Some remarks on Δm (Iλ)-summability on neutrosophic normed spaces

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Some remarks on $\Delta^m(I_\lambda)$ −summability on neutrosophic normed spaces

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Abstract

In the present paper, we use the difference operator Δ^m and $\lambda-$ summability to define some new summability concepts on neutrosophic normed spaces. We also introduce concepts of generalized limit point, and cluster point and obtain some relationships among these notions. Finally, we define generalized Cauchy sequences on these spaces and present a characterization of a new summability method that preserves linear operators on neutrosophic normed spaces.

Keywords: Neutrosophic normed spaces; lacunary convergence; and *I*-convergence.

1. Introduction

Statistical convergence was defined by Fast [9] and further studied in [5], [10], [11], [24] and [25]. "A sequence (x_k) of numbers is said to be statistical convergence to a number L if for each $\varepsilon > 0$, $\lim_n \frac{1}{n}|\{k \leq n: |x_k - L| \geq \varepsilon\}| = 0$ or $\delta(K_\varepsilon) = 0$, where K_ε by $\{k \leq n: |x_k - L| \geq \varepsilon\} \subseteq \mathbb{N}$".

In [20], the $(V,\lambda) - summability$ is defined as follows: "For any non-decreasing sequence of positive numbers $\lambda = (\lambda_n)$ with $\lambda_n \to \infty$ as $n \to \infty$ and $\lambda_{n+1} \leq \lambda_n + 1,\ \lambda_1 = 1$, the generalized de la Valée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n}\sum_{k\in I_n} x_k \quad \text{where } I_n = [\mathrm{n} - \lambda_n + 1, n].$$

A sequence $x = (x_k)$ is said to be $\lambda-$ $summable$ to a number L if $t_n(x) = \frac{1}{\lambda_n}\sum_{k\in I_n} x_k \to L$ as $n \to \infty$".

Statistical convergence is further generalized in [16], called $I - convergence$. Later, the idea is developed in [4], [6], [7], [8], [12], [15], [17], [18], [19] and [21]. For basic information on $I - convergence$, we recommend the paper [16].

Fuzzy sets were invented by Zadeh [28] and generalized by Atanassov [1], called intuitionistic fuzzy sets. Over the years, many applications of these sets can be found in [6]. [12]. [21]. [22] and [23]. Recently, Smarandache [27] introduced a generalization of an intuitionistic fuzzy set, called a neutrosophic set. For some recent works on these sets, we refer [2], [3], [13] and [14]. We aim in this paper, to define$\Delta^m(\lambda)$ −convergence and $\Delta^m(I_\lambda)$ −convergence on neutrosophic normed spaces as a generalization of ideal convergence. Later, we introduce the concepts of $\Delta^m(I_\lambda)$ −limit point,

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$\Delta^m(I_\lambda)$ −cluster point and obtain some relationships among these. We also define $\Delta^m(\lambda)$ − Cauchy, $\Delta^m(I_\lambda)$ −Cauchy sequences on these spaces and study their relations. Finally, we present a characterization for$\Delta^m(I_\lambda)$ −convergence preserving linear operators on neutrosophic normed spaces

2. Background and Preliminary

In this section, we review and give some definitions and results which form the basis for the present study.

Definition 2.1[16] "Let F be a vector space, $\mathcal{N} = \{\langle\vartheta,\ \mathcal{G}(\vartheta), \mathcal{B}(\vartheta), \mathcal{Y}(\vartheta)\rangle : \vartheta \in F\}$ be a normed space such that $\mathcal{N}: F \times \mathbb{R}^+ \to [0,1]$ and $\circ$, $\bullet$ respectively are continuous $t - norm$ and continuous $t - conorm$. Then a four touple $V = (F, \mathcal{N}, \circ, \bullet)$ is called a neutrosophic normed space (NNS) if the following conditions are satisfied.

For every $u, v\ \in F$ and $\lambda, \mu > 0$ and for every $\sigma \neq 0$ we have

(i) $0 \leq \mathcal{G}(u,\lambda) \leq 1,\ \ 0 \leq \mathcal{B}(u,\lambda) \leq 1,\ \ 0 \leq \mathcal{Y}(u,\lambda) \leq 1$ for every $\lambda \in \mathbb{R}^+$;
(ii) $\mathcal{G}(u,\lambda) + \mathcal{B}(u,\lambda) + \mathcal{Y}(u,\lambda) \leq 3$ for $\lambda \in \mathbb{R}^+$;
(iii) $\mathcal{G}(u,\lambda) = 1$ (for $\lambda > 0$) *if and only if* $u = 0$;
(iv) $\mathcal{G}(\sigma u,\lambda) = \mathcal{G}\left(u,\frac{\lambda}{|\sigma|}\right)$;
(v) $\mathcal{G}(u,\mu) \circ \mathcal{G}(v,\lambda) \leq \mathcal{G}(u+v,\lambda+\mu)$;
(vi) $\mathcal{G}(u,.)$ is a continuous non-decreasing function;
(vii) $\lim_{\lambda\to\infty}\mathcal{G}(u,\lambda) = 1$;
(viii) $\mathcal{B}(u,\lambda) = 0$ (for $\lambda > 0$) *if and only if* $u = 0$;
(ix) $\mathcal{B}(\sigma u,\lambda) = \mathcal{B}\left(u,\frac{\lambda}{|\sigma|}\right)$;
(x) $\mathcal{B}(u,\mu) \bullet \mathcal{B}(v,\lambda) \geq \mathcal{B}(u+v,\lambda+\mu)$;
(xi) $\mathcal{B}(u,.)$ is a continuous non-decreasing function;
(xii) $\lim_{\lambda\to\infty}\mathcal{B}(u,\lambda) = 0$;
(xiii) $\mathcal{Y}(u,\lambda) = 0$ (for $\lambda > 0$) *if and only if* $u = 0$;
(xiv) $\mathcal{Y}(\sigma u,\lambda) = \mathcal{Y}\left(u,\frac{\lambda}{|\sigma|}\right)$;
(xv) $\mathcal{Y}(u,\mu) \bullet \mathcal{Y}(v,\lambda) \geq \mathcal{Y}(u+v,\lambda+\mu)$;
(xvi) $\mathcal{Y}(u,.)$ is a continuous non-decreasing function;
(xvii) $\lim_{\lambda\to\infty}\mathcal{Y}(u,\lambda) = 0$ and
(xviii) If $\lambda \leq 0$, then $\mathcal{G}(u,\lambda) = 0,\ \ \mathcal{B}(u,\lambda) = 1$ and $\mathcal{Y}(u,\lambda) = 1$.

