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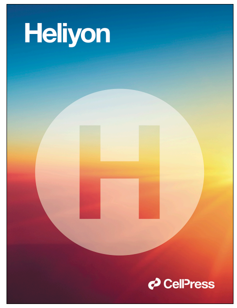

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Neutrosophic Robust Ratio Type Estimator for Estimating Finite Population Mean

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Abstract: Various authors have put their sincere efforts into proposing ratio estimators for estimating the population's mean and variance under different situations and sampling methods. But the problem arises when data is unstable, imprecise, ambiguous, incomplete and vague. In such situations, classical methods of estimation do not yield precise results, as these methods are not meant for such problems. Given this difficulty, Neutrosophic statistics are the only alternative as it deals with indeterminacy. So in this study, we have proposed a generalized Neutrosophic robust ratio type estimator which can be used to provide good results in such situations, as well as in the case of the presence of outliers. For the evaluation point of view, we have made use of real data set and simulation study to check the efficacy of our suggested estimators over the mentioned existed estimators.

KEYWORDS: Neutrosophic Data; Neutrosophic Ratio estimator; Neutrosophic Product estimator; Outliers; MSE; Efficiency

1 Introduction

In traditional statistical analysis, data is typically represented by precise numerical values. Numerous researchers have devoted their efforts to devising estimators for determining the mean and variance of a finite population, particularly when additional information is available. Utilizing supplementary information has consistently proven advantageous in enhancing precision, provided that the supplementary variables are positively or negatively correlated with the study variables. Researchers have extensively explored and innovated methods to improve the efficiency of ratio estimation for population means, either by refining existing approaches or proposing novel estimators. [3] introduced new ratio estimators for population means based on the coefficient of kurtosis, skewness, and correlation, along with their combinations. Similarly, [5] presented a novel information-based approach to ranked set sampling, along with sub-ratio estimators. [4] also suggested innovative mean estimators for ranked set sampling utilizing dual auxiliary variables. [16] addressed proportion estimation in ranked set sampling considering tie information, while [17] proposed EDF-based tests for exponentially paired ranked set sampling. Additionally, [18] proposed an efficient method for estimating cumulative distribution functions using moving extreme ranked set sampling, particularly applied to reliability analysis.

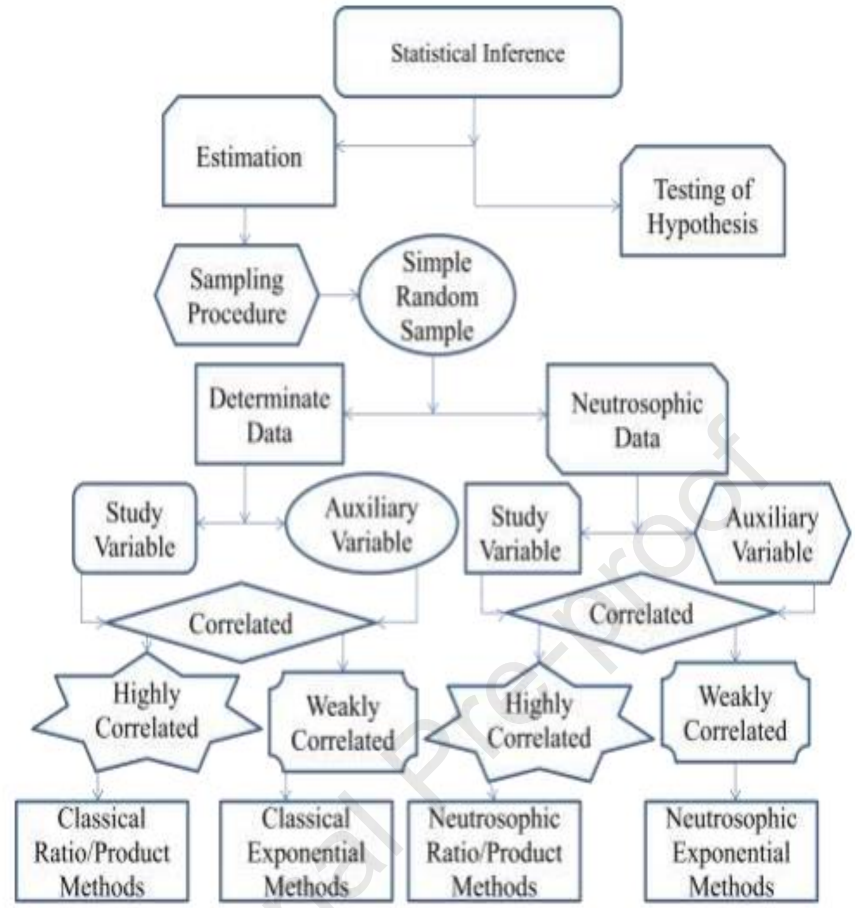
[15] conducted a simulation study on logarithmic ratio type estimators of population mean for simple random sampling. Conversely, [8] employed various generalized robust regression techniques for population mean estimation in simple random sampling, particularly when auxiliary information was contaminated with outliers. Furthermore, [14] introduced a robust family of estimators for population variance in both simple and stratified random sampling scenarios.

Recently, [9], [10] proposed exponential ratio type estimators for skewed data, particularly addressing outliers in estimating population medians in SRSWOR and stratified sampling designs, offering an alternative to regression estimators. In another line of development, authors have introduced new ratio estimators such as exponential ratio and product estimators, effective in situations devoid of indeterminacy. However, the inherent drawback of point estimation in survey sampling is its susceptibility to fluctuation due to sampling error across different samples, particularly problematic when data exhibits indeterminacy.

To address these challenges, this study focuses on developing Generalized Neutrosophic ratio product estimators for estimating population means, employing both OLS and Huber M estimation techniques to account for data indeterminacy and contamination by outliers. This approach, pioneered by Florentin Smarandache, provides a valuable tool for estimating parameters in sampling theory, yielding interval estimates to accommodate ambiguous, indeterminate, and uncertain data.

