^H$ are non-negative $BS_vNNs$.

**Step 1.** Assuming $\tilde{c}_j^H = \prec ([s_j, t_j, u_j]; \chi_j), ([s'_j, t'_j, u'_j]; \beta_j), ([s''_j, t''_j, u''_j]; \zeta_j), ([v_j, w_j, r_j]; \alpha_j), ([v'_j, w'_j, r'_j]; \varphi_j),$ $([v''_j, w''_j, r''_j]; \nu_j) \succ,$
$\tilde{x}_j^H = \prec ([x_j, y_j, z_j]; \phi_j), ([x'_j, y'_j, z'_j]; \gamma_j), ([x''_j, y''_j, z''_j]; \eta_j), ([g_j, h_j, k_j]; \theta_j), ([g'_j, h'_j, k'_j]; \kappa_j),$ $([g''_j, h''_j, k''_j]; \tau_j) \succ,$
$\tilde{a}_{ij}^H = \prec ([d_{ij}, e_{ij}, f_{ij}]; \xi_{ij}), ([d'_{ij}, e'_{ij}, f'_{ij}]; \psi_{ij}), ([d''_{ij}, e''_{ij}, f''_{ij}]; \Gamma_{ij}), ([m_{ij}, n_{ij}, p_{ij}]; \sigma_{ij}), ([m'_{ij}, n'_{ij}, p'_{ij}]; \iota_{ij}),$ $([m''_{ij}, n''_{ij}, p''_{ij}]; \mu_{ij}) \succ$ and
$\tilde{b}_i^H = \prec ([b_i, c_i, d_i]; \epsilon_j), ([b'_i, c'_i, d'_i]; \varepsilon_j), ([b''_i, c''_i, d''_i]; \phi_j), ([e_i, f_i, t_i]; \delta_j), ([e'_i, f'_i, t'_i]; \varrho_j), ([e''_i, f''_i, t''_i]; \omega_j) \succ,$ the $FBS_vNLP$ problem (7) can be transformed as follows:

$$\text{Maximize/Minimize } \begin{pmatrix} \sum_{j=1}^{n} \prec ([s_j, t_j, u_j]; \chi_j), ([s'_j, t'_j, u'_j]; \beta_j), ([s''_j, t''_j, u''_j]; \zeta_j), ([v_j, w_j, r_j]; \alpha_j), \\ ([v'_j, w'_j, r'_j]; \varphi_j), ([v''_j, w''_j, r''_j]; \nu_j) \succ \otimes \prec ([x_j, y_j, z_j]; \phi_j), ([x'_j, y'_j, z'_j]; \gamma_j), \\ ([x''_j, y''_j, z''_j]; \eta_j), ([g_j, h_j, k_j]; \theta_j), ([g'_j, h'_j, k'_j]; \kappa_j), ([g''_j, h''_j, k''_j]; \tau_j) \succ \end{pmatrix}; \tag{8}$$

subject to

$$\sum_{j=1}^{n} \prec ([d_{ij}, e_{ij}, f_{ij}]; \xi_{ij}), ([d'_{ij}, e'_{ij}, f'_{ij}]; \psi_{ij}), ([d''_{ij}, e''_{ij}, f''_{ij}]; \varGamma_{ij}), ([m_{ij}, n_{ij}, p_{ij}]; \sigma_{ij}), ([m'_{ij}, n'_{ij}, p'_{ij}]; \iota_{ij}),$$
$$([m''_{ij}, n''_{ij}, p''_{ij}]; \mu_{ij}) \succ \otimes \prec ([x_j, y_j, z_j]; \phi_j), ([x'_j, y'_j, z'_j]; \gamma_j), ([x''_j, y''_j, z''_j]; \eta_j), ([g_j, h_j, k_j]; \theta_j),$$
$$([g'_j, h'_j, k'_j]; \kappa_j), ([g''_j, h''_j, k''_j]; \tau_j) \succ$$
$$= \prec ([b_i, c_i, d_i]; \epsilon_j), ([b'_i, c'_i, d'_i]; \varepsilon_j), ([b''_i, c''_i, d''_i]; \phi_j), ([e_i, f_i, t_i]; \delta_j), ([e'_i, f'_i, t'_i]; \varrho_j), ([e''_i, f''_i, t''_i]; \omega_j) \succ,$$
$$\forall i = 1, 2, 3, \ldots, m.$$

where $\prec ([x_j, y_j, z_j]; \phi_j), ([x'_j, y'_j, z'_j]; \gamma_j), ([x''_j, y''_j, z''_j]; \eta_j), ([g_j, h_j, k_j]; \theta_j),$
$([g'_j, h'_j, k'_j]; \kappa_j), ([g''_j, h''_j, k''_j]; \tau_j) \succ$ are non-negative $TBS_vFN$, $\forall j = 1, 2, 3, \ldots, n$.

**Step 2.** Using product of non-negative $BS_vNNs$ (2.9) and assuming

$$\prec ([d_{ij}, e_{ij}, f_{ij}]; \xi_{ij}), ([d'_{ij}, e'_{ij}, f'_{ij}]; \psi_{ij}), ([d''_{ij}, e''_{ij}, f''_{ij}]; \varGamma_{ij}), ([m_{ij}, n_{ij}, p_{ij}]; \sigma_{ij}), ([m'_{ij}, n'_{ij}, p'_{ij}]; \iota_{ij}),$$
$$([m''_{ij}, n''_{ij}, p''_{ij}]; \mu_{ij}) \succ \otimes \prec ([x_j, y_j, z_j]; \phi_j), ([x'_j, y'_j, z'_j]; \gamma_j), ([x''_j, y''_j, z''_j]; \eta_j), ([g_j, h_j, k_j]; \theta_j),$$
$$([g'_j, h'_j, k'_j]; \kappa_j), ([g''_j, h''_j, k''_j]; \tau_j) \succ$$
$$= \prec ([d'_{ij}, e'_{ij}, f'_{ij}]; \xi_{ij} \wedge \phi_j), ([d''_{ij}, e''_{ij}, f''_{ij}]; \psi_{ij} \vee \gamma_j), ([d'''_{ij}, e'''_{ij}, f'''_{ij}]; \varGamma_{ij} \vee \eta_j), ([m'_{ij}, n'_{ij}, p'_{ij}]; \sigma_{ij} \vee \theta_j),$$
$$([m''_{ij}, n''_{ij}, p''_{ij}]; \iota_{ij} \wedge \kappa_j), ([m'''_{ij}, n'''_{ij}, p'''_{ij}]; \mu_{ij} \wedge \tau_j) \succ .$$

The $FBS_vNLP$ problem (8), can be transformed as follows:

$$\text{Maximize/Minimize} \begin{pmatrix} \sum_{j=1}^{n} (\prec ([s_j, t_j, u_j]; \chi_j), ([s'_j, t'_j, u'_j]; \beta_j), ([s''_j, t''_j, u''_j]; \zeta_j), ([v_j, w_j, r_j]; \alpha_j), \\ ([v'_j, w'_j, r'_j]; \varphi_j), ([v''_j, w''_j, r''_j]; \nu_j) \succ \otimes \prec ([x_j, y_j, z_j]; \phi_j), ([x'_j, y'_j, z'_j]; \gamma_j), \\ ([x''_j, y''_j, z''_j]; \eta_j), ([g_j, h_j, k_j]; \theta_j), ([g'_j, h'_j, k'_j]; \kappa_j), ([g''_j, h''_j, k''_j]; \tau_j) \succ) \end{pmatrix}; \tag{9}$$

subject to

$$\sum_{j=1}^{n} (\prec ([d'_{ij}, e'_{ij}, f'_{ij}]; \xi_{ij} \wedge \phi_j), ([d''_{ij}, e''_{ij}, f''_{ij}]; \psi_{ij} \vee \gamma_j), ([d'''_{ij}, e'''_{ij}, f'''_{ij}]; \varGamma_{ij} \vee \eta_j), ([m'_{ij}, n'_{ij}, p'_{ij}]; \sigma_{ij} \vee \theta_j),$$
$$([m''_{ij}, n''_{ij}, p''_{ij}]; \iota_{ij} \wedge \kappa_j), ([m'''_{ij}, n'''_{ij}, p'''_{ij}]; \mu_{ij} \wedge \tau_j) \succ$$
$$= \prec ([b_i, c_i, d_i]; \epsilon_j), ([b'_i, c'_i, d'_i]; \varepsilon_j), ([b''_i, c''_i, d''_i]; \phi_j), ([e_i, f_i, t_i]; \delta_j), ([e'_i, f'_i, t'_i]; \varrho_j), ([e''_i, f''_i, t''_i]; \omega_j) \succ,$$
$$\forall i = 1, 2, 3, \ldots, m.$$

where $\prec ([x_j, y_j, z_j]; \phi_j), ([x'_j, y'_j, z'_j]; \gamma_j), ([x''_j, y''_j, z''_j]; \eta_j), ([g_j, h_j, k_j]; \theta_j),$
$([g'_j, h'_j, k'_j]; \kappa_j), ([g''_j, h''_j, k''_j]; \tau_j) \succ$ are non-negative $TS_vBFN$, $\forall j = 1, 2, 3, \ldots, n$.

**Step 3.** Using arithmetic operations (2.9), above problem becomes:

$$\text{Maximize/Minimize} \begin{pmatrix} \sum_{j=1}^{n}(\prec([s_j,t_j,u_j];\chi_j),([s'_j,t'_j,u'_j];\beta_j),([s''_j,t''_j,u''_j];\zeta_j),([v_j,w_j,r_j];\alpha_j), \\ ([v'_j,w'_j,r'_j];\varphi_j),([v''_j,w''_j,r''_j];\nu_j)\succ\otimes\prec([x_j,y_j,z_j];\phi_j),([x'_j,y'_j,z'_j];\gamma_j), \\ ([x''_j,y''_j,z''_j];\eta_j),([g_j,h_j,k_j];\theta_j),([g'_j,h'_j,k'_j];\kappa_j),([g''_j,h''_j,k''_j];\tau_j)\succ) \end{pmatrix};$$

subject to

$$\begin{aligned}
&\textstyle\sum_{j=1}^{n} d'_{ij} = b_i, \quad \sum_{j=1}^{n} m'_{ij} = e_i,\\
&\textstyle\sum_{j=1}^{n} e'_{ij} = c_i, \quad \sum_{j=1}^{n} n'_{ij} = f_i,\\
&\textstyle\sum_{j=1}^{n} f'_{ij} = d_i, \quad \sum_{j=1}^{n} p'_{ij} = t_i,\\
&\wedge[\xi_{ij}\wedge\phi_j] = \epsilon_j, \quad \vee[\sigma_{ij}\vee\theta_j] = \delta_j,\\
&\textstyle\sum_{j=1}^{n} d''_{ij} = b'_i, \quad \sum_{j=1}^{n} m''_{ij} = e'_i,\\
&\textstyle\sum_{j=1}^{n} e''_{ij} = c'_i, \quad \sum_{j=1}^{n} n''_{ij} = f'_i,\\
&\textstyle\sum_{j=1}^{n} f''_{ij} = d'_i, \quad \sum_{j=1}^{n} p''_{ij} = t'_i,\\
&\vee[\psi_{ij}\vee\gamma_j] = \varepsilon_j, \quad \wedge[\iota_{ij}\wedge\kappa_j] = \varrho_j,\\
&\textstyle\sum_{j=1}^{n} d'''_{ij} = b''_i, \quad \sum_{j=1}^{n} m'''_{ij} = e''_i,\\
&\textstyle\sum_{j=1}^{n} e'''_{ij} = c''_i, \quad \sum_{j=1}^{n} n'''_{ij} = f''_i,\\
&\textstyle\sum_{j=1}^{n} f'''_{ij} = d''_i, \quad \sum_{j=1}^{n} p'''_{ij} = t''_i,\\
&\vee[\Gamma_{ij}\vee\eta_j] = \phi_j, \quad \wedge[\mu_{ij}\wedge\tau_j] = \omega_j,
\end{aligned} \tag{10}$$

$x_j \geq 0$, $y_j - x_j \geq 0$, $z_j - y_j \geq 0$, $x'_j \geq 0$, $y'_j - x'_j \geq 0$, $z'_j - y'_j \geq 0$, $x''_j \geq 0$, $y''_j - x''_j \geq 0$, $z''_j - y''_j \geq 0$, $g_j \geq 0$, $h_j - g_j \geq 0$, $k_j - h_j \geq 0$, $g'_j \geq 0$, $h'_j - g'_j \geq 0$, $k'_j - h'_j \geq 0$, $g''_j \geq 0$, $h''_j - g''_j \geq 0$, $k''_j - h''_j \geq 0$, $\forall j = 1, 2, 3, \ldots, n$.

and $\phi_j, \gamma_j, \eta_j \in [0, 1], \theta_j, \kappa_j, \tau_j \in [-1, 0]$.