Here, $\mathcal{N}(\mathcal{G},\mathcal{B},\mathcal{Y})$ is called the neutrosophic norm".

Some examples of neutrosophic normed spaces can be found in [16].

"A sequence (a_k) in Neutrosophic Normed Spaces V is said to $convergent$ if, for each $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer m and $\mathcal{L} \in F$ such that

$$\mathcal{G}(a_k - \mathcal{L},\lambda) > 1-\varepsilon, \mathcal{B}(a_k - \mathcal{L},\lambda) < \varepsilon \text{ and } \mathcal{Y}(a_k - \mathcal{L},\lambda) < \varepsilon \quad \text{for all } k \geq m.$$

This is equivalent to say

$$\lim_{k\to\infty}\mathcal{G}(a_k - \mathcal{L},\lambda) = 1, \lim_{k\to\infty}\mathcal{B}(a_k - \mathcal{L},\lambda) = 0 \text{ and } \lim_{k\to\infty}\mathcal{Y}(a_k - \mathcal{L},\lambda) = 0.$$

and we write in this case $\mathcal{N} - \lim_{k\to\infty} a_k = \mathcal{L}$."

"The sequence (a_k) is said to be $Cauchy$ if, for each $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer m and such that

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$\mathcal{G}(a_k - a_n, \lambda) > 1 - \varepsilon, \mathcal{B}(a_k - a_n, \lambda) < \varepsilon$ and $\mathcal{Y}(a_k - a_n, \lambda) < \varepsilon$ for all $k, n \geq m$."

For basic terminology on neutrosophic normed space, we refer to [14].

Let w denotes the set of all sequences in the neutrosophic normed space $V = (F, \mathcal{N}, \circ, \bullet)$. Define $\Delta^m: w \to w$ by

$$\Delta^0 a_k = a_k;$$

$$\Delta^1 a_k = a_k - a_{k+1};$$

$$\Delta^m a_k = \Delta^{m-1} (\Delta a_k) = \Delta^{m-1} (a_k - a_{k+1}) \quad m \geq 2 \text{ and for all } k \in \mathbb{N}.$$

Throughout the work V be an NNS; $\lambda = (\lambda_n)$ is a sequence as described above and $I \subseteq \wp(\mathbb{N})$ denotes an admissible ideal.

3. $\Delta^m(\boldsymbol{\lambda})$ −Convergence

Definition 3.1 Let $0 < \varepsilon < 1$ and $\mu > 0$. A sequence $x = (x_k)$ in V is said to be $\Delta^m(\lambda) - convergent$ in neutrosophic norm $\mathcal{N}$ if $\exists$, $\mathcal{L} \epsilon F$ and an $n_0 \in \mathbb{N}$ satisfying

$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}, \mu) > 1 - \varepsilon,$$

$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}, \mu) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}, \mu) < \varepsilon \quad \text{for } n \geq n_0,$$

and we write $\mathcal{N}_\lambda - \lim_k \Delta^m x_k = \mathcal{L}$..

Theorem 3.1 If $x = (x_k)$ be a $\Delta^m(\lambda) - convergent$ with $\mathcal{N}_\lambda - \lim_k \Delta^m x_k = \mathcal{L}$, then, $\mathcal{L}$ is unique.

Proof Assume, there exists $\mathcal{L}_1$ and $\mathcal{L}_2$ in V such that $\mathcal{L}_1 \neq \mathcal{L}_2$ and $\mathcal{N}_\lambda - \lim_k \Delta^m x_k = \mathcal{L}_1, \mathcal{N}_\lambda - \lim_k \Delta^m x_k = \mathcal{L}_2$. let $\varepsilon > 0$ and $\mu > 0$. Choose $\vartheta > 0$ such that

$$(1-\varepsilon) \circ (1-\varepsilon) > 1 - \vartheta \quad \text{and} \quad \varepsilon \bullet \varepsilon < \vartheta \tag{3.1}$$

Since, $\mathcal{N}_\lambda - \lim_k \Delta^m x_k = \mathcal{L}_1$, $\mathcal{N}_\lambda - \lim_k \Delta^m x_k = \mathcal{L}_2$ so for every $\varepsilon > 0$ and $\mu > 0$ $\exists$ n_1 and n_2 in $\mathbb{N}$ with

$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}_1, \mu) > 1 - \varepsilon,$$

$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}_1, \mu) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}_1, \mu) < \varepsilon \quad \text{for } n \geq n_1;$$

and

$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(x\Delta^m{}_k - \mathcal{L}_2, \mu) > 1 - \varepsilon,$$

$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}_2, \mu) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}_2, \mu) < \varepsilon \quad \text{for } n \geq n_2.$$

Case (i) $\displaystyle\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}_1, \mu) > 1 - \varepsilon$ and $\displaystyle\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}_2, \mu) > 1 - \varepsilon;$

Case (ii) $\displaystyle\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}_1, \mu) < \varepsilon$ and $\displaystyle\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}_2, \mu) < \varepsilon$ and

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Case (iii) $\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}_1, \mu) < \varepsilon$ and $\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}_2, \mu) < \varepsilon$.

Case (i) Let $n_0 = \max\{n_1, n_2\}$, then for $n \geq n_0$ we can find $p \in \mathbb{N}$ satisfying

$$\mathcal{G}\left(\Delta^m x_p - \mathcal{L}_1, \frac{\mu}{2}\right) > \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}\left(\Delta^m x_k - \mathcal{L}_1, \frac{\mu}{2}\right) > 1 - \varepsilon \quad \text{and}$$

$$\mathcal{G}\left(\Delta^m x_p - \mathcal{L}_2, \frac{\mu}{2}\right) > \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}\left(\Delta^m x_k - \mathcal{L}_2, \frac{\mu}{2}\right) > 1 - \varepsilon .$$

Now,

$$\mathcal{G}(\mathcal{L}_1 - \mathcal{L}_2, \mu) \geq \mathcal{G}\left(\Delta^m a_p - \mathcal{L}_1, \frac{\mu}{2}\right) \circ \mathcal{G}\left(\Delta^m a_p - \mathcal{L}_2, \frac{\mu}{2}\right) > (1-\varepsilon) \circ (1-\varepsilon) > 1 - \vartheta \tag{3.2}$$

Since ϑ is arbitrary and (3.2) holds for every $\mu > 0$, it follows that $\mathcal{G}(\mathcal{L}_1 - \mathcal{L}_2, \mu) = 1$, and therefore $\mathcal{L}_1 = \mathcal{L}_2$.