1.1. Neutrosophic Data

Neutrosophic analysis examines neutrality, encompassing the evaluation of truthfulness in viewpoints and the exploration of Neutrosophic sets, probabilities, logic, and statistics. Neutrosophic logic, a versatile tool applicable across various domains, addresses concerns arising from indeterminacy. Real-world data often exhibit inconsistencies, indeterminacies, and incompleteness. While fuzzy sets tackle uncertainties and intuitionistic fuzzy sets handle incomplete data, neither adequately address indeterminate data. To confront this challenge, researchers employ Neutrosophic sets, an extension of fuzzy and intuitionistic fuzzy sets, capable of representing partial and indeterminate data. Neutrosophic sets delineate three membership degrees: truth, indeterminacy, and falsity, accommodating data uncertainty. Neutrosophic statistical methods analyze such data, even when sample size is uncertain [6]. [1] and [6] have underscored the effectiveness of Neutrosophic statistics in uncertain systems. To our knowledge, no prior work has developed a robust ratio estimator for estimating finite populations under Neutrosophic statistics. This paper introduces an original robust ratio estimator tailored for estimating finite populations when dealing with interval, imprecise, uncertain, and indeterminate data. We compare the advantages of our proposed estimator with existing ones in classical statistics, anticipating greater efficiency in terms of Neutrosophic mean square error. [11] outline a methodology flowchart for selecting estimation procedures based on different conditions and circumstances. Given our study's focus on Neutrosophic data with high correlation and contamination by outliers, we employ Neutrosophic robust ratio estimators utilizing robust methods for population parameter estimation. The accompanying graph illustrates the flowchart of our proposed study:



2. Terminology

Let us consider a Neutrosophic random sample of size d_{+j} that is selected from a finite set of D units $(T_1, T_2, \ldots, T_D)$. Consider a_{+j} is the jth sample observation of our Neutrosophic data, which is of the form $a_{+j} \in \{a_{Lj}, a_{Uj}\}$ and similarly for auxiliary variable $b_{+j} \in \{b_{Lj}, b_{Uj}\}$. Since our Neutrosophic data is of interval form with vague values on both extremes, therefore, we have taken an average of these two levels to determine a crisp value between indeterminate values. Let $\bar{a}_{+j} \in \{\bar{a}_{Lj}, \bar{a}_{Aj}, \bar{a}_{Uj}\}$ is our Neutrosophic variable of interest and $\bar{b}_{+j} \in \{\bar{b}_{Lj}, \bar{b}_{Aj}, \bar{b}_{Uj}\}$ is our auxiliary Neutrosophic variable that is correlated with our study variable $\bar{a}_{+j}$, where $\bar{a}_{Aj}$ and $\bar{b}_{Aj}$ are the regulated parts (determinate parts) taken as an average of two extreme vague levels $\{\bar{a}_{Lj}, \bar{a}_{Uj}\}$ and $\{\bar{b}_{Lj}, \bar{b}_{Uj}\}$ respectively. The overall averages of the Neutrosophic collection of data are $\bar{A}$ and $\bar{B}$ and also $C_{a+j} \in \{C_{a+Lj}, C_{a+Aj}, C_{a+Uj}\}$ and $C_{b+j} \in \{C_{b+Lj}, C_{b+Aj}, C_{b+Uj}\}$ respectively, are the

Neutrosophic coefficients of variation for a and b. $\rho_{b+j,a+j}$ is Neutrosophic correlation coefficient between b and a Neutrosophic variables. Additionally, the structured formulas below provide the probability-weighted moment of the Neutrosophic variable, Downton's technique, and Gini's mean difference

$S_{pw}(b_{+j}) = \left(\sqrt{\pi}/D^2\right)\sum_{j=1}^{D}(2j-D-1)b_{+j}$, $D(b_{+j}) = \left(2\sqrt{\pi}/\left(D(D-1)\right)\right)\sum_{j=1}^{D}\left(j-(D+1/2)\right)b_{+j}$ and $G(b_{+j}) = (4/D-1)\sum_{j=1}^{D}\left((2j-D-1)/2D\right)b_{+j}$ respectively.`

3. Proposed Generalized Neutrosophic Ratio Product Estimator using OLS Method

We present a generalized estimator of the Neutrosophic ratio product for Neutrosophic data in this study, which is robust despite the existence of extreme values. The suggested estimator appears as follows:

$$\bar{a}_{S(RP+j)} = \left[\bar{a}_{+j} + k\lambda\left(\bar{b}_{+j} - \bar{V}\right)\right]\left[\frac{\bar{b}_{+j}\delta+\eta}{\bar{B}\delta+\eta}\right]^{\tau\theta} \tag{3.1}$$

where $\tau = \begin{cases} 1 & \textit{for product estimator} \\ -1 & \textit{for ratio estimator} \end{cases}$

δ and η are suitably chosen constants, θ is unknown constant. The population mean of the supplemental variable is assumed to be known in equation (3.1), and OLS (Ordinary Least Square) is used to obtain the value of $\lambda = \frac{S_{(b+j)(a+j)}}{S_{b+j}^2}$. Based on the selection of distinct values for δ, η and θ in equation (3.1), various members of this generalized class are listed in Table 1. Using the suggested generalized robust Neutrosophic ratio product estimator's OLS approach, we let the bias be determined together with the mean square error (MSE).

$$e_0 = \frac{\bar{a}_{+j} - \bar{A}}{\bar{A}}, \quad e_1 = \frac{\bar{b}_{+j} - \bar{B}}{\bar{B}}, \quad E\left(e_0^2\right) = \frac{1-f}{d_{+j}}C_{a+j}^2, \quad E\left(e_1^2\right) = \frac{1-f}{d_{+j}}C_{b+j}^2,$$

$$E\left(e_0e_1\right) = \frac{1-f}{d_{+i}}\rho_{b+j,a+j}C_{a+j}C_{b+j}, \quad f = \frac{d_{+j}}{D} \tag{3.2}$$

On transforming the equation (3.1) while using the equation (3.2), we have

$$\bar{a}_{S(RP+j)} = \left[\bar{A}(1+e_0) + \bar{B}\,\tau\lambda e_1\right]\left[1+\beta_j e_1\right]^{\tau\theta} \tag{3.3}$$

where $\beta_J = \frac{\delta\bar{B}}{\delta\bar{B}+\eta}$. While using the Taylor series expansion up to second order of $\left[1+\beta_j e_1\right]^{\tau\theta}$ for equation (3.3) and upon simplifying, we have

$$\bar{a}_{S(RP+j)} = \bar{A}\left(1+e_0+\tau\lambda Ze_1\right)\left[1+\tau\theta\beta_j e_1 + \frac{\tau\theta(\tau\theta-1)}{2!}\beta_j^2 e_1^2 + \ldots\right] \tag{3.4}$$

Consequently, the estimator's bias is

$$B\left(\bar{a}_{S(RP+j)}\right)=E\left(\bar{a}_{S(RP+j)}-\bar{A}\right)=\frac{1-f}{d_{+j}}\bar{A}$$
$$\left\{\left[\frac{\tau\theta(\tau\theta-1)}{2}\beta_j^2+\tau^2\theta\beta_j\lambda Z\right]C_{b+j}^2+\tau\theta\beta_j\rho_{b+j,a+j}C_{a+j}C_{b+j}\right\} \tag{3.5}$$

Additionally, the mean square error expression is obtained by utilizing the Taylor series approximation of the suggested estimator, which is provided in equation (3.1).

$$MSE\left(\bar{a}_{S(RP+j)}\right)=E\left(\bar{a}_{S(RP+j)}-\bar{A}\right)^2=E\left\{\bar{A}\left[e_0+\tau\left(\theta\beta_j+\lambda Z\right)e_1\right]\right\}^2$$
$$=\frac{1-f}{d_{+j}}\bar{U}\left\{C_{a+j}^2+2\tau\left(\theta\beta_j+\lambda Z\right)\rho_{b+j,a+j}C_{b+j}C_{a+j}+\tau^2\left(\theta\beta_j+\lambda Z\right)^2C_{b+j}^2\right\},\quad Z=\frac{\bar{B}}{\bar{A}}=\frac{1}{R} \tag{3.6}$$