**Step 4.** By using score function (2.4), the neutrosophic optimal solution of the $FBS_vNLP$ problem can be obtained by solving following CLP problem:

$$\text{Maximize/Minimize} \begin{pmatrix} S(\sum_{j=1}^{n}(\prec([s_j,t_j,u_j];\chi_j),([s'_j,t'_j,u'_j];\beta_j),([s''_j,t''_j,u''_j];\zeta_j),([v_j,w_j,r_j];\alpha_j), \\ ([v'_j,w'_j,r'_j];\varphi_j),([v''_j,w''_j,r''_j];\nu_j)\succ\otimes\prec([x_j,y_j,z_j];\phi_j),([x'_j,y'_j,z'_j];\gamma_j), \\ ([x''_j,y''_j,z''_j];\eta_j),([g_j,h_j,k_j];\theta_j),([g'_j,h'_j,k'_j];\kappa_j),([g''_j,h''_j,k''_j];\tau_j)\succ)) \end{pmatrix}; \tag{11}$$

subject to:

$\sum_{j=1}^{n} d'_{ij} = b_i, \quad \sum_{j=1}^{n} m'_{ij} = e_i,$

$\sum_{j=1}^{n} e'_{ij} = c_i, \quad \sum_{j=1}^{n} n'_{ij} = f_i,$

$\sum_{j=1}^{n} f'_{ij} = d_i, \quad \sum_{j=1}^{n} p'_{ij} = t_i,$

$\wedge[\xi_{ij} \wedge \phi_j] = \epsilon_j, \quad \vee[\sigma_{ij} \vee \theta_j] = \delta_j,$

$\sum_{j=1}^{n} d''_{ij} = b'_i, \quad \sum_{j=1}^{n} m''_{ij} = e'_i,$

$\sum_{j=1}^{n} e''_{ij} = c'_i, \quad \sum_{j=1}^{n} n''_{ij} = f'_i,$

$\sum_{j=1}^{n} f''_{ij} = d'_i, \quad \sum_{j=1}^{n} p''_{ij} = t'_i,$

$\vee[\psi_{ij} \vee \gamma_j] = \varepsilon_j, \quad \wedge[\iota_{ij} \wedge \kappa_j] = \varrho_j,$

$\sum_{j=1}^{n} d'''_{ij} = b''_i, \quad \sum_{j=1}^{n} m'''_{ij} = e''_i,$

$\sum_{j=1}^{n} e'''_{ij} = c''_i, \quad \sum_{j=1}^{n} n'''_{ij} = f''_i,$

$\sum_{j=1}^{n} f'''_{ij} = d''_i, \quad \sum_{j=1}^{n} p'''_{ij} = t''_i,$

$\vee[\Gamma_{ij} \vee \eta_j] = \phi_j, \quad \wedge[\mu_{ij} \wedge \tau_j] = \omega_j,$

$x_j \geq 0, \quad y_j - x_j \geq 0, \quad z_j - y_j \geq 0, \quad x'_j \geq 0, \quad y'_j - x'_j \geq 0, \quad z'_j - y'_j \geq 0, \quad x''_j \geq 0, \quad y''_j - x''_j \geq 0, \quad z''_j - y''_j \geq 0, \quad g_j \geq 0, \quad h_j - g_j \geq 0, \quad k_j - h_j \geq 0, \quad g'_j \geq 0, \quad h'_j - g'_j \geq 0, \quad k'_j - h'_j \geq 0, \quad g''_j \geq 0, \quad h''_j - g''_j \geq 0, \quad k''_j - h''_j \geq 0, \quad \forall j = 1, 2, 3, \ldots, n.$

and $\phi_j, \gamma_j, \eta_j \in [0, 1], \theta_j, \kappa_j, \tau_j \in [-1, 0]$.

**Step 5.** Solve the crisp LP problem (11), to find the optimal solution $\prec ([x_j, y_j, z_j]; \phi_j)$, $([x'_j, y'_j, z'_j]; \gamma_j), ([x''_j, y''_j, z''_j]; \eta_j), ([g_j, h_j, k_j]; \theta_j), ([g'_j, h'_j, k'_j]; \kappa_j), ([g''_j, h''_j, k''_j]; \tau_j) \succ$,
$\forall j = 1, 2, 3, \ldots, n$.

**Step 6.** Find the bipolar single-valued neutrosophic optimal solution $\tilde{x^*_{j.}}$ of $FBS_vNLP$ problem by putting the values of $([x_j, y_j, z_j]; \phi_j), ([x'_j, y'_j, z'_j]; \gamma_j), ([x''_j, y''_j, z''_j]; \eta_j), ([g_j, h_j, k_j]; \theta_j), ([g'_j, h'_j, k'_j]; \kappa_j)$, $([g''_j, h''_j, k''_j]; \tau_j)$ in $\tilde{x^{*.}_j} = \prec ([x_j, y_j, z_j]; \phi_j), ([x'_j, y'_j, z'_j]; \gamma_j), ([x''_j, y''_j, z''_j]; \eta_j), ([g_j, h_j, k_j]; \theta_j), ([g'_j, h'_j, k'_j]; \kappa_j)$, $([g''_j, h''_j, k''_j]; \tau_j), \succ$
$\forall j = 1, 2, 3, \ldots, n$.

**Step 7.** Find bipolar single-valued neutrosophic optimal value by putting the values of $\tilde{x^*_{j.}}$ obtained in Step 6, in $\sum_{j=1}^{n} \tilde{c_j}.^H \otimes \tilde{x}^H_{j.}$.

Thus, we state the existence condition for the optimal solution of bipolar single-valued neutrosophic LPP in the following Theorem:

**Theorem 3.1.** The solution of FBNLP problem

Maximize/Minimize

$$Z=\sum_{j=1}^{n} C_j^H \otimes X_j^H; \quad \text{subject to } \sum_{j=1}^{n} A_{ij}^H \otimes X_j^H = B_i^H, \quad \forall i=1,\ldots,m, \tag{12}$$

$X_j^H$ are non-negative TBNNs and $A_{ij}^H$, $C_j^H$, $B_i^H$ are TBNNs exists, when the solution of the associated crisp LPP

Maximize/Minimize

$$s(Z)=\begin{pmatrix} s(\sum_{j=1}^{n}(\prec([s_j,t_j,u_j];\chi_j),([s'_j,t'_j,u'_j];\beta_j),([s''_j,t''_j,u''_j];\zeta_j),([v_j,w_j,r_j];\alpha_j), \\ ([v'_j,w'_j,r'_j];\varphi_j),([v''_j,w''_j,r''_j];\nu_j)\succ\otimes\prec([x_j,y_j,z_j];\phi_j),([x'_j,y'_j,z'_j];\gamma_j), \\ ([x''_j,y''_j,z''_j];\eta_j),([g_j,h_j,k_j];\theta_j),([g'_j,h'_j,k'_j];\kappa_j),([g''_j,h''_j,k''_j];\tau_j)\succ)) \end{pmatrix};$$

subject to

$\sum_{j=1}^{n} d'_{ij}=b_i,\quad \sum_{j=1}^{n} e'_{ij}=c_i,\quad \sum_{j=1}^{n} f'_{ij}=d_i,\quad \sum_{j=1}^{n} d''_{ij}=b'_i,\quad \sum_{j=1}^{n} e''_{ij}=c'_i,\quad \sum_{j=1}^{n} f''_{ij}=d'_i,$

$\sum_{j=1}^{n} d'''_{ij}=b''_i,\quad \sum_{j=1}^{n} e'''_{ij}=c''_i,\quad \sum_{j=1}^{n} f'''_{ij}=d''_i,$

$\wedge[\xi_{ij}\wedge\phi_j]=\epsilon_j,\quad \vee[\psi_{ij}\vee\gamma_j]=\varepsilon_j,\quad \vee[\Gamma_{ij}\vee\eta_j]=\phi_j,$

$\sum_{j=1}^{n} m'_{ij}=e_i,\quad \sum_{j=1}^{n} n'_{ij}=f_i,\quad \sum_{j=1}^{n} p'_{ij}=t_i,\quad \sum_{j=1}^{n} m''_{ij}=e'_i,\quad \sum_{j=1}^{n} n''_{ij}=f'_i,\quad \sum_{j=1}^{n} p''_{ij}=t'_i,$

$\sum_{j=1}^{n} m'''_{ij}=e''_i,\quad \sum_{j=1}^{n} n'''_{ij}=f''_i, \sum_{j=1}^{n} p'''_{ij}=t''_i,$

$\vee[\sigma_{ij}\vee\theta_j]=\delta_j,\quad \wedge[\iota_{ij}\wedge\kappa_j]=\varrho_j,$

$\wedge[\mu_{ij}\wedge\tau_j]=\omega_j,$

$x_j\ge 0,\quad y_j-x_j\ge 0,\quad z_j-y_j\ge 0,\quad x'_j\ge 0,\quad y'_j-x'_j\ge 0,\quad z'_j-y'_j\ge 0,\quad x''_j\ge 0,\quad y''_j-x''_j\ge 0,\quad z''_j-y''_j\ge 0,\quad g_j\ge 0,\quad h_j-g_j\ge 0,\quad k_j-h_j\ge 0,\quad g'_j\ge 0,\quad h'_j-g'_j\ge 0,\quad k'_j-h'_j\ge 0,\quad g''_j\ge 0,\quad h''_j-g''_j\ge 0,\quad k''_j-h''_j\ge 0, \forall j=1,2,3,\ldots,n.$

and $\phi_j,\gamma_j,\eta_j\in[0,1],\theta_j,\kappa_j,\tau_j\in[-1,0]$.

$\forall i=1,\ldots,m$ exists. Otherwise, there is no guarantee that the bipolar single-valued neutrosophic optimal solution exists.

Proof. Straightforward.

## 4 Numerical Examples

In this section, we present numerical examples and models to illustrate the methodology given in Section 3.

**Example 4.1.** Minimize $\prec([2,3,5];0.7),([2,6,8];0.3),([2,5,7];0.2),([3,5,6];-0.7),([3,4,6];-0.1),([3,4,8];-0.2)\succ\otimes\tilde{x_1}\oplus\prec([3,6,9];0.8),([3,5,8];0.2),([3,6,8];0.3),([2,3,7];-0.8),([2,3,6];-0.2),([2,4,7];-0.3)\succ\otimes\tilde{x_2}$;
subject to
$\prec([1,3,4];0.9),([1,4,5];0.1),([2,3,4];0.2),([2,4,9];-0.9),([3,5,7];-0.2),([3,4,5];-0.3)\succ\otimes\tilde{x_1}$
$\oplus\prec([4,5,6];0.8),([4,5,7];0.1),([4,6,7];0.2),([5,6,7];-0.8),([5,6,8];-0.2),([4,6,9];-0.1)\succ\otimes\tilde{x_2}$
$=\prec([23,50,76];0.7),([23,59,98];0.2),([24,54,80];0.3),([24,42,105];-0.7),([23,40,113];-0.3),([19,48,98];-0.4)\succ$,
$\prec([0,4,9];0.9),([0,2,8];0.3),([1,4,7];0.2),([2,6,8];-0.8),([2,5,9];-0.3),([1,6,8];-0.2)\succ\otimes\tilde{x_1}$
$\oplus\prec([3,4,8];0.8),([3,5,9];0.2),([3,6,7];0.3),([1,2,3];-0.9),([2,3,4];-0.3),([3,5,7];-0.2)\succ\otimes\tilde{x_2}$
$=\prec([15,48,127];0.7),([15,47,137];0.3),([17,58,98];0.3),([8,28,74];-0.7),([10,25,95];-0.3),([13,48,105];-0.2)\succ$,
where $\tilde{x_1}=\prec([k_1,l_1,m_1];\chi_1),([k_1',l_1',m_1'];\beta_1),([k_1'',l_1'',m_1''];\zeta_1),([n_1,o_1,p_1];\alpha_1),([n_1',o_1',p_1'];\varphi_1),([n_1'',o_1'',p_1''];\nu_1)\succ$ and $\tilde{x_2}=\prec([k_2,l_2,m_2];\chi_2),([k_2',l_2',m_2'];\beta_2),([k_2'',l_2'',m_2''];\zeta_2),([n_2,o_2,p_2];\alpha_2),([n_2',o_2',p_2'];\varphi_2),([n_2'',o_2'',p_2''];\nu_2)\succ$ are non-negative $TBS_vNNs$.