Case (ii) As in case (i) for $n \geq n_0$ there exists $p \in \mathbb{N}$ such that

$$\mathcal{B}\left(\Delta^m x_p - \mathcal{L}_1, \frac{\mu}{2}\right) < \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}\left(\Delta^m x_k - \mathcal{L}_1, \frac{\mu}{2}\right) < \varepsilon \quad \text{and}$$

$$\mathcal{B}\left(\Delta^m x_p - \mathcal{L}_2, \frac{\mu}{2}\right) < \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}\left(\Delta^m x_k - \mathcal{L}_2, \frac{\mu}{2}\right) < \varepsilon.$$

Now,

$$\mathcal{B}(\mathcal{L}_1 - \mathcal{L}_2, \mu) < \mathcal{B}\left(\Delta^m a_p - \mathcal{L}_1, \frac{\mu}{2}\right) \circ \mathcal{B}\left(\Delta^m a_p - \mathcal{L}_2, \frac{\mu}{2}\right) < \varepsilon \bullet \varepsilon < \vartheta.$$

Since ϑ is arbitrary and above inequality holds for every $\mu > 0$, it follows that $\mathcal{B}(\mathcal{L}_1 - \mathcal{L}_2, \mu) = 0$, and therefore $\mathcal{L}_1 = \mathcal{L}_2$.

Case (iii) follows similarly to case (ii) ∎

Theorem 3.2 Let $x = (x_k)$, $y = (y_k)$ be two sequences in V such that $\mathcal{N}_\lambda - \lim_k \Delta^m x_k = \mathcal{L}_1$ and $\mathcal{N}_\lambda - \lim_k \Delta^m y_k = \mathcal{L}_2$, then

(i) $\mathcal{N}_\lambda - \lim_k (\Delta^m (x_k + y_k)) = \mathcal{L}_1 + \mathcal{L}_2$.
(ii) $\mathcal{N}_\lambda - \lim_k (\beta(\Delta^m x_k)) = \beta\mathcal{L}_1$ for $\beta \neq 0$

Proof. excluded. ∎

Theorem 3.3 For any $x = (x_k)$ with $\mathcal{N}_\lambda - \lim_k \Delta^m x_k = \mathcal{L}$, ∃ a subsequence $\left(x_{k_j}\right)$ of (x_k) with $\mathcal{N} - \lim_j \Delta^m x_{k_j} = \mathcal{L}$.

Proof Suppose $\mathcal{N}_\lambda - \lim_k \Delta^m x_k = \mathcal{L}$ holds. For each $\varepsilon > 0$ and $\mu > 0$, ∃ $n_0 \in \mathbb{N}$ satisfying

$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}, \mu) > 1 - \varepsilon \ ,$$

$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}, \mu) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}, \mu) < \varepsilon \quad \text{for } n \geq n_0 .$$

Now for $n \geq n_0$, we can select a $k_j \in I_n$ such that

$$\mathcal{G}\left(\Delta^m x_{k_j} - \mathcal{L}, \mu\right) > \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}, \mu) > 1 - \varepsilon;$$

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$$\mathcal{B}\left(\Delta^m x_{k_j} - \mathcal{L}, \mu\right) < \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}, \mu) < \varepsilon \quad \text{and}$$

$$\mathcal{Y}(\Delta^m x_k - \mathcal{L}, \mu) < \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}, \mu) < \varepsilon \, .$$

Thus, we have a subsequence $\left(x_{k_j}\right)$ of (x_k) such that $\mathcal{N} - \lim_j \Delta^m x_{k_j} = \mathcal{L}$. ■

4. $\Delta^m(\boldsymbol{I}_\lambda)$ − Convergence

Definition 4.1, Let $0 < \varepsilon < 1$ and $\mu > 0$. A sequence $x = (x_k)$ in V is said to be $\Delta^m(I_\lambda) - convergent$ with respect to $\mathcal{N}$ if $\exists$, $\mathcal{L} \epsilon F$ satisfying

$$A(\varepsilon, \mu) = \left\{ n \in \mathbb{N}: \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}, \mu) \leq 1 - \varepsilon \text{ or } \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}, \mu) \geq \varepsilon \text{ and } \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}, \mu) \geq \varepsilon \right\} \in I.$$

We denote it by $\Delta^m\left(I_{\mathcal{N}}^\lambda\right) - \lim_k x_k = \mathcal{L}$.

Example 4.1 Let $(F, \|.\|)$ be a normed space. Define $t - norm \circ$ and $t - conorm \bullet$ as follows. For $u, v \in [0,1]$, $u \circ v = uv$ and $u \bullet v = \min\{u + v, 1\}$. For $\mu > 0$ and $\mu > \|u\|$, define

$$\mathcal{G}(u, \lambda) = \frac{\mu}{\mu + \|u\|}, \quad \mathcal{B}(u, \lambda) = \frac{\|u\|}{\mu + \|u\|} \quad \text{and} \quad \mathcal{Y}(u, \lambda) = \frac{\|u\|}{\mu}$$

Then, $(F, \mathcal{N}, \circ, \bullet)$ is a neutrosophic normed space.

Let, $m = 0$, $\lambda = (\lambda_n) = (n)$ and chose $I = \{S \subseteq \mathbb{N}: \delta(S) = 0\}$. Construct a sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1 & \text{if } k = j^2 \text{ where } j \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases};$$

Then for $\varepsilon > 0$ and $\mu > 0$, the set

$$A(\varepsilon, \mu) = \left\{ n \in \mathbb{N}: \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{G}(\Delta^m x_k, \mu) \leq 1 - \varepsilon \text{ or} \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{B}(\Delta^m x_k, \mu) \geq \varepsilon \text{ and } \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{Y}(\Delta^m x_k, \mu) \geq \varepsilon \right\}$$

is contained in the set of squares whose natural density is zero. So, we have $\delta\big(A(\varepsilon, \mu)\big) = 0$ and therefore $\delta\big(A(\varepsilon, \mu)\big) \in I$. ■

Lemma 4.1 For $\varepsilon > 0$ and $\mu > 0$, the following are equivalents.