Table 1: Some members in the suggested class under OLS, based on product and ratio estimators

δ	η	λ	θ	Product estimator $\tau=1$	Ratio estimator $\tau=-1$
1	$G(b_{+j})$	λ	$\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)$	$\bar{a}_{S(P1+j)}=\left[\bar{a}_{+j}+\lambda\left(\bar{b}_{+j}-\bar{B}\right)\right]\left[\frac{\bar{b}_{+j}+G(b_{+j})}{\bar{B}+G(b_{+j})}\right]^{\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)}$	$\bar{a}_{S(R1+j)}=\left[\bar{a}_{+j}-\lambda\left(\bar{b}_{+j}-\bar{B}\right)\right]\left[\frac{\bar{B}+G(b_{+j})}{\bar{b}_{+j}+G(b_{+j})}\right]^{-\rho_{v+j,u+j}\left(\frac{C_{u+j}}{C_{v+j}}\right)}$
1	$D(b_{+j})$	λ	$\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)$	$\bar{a}_{S(P2+j)}=\left[\bar{a}_{+j}+\lambda\left(\bar{b}_{+j}-\bar{B}\right)\right]\left[\frac{\bar{b}_{+j}+D(b_{+j})}{\bar{B}+D(b_{+j})}\right]^{\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)}$	$\bar{a}_{S(R2+j)}=\left[\bar{a}_{+j}-\lambda\left(\bar{b}_{+j}-\bar{B}\right)\right]\left[\frac{\bar{B}+D(b_{+j})}{\bar{b}_{+j}+D(b_{+j})}\right]^{-\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)}$
1	$S_{pw}(b_{+j})$	λ	$\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)$	$\bar{a}_{S(P3+j)}=\left[\bar{a}_{+j}+\lambda\left(\bar{b}_{+j}-\bar{B}\right)\right]\left[\frac{\bar{b}_{+j}+S_{pw}(b_{+j})}{\bar{B}+S_{pw}(b_{+J})}\right]^{\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)}$	$\bar{a}_{S(R3+j)}=\left[\bar{a}_{+j}-\lambda\left(\bar{b}_{+j}-\bar{B}\right)\right]\left[\frac{\bar{B}+S_{pw}(b_{+j})}{\bar{b}_{+j}+S_{pw}(b_{+j})}\right]^{-\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)}$

4. Proposed Generalized Robust Neutrosophic Ratio product estimators under Neutrosophic data using Huber M Estimation Technique

It has been seen that real-life data is not always symmetrical but can have extreme values, which can distort the efficacy of the findings while utilizing classical methods, as these methods are not meant for that. Thus, this study focused on two issues. Firstly, we have proposed a generalized Neutrosophic ratio product estimator under Neutrosophic data using the classical method (OLS) and then Using the Huber M estimation technique in place of the OLS method, we have also proposed a generalized robust Neutrosophic ratio product estimator. Even when there are extreme values in the data, this procedure is reliable and will produce good results. The generalized robust Neutrosophic is provided as

$$\bar{a}_{P(RP+j)}=\left[\bar{a}_{+j}+\tau\lambda_{HM}\left(\bar{b}_{+j}-\bar{B}\right)\right]\left[\frac{\bar{b}_{+j}\delta+\eta}{\bar{B}\delta+\eta}\right]^{\tau\theta} \tag{4.1}$$

where λ_{HM} is obtained while using the Huber M estimation technique.

We intend to achieve valid results and valid inferences as the negative effects of outliers are reduced while using the Huber M estimation rather than OLS. The compromise between t^2 and $|t|$ is the function $\rho_{b+j,a+j}(t)$ which is used in Huber M estimator; t is the error term in regression model $a_{+i} = \kappa + \upsilon b_{+j} + t$, κ being the constant of the model. The function $\rho_{b+j,a+j}(t)$ has the form

$$\rho_{b+j,a+j}(t) = \begin{cases} t^2 & -r \le t \le r \\ 2r|t| - r^2 & t < -r \;\; or \;\; r < t \end{cases} \tag{4.2}$$

where r is a tuning constant that regulates the estimator's robustness. Regression coefficient value λ_{HM} is determined by minimizing

$$\sum_{j=1}^{d_{+j}} \rho_{b+j,a+j}\left(a_{+j} - \kappa - \upsilon b_{+j}\right) \tag{4.3}$$

in relation to κ and υ. We utilize (3.2) and convert it into (4.3) to obtain MSE along with the bias of the constructed generalized estimators using Huber M-estimation.

$$\bar{a}_{P(RP+j)} = \left[\bar{A}(1+e_0) + \bar{B}\tau\lambda_{HM} e_1\right]\left[1+\beta_j e_1\right]^{\tau\theta} \tag{4.4}$$

where $\beta_j = \dfrac{\delta\bar{B}}{\delta\bar{B}+\eta}$ and while using the Taylor series expansion up to second order of $\left[1+\beta_j e_1\right]^{\tau\theta}$ for equation (4.4) and upon simplifying, we have

$$\bar{a}_{P(RP+j)} = \bar{A}\left(1+e_0+\tau\lambda_{HM} Z e_1\right)\left[1+\tau\theta\beta_j e_1 + \frac{\tau\theta(\tau\theta-1)}{2!}\beta_j^2 e_1^2 + \dots\right] \tag{4.5}$$

Consequently, the estimator's bias is

$$B\left(\bar{a}_{P(RP+j)}\right) = E\left(\bar{a}_{RP+j} - \bar{A}\right) = \frac{1-f}{d_{+j}}\bar{A} \left\{\left[\frac{\tau\theta(\tau\theta-1)}{2}\beta_j^2 + \tau^2\theta\beta_j\lambda_{HM} Z\right]C_{b+j}^2 + \tau\theta\beta_j\rho_{b+j,a+j}C_{a+j}C_{b+j}\right\} \tag{4.6}$$

Also, upon using the Taylor series approximation of the proposed estimator given in equation (4.1), the mean square error expression is obtained as

$$MSE\left(\bar{a}_{P(RP+j)}\right) = E\left(\bar{a}_{P(RP+j)} - \bar{A}\right)^2 = E\left\{\bar{A}\left[e_0 + \tau\left(\theta\beta_j + \lambda_{HM} Z\right)e_1\right]\right\}^2 \tag{4.7}$$

$$= \frac{1-f}{d_{+j}}\bar{A}\left\{C_{a+j}^2 + 2\tau\left(\theta\beta_j + \lambda_{HM} Z\right)\rho_{b+j,a+j}C_{b+j}C_{a+j} + \tau^2\left(\theta\beta_j + \lambda_{HM} Z\right)^2 C_{b+j}^2\right\}, \quad Z = \frac{\bar{B}}{\bar{A}} = \frac{1}{R} \tag{4.8}$$

Equation (4.1) yields distinct robust Neutrosophic ratios and product estimators when various values of δ, η, θ and τ, such as non-conventional measures of dispersion and coefficient of variation, are substituted. These are actually members of the proposed Generalized class and are listed in Table 2. Since these measures are not sensitive to outliers, they will yield reliable results even in the presence of outliers in the data. In this section, the robust measure (non-parametric) of the regression coefficient and the various non-conventional measures of dispersion result in different estimators.