**Step 1.**

$$\text{Minimize}\begin{pmatrix}\prec([2,3,5];0.7),([2,6,8];0.3),([2,5,7];0.2),([3,5,6];-0.7),([3,4,6];-0.1),\\ ([3,4,8];-0.2)\succ\otimes\prec([k_1,l_1,m_1];\chi_1),([k_1',l_1',m_1'];\beta_1),([k_1'',l_1'',m_1''];\zeta_1),([n_1,o_1,p_1];\alpha_1),\\ ([n_1',o_1',p_1'];\varphi_1),([n_1'',o_1'',p_1''];\nu_1)\succ\oplus\prec([3,6,9];0.8),([3,5,8];0.2),([3,6,8];0.3),\\ ([2,3,7];-0.8),([2,3,6];-0.2),([2,4,7];-0.3)\succ\otimes\prec([k_2,l_2,m_2];\chi_2),([k_2',l_2',m_2'];\beta_2),\\ ([k_2'',l_2'',m_2''];\zeta_2),([n_2,o_2,p_2];\alpha_2),([n_2',o_2',p_2'];\varphi_2),([n_2'',o_2'',p_2''];\nu_2)\succ;\end{pmatrix};$$

subject to
$\prec([1,3,4];0.9),([1,4,5];0.1),([2,3,4];0.2),([2,4,9];-0.9),([3,5,7];-0.2),([3,4,5];-0.3)\succ\otimes$
$\prec([k_1,l_1,m_1];\chi_1),([k_1',l_1',m_1'];\beta_1),([k_1'',l_1'',m_1''];\zeta_1),([n_1,o_1,p_1];\alpha_1),([n_1',o_1',p_1'];\varphi_1),([n_1'',o_1'',p_1''];\nu_1)\succ$

$\oplus\prec([4,5,6];0.8),([4,5,7];0.1),([4,6,7];0.2),([5,6,7];-0.8),([5,6,8];-0.2),([4,6,9];-0.1)\succ\otimes$
$\prec([k_2,l_2,m_2];\chi_2),([k_2',l_2',m_2'];\beta_2),([k_2'',l_2'',m_2''];\zeta_2),([n_2,o_2,p_2];\alpha_2),([n_2',o_2',p_2'];\varphi_2),([n_2'',o_2'',p_2''];\nu_2)\succ$

$=\prec([23,50,76];0.7),([23,59,98];0.2),([24,54,80];0.3),([24,42,105];-0.7),([23,40,113];-0.3),([19,48,98];-0.4)\succ$,
$\prec([0,4,9];0.9),([0,2,8];0.3),([1,4,7];0.2),([2,6,8];-0.8),([2,5,9];-0.3),([1,6,8];-0.2)\succ\otimes$
$\prec([k_1,l_1,m_1];\chi_1),([k_1',l_1',m_1'];\beta_1),([k_1'',l_1'',m_1''];\zeta_1),([n_1,o_1,p_1];\alpha_1),([n_1',o_1',p_1'];\varphi_1),([n_1'',o_1'',p_1''];\nu_1)\succ$

$\oplus\prec([3,4,8];0.8),([3,5,9];0.2),([3,6,7];0.3),([1,2,3];-0.9),([2,3,4];-0.3),([3,5,7];-0.2)\succ\otimes$
$\prec([k_2,l_2,m_2];\chi_2),([k_2',l_2',m_2'];\beta_2),([k_2'',l_2'',m_2''];\zeta_2),([n_2,o_2,p_2];\alpha_2),([n_2',o_2',p_2'];\varphi_2),([n_2'',o_2'',p_2''];\nu_2)\succ$

$=\prec([15,48,127];0.7),([15,47,137];0.3),([17,58,98];0.3),([8,28,74];-0.7),([10,25,95];-0.3),([13,48,105];-0.2)\succ$,
where $\tilde{x_1}=\prec([k_1,l_1,m_1];\chi_1),([k_1',l_1',m_1'];\beta_1),([k_1'',l_1'',m_1''];\zeta_1),([n_1,o_1,p_1];\alpha_1),([n_1',o_1',p_1'];\varphi_1),([n_1'',o_1'',p_1''];\nu_1)\succ$ and $\tilde{x_2}=\prec([k_2,l_2,m_2];\chi_2),([k_2',l_2',m_2'];\beta_2),([k_2'',l_2'',m_2''];\zeta_2),([n_2,o_2,p_2];\alpha_2),([n_2',o_2',p_2'];\varphi_2),([n_2'',o_2'',p_2''];\nu_2)\succ$ are non-negative $TBS_vNNs$.

**Step 2.**

Minimize $\prec([2k_1, 3l_1, 5m_1]; 0.7 \wedge \chi_1), ([2k_1', 6l_1', 8m_1']; 0.3 \vee \beta_1), ([2k_1'', 5l_1'', 7m_1'']; 0.2 \vee \zeta_1), ([3n_1, 5o_1, 6p_1]; -0.7 \vee \alpha_1), ([3n_1', 4o_1', 6p_1']; -0.1 \wedge \varphi_1), ([3n_1'', 4o_1'', 8p_1'']; -0.2 \wedge \nu_1) \succ$
$\oplus \prec([3k_2, 6l_2, 9m_2]; 0.8 \wedge \chi_2), ([3k_2, 5l_2, 8m_2]; 0.2 \vee \beta_2), ([3k_2, 6l_2, 8m_2]; 0.3 \vee \zeta_2), ([2n_2, 3o_2, 7p_2]; -0.8 \vee \alpha_2), ([2n_2, 3o_2, 6p_2]; -0.2 \wedge \varphi_2), ([2n_2, 4o_2, 7p_2]; -0.3 \wedge \nu_2) \succ;$
subject to
$\prec([k_1, 3l_1, 4m_1]; 0.9 \wedge \chi_1), ([k_1', 4l_1', 5m_1']; 0.1 \vee \beta_1), ([2k_1'', 3l_1'', 4m_1'']; 0.2 \vee \zeta_1), ([2n_1, 4o_1, 9p_1]; -0.9 \vee \alpha_1), ([3n_1', 5o_1', 7p_1']; -0.2 \wedge \varphi_1), ([3n_1'', 4o_1'', 5p_1'']; -0.3 \wedge \nu_1) \succ \oplus$
$\prec([4k_2, 5l_2, 6m_2]; 0.8 \wedge \chi_2), ([4k_2', 5l_2', 7m_2']; 0.1 \vee \beta_2), ([4k_2'', 6l_2'', 7m_2'']; 0.2 \vee \zeta_2), ([5n_2, 60_2, 7p_2]; -0.8 \vee \alpha_2), ([5n_2', 60_2', 8p_2']; -0.2 \wedge \varphi_2), ([4n_2'', 60_2'', 9p_2'']; -0.1 \wedge \nu_2) \succ$
$= \prec([23, 50, 76]; 0.7), ([23, 59, 98]; 0.2), ([24, 54, 80]; 0.3), ([24, 42, 105]; -0.7), ([23, 40, 113]; -0.3), ([19, 48, 98]; -0.4) \succ,$
$\prec([0k_1, 4l_1, 9m_1]; 0.9 \wedge \chi_1), ([0k_1', 2l_1', 8m_1']; 0.3 \vee \beta_1), ([1k_1'', 4l_1'', 7m_1'']; 0.2 \vee \zeta_1), ([2n_1, 6o_1, 8p_1]; -0.8 \vee \alpha_1), ([2n_1', 5o_1', 9p_1']; -0.3 \wedge \varphi_1), ([1n_1'', 6o_1'', 8p_1'']; -0.2 \wedge \nu_1) \succ \oplus$
$\prec([3k_2, 4l_2, 8m_2]; 0.8 \wedge \chi_2), ([3k_2', 5l_2', 9m_2']; 0.2 \vee \beta_2), ([3k_2'', 6l_2'', 7m_2'']; 0.3 \vee \zeta_2), ([1n_2, 20_2, 3p_2]; -0.9 \vee \alpha_2), ([2n_2', 30_2', 4p_2']; -0.3 \wedge \varphi_2), ([3n_2'', 50_2'', 7p_2'']; -0.2 \wedge \nu_2) \succ$
$= \prec([15, 48, 127]; 0.7), ([15, 47, 137]; 0.3), ([17, 58, 98]; 0.3), ([8, 28, 74]; -0.7), ([10, 25, 95]; -0.3), ([13, 48, 105]; -0.2) \succ,$
$k_1 \geq 0, \quad l_1 - k_1 \geq 0, \quad m_1 - l_1 \geq 0, k_1' \geq 0, \quad l_1' - k_1' \geq 0, \quad m_1' - l_1' \geq 0, \quad k_1'' \geq 0, \quad l_1'' - k_1'' \geq 0, \quad m_1'' - l_1'' \geq 0, \quad n_1 \geq 0, \quad o_1 - n_1 \geq 0, \quad p_1 - o_1 \geq 0, \quad n_1' \geq 0, \quad o_1' - n_1' \geq 0, \quad p_1' - o_1' \geq 0, \quad n_1'' \geq 0, \quad o_1'' - n_1'' \geq 0, \quad p_1'' - o_1'' \geq 0, \ k_2 \geq 0, \quad l_2 - k_2 \geq 0, \quad m_2 - l_2 \geq 0, k_2' \geq 0, \quad l_2' - k_2' \geq 0, \quad m_2' - l_2' \geq 0, \quad k_2'' \geq 0, \quad l_2'' - k_2'' \geq 0, \quad m_2'' - l_2'' \geq 0, \quad n_2 \geq 0, \quad o_2 - n_2 \geq 0, \quad p_2 - o_2 \geq 0, \quad n_2' \geq 0, \quad o_2' - n_2' \geq 0, \quad p_2' - o_2' \geq 0, \quad n_2'' \geq 0, \quad o_2'' - n_2'' \geq 0, \quad p_2'' - o_2'' \geq 0,$
here
$\chi_1, \beta_1, \zeta_1, \chi_2, \beta_2, \zeta_2 \in [0, 1], \quad \alpha_1, \varphi_1, \nu_1, \alpha_2, \varphi_2, \nu_2 \in [-1, 0].$

**Step 3.**

Minimize $\prec([2k_1 + 3k_2, 3l_1 + 6l_2, 5m_1 + 9m_2]; \wedge[(0.7 \wedge \chi_1) \wedge (0.8 \wedge \chi_2)]), ([2k_1' + 3k_2', 6l_1' + 5l_2', 8m_1' + 8m_2']; \vee[(0.3 \vee \beta_1) \vee (0.2 \vee \beta_2)]), ([2k_1'' + 3k_2'', 5l_1'' + 6l_2'', 7m_1'' + 8m_2'']; \vee[(0.2 \vee \zeta_1) \vee (0.3 \vee \zeta_2)]), ([3n_1 + 2n_2, 5o_1 + 3o_2, 6p_1 + 7p_2]; \vee[(-0.7 \vee \alpha_1) \vee (-0.8 \vee \alpha_2)]), ([3n_1' + 2n_2', 4o_1' + 3o_2', 6p_1' + 6p_2']; \wedge[(-0.1 \wedge \varphi_1) \wedge (-0.2 \wedge \varphi_2)]), ([3n_1'' + 2n_2'', 4o_1'' + 4o_2'', 8p_1'' + 7p_2'']; \wedge[(-0.2 \wedge \nu_1) \wedge (-0.3 \wedge \nu_2)]) \succ;$
subject to
$\prec([k_1 + 4k_2, 3l_1 + 5l_2, 4m_1 + 6m_2]; \wedge[(0.9 \wedge \chi_1) \wedge (0.8 \wedge \chi_2)]), ([k_1' + 4k_2', 4l_1' + 5l_2', 5m_1' + 7m_2']; \vee[(0.1 \vee \beta_1) \vee (0.1 \vee \beta_2)]), ([2k_1'' + 4k_2'', 3l_1'' + 6l_2'', 4m_1'' + 7m_2'']; \vee[(0.2 \vee \zeta_1) \vee (0.2 \vee \zeta_2)]), ([2n_1 + 5n_2, 4o_1 + 6o_2, 9p_1 + 7p_2]; \vee[(-0.9 \vee \alpha_1) \vee (-0.8 \vee \alpha_2)]), ([3n_1' + 5n_2', 5o_1' + 6o_2', 7p_1' + 8p_2']; \wedge[(-0.2 \wedge \varphi_1) \wedge (-0.2 \wedge \varphi_2)]), ([3n_1'' + 4n_2'', 4o_1'' + 6o_2'', 5p_1'' + 9p_2'']; \wedge[(-0.3 \wedge \nu_1) \wedge (-0.1 \wedge \nu_2)]) \succ$
$= \prec([23, 50, 76]; 0.7), ([23, 59, 98]; 0.2), ([24, 54, 80]; 0.3), ([24, 42, 105]; -0.7), ([23, 40, 113]; -0.3), ([19, 48, 98]; -0.4) \succ,$
$\prec([0k_1 + 3k_2, 4l_1 + 4l_2, 9m_1 + 8m_2]; \wedge[(0.9 \wedge \chi_1) \wedge (0.8 \wedge \chi_2)]), ([0k_1' + 3k_2', 2l_1' + 5l_2', 8m_1' + 9m_2']; \vee[(0.3 \vee \beta_1) \vee (0.2 \vee \beta_2)]), ([1k_1'' + 3k_2'', 4l_1'' + 6l_2'', 7m_1'' + 7m_2'']; \vee[(0.2 \vee \zeta_1) \vee (0.3 \vee \zeta_2)]), ([2n_1 + 1n_2, 6o_1 + 2o_2, 8p_1 + 3p_2]; \vee[(-0.8 \vee \alpha_1) \vee (-0.9 \vee \alpha_2)]), ([2n_1' + 2n_2', 5o_1' + 3o_2', 9p_1' + 4p_2']; \wedge[(-0.3 \wedge \varphi_1) \wedge (-0.3 \wedge \varphi_2)]), ([n_1'' + 3n_2'', 6o_1'' + 5o_2'', 8p_1'' + 7p_2'']; \wedge[(-0.2 \wedge \nu_1) \wedge (-0.2 \wedge \nu_2)]) \succ$
$= \prec([15, 48, 127]; 0.7), ([15, 47, 137]; 0.3), ([17, 58, 98]; 0.3), ([8, 28, 74]; -0.7), ([10, 25, 95]; -0.3), ([13, 48, 105]; -0.2) \succ,$
$k_1 \geq 0, \quad l_1 - k_1 \geq 0, \quad m_1 - l_1 \geq 0, k_1' \geq 0, \quad l_1' - k_1' \geq 0, \quad m_1' - l_1' \geq 0, \quad k_1'' \geq 0, \quad l_1'' - k_1'' \geq 0, \quad m_1'' - l_1'' \geq 0, \quad n_1 \geq 0, \quad o_1 - n_1 \geq 0, \quad p_1 - o_1 \geq 0, \quad n_1' \geq 0, \quad o_1' - n_1' \geq 0, \quad p_1' - o_1' \geq 0, \quad n_1'' \geq 0, \quad o_1'' - n_1'' \geq 0, \quad p_1'' - o_1'' \geq 0, \ k_2 \geq 0, \quad l_2 - k_2 \geq 0, \quad m_2 - l_2 \geq 0, k_2' \geq 0, \quad l_2' - k_2' \geq 0, \quad m_2' - l_2' \geq 0,$