(*i*) $\Delta^m\left(I_{\mathcal{N}}^\lambda\right) - \lim_k x_k = \mathcal{L}$;

(*ii*) $$\left\{ n \in \mathbb{N}: \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}, \mu) \leq 1 - \varepsilon \right\} \in I; \left\{ n \in \mathbb{N}: \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}, \mu) \geq \varepsilon \right\} \in I \text{ and } \left\{ n \in \mathbb{N}: \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}, \mu) \geq \varepsilon \right\} \in I.$$

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(iii) $$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}, \mu) > 1-\varepsilon\,, \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}, \mu) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}, \mu) < \varepsilon\right\} \in \mathcal{F}(I)$$

(iv) $$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}, \mu) > 1-\varepsilon\right\} \in \mathcal{F}(I); \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}, \mu) < \varepsilon\right\} \in \mathcal{F}(I) \text{ and}$$
$$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}, \mu) < \varepsilon\right\} \in \mathcal{F}(I)$$

(v) $I_\lambda - \lim_k \mathcal{G}(\Delta^m x_k - \mathcal{L}, \mu) = 1$; $I_\lambda - \lim_k \mathcal{B}(\Delta^m x_k - \mathcal{L}, \mu) = 0$ and $I_\lambda - \lim_k \mathcal{Y}(\Delta^m x_k - \mathcal{L}, \mu) = 0$.

Proof. Omitted ■

Next Theorem provided the uniqueness of $I_{\mathcal{N}}^{\theta} - \lim_k x_k$ provided it exists.

Theorem 4.1 If $x = (x_k)$ is a sequence in V such that $\Delta^m(I_{\mathcal{N}}^{\lambda}) - \lim_k x_k = \mathcal{L}_1$ and $\Delta^m(I_{\mathcal{N}}^{\lambda}) - \lim_k x_k = \mathcal{L}_2$, then $\mathcal{L}_1 = \mathcal{L}_2$.

Proof: Suppose that $\mathcal{L}_1 \neq \mathcal{L}_2$ and let $\varepsilon > 0$. Choose $\gamma > 0$ such that

$$(1-\varepsilon)\circ(1-\varepsilon) > 1-\gamma \quad \text{and} \quad \varepsilon \bullet \varepsilon < \gamma \tag{4.1}$$

For $\mu > 0$, define.

$$K_{\mathcal{G}_1}(\varepsilon,\mu) = \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}\left(\Delta^m x_k - \mathcal{L}_1, \frac{\mu}{2}\right) \le 1-\varepsilon\right\},$$
$$K_{\mathcal{G}_2}(\varepsilon,\mu) = \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}\left(\Delta^m x_k - \mathcal{L}_2, \frac{\mu}{2}\right) \le 1-\varepsilon\right\};$$

$$K_{\mathcal{B}_1}(\varepsilon,\mu) = \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}\left(\Delta^m x_k - \mathcal{L}_1, \frac{\mu}{2}\right) \ge \varepsilon\right\}, \quad K_{\mathcal{B}_2}(\varepsilon,\mu) = \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}\left(\Delta^m x_k - \mathcal{L}_2, \frac{\mu}{2}\right) \ge \varepsilon\right\};$$

$$K_{\mathcal{Y}_1}(\varepsilon,\mu) = \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}\left(\Delta^m x_k - \mathcal{L}_1, \frac{\mu}{2}\right) \ge \varepsilon\right\} \text{ and } K_{\mathcal{Y}_2}(\varepsilon,\mu) = \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}\left(\Delta^m x_k - \mathcal{L}_2, \frac{\mu}{2}\right) \ge \varepsilon\right\}.$$

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Since $\Delta^m(I_{\mathcal{N}}^{\lambda}) - \lim_k x_k = \mathcal{L}_1$ and $\Delta^m(I_{\mathcal{N}}^{\lambda}) - \lim_k x_k = \mathcal{L}_2$ so by Lemma 3.1, we respectively have the sets $K_{\mathcal{G}_1}(\varepsilon,\mu)$; $K_{\mathcal{B}_1}(\varepsilon,\mu)$; $K_{\mathcal{Y}_1}(\varepsilon,\mu)$ and $K_{\mathcal{G}_2}(\varepsilon,\mu)$; $K_{\mathcal{B}_2}(\varepsilon,\mu)$ and $K_{\mathcal{Y}_2}(\varepsilon,\mu)$ belongs to I. Define a set $K_{\mathcal{N}}(\varepsilon,\mu)$ by

$$K_{\mathcal{N}}(\varepsilon,\mu) = \Big\{\Big\{\{K_{\mathcal{G}_1}(\varepsilon,\mu)\} \cup \{K_{\mathcal{G}_2}(\varepsilon,\mu)\}\Big\} \cap \Big\{\{K_{\mathcal{B}_1}(\varepsilon,\mu)\} \cup \{K_{\mathcal{B}_2}(\varepsilon,\mu)\}\Big\} \cap \Big\{\{K_{\mathcal{Y}_1}(\varepsilon,\mu)\} \cup \{K_{\mathcal{Y}_2}(\varepsilon,\mu)\}\Big\}\Big\};$$

then $K_{\mathcal{N}}(\varepsilon,\mu) \in I$ and therefore $\emptyset \neq K_{\mathcal{N}}^{C}(\varepsilon,\mu) \in \mathcal{F}(I)$. Let $n \in K_{\mathcal{N}}^{C}(\varepsilon,\mu)$, then

(i) $n \in \mathbb{N} - \Big\{\{K_{\mathcal{G}_1}(\varepsilon,\mu)\} \cup \{K_{\mathcal{G}_2}(\varepsilon,\mu)\}\Big\}$

(ii) $n \in \mathbb{N} - \Big\{\{K_{\mathcal{B}_1}(\varepsilon,\mu)\} \cup \{K_{\mathcal{B}_2}(\varepsilon,\mu)\}\Big\}$

(iii) $n \in \mathbb{N} - \Big\{\{K_{\mathcal{Y}_1}(\varepsilon,\mu)\} \cup \{K_{\mathcal{Y}_2}(\varepsilon,\mu)\}\Big\}$.