Table 2: Some members in the proposed class under Huber M, based on product and ratio estimators

δ	η	λ_{HM}	θ	Product estimator $k=1$	Ratio estimator $k=-1$
1	$G(b_{+j})$	λ_{HM}	$\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)$	$\bar{a}_{P(P1+j)}=\left[\bar{a}_{+j}+\lambda_{HM}\left(\bar{b}_{+j}-\bar{B}\right)\right]\left[\frac{\bar{b}_{+j}+G(b_{+j})}{\bar{B}+G(b_{+j})}\right]^{\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)}$	$\bar{a}_{P(R1+j)}=\left[\bar{a}_{+j}-\lambda_{HM}\left(\bar{b}_{+j}-\bar{B}\right)\right]\left[\frac{\bar{B}+G(b_{+j})}{\bar{b}_{+j}+G(b_{+j})}\right]^{-\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)}$
1	$D(b_{+j})$	λ_{HM}	$\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)$	$\bar{a}_{P(P2+j)}=\left[\bar{a}_{+j}+\lambda_{HM}\left(\bar{b}_{+j}-\bar{B}\right)\right]\left[\frac{\bar{b}_{+j}+D(b_{+j})}{\bar{B}+D(b_{+j})}\right]^{\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)}$	$\bar{a}_{P(R2+j)}=\left[\bar{a}_{+j}-\lambda_{HM}\left(\bar{b}_{+j}-\bar{B}\right)\right]\left[\frac{\bar{B}+D(b_{+j})}{\bar{b}_{+j}+D(b_{+j})}\right]^{-\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)}$
1	$S_{pw}(b_{+j})$	λ_{HM}	$\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)$	$\bar{a}_{P(P3+j)}=\left[\bar{a}_{+j}+\lambda_{HM}\left(\bar{b}_{+j}-\bar{B}\right)\right]\left[\frac{\bar{b}_{+j}+S_{pw}(b_{+j})}{\bar{B}+S_{pw}(b_{+J})}\right]^{\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)}$	$\bar{a}_{P(R3+j)}=\left[\bar{a}_{+j}-\lambda_{HM}\left(\bar{b}_{+j}-\bar{B}\right)\right]\left[\frac{\bar{B}+S_{pw}(b_{+j})}{\bar{b}_{+j}+S_{pw}(b_{+j})}\right]^{-\rho_{b+j,a+j}\left(\frac{C_{a+j}}{C_{b+j}}\right)}$

5. Efficiency Comparison

In this section, we will provide the theoretical efficiency comparison between the suggested generalized class while using the OLS given in equation (3.1) and the generalized class while adopting the robust measure Huber M estimation mentioned in equation (4.1), while utilizing the same auxiliary information in both cases. So, we will theoretically show how Huber M will provide more proficient results in the presence of extreme values in the data than the traditional method of OLS, as this is sensitive to outliers. So, for $\bar{a}_{P(RP+j)}$ to be proficient than $\bar{a}_{S(RP+j)}$, we have

$$\Rightarrow 2\tau\left(\theta\beta_j+\lambda_{HM}Z\right)\rho_{b+j,a+j}C_{b+j}C_{a+j}+\tau^2\left(\theta\beta_j+\lambda_{HM}Z\right)^2C_{b+j}^2< \\ 2\tau\left(\theta\beta_j+\lambda Z\right)\rho_{b+j,a+j}C_{b+j}C_{a+j}+\tau^2\left(\theta\beta_j+\lambda Z\right)^2C_{b+j}^2 \quad (5.1)$$

$$\Rightarrow 2\rho_{b+j,a+j}C_{b+j}C_{a+j}\tau\left[\theta\beta_j+\lambda_{HM}Z-\theta\beta_j-\lambda Z\right]+C_{b+j}^2\tau^2\left[\left(\theta\beta_j+\lambda_{HM}Z\right)^2-\left(\theta\beta_j+\lambda Z\right)^2\right]<0 \quad (5.2)$$

$$\Rightarrow 2\rho_{b+j,a+j}C_{b+j}C_{a+j}Z\tau\left[\lambda_{HM}-\lambda\right]+C_{b+j}^2\tau^2\left[\left(\theta\beta_j+\lambda_{HM}Z\right)-\left(\theta\beta_j+\lambda Z\right)\right]\left[\theta\beta_j+\lambda_{HM}Z+\theta\beta_j+\lambda Z\right]<0$$
(5.3)

$$\Rightarrow 2W\theta\left[\lambda_{HM}-\lambda\right]+\tau\left[\left(\lambda_{HM}-\lambda\right)Z\right]\left[2\theta\beta_j+Z\left(\lambda_{HM}+\lambda\right)\right]<0 \quad (5.4)$$

$$\Rightarrow W\left[\lambda_{HM}-\lambda\right]2\theta+\tau\left(2\theta\beta_j+Z\left(\lambda_{HM}+\lambda\right)\right)<0 \quad (5.5)$$

$$\Rightarrow W\left[\lambda_{HM}-\lambda\right]2\theta\left(1+\tau\beta_j\right)+Z\tau\left(\lambda_{HM}+\lambda\right)<0 \quad (5.6)$$

$$\Rightarrow \left[\lambda_{HM}-\lambda\right]\left[2\theta\left(1+\tau\beta_j\right)+Z\tau\left(\lambda_{HM}+\lambda\right)\right]<0 \quad (5.7)$$

Since, $Z > 0$, either $\lambda_{HM} - \lambda < 0$ and $2\theta(1+\tau\beta_j) + Z\tau(\lambda_{HM} + \lambda) < 0$. This implies that

$$\Rightarrow \lambda_{HM} < \lambda \text{ and } 2\theta(1+\tau\beta_j) > -Z\tau(\lambda_{HM} + \lambda) < 0 \tag{5.8}$$

Or $\lambda_{HM} > \lambda$ and $2\theta(1+\tau\beta_j) < -Z\tau(\lambda_{HM} + \lambda) < 0$ (5.9)

From the above theoretical comparison, the generalized class in which the robust measure Huber M is adopted would be more proficient than the generalized class in which Classical OLS is adopted when the conditions given in Equation (5.8) or (5.9) are satisfied.