$k_2'' \geq 0, \quad l_2'' - k_2'' \geq 0, \quad m_2'' - l_2'' \geq 0, \quad n_2 \geq 0, \quad o_2 - n_2 \geq 0, \quad p_2 - o_2 \geq 0, \quad n_2' \geq 0, \quad o_2' - n_2' \geq 0, \quad p_2' - o_2' \geq 0, \quad n_2'' \geq 0, \quad o_2'' - n_2'' \geq 0, \quad p_2'' - o_2'' \geq 0,$
here
$\chi_1, \beta_1, \zeta_1, \chi_2, \beta_2, \zeta_2 \in [0,1], \quad \alpha_1, \varphi_1, \nu_1, \alpha_2, \varphi_2, \nu_2 \in [-1,0].$

**Step 4.**

By using definition of blue score function the above $BS_vN$ linear programming problem can be converted into CL programming problem:
Minimize $\frac{1}{6}([2k_1 + 3k_2, 3l_1 + 6l_2, 5m_1 + 9m_2] + 1 - [2k_1' + 3k_2', 6l_1' + 5l_2', 8m_1' + 8m_2'] + 1 - [2k_1'' + 3k_2'', 5l_1'' + 6l_2'', 7m_1'' + 8m_2''] + 1 + [3n_1 + 2n_2, 5o_1 + 3o_2, 6p_1 + 7p_2] - [3n_1' + 2n_2', 4o_1' + 3o_2', 6p_1' + 6p_2'] - [3n_1'' + 2n_2'', 4o_1'' + 4o_2'', 8p_1'' + 7p_2''])$;
subject to:

$$
\begin{array}{ll}
k_1 + 4k_2 = 23, & 0k_1 + 3k_2 = 15, \\
3l_1 + 5l_2 = 50, & 4l_1 + 4l_2 = 48, \\
4m_1 + 6m_2 = 76, & 9m_1 + 8m_2 = 127, \\
\wedge[(0.9 \wedge \chi_1) \wedge (0.8 \wedge \chi_2)] = 0.7, & \wedge[(0.9 \wedge \chi_1) \wedge (0.8 \wedge \chi_2)] = 0.7, \\
k_1' + 4k_2' = 23, & 0k_1' + 3k_2' = 15, \\
4l_1' + 5l_2' = 59, & 2l_1' + 5l_2' = 47, \\
5m_1' + 7m_2' = 98, & 8m_1' + 9m_2' = 137, \\
\vee[(0.1 \vee \beta_1) \vee (0.1 \vee \beta_2)] = 0.2 & \vee[(0.3 \vee \beta_1) \vee (0.2 \vee \beta_2)] = 0.3 \\
2k_1'' + 4k_2'' = 24, & k_1'' + 3k_2'' = 17, \\
3l_1'' + 6l_2'' = 54, & 4l_1'' + 6l_2'' = 58, \\
4m_1'' + 7m_2'' = 80, & 7m_1'' + 7m_2'' = 98, \\
\vee[(0.2 \vee \zeta_1) \vee (0.2 \vee \zeta_2)] = 0.3, & \vee[(0.2 \vee \zeta_1) \vee (0.3 \vee \zeta_2)] = 0.3, \\
2n_1 + 5n_2 = 24, & 2n_1 + n_2 = 8, \\
4o_1 + 6o_2 = 42, & 6o_1 + 2o_2 = 28, \\
6p_1 + 7p_2 = 105, & 8p_1 + 3p_2 = 74, \\
\vee[(-0.9 \vee \alpha_1) \vee (-0.8 \vee \alpha_2)] = -0.7, & \vee[(-0.8 \vee \alpha_1) \vee (-0.9 \vee \alpha_2)] = -0.7, \\
3n_1' + 5n_2' = 23, & 2n_1' + 2n_2' = 10, \\
5o_1' + 6o_2' = 40, & 5o_1' + 3o_2' = 25, \\
7p_1' + 8p_2' = 113, & 9p_1' + 4p_2' = 95, \\
\wedge[(-0.2 \wedge \varphi_1) \wedge (-0.2 \wedge \varphi_2)] = -0.3, & \wedge[(-0.3 \wedge \varphi_1) \wedge (-0.3 \wedge \varphi_2)] = -0.3, \\
3n_1'' + 4n_2'' = 19, & n_1'' + 3n_2'' = 13, \\
4o_1'' + 6o_2'' = 48, & 6o_1'' + 5o_2'' = 48, \\
5p_1'' + 9p_2'' = 98, & 8p_1'' + 7p_2'' = 105, \\
\wedge[(-0.3 \wedge \nu_1) \wedge (-0.1 \wedge \nu_2)] = -0.4, & \wedge[(-0.2 \wedge \nu_1) \wedge (-0.2 \wedge \nu_2)] = -0.2,
\end{array}
$$

$k_1 \geq 0, \quad l_1 - k_1 \geq 0, \quad m_1 - l_1 \geq 0, k'_1 \geq 0, \quad l'_1 - k'_1 \geq 0, \quad m'_1 - l'_1 \geq 0, \quad k''_1 \geq 0, \quad l''_1 - k''_1 \geq 0, \quad m''_1 - l''_1 \geq 0, \quad n_1 \geq 0, \quad o_1 - n_1 \geq 0, \quad p_1 - o_1 \geq 0, \quad n'_1 \geq 0, \quad o'_1 - n'_1 \geq 0, \quad p'_1 - o'_1 \geq 0, \quad n''_1 \geq 0, \quad o''_1 - n''_1 \geq 0, \quad p''_1 - o''_1 \geq 0, \ k_2 \geq 0, \quad l_2 - k_2 \geq 0, \quad m_2 - l_2 \geq 0, k'_2 \geq 0, \quad l'_2 - k'_2 \geq 0, \quad m'_2 - l'_2 \geq 0, k''_2 \geq 0, \ l''_2 - k''_2 \geq 0, \quad m''_2 - l''_2 \geq 0, \quad n_2 \geq 0, \quad o_2 - n_2 \geq 0, \quad p_2 - o_2 \geq 0, \quad n'_2 \geq 0, \quad o'_2 - n'_2 \geq 0, p'_2 - o'_2 \geq 0, \quad n''_2 \geq 0, \quad o''_2 - n''_2 \geq 0, \quad p''_2 - o''_2 \geq 0,$
here
$\chi_1, \beta_1, \zeta_1, \chi_2, \beta_2, \zeta_2 \in [0, 1], \quad \alpha_1, \varphi_1, \nu_1, \alpha_2, \varphi_2, \nu_2 \in [-1, 0].$

**Step 5.**

The optimal solution of the crisp linear programming problem is $k_1 = 3, l_1 = 5, m_1 = 7, k'_1 = 3, l'_1 = 6, m'_1 = 7, k''_1 = 2, l''_1 = 4, m''_1 = 6, k_2 = 5, l_2 = 7, m_2 = 8, k'_2 = 5, l'_2 = 7, m'_2 = 9, k''_2 = 5, l''_2 = 7, m''_2 = 8, n_1 = 2, o_1 = 3, p_1 = 7, n'_1 = 1, o'_1 = 2, p'_1 = 7, n''_1 = 1, o''_1 = 3, p''_1 = 7, n_2 = 4, o_2 = 5, p_2 = 6, n'_2 = 4, o'_2 = 5, p'_2 = 8, n''_2 = 4, o''_2 = 6, p''_2 = 7, \chi_1 = 0.8, \beta_1 = 0.2, \zeta_1 = 0.3, \chi_2 = 0.7, \beta_2 = 0.2, \zeta_2 = 0.1, \alpha_1 = -0.8, \varphi_1 = -0.3, \nu_1 = -0.2, \alpha_2 = -0.7, \varphi_2 = -0.2, \nu_2 = -0.1.$

**Step 6.**

The exact optimal solution is $\tilde{x_1} = \prec ([3, 5, 7]; 0.8), ([3, 6, 7]; 0.2), ([2, 4, 6]; 0.3), ([2, 3, 7]; -0.8), ([1, 2, 7]; -0.3), ([1, 3, 7]; -0.2) \succ, \tilde{x_2} = \prec ([5, 7, 8]; 0.7), ([5, 7, 9]; 0.2), ([5, 7, 8]; 0.1), ([4, 5, 6]; -0.7), ([4, 5, 8]; -0.3), ([4, 6, 9]; -0.1) \succ$ .

**Step 7.**

The bipolar single-valued neutrosophic optimal value of fully bipolar single-valued neutrosophic linear programming problem is
$\prec ([21, 57, 107]; 0.7), ([21, 71, 128]; 0.3), ([19, 62, 106]; 0.3), ([14, 30, 84]; -0.7), ([11, 23, 90]; -0.3), ([11, 36, 119]; -0.3) \succ$.

**Example 4.2. Farming Problem.** A farmer contains cattle like cows and goats in his form and he cells ghee and meat. The price of ghee per kg is Rs. $\prec ([5, 7, 9]; 0.9, 0.4, 0.3), ([6, 7, 8]; -0.9, -0.1, -0.2) \succ$ and for meat is Rs. $\prec ([4, 6, 8]; 0.7, 0.3, 0.3), ([3, 7, 9]; -0.8, -0.4, -0.1) \succ$ the maximum production of ghee is
$\prec ([7, 17, 38]; 0.8, 0.4, 0.4), ([13, 50, 112]; -0.7, -0.3, -0.4) \succ$ kg per day and meat is
$\prec ([9, 32, 67]; 0.6, 0.3, 0.4), ([14, 48, 120]; -0.6, -0.3, -0.4) \succ$ kg per day. The ghee and meat (in kg) of both cattle's are given in Tab. 1.

**Table 1:** Farming problem

| Material | Ghee | Meat | Maximum production per day (in kg) |
|---|---|---|---|
| Cow | $\prec ([2, 3, 5]; 0.9, 0.4, 0.2), ([1, 5, 8]; -0.7, -0.3, -0.1) \succ$ | $\prec ([3, 4, 6]; 0.9, 0.3, 0.4), ([4, 5, 7]; -0.9, -0.2, -0.2) \succ$ | $\prec ([7, 17, 38]; 0.8, 0.4, 0.4), ([13, 50, 112]; -0.7, -0.3, -0.4) \succ$ |
| Goat | $\prec ([4, 8, 10]; 0.6, 0.2, 0.1), ([2, 4, 8]; -0.7, -0.2, -0.4) \succ$ | $\prec ([1, 4, 9]; 0.7, 0.1, 0.1), ([2, 6, 8]; -0.6, -0.2, -0.2) \succ$ | $\prec ([9, 32, 67]; 0.6, 0.3, 0.4), ([14, 48, 120]; -0.6, -0.3, -0.4) \succ$ |

We have to maximize the profit. Let $\tilde{x_1}$ and $\tilde{x_1}$ be ghee and meat in kg. Then the bipolar single-valued neutrosophic linear programming problem becomes:
Maximize $\prec ([5, 7, 9]; 0.9, 0.4, 0.3), ([6, 7, 8]; -0.9, -0.1, -0.2) \succ \otimes \tilde{x_1}$
$\oplus \prec ([4, 6, 8]; 0.7, 0.3, 0.3), ([3, 7, 9]; -0.8, -0.4, -0.1) \succ \otimes \tilde{x_2};$

subject to
$\prec([2,3,5];0.9,0.4,0.2),([1,5,8];-0.7,-0.3,-0.1)\succ\otimes\tilde{x_1}\oplus\prec([3,4,6];0.9,0.3,0.4),([4,5,7];$
$-0.9,-0.2,-0.2)\succ$

$\otimes\tilde{x_2}=\prec([7,17,38];0.8,0.4,0.4),([13,50,112];-0.7,-0.3,-0.4)\succ,$
$\prec([4,8,10];0.6,0.2,0.1),([2,4,8];-0.7,-0.2,-0.4)\succ\otimes\tilde{x_1}\oplus\prec([1,4,9];0.7,0.1,0.1),([2,6,8];$
$-0.6,-0.2,-0.2)\succ$

$\otimes\tilde{x_2}=\prec([9,32,67];0.6,0.3,0.4),([14,48,120];-0.6,-0.3,-0.4)\succ,$
where $\tilde{x_1}=\prec([k_1,l_1,m_1];\chi_1,\beta_1,\zeta_1),([n_1,o_1,p_1];\alpha_1,\varphi_1,\nu_1)\succ$
and $\tilde{x_2}=\prec([k_2,l_2,m_2];\chi_2,\beta_2,\zeta_2),$
$([n_2,o_2,p_2];\alpha_2,\varphi_2,\nu_2)\succ$ are non-negative $TBS_vNNs$.