Suppose that (i) holds, then $n \notin \{K_{\mathcal{G}_1}(\varepsilon,\mu)\} \cup \{K_{\mathcal{G}_2}(\varepsilon,\mu)\}$ which gives $n \notin K_{\mathcal{G}_1}(\varepsilon,\mu)$ and $n \notin K_{\mathcal{G}_2}(\varepsilon,\mu)$. This implies that

$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}\left(\Delta^m x_k - \mathcal{L}_1, \frac{\mu}{2}\right) > 1-\varepsilon \quad \text{and} \quad \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}\left(\Delta^m x_k - \mathcal{L}_2, \frac{\mu}{2}\right) > 1-\varepsilon \tag{4.2}$$

Clearly, we will get a $p \in \mathbb{N}$ such that

$$\mathcal{G}\left(\Delta^m x_p - \mathcal{L}_1, \frac{\mu}{2}\right) > \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}\left(\Delta^m x_k - \mathcal{L}_1, \frac{\mu}{2}\right) > 1-\varepsilon \quad \text{and}$$

$$\mathcal{G}\left(\Delta^m x_p - \mathcal{L}_2, \frac{\mu}{2}\right) > \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}\left(\Delta^m x_k - \mathcal{L}_2, \frac{\mu}{2}\right) > 1-\varepsilon.$$

Now,

$$\mathcal{G}(\mathcal{L}_1 - \mathcal{L}_2, \mu) \geq \mathcal{G}\left(\Delta^m x_p - \mathcal{L}_1, \frac{\mu}{2}\right) \circ \mathcal{G}\left(\Delta^m x_p - \mathcal{L}_2, \frac{\mu}{2}\right) > (1-\varepsilon) \circ (1-\varepsilon) > 1-\gamma \tag{4.3}$$

(using (4.1) and (4.2))

Since γ is arbitrary and (4.3) holds for every $\mu > 0$, it follows that $\mathcal{G}(\mathcal{L}_1 - \mathcal{L}_2, \mu) = 1$, and therefore $\mathcal{L}_1 = \mathcal{L}_2$.

We now assume (ii) holds, then $n \notin K_{\mathcal{B}_1}(\varepsilon,\mu)$ and $n \notin K_{\mathcal{B}_2}(\varepsilon,\mu)$, and therefore we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}\left(\Delta^m x_k - \mathcal{L}_1, \frac{\mu}{2}\right) < \varepsilon \quad \text{and} \quad \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}\left(\Delta^m x_k - \mathcal{L}_2, \frac{\mu}{2}\right) < \varepsilon \tag{4.4}$$

Using the same technique as above, we get $p \in \mathbb{N}$ such that

$$\mathcal{B}\left(\Delta^m x_p - \mathcal{L}_1, \frac{\mu}{2}\right) < \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}\left(\Delta^m x_k - \mathcal{L}_1, \frac{\mu}{2}\right) < \varepsilon \quad \text{and}$$

$$\mathcal{B}\left(\Delta^m x_p - \mathcal{L}_2, \frac{\mu}{2}\right) < \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}\left(\Delta^m x_k - \mathcal{L}_2, \frac{\mu}{2}\right) < \varepsilon.$$

Now,

$$\mathcal{B}(\mathcal{L}_1 - \mathcal{L}_2, \lambda) \leq \mathcal{B}\left(\Delta^m x_p - \mathcal{L}_1, \frac{\mu}{2}\right) \bullet \mathcal{B}\left(\Delta^m x_p - \mathcal{L}_2, \frac{\mu}{2}\right) < \varepsilon \bullet \varepsilon < \gamma \tag{4.5}$$

(using (4.1) and (4.4))

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As γ is arbitrary and (4.5) holds for every $\mu > 0$, we must have $\mathcal{B}(\mathcal{L}_1 - \mathcal{L}_2, \mu) = 0$, which gives immediately $\mathcal{L}_1 = \mathcal{L}_2$.

Finally, assume that (iii) holds. It follows that $n \notin K_{y_1}(\mu, \lambda)$, and $n \notin K_{y_2}(\mu, \lambda)$and therefore we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}\left(\Delta^m x_k - \mathcal{L}_1, \frac{\mu}{2}\right) < \varepsilon \quad \text{and} \quad \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}\left(\Delta^m x_k - \mathcal{L}_2, \frac{\mu}{2}\right) < \varepsilon$$

As above, we get $p \in \mathbb{N}$ such that

$$\mathcal{Y}\left(\Delta^m x_p - \mathcal{L}_1, \frac{\mu}{2}\right) < \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}\left(\Delta^m x_k - \mathcal{L}_1, \frac{\mu}{2}\right) < \varepsilon \quad \text{and}$$

$$\mathcal{Y}\left(\Delta^m x_p - \mathcal{L}_2, \frac{\mu}{2}\right) < \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}\left(\Delta^m x_k - \mathcal{L}_2, \frac{\mu}{2}\right) < \varepsilon.$$

Now,

$$\mathcal{Y}(\mathcal{L}_1 - \mathcal{L}_2, \mu) \le \mathcal{Y}\left(\Delta^m x_p - \mathcal{L}_1, \frac{\mu}{2}\right) \bullet \mathcal{Y}\left(\Delta^m x_p - \mathcal{L}_2, \frac{\mu}{2}\right) < \varepsilon \bullet \varepsilon < \gamma \tag{4.6}$$

Since γ is arbitrary and (4.6) holds for every $\mu > 0$, we must have $\mathcal{Y}(\mathcal{L}_1 - \mathcal{L}_2, \mu) = 0$ and therefore we have $\mathcal{L}_1 = \mathcal{L}_2$.. ■

Theorem 4.2 If $x = (x_k)$, $y = (y_k)$ be two sequences in Vsuch that $\Delta^m(I_{\mathcal{N}}^\lambda) - \lim_k x_k = \mathcal{L}_1$ and $\Delta^m(I_{\mathcal{N}}^\lambda) - \lim_k y_k = \mathcal{L}_2$, then

(i) $\Delta^m\left(I_{\mathcal{N}}^\lambda\right) - \lim_k(x_k + y_k) = \mathcal{L}_1 + \mathcal{L}_2$

(ii) $\Delta^m\left(I_{\mathcal{N}}^\lambda\right) - \lim_k(\beta\, x_k) = \beta\mathcal{L}_1$ for $\beta \ne 0$

Proof. Omitted. ■

Next Theorem provide the relation between $\Delta^m(\mathcal{N}_{\lambda)} - convergence$ and $\Delta^m(I_{\mathcal{N}}^\lambda) - convergence$.

Theorem 4.3 For $x = (x_k)$ if $\Delta^m(\mathcal{N}_\lambda) - \lim_k x_k = \mathcal{L}$, then, $\Delta^m(I_{\mathcal{N}}^\lambda) - \lim_k x_k = \mathcal{L}$.

Proof Suppose $\Delta^m(\mathcal{N}_\lambda) - \lim_k x_k = \mathcal{L}$ holds. Then for $\varepsilon > 0$ and $\mu > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t

$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}, \mu) > 1 - \varepsilon\ , \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}, \mu) < \varepsilon \ \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}, \mu) < \varepsilon \quad \text{for } n \ge n_0\,.$$

Let,

$$A(\varepsilon, \mu) = \left\{ n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}, \mu) \le 1 - \varepsilon \ \text{ or } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}, \mu) \ge \varepsilon \ \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}, \mu) \ge \varepsilon \right\},$$

then it is clear that $A(\varepsilon, \mu) \subseteq \{1, 2, 3, \cdots n_0 - 1\}$ so in I. This implies $\Delta^m(I_{\mathcal{N}}^\lambda) - \lim_k x_k = \mathcal{L}$. ■

Theorem 4.4 If $x = (x_k)$ be $\Delta^m(\lambda) - convergent$ in V and $y = (y_k)$ be another sequence in V such that $\{n \in \mathbb{N}: \Delta^m x_k \ne \Delta^m y_k \text{ for some } k \in I_n\} \in I$, then (y_k) is $\Delta^m(I_\lambda) - convergent$ to the same limit.