6. Numerical Study

Given its novelty, and to our knowledge, the concept of Neutrosophic ratio type estimators remains unexplored. We conducted a comparison of the mean square errors between the proposed generalized class employing OLS and the same class employing Huber M estimation techniques, using identical auxiliary information in both scenarios. For our numerical demonstration, we utilized real-world interval data on temperature, reflecting its Neutrosophic nature due to the vagueness in daily temperature readings [2]. This data spans six years and was sourced from publicly available weather websites, specifically focusing on the temperature in Lahore, Punjab, Pakistan, from 2014 to 2019 (see Table 3). Ethical approval was unnecessary as the data was openly accessible online.

As outlined in subsection 1.1, we employed Neutrosophic data and computed the central value, serving as the determinate average representative of the time period spanning 1 to 6 years. Neutrosophic averages for each month over the six-year period were calculated using the lower and upper limits of temperature, forming the Neutrosophic component of the data for each respective year. The lowest and upper limits of the temperature for each month over a period of six years were measured as the neutrosophic averages. These represent the neutrosophic portion of the data for 'a' corresponding to known 'b' year, with the month-wise total averages throughout the full six years being considered as neutrosophic data $(Temp,\ a_{+j} \in \{a_{Lj}, a_{Uj}\})$ corresponding to time (in years 'b') as the independent determinate variable and the central value 'a_{Aj}' mentioned in section 2 is part of the data for which indeterminacy is considered to be zero. $\overline{B}$ represents the average of the data collected over six (6) years and is actually the same for all lower, central and upper limits of the corresponding Neutrosophic data. C_{b+j}, $G(b_{+j})$, $D(b_{+j})$ and $S_{pw}(b_{+j})$ are the coefficient of variation, Gini's mean difference, Downton's method and probability weighted moment of the auxiliary variable respectively. C_{b+j} is the coefficient of variation of study variable.

Table 4 presents the mean square error and bias of the proposed ratio estimators employing the OLS method, while Table 5 presents the corresponding metrics for the proposed product estimators. When adopting the Huber M estimation technique, Tables 6 and 7 detail the performance of both proposed ratio and product estimators respectively. Finally, we calculated the relative efficiency of the proposed ratio estimators using the Huber M method compared to those using the OLS method, detailed in Table 8. Similarly, Table 9 displays the relative efficiency of the proposed product estimators using the Huber M method compared to those using the OLS method. For comprehensive details, refer to the tables in the Appendix.

Table 3: Population's Characteristics for Single Auxiliary Variable

Parameters	Source (Data): Temperature of Lahore, Punjab, Pakistan from the year 2014 to 2019								
	Population available: D = 30 (years); sample taken: d= 6 (years)								
	No. of Year	Average temperature (Max, Average, Min)							
$\bar{B}$	$\bar{A}_{+j}$	C_{a+j}	C_{b+j}	$\rho_{b+j,a+j}$	λ	λ_{HM}	$G(b_{+j})$	$D(b_{+j})$	$S_{pw}(b_{+j})$
(3.5, 3.5, 3.5)	(64,54,44)	(0.0360,0.0205,0.0345)	(0.53, 0.53, 0.53)	(0.40,0.31,0.23)	(0.497,0.185,0.188)	(0.416,0.106,0.113)	(2.356, 2.0506, 1.956)	(1.890, 1.823, 1.709)	(2.567, 2.256, 2.150)
	(72,61,50)	(0.0529,0..0440,0.0424)		(0.18,0.09,0.14)	(0.370,0.130,0.160)	(0.260,0.099,0.107)			
	(80,69,58)	(0.0454,0.0448,0.0424)		(0.43,0.28,0.17)	(0.842,0.467,0.225)	(0.665,0.298,0.156)			
	(94,82,69)	(0.0291,0.0274,0.0293)		(0.60,0.60,0.55)	(0.885,0.727,0.599)	(0.678,0.547,0.357)			
	(102,90,77)	(0.0187,0.0205,0.0282)		(-0.04,0.08,0.04)	(-0.041,0.080,0.047)	(-0.021,0.050,0.029)			
	(103,92,81)	(0.0316,0.0264,0.0272)		(-0.23,-0.24,-0.08)	(-0.404,-0.314,-0.095)	(-0.234,-0.157,-0.047)			
	(95,87,80)	(0.0171,0.0148,0.0151)		(-0.60,-0.56,-0.49)	(-0.525,-0.389,-0.319)	(-0.313,-0.198,-0.167)			
	(95,87,80)	(0.0141,0.0169,0.0108)		(-0.59,-0.59,-0.53)	(-0.426,-0.468,-0.247)	(-0.219,-0.202,-0.116)			
	(94,85,77)	(0.0231,0.0216,0.0254)		(0.09,0.05,0.01)	(0.105,0.049,0.011)	(0.068,0.023,0.007)			
	(90,79,68)	(0.0277,0.0312,0.0338)		(-0.20,-0.38,-0.39)	(-0.269,-0.519,-0.483)	(-0.158,-0.301,-0.250)			
	(79,66,55)	(0.0233,0.0187,0.0260)		(-0.55,-0.20,0.42)	(-0.546,-0.133,0.324)	(-0.298,-0.079,0.178)			
	(69,56,45)	(0.050,0.0508,0.0341)		(-0.05,-0.21,-0.14)	(-0.093,-0.322,-0.116)	(-0.046,-0.168,-0.069)			

7. Simulation Study

To validate the theoretical efficiency criteria and assess the performance of the proposed Neutrosophic estimator against alternative methods, we simulated a Neutrosophic dataset using parameters outlined in the research conducted by [19]. Our simulation involved generating Neutrosophic data, assuming that the primary and auxiliary random variables followed Neutrosophic normal distributions. Thus $A \sim NN(\mu_a, \sigma_a^2)$; $A \in (A_{Lj}, A_{Uj})$, $\mu_a \in (\mu_{aLj}, \mu_{aUj})$, $\sigma_a^2 \in (\sigma_{a\,Lj}^2, \sigma_{a\,Uj}^2)$ and $B \sim NN(\mu_b, \sigma_b^2)$; $B \in (B_{Lj}, B_{Uj})$, $\mu_b \in (\mu_{bLj}, \mu_{bUj})$, $\sigma_b^2 \in (\sigma_{b\,Lj}^2, \sigma_{b\,Uj}^2)$. For the numerical illustration, we have taken $A \sim NN([76.0, 84.9], [(12.9)^2, (17.2)^2])$, where $\mu_a \in (76.0, 84.9)$, $\sigma_a \in (12.9, 17.2)$ and $B \sim NN([171.2, 1840.4], [(5.8)^2, (6.7)^2])$, where $\mu_b \in (171.2, 180.4)$, $\sigma_b \in (5.8, 6.7)$ and generated 1000 normal random observation for both the variables. The descriptive statistics for the simulated data is presented below in Table 10

Table 10: Descriptive Statistics of the simulated data for the Neutrosophic data

Parameters	Neutrosophic Value	Parameters	Neutrosophic Value
D	[1000, 1000]	C_b	[0.0332, 0.0369]
d	[20, 20]	$\beta_{1(b)}$	[0.0020, 0.0051]
μ_a	[76.20, 85.63]	$\beta_{2(b)}$	[3.0227, 2.9539]
μ_b	[171.08, 180.34]	$Q_{1(b)}$	[167.3941, 176.1144]
σ_a	[12.79, 17.37]	$M_{d(b)}$	[170.9067, 180.3451]
σ_b	[5.67, 6.65]	$Q_{3(b)}$	[174.9269, 184.7586]
C_a	[0.1679, 0.2028]	ρ_{ba}	[0.01933, 0.00703]
λ	[3.4967, 4.67908]	λ_{Huber}	[1.69875, 2.45670]
$G_{(b)}$	[155.4467, 161.4538]	$D_{(b)}$	[140.89167, 146.7890]
$S_{pw(b)}$	[163.78954, 168.4567]		

Table 11 given below represents the Neutrosophic MSEs of different competing along with the suggested estimators of population mean.