**Step 1.**

$$\text{Maximize}\ \begin{pmatrix}\prec([5,7,9];0.9,0.4,0.3),([6,7,8];-0.9,-0.1,-0.2)\succ\otimes\prec([k_1,l_1,m_1];\chi_1,\beta_1,\zeta_1),\\ ([n_1,o_1,p_1];\alpha_1,\varphi_1,\nu_1)\succ\oplus\prec([4,6,8];0.7,0.3,0.3),([3,7,9];-0.8,-0.4,-0.1)\succ\otimes\\ \prec([k_2,l_2,m_2];\chi_2,\beta_2,\zeta_2),([n_2,o_2,p_2];\alpha_2,\varphi_2,\nu_2)\succ\end{pmatrix};$$

subject to
$\prec([2,3,5];0.9,0.4,0.2),([1,5,8];-0.7,-0.3,-0.1)\succ\otimes\prec([k_1,l_1,m_1];\chi_1,\beta_1,\zeta_1),([n_1,o_1,p_1];$
$\alpha_1,\varphi_1,\nu_1)\succ$
$\oplus\prec([3,4,6];0.9,0.3,0.4),([4,5,7];-0.9,-0.2,-0.2)\succ\otimes\prec([k_2,l_2,m_2];\chi_2,\beta_2,\zeta_2),$
$([n_2,o_2,p_2];\alpha_2,\varphi_2,\nu_2)\succ$

$=\prec([7,17,38];0.8,0.4,0.4),([13,50,112];-0.7,-0.3,-0.4)\succ,$
$\prec([4,8,10];0.6,0.2,0.1),([2,4,8];-0.7,-0.2,-0.4)\succ\otimes\prec([k_1,l_1,m_1];\chi_1,\beta_1,\zeta_1),([n_1,o_1,p_1];$
$\alpha_1,\varphi_1,\nu_1)\succ$
$\oplus\prec([1,4,9];0.7,0.1,0.1),([2,6,8];-0.6,-0.2,-0.2)\succ\otimes\prec([k_2,l_2,m_2];\chi_2,\beta_2,\zeta_2),$
$([n_2,o_2,p_2];\alpha_2,\varphi_2,\nu_2)\succ$

$=\prec([9,32,67];0.6,0.3,0.4),([14,48,120];-0.6,-0.3,-0.4)\succ,$
where $\tilde{x_1}=\prec([k_1,l_1,m_1];\chi_1,\beta_1,\zeta_1),([n_1,o_1,p_1];\alpha_1,\varphi_1,\nu_1)\succ$
and $\tilde{x_2}=\prec([k_2,l_2,m_2];\chi_2,\beta_2,\zeta_2),([n_2,o_2,p_2];\alpha_2,\varphi_2,\nu_2)\succ$ are non-negative $TBS_vNNs$.

**Step 2.**

$$\text{Maximize}\ \begin{pmatrix}\prec([5k_1,7l_1,9m_1];0.9\wedge\chi_1,0.4\vee\beta_1,0.3\vee\zeta_1),([6n_1,7o_1,8p_1];-0.9\vee\alpha_1,-0.1\wedge\varphi_1,\\ -0.2\wedge\nu_1)\succ\\ \oplus\prec([4k_2,6l_2,8m_2];0.7\wedge\chi_2,0.3\vee\beta_2,0.3\vee\zeta_2),([3n_2,7o_2,9p_2];-0.8\vee\alpha_2,-0.4\wedge\varphi_2,\\ -0.1\wedge\nu_2)\succ\end{pmatrix};$$

subject to
$\prec([2k_1,3l_1,5m_1];0.9\wedge\chi_1,0.4\vee\beta_1,0.2\vee\zeta_1),([1n_1,5o_1,8p_1];-0.7\vee\alpha_1,-0.3\wedge\varphi_1,-0.1\wedge\nu_1)\succ\oplus$
$\prec([3k_2,4l_2,6m_2];0.9\wedge\chi_2,0.3\vee\beta_2,0.4\vee\zeta_2),([4n_2,50_2,7p_2];-0.9\vee\alpha_2,-0.2\wedge\varphi_2,-0.2\wedge\nu_2)\succ$
$=\prec([7,17,38];0.8,0.4,0.4),([13,50,112];-0.7,-0.3,-0.4)\succ,$
$\prec([4k_1,8l_1,10m_1]0.6\wedge\chi_1,0.2\vee\beta_1,0.1\vee\zeta_1),([2n_1,4o_1,8p_1];-0.7\vee\alpha_1,-0.2\wedge\varphi_1,-0.4\wedge\nu_1)\succ\oplus$
$\prec([k_2,4l_2,9m_2];0.7\wedge\chi_2,0.1\vee\beta_2,0.1\vee\zeta_2),([2n_2,60_2,8p_2];-0.6\vee\alpha_2,-0.2\wedge\varphi_2,-0.2\wedge\nu_2)\succ$
$=\prec([9,32,67];0.6,0.3,0.4),([14,48,120];-0.6,-0.3,-0.4)\succ,$

$$k_1\geq 0,\quad l_1-k_1\geq 0,\quad m_1-l_1\geq 0,\quad n_1\geq 0,\quad o_1-n_1\geq 0,\quad p_1-o_1\geq 0,$$
$$k_2\geq 0,\quad l_2-k_2\geq 0,\quad m_2-l_2\geq 0,\quad n_2\geq 0,\quad o_2-n_2\geq 0,\quad p_2-o_2\geq 0,$$

here
$\chi_1, \beta_1, \zeta_1, \chi_2, \beta_2, \zeta_2 \in [0,1], \quad \alpha_1, \varphi_1, \nu_1, \alpha_2, \varphi_2, \nu_2 \in [-1,0]$.

**Step 3.**

Maximize $\prec ([5k_1+4k_2, 7l_1+6l_2, 9m_1+8m_2]; \wedge[(0.9\wedge\chi_1)\wedge(0.7\wedge\chi_2)], \vee[(0.4\vee\beta_1)\vee(0.3\vee\beta_2)],$
$\vee[(0.3\vee\zeta_1)\vee(0.3\vee\zeta_2)]), ([6n_1+3n_2, 7o_1+7o_2, 8p_1+9p_2]; \vee[(-0.9\vee\alpha_1)\vee(-0.8\vee\alpha_2)],$
$\wedge[(-0.1\wedge\varphi_1)\wedge(-0.4\wedge\varphi_2)], \wedge[(-0.2\wedge\nu_1)\wedge(-0.1\wedge\nu_2)]) \succ;$
subject to
$\prec ([2k_1+3k_2, 3l_1+4l_2, 5m_1+6m_2]; \wedge[(0.9\wedge\chi_1)\wedge(0.9\wedge\chi_2)], \vee[(0.4\vee\beta_1)\vee(0.3\vee\beta_2)],$
$\vee[(0.2\vee\zeta_1)\vee(0.4\vee\zeta_2)]), ([n_1+4n_2, 5o_1+5o_2, 8p_1+7p_2]; \vee[(-0.7\vee\alpha_1)\vee(-0.9\vee\alpha_2)],$
$\wedge[(-0.3\wedge\varphi_1)\wedge(-0.2\wedge\varphi_2)], \wedge[(-0.1\wedge\nu_1)\wedge(-0.2\wedge\nu_2)]) \succ$
$= \prec ([7,17,38]; 0.8, 0.4, 0.4), ([13,50,112]; -0.7, -0.3, -0.4) \succ,$
$\prec ([4k_1+k_2, 8l_1+4l_2, 10m_1+9m_2]; \wedge[(0.6\wedge\chi_1)\wedge(0.7\wedge\chi_2)], \vee[(0.2\vee\beta_1)\vee(0.1\vee\beta_2)],$
$\vee[(0.1\vee\zeta_1)\vee(0.1\vee\zeta_2)]), ([2n_1+2n_2, 4o_1+6o_2, 8p_1+8p_2]; \vee[(-0.7\vee\alpha_1)\vee(-0.6\vee\alpha_2)],$
$\wedge[(-0.2\wedge\varphi_1)\wedge(-0.2\wedge\varphi_2)], \wedge[(-0.4\wedge\nu_1)\wedge(-0.2\wedge\nu_2)]) \succ$
$= \prec ([9,32,67]; 0.6, 0.3, 0.4), ([14,48,120]; -0.6, -0.3, -0.4) \succ,$

$$k_1 \geq 0, \quad l_1 - k_1 \geq 0, \quad m_1 - l_1 \geq 0, \quad n_1 \geq 0, \quad o_1 - n_1 \geq 0, \quad p_1 - o_1 \geq 0,$$
$$k_2 \geq 0, \quad l_2 - k_2 \geq 0, \quad m_2 - l_2 \geq 0, \quad n_2 \geq 0, \quad o_2 - n_2 \geq 0, \quad p_2 - o_2 \geq 0,$$

here
$\chi_1, \beta_1, \zeta_1, \chi_2, \beta_2, \zeta_2 \in [0,1], \quad \alpha_1, \varphi_1, \nu_1, \alpha_2, \varphi_2, \nu_2 \in [-1,0]$.

**Step 4.**

By using definition of score function the above $BS_vN$ linear programming problem can be converted into CL programming problem

Maximize $\frac{1}{6}([5k_1+4k_2, 7l_1+6l_2, 9m_1+8m_2] + 1 - [5k_1+4k_2, 7l_1+6l_2, 9m_1+8m_2] + 1 - [5k_1+4k_2, 7l_1+6l_2, 9m_1+8m_2] + 1 + [6n_1+3n_2, 7o_1+7o_2, 8p_1+9p_2] - [6n_1+3n_2, 7o_1+7o_2, 8p_1+9p_2] - [6n_1+3n_2, 7o_1+7o_2, 8p_1+9p_2]);$

$$\text{subject to} \quad \begin{array}{ll} 2k_1+3k_2=7, & 4k_1+k_2=9, \\ 3l_1+4l_2=17, & 8l_1+4l_2=32, \\ 5m_1+6m_2=38, & 10m_1+9m_2=67, \\ \wedge[(0.9\wedge\chi_1)\wedge(0.9\wedge\chi_2)]=0.8, & \wedge[(0.6\wedge\chi_1)\wedge(0.7\wedge\chi_2)]=0.6, \\ \vee[(0.4\vee\beta_1)\vee(0.3\vee\beta_2)]=0.4, & \vee[(0.2\vee\beta_1)\vee(0.1\vee\beta_2)]=0.3, \\ \vee[(0.2\vee\zeta_1)\vee(0.4\vee\zeta_2)]=0.4, & \vee[(0.1\vee\zeta_1)\vee(0.1\vee\zeta_2)]=0.4, \\ n_1+4n_2=13, & 2n_1+2n_2=14, \\ 5o_1+5o_2=50, & 4o_1+6o_2=48, \\ 8p_1+7p_2=112, & 8p_1+8p_2=120, \\ \vee[(-0.7\vee\alpha_1)\vee(-0.9\vee\alpha_2)]=-0.7, & \vee[(-0.7\vee\alpha_1)\vee(-0.6\vee\alpha_2)]=-0.6, \\ \wedge[(-0.3\wedge\varphi_1)\wedge(-0.2\wedge\varphi_2)]=-0.3, & \wedge[(-0.2\wedge\varphi_1)\wedge(-0.2\wedge\varphi_2)]=-0.3, \\ \wedge[(-0.1\wedge\nu_1)\wedge(-0.2\wedge\nu_2)]=-0.4, & \wedge[(-0.4\wedge\nu_1)\wedge(-0.2\wedge\nu_2)]=-0.4, \end{array}$$

$$k_1 \geq 0, \quad l_1 - k_1 \geq 0, \quad m_1 - l_1 \geq 0, \quad n_1 \geq 0, \quad o_1 - n_1 \geq 0, \quad p_1 - o_1 \geq 0,$$
$$k_2 \geq 0, \quad l_2 - k_2 \geq 0, \quad m_2 - l_2 \geq 0, \quad n_2 \geq 0, \quad o_2 - n_2 \geq 0, \quad p_2 - o_2 \geq 0,$$

here
$\chi_1, \beta_1, \zeta_1, \chi_2, \beta_2, \zeta_2 \in [0, 1], \quad \alpha_1, \varphi_1, \nu_1, \alpha_2, \varphi_2, \nu_2 \in [-1, 0].$

**Step 5.**

The optimal solution of the crisp linear programming problem is $k_1 = 2, l_1 = 3, m_1 = 4, k_2 = 1, l_2 = 2, m_2 = 3, n_1 = 5, o_1 = 6, p_1 = 7, n_2 = 2, o_2 = 4, p_2 = 8, \chi_1 = 0.8, \beta_1 = 0.3, \zeta_1 = 0.4, \chi_2 = 0.9, \beta_2 = 0.2, \zeta_2 = 0.3, \alpha_1 = -0.8, \varphi_1 = -0.2, \nu_1 = -0.3, \alpha_2 = -0.8, \varphi_2 = -0.3, \nu_2 = -0.4$.