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Proof Let $\Delta^m(\mathcal{N}_\lambda) - \lim_k x_k = \mathcal{L}$. For each $\varepsilon > 0$ and $\mu > 0$, $\exists\ n_0 \in \mathbb{N}$ s.t

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{G}(\Delta^m x_k - \mathcal{L},\mu) > 1-\varepsilon\ ,$$

$$\frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{B}(\Delta^m x_k - \mathcal{L},\mu) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{Y}(\Delta^m x_k - \mathcal{L},\mu) < \varepsilon \quad \text{for } n \geq n_0 .$$

Thus

$$A(\varepsilon,\mu) = \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{G}(\Delta^m x_k - \mathcal{L},\mu) \leq 1-\varepsilon \text{ or } \frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{B}(\Delta^m x_k - \mathcal{L},\mu) \geq \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{Y}(\Delta^m x_k - \mathcal{L},\mu) \geq \varepsilon\right\} \subseteq \{1, 2.3. \cdots n_0 - 1\} \in I.$$

This implies that

$$A^C(\varepsilon,\mu) = \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{G}(\Delta^m x_k - \mathcal{L},\mu) > 1-\varepsilon, \frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{B}(\Delta^m x_k - \mathcal{L},\mu) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{Y}(\Delta^m x_k - \mathcal{L},\mu) < \varepsilon\right\} \notin I.$$

Now, for $0 < \varepsilon < 1$ and $\mu > 0$ we have

$$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{G}(\Delta^m y_k - \mathcal{L},\mu) \leq 1-\varepsilon \text{ or } \frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{B}(\Delta^m y_k - \mathcal{L},\mu) \geq \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{Y}(\Delta^m y_k - \mathcal{L},\mu) \geq \varepsilon\right\}$$
$$\subseteq \{n \in \mathbb{N}: \Delta^m x_k \neq \Delta^m y_k \text{ for some } k \in I_n\} \cup A(\varepsilon,\mu).$$

Since, $A(\varepsilon,\mu)$ and $\{n \in \mathbb{N}: \Delta^m x_k \neq \Delta^m y_k \text{ for some } k \in I_n\}$ are sets in I so $\{n \in \mathbb{N}: \Delta^m x_k \neq \Delta^m y_k \text{ for some } k \in I_n\} \cup A(\varepsilon,\mu) \in I$, which immediately gives

$$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{G}(\Delta^m y_k - \mathcal{L},\mu) \leq 1-\varepsilon \text{ or } \frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{B}(\Delta^m y_k - \mathcal{L},\mu) \geq \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n}\mathcal{Y}(\Delta^m y_k - \mathcal{L},\mu) \geq \varepsilon\right\} \in I;$$

and therefore (y_k) is $\Delta^m(I_\lambda) - convergent$. ∎.

5. $\Delta^m(I_\lambda)$ – limit points and $\Delta^m(I_\lambda)$ – cluster points

Definition 5.1 An element $\mathcal{L}_0$ is said to be a $\Delta^m - limit\ point$ of (x_k) w.r.t $\mathcal{N}$ if $\exists$ a subsequence of $(\Delta^m x_k)$ that is convergent to $\mathcal{L}_0$ w.r.t $\mathcal{N}$.

Definition 5.2 An element $\mathcal{L}_0$ is said to be a $\Delta^m(I_\lambda) - limit\ point$ of (x_k) if $\exists$, $K = \{k_1 < k_2 < \cdots k_j < \cdots\} \subseteq \mathbb{N}$ s.t the set $K' = \{r \in \mathbb{N}: k_j \in I_r\} \notin I$ and $\mathcal{N}_\lambda - \lim_j \Delta^m x_{k_j} = \mathcal{L}_0$.

Let $\Lambda_{\mathcal{N}}^{\lambda}(I,x)$ denotes the set of all $\Delta^m(I_\lambda) - limit\ points$ of $x = (x_k)$.

Definition 5.3 An element $\mathcal{L}_0$ is said to be a $\Delta^m(I_\lambda) - cluster\ point$ of (x_k) if $0 < \varepsilon < 1$ and $\mu > 0$,

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$$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}_0, \mu) > 1-\varepsilon, \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}_0, \mu) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}_0, \mu) < \varepsilon\right\} \notin I.$$

Let $\Gamma_{\mathcal{N}}^{\lambda}(I,x)$ denotes the set of $\Delta^m(I_\lambda) - cluster\ points$ of the sequence $x = (x_k)$.

Theorem 5.1 For $x = (x_k)$ in V, $\Lambda_{\mathcal{N}}^{\lambda}(I,x) \subseteq \Gamma_{\mathcal{N}}^{\lambda}(I,x)$.

Proof Let $\mathcal{L}_0 \in \Lambda_{\mathcal{N}}^{\lambda}(I,x)$, $\exists$ set $K = \{k_1 < k_2 < \cdots k_j < \cdots\} \subseteq \mathbb{N}$ s.t. $K' = \{r \in \mathbb{N}: k_j \in I_r\} \notin I$, $\mathcal{N}_\lambda - \lim_j \Delta^m x_{k_j} = \mathcal{L}_0$. So for $0 < \varepsilon < 1$ and $\mu > 0$, $\exists, j_0 \in \mathbb{N}$ s.t. for $j \geq j_0$

$$\frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{G}\left(\Delta^m x_{k_j} - \mathcal{L}_0, \mu\right) > 1-\varepsilon, \quad \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{B}\left(\Delta^m x_{k_j} - \mathcal{L}_0, \mu\right) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{Y}\left(\Delta^m x_{k_j} - \mathcal{L}_0, \mu\right) < \varepsilon$$

This implies that

$$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}_0, \mu) > 1-\varepsilon, \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}_0, \mu) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}_0, \mu) < \varepsilon\right\} \subseteq K' - \{k_1 < k_2 < \cdots k_{j_0}\}$$

Since I is admissible so $K' - \{k_1 < k_2 < \cdots k_{n_0}\} \notin I$, which immediately gives

$$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}_0, \mu) > 1-\varepsilon, \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}_0, \mu) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}_0, \mu) < \varepsilon\right\} \notin I;$$

and therefore $\mathcal{L}_0 \in \Gamma_{\mathcal{N}}^{\lambda}(I,x)$. ∎

Theorem 5.2 For $x = (x_k)$ in V, $\Gamma_{\mathcal{N}}^{\lambda}(I,x)$ is a closed set..