Table 11: Neutrosophic MSEs of different competing and suggested estimators

S. No.	Estimators	MSEs	S. No.	Estimators	MSEs
1	t_0	[8.019213, 14.77799]	17	$t_{14D}(\kappa = 1, \upsilon = 1)$	[17.42754, 28.00579]
2	t_{RD}	[17.39673, 27.98680]	18	t_{15D}	[8.016216, 14.77726]
3	t_{1D}	[17.39673, 27.98681]	19	t_{pD}	[7.864525, 13.82184]

4	t_{2D}	[8.066852, 14.8812]	20	$\bar{a}_{S(R1+j)}$ *OLS*	[6.87546, 11.1345]
5	t_{3D}	[17.39709, 27.98701]	21	$\bar{a}_{S(R2+j)}$ *OLS*	[6.01783, 10.67835]
6	t_{4D}	[17.39674, 27.98681]	22	$\bar{a}_{S(R3+j)}$ *OLS*	[7.016890, 12.98756]
7	t_{5D}	[17.39674, 27.98681]	23	$\bar{u}_{S(R1+j)}$ *HUBER*	[5.018956, 10.6783]
8	t_{6D}	[17.3978, 27.98741]	24	$\bar{a}_{S(R2+j)}$ *HUBER*	[4.76890, 8.89051]
9	t_{7D}	[17.42703, 28.00546]	25	$\bar{a}_{S(R3+j)}$ *HUBER*	[5.679501, 11.01247]
10	t_{8D}	[17.4277, 28.00592]	26	$\bar{a}_{S(P1+j)}$ *OLS*	[6.91017, 11.67897]
11	t_{9D}	[17.39673, 27.98681]	27	$\bar{a}_{S(P2+j)}$ *OLS*	[5.89912, 11.02345]
s12	t_{10D}	[17.4517, 28.01563]	28	$\bar{a}_{S(P3+j)}$ *OLS*	[7.023745, 12.98751]
13	t_{11D}	[17.42734, 28.00569]	29	$\bar{a}_{S(P1+j)}$ *HUBER*	[5.55789, 10.87654]
14	t_{12D}	[17.45602, 28.02323]	30	$\bar{a}_{S(P2+j)}$ *HUBER*	[5.014326, 10.09872]
15	t_{13D}	[17.40314, 27.99058]	31	$\bar{a}_{S(P3+j)}$ *HUBER*	[6.90178, 11.67845]
16	$t_{14D}(\kappa=1,\upsilon=0)$	[17.42736, 28.00569]			

8. Discussion

In this study we focus on developing the Generalized Neutrosophic ratio product estimators for estimating the population mean by using both OLS and Huber M estimation techniques when data is both indeterminate and contaminated with outliers. And upon substituting the various values of δ ,η ,θ , and τ ,we get different ratio and product estimators. The bias and mean square error formulae are obtained using OLS and Huber M estimation methods and finally compared. From Table 4, Table 6 and Table 11, we come to conclude that Neutrosophic ratio estimators using Huber M estimation perform much better than Neutrosophic ratio estimators using the OLS method. Also, from Table 5, Table 7 and Table 11, we came to conclude that Neutrosophic product estimators using Huber M estimation perform much better than Neutrosophic product estimators using the OLS method. Also from the results of Table 11, which were generated by using the simulation study and the estimators mentioned were suggested by various researchers for reference see [13]. From the given table, we conclude that the estimators suggested by [13] out class the other estimators. But however upon substituting the various values of δ ,η ,θ , and τ ,we get different ratio and product estimators using both OLS and Huber M estimation methods and also actually incorporating the value of η as Non-Conventional Measures of Dispersion such as Gini's mean difference, Downton's method and probability weighted moment as auxiliary information which actually perform much better in skewed population under study and finally outclass all the existing estimators in terms of efficiency.

9. Conclusion

From both numerical evaluation and simulation study results of existing and suggested estimators mentioned in various tables we came to conclude that either using OLS or Huber, our suggested estimators out class the previous estimators mentioned in this study on the basis of the efficiency in terms of their mean square error. However, both suggested ratio and product estimators upon using the Huber M method outclass the suggested ratio or product estimator while using the OLS method. Hence, we strongly recommend that our proposed Generalized Neutrosophic ratio product estimator perform better in the situation mentioned in this study. We can also generate other different Neutrosophic ratio estimators or Neutrosophic product estimators for achieving greater efficiency by using other different robust regression techniques while estimating the parameters of the population through survey sampling

Abbreviations

(Ordinary Least Square), MSE (mean square error)

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Appendix

Table 4: Mean square error and Bias of Proposed Ratio Estimators using OLS Method

$d=6$	$M\left(\bar{a}_{S(R1+j)}\right)$	$B\left(\bar{a}_{S(R1+i)}\right)$	$M\left(\bar{a}_{S(R2+i)}\right)$	$B\left(\bar{a}_{S(R2+j)}\right)$	$M\left(\bar{a}_{S(R3+j)}\right)$	$B\left(\bar{a}_{S(R3+j)}\right)$
Jan	0.0097	0.0116	0.0098	0.0138	0.0109	0.0096
	0.0028	0.0043	0.0028	0.0050	0.0040	0.0028
	0.0066	0.0044	0.0066	0.0051	0.0041	0.0066
Feb	0.0256	0.0086	0.0257	0.0101	0.0080	0.0256
	0.0153	0.0030	0.0153	0.0035	0.0028	0.0153
	0.0115	0.0037	0.0116	0.0044	0.0034	0.0115
Mar	0.0189	0.0199	0.0191	0.0235	0.0186	0.0188
	0.0171	0.0109	0.0172	0.0129	0.0102	0.0171
	0.0133	0.0052	0.0133	0.0062	0.0049	0.0133
April	0.0080	0.0209	0.0082	0.0246	0.0194	0.0079
	0.0062	0.0171	0.0063	0.0202	0.0159	0.0061
	0.0062	0.0141	0.0064	0.0166	0.0131	0.0061
May	0.0046	-0.0009	0.0046	-0.0011	-0.0009	0.0046
	0.0049	0.0018	0.0049	0.0022	0.0017	0.0049
	0.0080	0.0011	0.0080	0.0013	0.0010	0.0080
June	0.0129	-0.0091	0.0130	-0.0107	-0.0085	0.0129
	0.0080	-0.0071	0.0081	-0.0084	-0.0066	0.0080
	0.0078	-0.0022	0.0078	-0.0025	-0.002	0.0078