**Step 6.**

The exact optimal solution is $\tilde{x_1} = \prec ([2, 3, 4]; 0.8, 0.3, 0.4), ([5, 6, 7]; -0.8, -0.2, -0.3) \succ$, $\tilde{x_2} = \prec ([1, 2, 3]; 0.9, 0.2, 0.3), ([2, 4, 8]; -0.8, -0.3, -0.4) \succ$.

**Step 7.**

The bipolar single-valued neutrosophic optimal value of fully bipolar single-valued neutrosophic linear programming problem is:
$\prec ([14, 33, 60]; 0.7, 0.4, 0.4), ([36, 70, 128]; -0.8, -0.4, -0.4) \succ$.

**Example 4.3. Maximize Profit Problem.** A company contains two plants namely, plant X and plant Y. It produces two products mobiles and LCDs by using raw material. The maximum working capacity for plant X is $\prec ([24, 81, 131, 190]; 0.4, 0.6, 0.5), ([24, 66, 128, 229]; -0.4, -0.6, -0.5) \succ$ hrs per week and for plant Y is $\prec ([20, 71, 134, 196]; 0.5, 0.6, 0.5), ([44, 90, 152, 218]; -0.5, -0.6, -0.6) \succ$ hrs per week.

For maximum production of mobiles and LCDs the raw material are given in Tab. 2.

**Table 2:** Maximize profit problem

| Raw material | Mobiles | LCDs | Maximum production per week |
|---|---|---|---|
| Plant X | $\prec ([2, 5, 7, 10]; 0.4, 0.5, 0.3),$ $([3, 6, 9, 11]; -0.6, -0.5, -0.4) \succ$ | $\prec ([2, 6, 8, 10]; 0.6, 0.5, 0.4),$ $([3, 5, 7, 13]; -0.4, -0.5, -0.3) \succ$ | $\prec ([24, 81, 131, 190]; 0.4, 0.6, 0.5),$ $([24, 66, 128, 229]; -0.4, -0.6, -0.5) \succ$ |
| Plant Y | $\prec ([5, 9, 10, 11]; 0.7, 0.5, 0.6),$ $([9, 10, 11, 12]; -0.7, -0.4, -0.6) \succ$ | $\prec ([1, 4, 7, 10]; 0.6, 0.4, 0.3),$ $([2, 5, 8, 11]; -0.6, -0.4, -0.6) \succ$ | $\prec ([20, 71, 134, 196]; 0.5, 0.6, 0.5),$ $([44, 90, 152, 218]; -0.5, -0.6, -0.6) \succ$ |

The cost of each mobile Rs. $\prec ([5, 7, 9, 11]; 0.5, 0.4, 0.3), ([6, 8, 10, 12]; -0.6, -0.4, -0.5) \succ$ and LCD Rs. $\prec ([5, 8, 11, 14]; 0.6, 0.5, 0.5), ([7, 9, 11, 13]; -0.7, -0.5, -0.6) \succ$. How can the company maximize the profit by producing mobiles and LCDs in the market. Let $\tilde{x_1}$ and $\tilde{x_1}$ be the production of mobiles and LCDs in h.

Then the bipolar single-valued neutrosophic linear programming problem becomes:

Maximize $\prec ([5, 7, 9, 11]; 0.5, 0.4, 0.3), ([6, 8, 10, 12]; -0.6, -0.4, -0.5) \succ \otimes \tilde{x_1} \oplus \prec ([5, 8, 11, 14]; 0.6, 0.5, 0.5), ([7, 9, 11, 13]; -0.7, -0.5, -0.6) \succ \otimes \tilde{x_2}$;
subject to
$\prec ([2, 5, 7, 10]; 0.4, 0.5, 0.3), ([3, 6, 9, 11]; -0.6, -0.5, -0.4) \succ \otimes \tilde{x_1} \oplus \prec ([2, 6, 8, 10]; 0.6, 0.5, 0.4), ([3, 5, 7, 13]; -0.4, -0.5, -0.3) \succ \otimes \tilde{x_2}$
$= \prec ([24, 81, 131, 190]; 0.4, 0.6, 0.5), ([24, 66, 128, 229]; -0.4, -0.6, -0.5) \succ$,
$\prec ([5, 9, 10, 11]; 0.7, 0.5, 0.6), ([9, 10, 11, 12]; -0.7, -0.4, -0.6) \succ \otimes \tilde{x_1} \oplus \prec ([1, 4, 7, 10]; 0.6, 0.4, 0.3),$

$([2,5,8,11];-0.6,-0.4,-0.6)\succ\otimes\tilde{x_2}$
$=\prec([20,71,134,196];0.5,0.6,0.5),([44,90,152,218];-0.5,-0.6,-0.6)\succ,$
where $\tilde{x_1}=\prec([k_1,l_1,m_1,s_1];\chi_1,\beta_1,\zeta_1),([n_1,o_1,p_1,t_1];\alpha_1,\varphi_1,\nu_1)\succ$
and $\tilde{x_2}=\prec([k_2,l_2,m_2,s_2];\chi_2,\beta_2,\zeta_2),([n_2,o_2,p_2,t_2];\alpha_2,\varphi_2,\nu_2)\succ$ are non-negative $T_rS_vBNNs$.

**Step 1.**

$$\text{Maximize}\ \begin{pmatrix}\prec([5,7,9,11];0.5,0.4,0.3),([6,8,10,12];-0.6,-0.4,-0.5)\succ\otimes\prec([k_1,l_1,m_1,s_1];\\ \chi_1,\beta_1,\zeta_1),\\ ([n_1,o_1,p_1,t_1];\alpha_1,\varphi_1,\nu_1)\succ\oplus\prec([5,8,11,14];0.6,0.5,0.5),([7,9,11,13];\\ -0.7,-0.5,-0.6)\succ\\ \otimes\prec([k_2,l_2,m_2,s_2];\chi_2,\beta_2,\zeta_2),([n_2,o_2,p_2,t_2];\alpha_2,\varphi_2,\nu_2)\succ\end{pmatrix};$$

subject to
$\prec([2,5,7,10];0.4,0.5,0.3),([3,6,9,11];-0.6,-0.5,-0.4)\succ\otimes\prec([k_1,l_1,m_1,s_1];\chi_1,\beta_1,\zeta_1),$
$([n_1,o_1,p_1,t_1];\alpha_1,\varphi_1,\nu_1)\succ\oplus\prec([2,6,8,10];0.6,0.5,0.4),([3,5,7,13];-0.4,-0.5,-0.3)\succ\otimes$
$\prec([k_2,l_2,m_2,s_2];\chi_2,\beta_2,\zeta_2),([n_2,o_2,p_2,t_2];\alpha_2,\varphi_2,\nu_2)\succ$
$=\prec([24,81,131,190];0.4,0.6,0.5),([24,66,128,229];-0.4,-0.6,-0.5)\succ,$
$\prec([5,9,10,11];0.7,0.5,0.6),([9,10,11,12];-0.7,-0.4,-0.6)\succ\otimes\prec([k_1,l_1,m_1,s_1];\chi_1,\beta_1,\zeta_1),$
$([n_1,o_1,p_1,t_1];\alpha_1,\varphi_1,\nu_1)\succ\oplus\prec([1,4,7,10];0.6,0.4,0.3),([2,5,8,11];-0.6,-0.4,-0.6)\succ\otimes$
$\prec([k_2,l_2,m_2,s_2];\chi_2,\beta_2,\zeta_2),([n_2,o_2,p_2,t_2];\alpha_2,\varphi_2,\nu_2)\succ$
$=\prec([20,71,134,196];0.5,0.6,0.5),([44,90,152,218];-0.5,-0.6,-0.6)\succ,$
where $\tilde{x_1}=\prec([k_1,l_1,m_1,s_1];\chi_1,\beta_1,\zeta_1),([n_1,o_1,p_1,t_1];\alpha_1,\varphi_1,\nu_1)\succ$
and $\tilde{x_2}=\prec([k_2,l_2,m_2,s_2];\chi_2,\beta_2,\zeta_2),([n_2,o_2,p_2,t_2];\alpha_2,\varphi_2,\nu_2)\succ$ are non-negative $T_rBS_vNNs$.

**Step 2.**

$$\text{Maximize}\ \begin{pmatrix}\prec([5k_1,7l_1,9m_1,11s_1];0.5\wedge\chi_1,0.4\vee\beta_1,0.3\vee\zeta_1),([6n_1,8o_1,10p_1,12t_1];-0.6\vee\alpha_1,\\ -0.4\wedge\varphi_1,-0.5\wedge\nu_1)\succ\oplus\prec([5k_2,8l_2,11m_2,14s_1];0.6\wedge\chi_2,0.5\vee\beta_2,0.5\vee\zeta_2),\\ ([7n_2,9o_2,11p_2,13t_2];-0.7\vee\alpha_2,-0.5\wedge\varphi_2,-0.6\wedge\nu_2)\succ\end{pmatrix};$$

subject to
$\prec([2k_1,5l_1,7m_1,10s_1];0.4\wedge\chi_1,0.5\vee\beta_1,0.3\vee\zeta_1),([3n_1,6o_1,9p_1,11t_1];-0.6\vee\alpha_1,-0.5\wedge\varphi_1,$
$-0.4\wedge\nu_1)\succ$
$\oplus\prec([2k_2,6l_2,8m_2,10s_1];0.6\wedge\chi_2,0.5\vee\beta_2,0.4\vee\zeta_2),([3n_2,5o_2,7p_2,13t_2];-0.4\vee\alpha_2,$
$-0.5\wedge\varphi_2,-0.3\wedge\nu_2)\succ$

$=\prec([24,81,131,190];0.4,0.6,0.5),([24,66,128,229];-0.4,-0.6,-0.5)\succ,$
$\prec([5k_1,9l_1,10m_1,11s_1]0.7\wedge\chi_1,0.5\vee\beta_1,0.6\vee\zeta_1),([9n_1,10o_1,11p_1,12t_1];-0.7\vee\alpha_1,-0.4\wedge\varphi_1,$
$-0.6\wedge\nu_1)\succ$
$\oplus\prec([k_2,4l_2,7m_2,10s_1];0.6\wedge\chi_2,0.4\vee\beta_2,0.3\vee\zeta_2),([2n_2,5o_2,8p_2,11t_2];-0.6\vee\alpha_2,$
$-0.4\wedge\varphi_2,-0.6\wedge\nu_2)\succ$

$=\prec([20,71,134,196];0.5,0.6,0.5),([44,90,152,218];-0.5,-0.6,-0.6)\succ,$

$$k_1\ge0,\quad l_1-k_1\ge0,\quad m_1-l_1\ge0,\quad s_1-m_1\ge0,\quad n_1\ge0,\quad o_1-n_1\ge0,\quad p_1-o_1\ge0,\quad t_1-p_1\ge0,$$
$$k_2\ge0,\quad l_2-k_2\ge0,\quad m_2-l_2\ge0,\quad s_2-m_2\ge0,\quad n_2\ge0,\quad o_2-n_2\ge0,\quad p_2-o_2\ge0,\quad t_2-p_2\ge0,$$

here
$\chi_1,\beta_1,\zeta_1,\chi_2,\beta_2,\zeta_2\in[0,1],\quad \alpha_1,\varphi_1,\nu_1,\alpha_2,\varphi_2,\nu_2\in[-1,0].$

**Step 3.**

Maximize $\prec([5k_1+5k_2, 7l_1+8l_2, 9m_1+11m_2, 11s_1+14s_2]; \wedge[(0.5\wedge\chi_1)\wedge(0.6\wedge\chi_2)], \vee[(0.4\vee\beta_1)\vee(0.5\vee\beta_2)], \vee[(0.3\vee\zeta_1)\vee(0.5\vee\zeta_2)]), ([6n_1+7n_2, 8o_1+9o_2, 10p_1+11p_2, 12t_1+13t_2]; \vee[(-0.6\vee\alpha_1)\vee(-0.7\vee\alpha_2)], \wedge[(-0.4\wedge\varphi_1)\wedge(-0.5\wedge\varphi_2)], \wedge[(-0.5\wedge\nu_1)\wedge(-0.6\wedge\nu_2)])\succ$;

subject to

$\prec([2k_1+2k_2, 5l_1+6l_2, 7m_1+8m_2, 10s_1+10s_2]; \wedge[(0.4\wedge\chi_1)\wedge(0.6\wedge\chi_2)], \vee[(0.5\vee\beta_1)\vee(0.5\vee\beta_2)], \vee[(0.3\vee\zeta_1)\vee(0.4\vee\zeta_2)]), ([3n_1+3n_2, 6o_1+5o_2, 9p_1+7p_2, 11t_1+13t_2]; \vee[(-0.6\vee\alpha_1)\vee(-0.4\vee\alpha_2)], \wedge[(-0.5\wedge\varphi_1)\wedge(-0.5\wedge\varphi_2)], \wedge[(-0.4\wedge\nu_1)\wedge(-0.3\wedge\nu_2)])\succ$

$=\prec([24, 81, 131, 190]; 0.4, 0.6, 0.5), ([24, 66, 128, 229]; -0.4, -0.6, -0.5)\succ$,

$\prec([5k_1+k_2, 9l_1+4l_2, 10m_1+7m_2, 11s_1+10s_2]; \wedge[(0.7\wedge\chi_1)\wedge(0.6\wedge\chi_2)], \vee[(0.5\vee\beta_1)\vee(0.4\vee\beta_2)], \vee[(0.6\vee\zeta_1)\vee(0.3\vee\zeta_2)]), ([9n_1+2n_2, 10o_1+5o_2, 11p_1+8p_2, 12t_1+11t_2]; \vee[(-0.7\vee\alpha_1)\vee(-0.6\vee\alpha_2)], \wedge[(-0.4\wedge\varphi_1)\wedge(-0.4\wedge\varphi_2)], \wedge[(-0.6\wedge\nu_1)\wedge(-0.6\wedge\nu_2)])\succ$

$=\prec=\prec([20, 71, 134, 196]; 0.5, 0.6, 0.5), ([44, 90, 152, 218]; -0.5, -0.6, -0.6)\succ$,

$$k_1\ge 0,\quad l_1-k_1\ge 0,\quad m_1-l_1\ge 0,\quad s_1-m_1\ge 0,\quad n_1\ge 0,\quad o_1-n_1\ge 0,\quad p_1-o_1\ge 0,\quad t_1-p_1\ge 0,$$

$$k_2\ge 0,\quad l_2-k_2\ge 0,\quad m_2-l_2\ge 0,\quad s_2-m_2\ge 0,\quad n_2\ge 0,\quad o_2-n_2\ge 0,\quad p_2-o_2\ge 0,\quad t_2-p_2\ge 0,$$

here

$\chi_1, \beta_1, \zeta_1, \chi_2, \beta_2, \zeta_2 \in [0,1], \quad \alpha_1, \varphi_1, \nu_1, \alpha_2, \varphi_2, \nu_2 \in [-1,0]$.