Proof. Let $\mathcal{L}_0 \in \overline{\Gamma_{\mathcal{N}}^{\lambda}(I,x)}$ where bar denotes the closure of the set. Let, $0 < \varepsilon < 1$ and $\mu > 0$. Since $\mathcal{L}_0 \in \overline{\Gamma_{\mathcal{N}}^{\lambda}(I,x)}$ so $\Gamma_{\mathcal{N}}^{\lambda}(I,x) \cap O(\mathcal{L}_0, \varepsilon, \mu)$ where $O(\mathcal{L}_0, \varepsilon, \mu) = \{\mathcal{L} \in V: \mathcal{G}(\mathcal{L} - \mathcal{L}_0, \mu) > 1-\varepsilon$, $\mathcal{B}(\mathcal{L} - \mathcal{L}_0, \mu) < \varepsilon$ and $\mathcal{Y}(\mathcal{L} - \mathcal{L}_0, \mu) < \varepsilon\}$ denotes the open neighborhood of $\mathcal{L}_0$. Let $\mathcal{L}_1 \in \Gamma_{\mathcal{N}}^{\lambda}(I,x) \cap O(\mathcal{L}_0, \varepsilon, \mu)$. Choose $\gamma > 0$ such that $O(\mathcal{L}_1, \gamma, \mu) \subseteq O(\mathcal{L}_0, \varepsilon, \mu)$. Now we have

$$A = \left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}_0, \mu) > 1-\varepsilon, \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}_0, \mu) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}_0, \mu) < \varepsilon\right\} \supseteq$$

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$$\left\{r \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{G}(\Delta^m x_k - \mathcal{L}_1, \mu) > 1-\gamma, \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{B}(\Delta^m x_k - \mathcal{L}_1, \mu) < \gamma \text{ and } \frac{1}{\lambda_n}\sum_{j\in I_n} \mathcal{Y}(\Delta^m x_k - \mathcal{L}_1, \mu) < \gamma\right\} = B.$$

As $\mathcal{L}_1 \in \Gamma_{\mathcal{N}}^{\lambda}(I, x)$ so $B \notin I$, and consequently $A \notin I$. Hence $\mathcal{L}_0 \in \Gamma_{\mathcal{N}}^{\lambda}(I, x)$ and therefore$\Gamma_{\mathcal{N}}^{\lambda}(I, x)$ is a closed set in V. ■

6. $\Delta^m(\lambda)$ −Cauchy and $\Delta^m(I_\lambda)$ −Cauchy sequences

Definition 6.1 A sequence $x = (x_k)$ in V is said to be $\Delta^m(\lambda) - Cauchy$ w.r.t $\mathcal{N}$ if ∃ positive integers p and n_0 s.t.

$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - x_p, \mu) > 1-\varepsilon,$$
$$\frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - x_p, \mu) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - x_p, \mu) < \varepsilon \text{ for } n \geq n_0$$

Definition 6.2 Let $0 < \varepsilon < 1$ and $\mu > 0$. A sequence $x = (x_k)$ in V is said to be $\Delta^m(I_\lambda) - Cauchy$ w.r.t $\mathcal{N}$ if ∃ positive integers p s.t

$$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - x_p, \mu) \leq 1-\varepsilon \text{ or } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - x_p, \mu) \geq \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - x_p, \mu) \geq \varepsilon\right\} \in I$$

or equivalently

$$\left\{n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - x_p, \mu) > 1-\varepsilon, \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - x_p, \mu) < \varepsilon \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - x_p, \mu) < \varepsilon\right\} \in \mathcal{F}(I).$$

Definition 6.3 Let, $0 < \varepsilon < 1$ and $\mu > 0$. A sequence $x = (x_k)$ in V is said to be $\Delta^m(I_\lambda^*) - Cauchy$ w.r.t $\mathcal{N}$ if ∃, $K = \{k_1 < k_2 < \cdots k_j < \cdots\} \subseteq \mathbb{N}$ s.t. $K' = \{n \in \mathbb{N}: k_j \in I_n\} \in \mathcal{F}(I)$ and (x_{k_j}) is $\Delta^m(\lambda) - Cauchy$ sequence w.r.t $\mathcal{N}$.

The following two Theorems are analogs of Theorem 4.3, and Theorem 3.3 respectively.

Theorem 6.1 2 If $x = (x_k)$ in V is $\Delta^m(\lambda) - Cauchy$ then it is $\Delta^m(I_\lambda) - Cauchy$.

Theorem 6.2 If $x = (x_k)$ in V is $\Delta^m(\lambda) - Cauchy$ then ∃ a subsequence of $(\Delta^m x_k)$ that is $\mathcal{N} - Cauchy$.

Theorem 6.3 If $x = (x_k)$ in V is $\Delta^m(I_\lambda^*) - Cauchy$, then it is $\Delta^m(I_\lambda) - Cauchy$.

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7. $\Delta^m(I_\lambda)$ −Convergence preserving linear operators

In this section, we define continuous and sequentially continuous mappings and present a characterization of $I_\lambda - convergence$ preserving linear operators.

Definition 7.1 A mapping $T: V \to V$ is said to be continuous at $x_0 \in V$ w.r.t $\mathcal{N}$ if for $0 < \varepsilon < 1$ and $\mu > 0$, $\exists\ 0 < \varepsilon' < 1$ and $\gamma > 0$ s.t. for all $x \in V$,

$$\mathcal{G}(x - x_0, \gamma) > 1 - \varepsilon', \quad \mathcal{B}(x - x_0, \gamma) < \varepsilon' \quad \text{and} \quad \mathcal{Y}(x - x_0, \gamma) < \varepsilon'$$

imply that

$$\mathcal{G}(T(x) - T(x_0), \mu) > 1 - \varepsilon, \quad \mathcal{B}(T(x) - T(x_0), \mu) < \varepsilon \quad \text{and} \quad \mathcal{Y}(T(x) - T(x_0), \mu) < \varepsilon.$$

If This is continuous at every point of the space V, then we call T continuous on V.