July	0.0028	-0.0118	0.0029	-0.0139	-0.0110	0.0027
	0.0020	-0.0087	0.0020	-0.0103	-0.0081	0.0020
	0.0020	-0.0072	0.0020	-0.0085	-0.0067	0.0020
Aug	0.0019	-0.0096	0.0020	-0.0113	-0.0089	0.0019
	0.0025	-0.0105	0.0026	-0.0124	-0.0098	0.0025
	0.0010	-0.0056	0.0010	-0.0066	-0.0052	0.0010
Sep	0.0065	0.0024	0.0065	0.0028	0.0022	0.0065
	0.0051	0.0011	0.0051	0.0013	0.0011	0.0051
	0.0065	0.0002	0.0065	0.0003	0.0002	0.0065
Oct	0.0087	-0.0061	0.0088	-0.0072	-0.0057	0.0087
	0.0091	-0.0112	0.0092	-0.0133	-0.0105	0.0091
	0.0091	-0.0108	0.0092	-0.0127	-0.0100	0.0091
Nov	0.0045	-0.0122	0.0046	-0.0143	-0.0113	0.0045
	0.0029	-0.0030	0.0029	-0.0036	-0.0028	0.0029
	0.0043	0.0075	0.0043	0.0089	0.0070	0.0043
Dec	0.0224	-0.0021	0.0224	-0.0025	-0.0020	0.0224
	0.0183	-0.0072	0.0183	-0.0085	-0.0067	0.0182
	0.0067	-0.0026	0.0067	-0.0031	-0.0024	0.0067

Table 5: Mean square error and Bias of Proposed Product Estimators using OLS Method

$d = 6$	$M\left(\bar{a}_{S(R1+j)}\right)$	$B\left(\bar{a}_{S(R1+j)}\right)$	$M\left(\bar{a}_{S(R2+j)}\right)$	$B\left(\bar{a}_{S(R2+j)}\right)$	$M\left(\bar{a}_{S(R3+j)}\right)$	$B\left(\bar{a}_{S(R3+j)}\right)$
Jan	0.0207	-0.009	0.0212	-0.0108	0.0205	-0.0083
	0.0046	-0.0038	0.0047	-0.0046	0.0045	-0.0036
	0.0089	-0.0038	0.0090	-0.0045	0.0088	-0.0035
Feb	0.0311	-0.0073	0.0313	-0.0087	0.0310	-0.0067
	0.0161	-0.0028	0.0161	-0.0033	0.0161	-0.0026
	0.0130	-0.0033	0.0131	-0.004	0.0130	-0.0031
Mar	0.0442	-0.0138	0.0453	-0.0167	0.0438	-0.0127
	0.0261	-0.0087	0.0265	-0.0104	0.0260	-0.0081
	0.0158	-0.0046	0.0159	-0.0055	0.0158	-0.0043
April	0.0318	-0.0151	0.0328	-0.0182	0.0314	-0.0139
	0.0246	-0.0126	0.0254	-0.0152	0.0243	-0.0117
	0.0211	-0.0105	0.0217	-0.0126	0.0208	-0.0097
May	0.0047	0.0009	0.0047	0.0011	0.0047	0.0009
	0.0051	-0.0018	0.0051	-0.0021	0.0051	-0.0017
	0.0080	-0.0011	0.0080	-0.0012	0.0080	-0.0010
June	0.0174	0.0102	0.0176	0.0119	0.0174	0.0095
	0.0111	0.0078	0.0112	0.0092	0.0110	0.0073
	0.0081	0.0022	0.0081	0.0026	0.0081	0.0021
July	0.0111	0.0138	0.0114	0.0161	0.0109	0.0129
	0.0069	0.0099	0.0072	0.0116	0.0069	0.0093

	0.0056	0.0081	0.0058	0.0095	0.0056	0.0075
Aug	0.0074	0.0109	0.0076	0.0128	0.0073	0.0102
	0.0097	0.0122	0.0100	0.0143	0.0096	0.0114
	0.0032	0.0061	0.0033	0.0072	0.0031	0.0057
Sep	0.0068	-0.0023	0.0068	-0.0028	0.0068	-0.0022
	0.0052	-0.0011	0.0052	-0.0013	0.0052	-0.0010
	0.0065	-0.0002	0.0065	-0.0003	0.0065	-0.0002
Oct	0.0110	0.0066	0.0111	0.0078	0.0110	0.0062
	0.0185	0.0135	0.0189	0.0158	0.0183	0.0127
	0.0189	0.0131	0.0193	0.0153	0.0188	0.0123
Nov	0.0153	0.0148	0.0157	0.0173	0.0151	0.0138
	0.0037	0.0032	0.0037	0.0038	0.0037	0.0030
	0.0097	-0.0062	0.0100	-0.0074	0.0096	-0.0058
Dec	0.0227	0.0022	0.0228	0.0026	0.0227	0.0021
	0.0235	0.0085	0.0238	0.0099	0.0235	0.0079
	0.0076	0.0028	0.0076	0.0033	0.0076	0.0026

Table 6: Mean square error and Bias of Proposed Ratio Estimators using Huber M Estimation Method

$d = 6$	$M\left(\bar{a}_{S(R1+j)}\right)$	$B\left(\bar{a}_{S(R1+j)}\right)$	$M\left(\bar{a}_{S(R2+j)}\right)$	$B\left(\bar{a}_{S(R2+j)}\right)$	$M\left(\bar{a}_{S(R3+j)}\right)$	$B\left(\bar{a}_{S(R3+j)}\right)$
Jan	0.0094	0.0115	0.0136	0.0095	0.0094	0.0107
	0.0027	0.0042	0.0050	0.0027	0.0027	0.0039
	0.0065	0.0043	0.0051	0.0065	0.0065	0.0040
Feb	0.0254	0.0084	0.0100	0.0255	0.0254	0.0079
	0.0152	0.0030	0.0035	0.0152	0.0152	0.0028
	0.0115	0.0036	0.0043	0.0115	0.0115	0.0034
Mar	0.0181	0.0194	0.0230	0.0182	0.0180	0.0181
	0.0167	0.0106	0.0125	0.0167	0.0167	0.0099
	0.0132	0.0051	0.0061	0.0132	0.0132	0.0048
April	0.0071	0.0203	0.0241	0.0073	0.0071	0.0189
	0.0055	0.0167	0.0197	0.0056	0.0054	0.0155
	0.0055	0.0135	0.0160	0.0055	0.0054	0.0126
May	0.0046	-0.0009	-0.0011	0.0046	0.0046	-0.0009
	0.0049	0.0018	0.0021	0.0049	0.0049	0.0017
	0.0079	0.0011	0.0013	0.0079	0.0079	0.0010
June	0.0127	-0.0093	-0.0109	0.0127	0.0127	-0.0086
	0.0079	-0.0072	-0.0085	0.0079	0.0079	-0.0067
	0.0077	-0.0022	-0.0026	0.0077	0.0077	-0.0020
July	0.0024	-0.0121	-0.0142	0.0024	0.0023	-0.0113
	0.0017	-0.0090	-0.0106	0.0017	0.0017	-0.0084
	0.0018	-0.0073	-0.0087	0.0018	0.0018	-0.0068