**Step 4.**

By using definition of score function the above $BS_vN$ linear programming problem can be converted into CL programming problem

Maximize $\frac{1}{6}([5k_1+5k_2, 7l_1+8l_2, 9m_1+11m_2, 11s_1+14s_2]+1-[5k_1+5k_2, 7l_1+8l_2, 9m_1+11m_2, 11s_1+14s_2]+1-[5k_1+5k_2, 7l_1+8l_2, 9m_1+11m_2, 11s_1+14s_2]+1+[6n_1+7n_2, 8o_1+9o_2, 10p_1+11p_2, 12t_1+13t_2]-[6n_1+7n_2, 8o_1+9o_2, 10p_1+11p_2, 12t_1+13t_2]-[6n_1+7n_2, 8o_1+9o_2, 10p_1+11p_2, 12t_1+13t_2])$;

subject to:

$$
\begin{aligned}
&2k_1+2k_2=24, && 5k_1+k_2=20,\\
&5l_1+6l_2=81, && 9l_1+4l_2=71,\\
&7m_1+8m_2=131, && 10m_1+7m_2=134,\\
&10s_1+10s_2=190, && 11s_1+10s_2=196,\\
&\wedge[(0.4\wedge\chi_1)\wedge(0.6\wedge\chi_2)]=0.4, && \wedge[(0.7\wedge\chi_1)\wedge(0.6\wedge\chi_2)]=0.5,\\
&\vee[(0.5\vee\beta_1)\vee(0.5\vee\beta_2)]=0.6, && \vee[(0.5\vee\beta_1)\vee(0.4\vee\beta_2)]=0.6,\\
&\vee[(0.3\vee\zeta_1)\vee(0.4\vee\zeta_2)]=0.5, && \vee[(0.6\vee\zeta_1)\vee(0.3\vee\zeta_2)]=0.5,\\
&3n_1+3n_2=24, && 9n_1+2n_2=44,\\
&6o_1+5o_2=66, && 10o_1+5o_2=90,\\
&9p_1+7p_2=128, && 11p_1+8p_2=152,\\
&11t_1+13t_2=229, && 12t_1+11t_2=218,\\
&\vee[(-0.6\vee\alpha_1)\vee(-0.4\vee\alpha_2)]=-0.4, && \vee[(-0.7\vee\alpha_1)\vee(-0.6\vee\alpha_2)]=-0.5,\\
&\wedge[(-0.5\wedge\varphi_1)\wedge(-0.5\wedge\varphi_2)]=-0.6, && \wedge[(-0.4\wedge\varphi_1)\wedge(-0.4\wedge\varphi_2)]=-0.6,\\
&\wedge[(-0.4\wedge\nu_1)\wedge(-0.3\wedge\nu_2)]=-0.5, && \wedge[(-0.6\wedge\nu_1)\wedge(-0.6\wedge\nu_2)]=-0.6,
\end{aligned}
$$

$$k_1 \geq 0, \quad l_1 - k_1 \geq 0, \quad m_1 - l_1 \geq 0, \quad n_1 \geq 0, \quad o_1 - n_1 \geq 0, \quad p_1 - o_1 \geq 0,$$
$$k_2 \geq 0, \quad l_2 - k_2 \geq 0, \quad m_2 - l_2 \geq 0, \quad n_2 \geq 0, \quad o_2 - n_2 \geq 0, \quad p_2 - o_2 \geq 0,$$

here
$\chi_1, \beta_1, \zeta_1, \chi_2, \beta_2, \zeta_2 \in [0,1], \quad \alpha_1, \varphi_1, \nu_1, \alpha_2, \varphi_2, \nu_2 \in [-1,0].$

**Step 5.**
The optimal solution of the crisp linear programming problem is $k_1 = 2, l_1 = 3, m_1 = 5, s_1 = 6, k_2 = 10, l_2 = 11, m_2 = 12, s_2 = 13, n_1 = 4, o_1 = 6, p_1 = 8, t_1 = 9, n_2 = 4, o_2 = 6, p_2 = 8, t_2 = 10, \chi_1 = 0.5, \beta_1 = 0.6, \zeta_1 = 0.4, \chi_2 = 0.7, \beta_2 = 0.6, \zeta_2 = 0.5, \alpha_1 = -0.7, \varphi_1 = -0.4, \nu_1 = -0.5, \alpha_2 = -0.5, \varphi_2 = -0.6, \nu_2 = -0.4$.

**Step 6.**
The exact optimal solution is $\tilde{x_1} = \prec ([2,3,5,6]; 0.5, 0.6, 0.4), ([4,6,8,9]; -0.7, -0.4, -0.5) \succ$,
$\tilde{x_2} = \prec ([10,11,12,13]; 0.7, 0.6, 0.5), ([4,6,8,10]; -0.5, -0.6, -0.4) \succ$.

**Step 7.**
The bipolar single-valued neutrosophic optimal value of fully bipolar single-valued neutrosophic linear programming problem is
$\prec ([60,89,177,248]; 0.5, 0.6, 0.5), ([52,102,168,238]; -0.5, -0.6, -0.6) \succ$.

**Example 4.4.** Maximize $\prec ([3,5,6,8]; 0.6, 0.5, 0.4), ([9,10,11,12]; -0.7, -0.5, -0.3) \succ \otimes \tilde{x_1}$
$\oplus \prec ([5,8,10,14]; 0.3, 0.6, 0.6), ([7,9,11,13]; -0.4, -0.5, -0.6) \succ \otimes \tilde{x_2}$;
subject to
$\prec ([3,5,6,8]; 0.6, 0.5, 0.4), ([9,10,11,12]; -0.7, -0.5, -0.3) \succ \otimes \tilde{x_1} \oplus \prec ([5,8,10,14]; 0.3, 0.6, 0.6),$
$([7,9,11,13]; -0.4, -0.5, -0.6) \succ \otimes \tilde{x_2}$
$= \prec ([18,42,68,116]; 0.3, 0.6, 0.6), ([73,96,154,200]; -0.4, -0.5, -0.6) \succ$,
$\prec ([5,8,10,14]; 0.3, 0.6, 0.6), ([7,9,11,13]; -0.4, -0.5, -0.6) \succ \otimes \tilde{x_1} \oplus \prec ([3,5,6,8]; 0.6, 0.5, 0.4),$
$([9,10,11,12]; -0.7, -0.5, -0.3) \succ \otimes \tilde{x_2}$
$= \prec ([14,36,60,104]; 0.3, 0.6, 0.6), ([71,114,154,200]; -0.4, -0.5, -0.6) \succ$,
where $\tilde{x_1} = \prec ([k_1, l_1, m_1, s_1]; 0.8, 0.2, 0.3), ([n_1, o_1, p_1, t_1]; -0.6, -0.4, -0.4) \succ$
and $\tilde{x_2} = \prec ([k_2, l_2, m_2, s_2]; 0.8, 0.2, 0.3), ([n_2, o_2, p_2, t_2]; -0.6, -0.4, -0.4) \succ$
are non-negative $T_rBS_vNNs$.

**Step 1.**

$$\text{Maximize} \begin{pmatrix} \prec ([3,5,6,8]; 0.6, 0.5, 0.4), ([9,10,11,12]; -0.7, -0.5, -0.3) \succ \otimes \\ \prec ([k_1, l_1, m_1, s_1]; 0.8, 0.2, 0.3), ([n_1, o_1, p_1, t_1]; -0.6, -0.4, -0.4) \succ \oplus \\ \prec ([5,8,10,14]; 0.3, 0.6, 0.6), ([7,9,11,13]; -0.4, -0.5, -0.6) \succ \otimes \\ \prec ([k_2, l_2, m_2, s_2]; 0.8, 0.2, 0.3), ([n_2, o_2, p_2, t_2]; -0.6, -0.4, -0.4) \succ \end{pmatrix};$$

subject to
$\prec ([3,5,6,8]; 0.6, 0.5, 0.4), ([9,10,11,12]; -0.7, -0.5, -0.3) \succ \otimes \prec ([k_1, l_1, m_1, s_1]; 0.8, 0.2, 0.3),$
$([n_1, o_1, p_1, t_1]; -0.6, -0.4, -0.4) \succ \oplus \prec ([5,8,10,14]; 0.3, 0.6, 0.6), ([7,9,11,13]; -0.4, -0.5, -0.6) \succ \otimes$
$\prec ([k_2, l_2, m_2, s_2]; 0.8, 0.2, 0.3), ([n_2, o_2, p_2, t_2]; -0.6, -0.4, -0.4) \succ$
$= \prec ([18,42,68,116]; 0.3, 0.6, 0.6), ([73,96,154,200]; -0.4, -0.5, -0.6) \succ$,
$\prec ([5,8,10,14]; 0.3, 0.6, 0.6), ([7,9,11,13]; -0.4, -0.5, -0.6) \succ \otimes \prec ([k_1, l_1, m_1, s_1]; 0.8, 0.2, 0.3),$
$([n_1, o_1, p_1, t_1]; -0.6, -0.4, -0.4) \succ \oplus \prec ([3,5,6,8]; 0.6, 0.5, 0.4), ([9,10,11,12]; -0.7, -0.5, -0.3) \succ \otimes$
$\prec ([k_2, l_2, m_2, s_2]; 0.8, 0.2, 0.3), ([n_2, o_2, p_2, t_2]; -0.6, -0.4, -0.4) \succ$
$= \prec ([14,36,60,104]; 0.3, 0.6, 0.6), ([71,114,154,200]; -0.4, -0.5, -0.6) \succ$,
where $\tilde{x_1} = \prec ([k_1, l_1, m_1, s_1]; 0.8, 0.2, 0.3), ([n_1, o_1, p_1, t_1]; -0.6, -0.4, -0.4) \succ$

and $\tilde{x_2} = \prec ([k_2, l_2, m_2, s_2]; 0.8, 0.2, 0.3), ([n_2, o_2, p_2, t_2]; -0.6, -0.4, -0.4) \succ$ are non-negative $T_rBS_vNNs$.