Definition 7.2 A mapping $T: V \to V$ is said to be sequentially continuous at $x_0 \in V$ w.r.t $\mathcal{N}$ and Δ^m if for any sequence $x = (x_k)$ in V with $\mathcal{N} - \lim_k \Delta^m x_k = x_0$ implies that $\mathcal{N} - \lim_k T(\Delta^m x_k) = T(x_0)$.

Theorem 7.1 Let V be an NNS. A mapping $T: V \to V$ is continuous w.r.t $\mathcal{N}$if and only if it is sequentially continuous with respect to the neutrosophic norm $\mathcal{N}$.

Proof. Omitted.

Definition 7.3 A mapping $T: V \to V$ is said to preserve $\Delta^m(I_\lambda) - convergence$ in Vif $\Delta^m(I_{\mathcal{N}}^{\lambda}) - \lim_k x_k = x_0$ implies $\Delta^m(I_{\mathcal{N}}^{\lambda}) - \lim_k T(x_k) = T(x_0)$ for every sequence $x = (x_k)$ in V which is $\Delta^m(I_\lambda) - convergent$ to $x_0 \in V$.

Theorem 7.2. A linear operator $T: V \to V$ preserves $\Delta^m(I_\lambda) - convergence$ in Vif and only if $T$is continuous on V.

Proof First suppose that $T$is continuous on V. Let $x = (x_k)$ be any sequence in V such that $\Delta^m(I_{\mathcal{N}}^{\lambda}) - \lim_k x_k = x_0$. For each $0 < \varepsilon < 1$ and $\mu > 0$, there exists $0 < \varepsilon' < 1$ and $\gamma > 0$ such that for all $x \in V$,

$$\mathcal{G}(x - x_0, \gamma) > 1 - \varepsilon', \quad \mathcal{B}(x - x_0, \gamma) < \varepsilon' \quad \text{and} \quad \mathcal{Y}(x - x_0, \gamma) < \varepsilon'$$

imply that

$$\mathcal{G}(T(x) - T(x_0), \mu) > 1 - \varepsilon, \quad \mathcal{B}(T(x) - T(x_0), \mu) < \varepsilon \quad \text{and} \quad \mathcal{Y}(T(x) - T(x_0), \mu) < \varepsilon.$$

If,

$$O(x_0, \varepsilon', \gamma) = \{x \in V: \mathcal{G}(x - x_0, \gamma) > 1 - \varepsilon', \mathcal{B}(x - x_0, \gamma) < \varepsilon' \quad \text{and} \quad \mathcal{Y}(x - x_0, \mu) < \varepsilon'\}$$

$O(T(x_0), \varepsilon, \mu) = \{T(x) \in V: \mathcal{G}(T(x) - T(x_0), \mu) > 1 - \varepsilon,\ \mathcal{B}(T(x) - T(x_0), \mu) < \varepsilon$ and $\mathcal{Y}(T(x) - T(x_0), \mu) < \varepsilon.\}$

denotes open balls centered at x_0 and $T(x_0)$ respectively, then for all $x \in V$, if $x \in O(x_0, \varepsilon', \gamma)$ then $T(x) \in O(T(x_0), \varepsilon, \mu)$. But then we have

$$A(\varepsilon', \gamma) = \left\{ n \in \mathbb{N}: \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{G}(\Delta^m x_k - x_0, \gamma) > 1 - \varepsilon', \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{B}(\Delta^m x_k - x_0, \gamma) < \varepsilon' \text{ and } \frac{1}{\lambda_n} \sum_{k \in I_n} \mathcal{Y}(\Delta^m x_k - x_0, \gamma) < \varepsilon' \right\}$$

Doi : https://doi.org/10.54216/IJNS.190105
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$$\subseteq \left\{ \begin{array}{c} n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(T(\Delta^m x_k) - T(x_0), \mu) > 1 - \varepsilon \; , \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(T(\Delta^m x_k) - T(x_0), \mu) < \varepsilon \text{ and} \\ \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(T(\Delta^m x_k) - T(x_0), \mu) < \varepsilon \end{array} \right\}$$

$$= B(\varepsilon, \mu).$$

Since $A(\varepsilon', \gamma) \in \mathcal{F}(I)$ therefore $B(\varepsilon, \mu) \in \mathcal{F}(I)$. Hence, we have $\Delta^m(I_{\mathcal{N}}^{\lambda}) - \lim_k T(x_k) = T(x_0)$.

Conversely, suppose that the operator $T: V \to V$ preserves $\Delta^m(I_\lambda) - convergence$ in V. We prove that T is continuous. Suppose that $T$is not continuous at some point x_0 in V. Then there is some $0 < \varepsilon < 1$ and $\mu > 0$ such that, for all $0 < \varepsilon' < 1$ and $\gamma > 0$ we have if $x \in O(x_0, \varepsilon', \gamma)$ then $T(x) \notin O(T(x_0), \varepsilon, \mu)$ where $x \in V$. Now we can have a sequence $x = (x_k)$ such that $\mathcal{N} - \lim_k \Delta^m x_k = x_0$ but $\mathcal{N} - \lim_k T(\Delta^m x_k) \neq T(x_0)$. This gives

$$D(\varepsilon', \gamma) = \left\{ n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(\Delta^m x_k - x_0, \gamma) > 1 - \varepsilon', \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(\Delta^m x_k - x_0, \gamma) < \varepsilon' \text{ and } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(\Delta^m x_k - x_0, \gamma) < \varepsilon' \right\}$$

$$\subseteq \left\{ \begin{array}{c} n \in \mathbb{N}: \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{G}(T(\Delta^m x_k) - T(x_0), \mu) \leq 1 - \varepsilon \text{ or } \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{B}(T(\Delta^m x_k) - T(x_0), \mu) \geq \varepsilon \text{ and} \\ \frac{1}{\lambda_n}\sum_{k\in I_n} \mathcal{Y}(T(\Delta^m x_k) - T(x_0), \mu) \geq \varepsilon \end{array} \right\}$$

$$= E(\varepsilon, \mu).$$

Since $D(\varepsilon', \gamma) \in \mathcal{F}(I)$ so $E(\varepsilon, \mu) \in \mathcal{F}(I)$ and therefore we have $\Delta^m(I_{\mathcal{N}}^{\lambda}) - \lim_k T(x_k) \neq T(x_0)$. ■

Conclusion

This paper used difference operators to define some new summability concepts on neutrosophic normed spaces. The paper introduced the concepts of generalized limit point, cluster point and obtain some relationships among these notions. In addition, the paper defined generalized Cauchy sequences on these spaces and present a characterization of new summability method that preserve linear operators on neutrosophic normed spaces.

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