**Step 2.**

$$\text{Maximize} \begin{pmatrix} \prec ([3k_1, 5l_1, 6m_1, 8s_1]; 0.6 \wedge 0.8, 0.5 \vee 0.2, 0.4 \vee 0.3), \\ ([9n_1, 10o_1, 11p_1, 12t_1]; -0.7 \vee -0.6, -0.5 \wedge -0.4, -0.3 \wedge -0.4) \succ \\ \oplus \prec ([5k_2, 8l_2, 10m_2, 14s_2]; 0.3 \wedge 0.8, 0.6 \vee 0.2, 0.6 \vee 0.3), \\ ([7n_2, 9o_2, 11p_2, 13t_2]; -0.4 \vee -0.6, -0.5 \wedge -0.4, -0.6 \wedge -0.4) \succ \end{pmatrix};$$

subject to

$\prec ([3k_1, 5l_1, 6m_1, 8s_1]; 0.6 \wedge 0.8, 0.5 \vee 0.2, 0.4 \vee 0.3), ([9n_1, 10o_1, 11p_1, 12t_1]; -0.7 \vee -0.6, -0.5 \wedge -0.4, -0.3 \wedge -0.4) \succ \oplus \prec ([5k_2, 8l_2, 10m_2, 14s_2]; 0.3 \wedge 0.8, 0.6 \vee 0.2, 0.6 \vee 0.3), ([7n_2, 9o_2, 11p_2, 13t_2]; -0.4 \vee -0.6, -0.5 \wedge -0.4, -0.6 \wedge -0.4) \succ$
$= \prec ([18, 42, 68, 116]; 0.3, 0.6, 0.6), ([73, 96, 154, 200]; -0.4, -0.5, -0.6) \succ,$
$\prec ([5k_2, 8l_2, 10m_2, 14s_2]; 0.3 \wedge 0.8, 0.6 \vee 0.2, 0.6 \vee 0.3), ([7n_2, 9o_2, 11p_2, 13t_2]; -0.4 \vee -0.6, -0.5 \wedge -0.4, -0.6 \wedge -0.4) \succ \oplus \prec ([3k_1, 5l_1, 6m_1, 8s_1]; 0.6 \wedge 0.8, 0.5 \vee 0.2, 0.4 \vee 0.3), ([9n_1, 10o_1, 11p_1, 12t_1]; -0.7 \vee -0.6, -0.5 \wedge -0.4, -0.3 \wedge -0.4)$
$= \prec ([14, 36, 60, 104]; 0.3, 0.6, 0.6), ([71, 114, 154, 200]; -0.4, -0.5, -0.6) \succ,$

$$k_1 \geq 0, \quad l_1 - k_1 \geq 0, \quad m_1 - l_1 \geq 0, \quad s_1 - m_1 \geq 0, \quad n_1 \geq 0, \quad o_1 - n_1 \geq 0, \quad p_1 - o_1 \geq 0, \quad t_1 - p_1 \geq 0,$$
$$k_2 \geq 0, \quad l_2 - k_2 \geq 0, \quad m_2 - l_2 \geq 0, \quad s_2 - m_2 \geq 0, \quad n_2 \geq 0, \quad o_2 - n_2 \geq 0, \quad p_2 - o_2 \geq 0, \quad t_2 - p_2 \geq 0,$$

**Step 3.**

Maximize $\prec ([3k_1 + 5k_2, 5l_1 + 8l_2, 6m_1 + 10m_2, 8s_1 + 14s_2]; \wedge[(0.6 \wedge 0.8) \wedge (0.3 \wedge 0.8)], \vee[(0.5 \vee 0.2) \vee (0.6 \vee 0.2)],$
$\vee[(0.4 \vee 0.3) \vee (0.6 \vee 0.3)]), ([9n_1 + 7n_2, 10o_1 + 9o_2, 11p_1 + 11p_2, 12t_1 + 13t_2]; \vee[(-0.7 \vee -0.6 \vee (-0.4 \vee -0.6)], \wedge[(-0.5 \wedge -0.4) \wedge (-0.5 \wedge -0.4)], \wedge[(-0.3 \wedge -0.4) \wedge (-0.6 \wedge -0.4)]) \succ;$
subject to
$\prec ([3k_1 + 5k_2, 5l_1 + 8l_2, 6m_1 + 10m_2, 8s_1 + 14s_2]; \wedge[(0.6 \wedge 0.8) \wedge (0.3 \wedge 0.8)], \vee[(0.5 \vee 0.2) \vee (0.6 \vee 0.2)],$
$\vee[(0.4 \vee 0.3) \vee (0.6 \vee 0.3)]), ([9n_1 + 7n_2, 10o_1 + 9o_2, 11p_1 + 11p_2, 12t_1 + 13t_2]; \vee[(-0.7 \vee -0.6 \vee (-0.4 \vee -0.6)], \wedge[(-0.5 \wedge -0.4) \wedge (-0.5 \wedge -0.4)], \wedge[(-0.3 \wedge -0.4) \wedge (-0.6 \wedge -0.4)]) \succ$
$= \prec ([18, 42, 68, 116]; 0.3, 0.6, 0.6), ([73, 96, 154, 200]; -0.4, -0.5, -0.6) \succ,$
$\prec ([5k_1 + 3k_2, 8l_1 + 5l_2, 10m_1 + 6m_2, 14s_1 + 8s_2]; \wedge[(0.3 \wedge 0.8) \wedge (0.6 \wedge 0.8)], \vee[(0.6 \vee 0.2) \vee (0.5 \vee 0.2)],$
$\vee[(0.6 \vee 0.3) \vee (0.4 \vee 0.3)]), ([7n_1 + 9n_2, 9o_1 + 10o_2, 11p_1 + 11p_2, 13t_1 + 12t_2]; \vee[(-0.4 \vee -0.6) \vee (-0.7 \vee -0.6)], \wedge[(-0.5 \wedge -0.4) \wedge (-0.5 \wedge -0.4)], \wedge[(-0.6 \wedge -0.4) \wedge (-0.3 \wedge -0.4)]) \succ$
$= \prec ([14, 36, 60, 104]; 0.3, 0.6, 0.6),$
$([71, 114, 154, 200]; -0.4, -0.5, -0.6) \succ,$

$$k_1 \geq 0, \quad l_1 - k_1 \geq 0, \quad m_1 - l_1 \geq 0, \quad s_1 - m_1 \geq 0, \quad n_1 \geq 0, \quad o_1 - n_1 \geq 0, \quad p_1 - o_1 \geq 0, \quad t_1 - p_1 \geq 0,$$
$$k_2 \geq 0, \quad l_2 - k_2 \geq 0, \quad m_2 - l_2 \geq 0, \quad s_2 - m_2 \geq 0, \quad n_2 \geq 0, \quad o_2 - n_2 \geq 0, \quad p_2 - o_2 \geq 0, \quad t_2 - p_2 \geq 0,$$

**Step 4.**

By using definition of score function the above $BS_vN$ linear programming problem can be converted into CL programming problem

Maximize $\frac{1}{6}([3k_1 + 5k_2, 5l_1 + 8l_2, 6m_1 + 10m_2, 8s_1 + 14s_2] + 1 - [3k_1 + 5k_2, 5l_1 + 8l_2, 6m_1 + 10m_2, 8s_1 + 14s_2] + 1 - [3k_1 + 5k_2, 5l_1 + 8l_2, 6m_1 + 10m_2, 8s_1 + 14s_2] + 1 + [9n_1 + 7n_2, 10o_1 + 9o_2,$

$11p_1 + 11p_2, 12t_1 + 13t_2] - [9n_1 + 7n_2, 10o_1 + 9o_2, 11p_1 + 11p_2, 12t_1 + 13t_2] - [9n_1 + 7n_2, 10o_1 + 9o_2, 11p_1 + 11p_2, 12t_1 + 13t_2]$);

$$\begin{aligned}
\text{subject to}\quad & 3k_1 + 5k_2 = 18, && 5k_1 + 3k_2 = 14,\\
& 5l_1 + 8l_2 = 42, && 8l_1 + 5l_2 = 36,\\
& 6m_1 + 10m_2 = 68, && 10m_1 + 6m_2 = 60,\\
& 8s_1 + 14s_2 = 116, && 14s_1 + 8s_2 = 104,\\
& \wedge[(0.6 \wedge 0.8) \wedge (0.3 \wedge 0.8)] = 0.3, && \wedge[(0.3 \wedge 0.8) \wedge (0.6 \wedge 0.8)] = 0.3,\\
& \vee[(0.5 \vee 0.2) \vee (0.6 \vee 0.2)] = 0.6, && \vee[(0.6 \vee 0.2) \vee (0.5 \vee 0.2)] = 0.6,\\
& \vee[(0.4 \vee 0.3) \vee (0.6 \vee 0.3)] = 0.6, && \vee[(0.6 \vee 0.3) \vee (0.4 \vee 0.3)] = 0.6,\\
& 9n_1 + 7n_2 = 73, && 7n_1 + 9n_2 = 71,\\
& 10o_1 + 9o_2 = 96, && 9o_1 + 10o_2 = 114,\\
& 11p_1 + 11p_2 = 154, && 11p_1 + 11p_2 = 154,\\
& 12t_1 + 13t_2 = 200, && 13t_1 + 12t_2 = 200,\\
& \vee[(-0.7 \vee -0.6 \vee (-0.4 \vee -0.6)] = -0.4, && \vee[(-0.4 \vee -0.6) \vee (-0.7 \vee -0.6)] = -0.4,\\
& \wedge[(-0.5 \wedge -0.4) \wedge (-0.5 \wedge -0.4)] = -0.5, && \wedge[(-0.5 \wedge -0.4) \wedge (-0.5 \wedge -0.4)] = -0.5,\\
& \wedge[(-0.3 \wedge -0.4) \wedge (-0.6 \wedge -0.4)] = -0.6, && \wedge[(-0.6 \wedge -0.4) \wedge (-0.3 \wedge -0.4)] = -0.6,
\end{aligned}$$

$$k_1 \geq 0,\quad l_1 - k_1 \geq 0,\quad m_1 - l_1 \geq 0,\quad n_1 \geq 0,\quad o_1 - n_1 \geq 0,\quad p_1 - o_1 \geq 0,$$
$$k_2 \geq 0,\quad l_2 - k_2 \geq 0,\quad m_2 - l_2 \geq 0,\quad n_2 \geq 0,\quad o_2 - n_2 \geq 0,\quad p_2 - o_2 \geq 0,$$

**Step 5.**

The optimal solution of the crisp linear programming problem is $k_1 = 1, l_1 = 2, m_1 = 3, s_1 = 4, k_2 = 3, l_2 = 4, m_2 = 5, s_2 = 5, n_1 = 5, o_1 = 6, p_1 = 7, t_1 = 8, n_2 = 4, o_2 = 6, p_2 = 7, t_2 = 8$.

**Step 6.**

The exact optimal solution is $\tilde{x_1} = \prec ([1,2,3,4]; 0.8, 0.2, 0.3), ([5,6,7,8]; -0.6, -0.4, -0.4) \succ$, $\tilde{x_2} = \prec ([3,4,5,6]; 0.8, 0.2, 0.3), ([4,6,7,8]; -0.6, -0.4, -0.4) \succ$.

**Step 7.**

The bipolar single-valued neutrosophic optimal value of fully bipolar single-valued neutrosophic linear programming problem is
$\prec ([18, 42, 68, 116]; 0.3, 0.6, 0.6), ([73, 96, 154, 200]; -0.4, -0.5, -0.6) \succ$.

## 5 Comparative Analysis

In this section, Khalifa et al.'s method [38] has been compared with our proposed method in Section 3. Bipolar single-valued neutrosophic optimal solution of the Example (4.4) is $\tilde{x_1} = \prec ([1,2,3,4]; 0.8, 0.2, 0.3), ([5,6,7,8]; -0.6, -0.4, -0.4) \succ, \tilde{x_2} = \prec ([3,4,5,6]; 0.8, 0.2, 0.3), ([4,6,7,8]; -0.6, -0.4, -0.4) \succ$ and bipolar single-valued neutrosophic optimal value $\prec ([18, 42, 68, 116]; 0.3,$

$0.6, 0.6), ([73, 96, 154, 200]; -0.4, -0.5, -0.6) \succ$. By restricting the bipolarity membership part of Example (4.4), the problem converts into FNLP problem;

Maximize $\left(([3,5,6,8]; 0.6, 0.5, 0.4) \otimes \tilde{x_1} \oplus ([5, 8, 10, 14]; 0.3, 0.6, 0.6) \otimes \tilde{x_2}\right)$;

subject to

$$\prec ([3,5,6,8]; 0.6, 0.5, 0.4) \succ \otimes \tilde{x_1} \oplus \prec ([5, 8, 10, 14]; 0.3, 0.6, 0.6) \succ \otimes \tilde{x_2}$$

$$= \prec ([16, 18, 22, 30]; 0.8, 0.2, 0.3) \succ,$$

$$\prec ([5, 8, 10, 14]; 0.3, 0.6, 0.6) \succ \otimes \tilde{x_1} \oplus \prec ([3,5,6,8]; 0.6, 0.5, 0.4) \succ \otimes \tilde{x_2}$$

$$= \prec ([13, 15, 18, 24]; 0.8, 0.2, 0.3) \succ,$$

where $\tilde{x_1} = ([k_1, l_1, m_1, s_1]; 0.8, 0.2, 0.3)$ and $\tilde{x_2} = ([k_2, l_2, m_2, s_2]; 0.8, 0.2, 0.3)$ are non-negative trapezoidal bipolar single-valued neutrosophic numbers.

Neutrosophic optimal solution and neutrosophic optimal value attained by Khalifa et al.'s method [38] are $\tilde{x_1} = ([3.3, 4, 4.6, 6]; 0.8, 0.2, 0.3), \tilde{x_2} = ([6.3, 7, 8.3, 12]; 0.8, 0.2, 0.3)$ and $([41.4, 78, 119, 220.8]; 0.3, 0.6, 0.6)$, respectively. These values of $\tilde{x_1}$ and $\tilde{x_2}$ have been not satisfied the constraints. So, our proposed method is accurately solved tha Khalifa et al.'s [38]. problem in an bipolar single-valued neutrosophic environment and satisfied the constraints.

Furthermore, our proposed method is generalization of Khalifa et al.'s [38] method by using this method all the shortcoming of Khalifa et al.'s [38] method are removed.

## 6 Conclusions

A bipolar single-valued neutrosophic model, an extension of bipolar fuzzy model, is a powerful tool to deal with vagueness. In this research article, we have solved fully bipolar single-valued neutrosophic linear programming problems with equality constraints. We have applied a score function to transform bipolar single-valued neutrosophic numbers into its equivalent crisp problem. Further, we have solved the numerical examples and practical models by using the proposed method. The obtained solutions satisfy the given constraints of the FBSvNLP problems, which shows that the suggested method is reliable. In future, this work can be extended to (1) Complex bipolar neutrosophic LPP; (2) Complex spherical neutrosophic LPP.

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