Aug	0.0016	-0.0098	-0.0116	0.0016	0.0016	-0.0092
	0.0021	-0.0109	-0.0128	0.0021	0.0021	-0.0101
	0.0009	-0.0057	-0.0067	0.0009	0.0009	-0.0053
Sep	0.0065	0.0024	0.0028	0.0065	0.0065	0.0022
	0.0051	0.0011	0.0013	0.0051	0.0051	0.0011
	0.0065	0.0002	0.0003	0.0065	0.0065	0.0002
Oct	0.0086	-0.0062	-0.0073	0.0086	0.0086	-0.0057
	0.0086	-0.0116	-0.0137	0.0086	0.0086	-0.0108
	0.0086	-0.0112	-0.0132	0.0086	0.0086	-0.0104
Nov	0.0039	-0.0126	-0.0148	0.0040	0.0039	-0.0118
	0.0029	-0.0030	-0.0036	0.0029	0.0029	-0.0028
	0.0040	0.0073	0.0087	0.0040	0.0040	0.0068
Dec	0.0224	-0.0021	-0.0025	0.0224	0.0224	-0.0020
	0.0180	-0.0074	-0.0088	0.0180	0.0180	-0.0069
	0.0067	-0.0027	-0.0031	0.0067	0.0067	-0.0025

Table 7: Mean square error and Bias of Proposed Product Estimators using Huber M Estimation Method

$d=6$	$M\left(\bar{a}_{S(R1+j)}\right)$	$B\left(\bar{a}_{S(R1+j)}\right)$	$M\left(\bar{a}_{S(R2+j)}\right)$	$B\left(\bar{a}_{S(R2+ij)}\right)$	$M\left(\bar{a}_{S(R3+j)}\right)$	$B\left(\bar{a}_{S(R3+j)}\right)$
Jan	0.0193	-0.0091	0.0197	-0.0110	0.0191	-0.0085
	0.0040	-0.0039	0.0041	-0.0046	0.0040	-0.0036
	0.0082	-0.0039	0.0083	-0.0046	0.0082	-0.0036
Feb	0.0298	-0.0074	0.0300	-0.0088	0.0298	-0.0069
	0.0159	-0.0028	0.0160	-0.0033	0.0159	-0.0026
	0.0126	-0.0034	0.0127	-0.0040	0.0126	-0.0031
Mar	0.0401	-0.0143	0.0411	-0.0172	0.0397	-0.0132
	0.0237	-0.0090	0.024	-0.0108	0.0235	-0.0084
	0.0152	-0.0047	0.0153	-0.0056	0.0152	-0.0043
April	0.0274	-0.0156	0.0284	-0.0188	0.0271	-0.0144
	0.0210	-0.0131	0.0217	-0.0157	0.0208	-0.0121
	0.0166	-0.0110	0.0171	-0.0132	0.0164	-0.0102
May	0.0047	0.0009	0.0047	0.0011	0.0047	0.0009
	0.0050	-0.0018	0.0050	-0.0021	0.0050	-0.0017
	0.0080	-0.0011	0.0080	-0.0012	0.0080	-0.0010
June	0.0160	0.0100	0.0162	0.0117	0.0160	0.0093
	0.0100	0.0077	0.0101	0.0090	0.0099	0.0072
	0.0080	0.0022	0.0080	0.0026	0.0080	0.0021
July	0.0086	0.0135	0.0089	0.0158	0.0084	0.0126
	0.0051	0.0097	0.0053	0.0114	0.0051	0.0091
	0.0044	0.0079	0.0045	0.0093	0.0043	0.0074
Aug	0.0054	0.0107	0.0056	0.0125	0.0053	0.0099
	0.0067	0.0118	0.0070	0.0139	0.0066	0.0111
	0.0023	0.0060	0.0024	0.0070	0.0023	0.0056
Sep	0.0067	-0.0023	0.0067	-0.0028	0.0067	-0.0022
	0.0052	-0.0011	0.0052	-0.0013	0.0052	-0.0010

	0.0065	-0.0002	0.0065	-0.0003	0.0065	-0.0002
Oct	0.0103	0.0065	0.0104	0.0077	0.0103	0.0061
	0.0155	0.0132	0.0158	0.0154	0.0154	0.0123
	0.0154	0.0127	0.0158	0.0149	0.0153	0.0119
Nov	0.0116	0.0143	0.0120	0.0168	0.0115	0.0134
	0.0035	0.0032	0.0035	0.0037	0.0034	0.0030
	0.0079	-0.0064	0.0081	-0.0077	0.0078	-0.0060
Dec	0.0226	0.0022	0.0226	0.0026	0.0226	0.0020
	0.0217	0.0083	0.0219	0.0097	0.0216	0.0077
	0.0073	0.0028	0.0073	0.0033	0.0073	0.0026

Table 8: Relative efficiencies of proposed ratio estimators using OLS method with Proposed Ratio estimators using Huber M method

$d=6$	$M\left(\bar{a}_{S(R1+j)}\right)/M\left(\bar{a}_{P(R1+j)}\right)$	$M\left(\bar{a}_{S(R2+j)}\right)/M\left(\bar{a}_{P(R2+j)}\right)$	$M\left(\bar{a}_{S(R3+j)}\right)/M\left(\bar{a}_{P(R3+j)}\right)$
Jan	103.096	103.377	102.982
	103.478	103.937	103.292
	101.765	101.992	101.673
Feb	100.894	100.996	100.853
	100.185	100.205	100.177
	100.571	100.639	100.544
Mar	104.544	104.974	104.367
	102.560	102.871	102.435
	100.805	100.898	100.768
April	111.781	112.778	111.359
	112.360	113.432	111.907
	113.615	115.249	112.941
May	100.055	100.063	100.052
	100.201	100.226	100.191
	100.050	100.057	100.048
June	101.823	102.064	101.726
	102.119	102.432	101.992
	100.224	100.258	100.211
July	117.564	119.616	116.710
	115.755	117.951	114.854
	110.763	112.253	110.155
Aug	118.280	120.787	117.247
	119.068	122.137	117.813
	113.778	115.857	112.930
Sep	100.240	100.270	100.228
	100.086	100.099	100.080
	100.003	100.004	100.003
Oct	101.347	101.523	101.276
	105.933	106.656	105.639
	106.146	107.020	105.792

Nov	114.490	116.385	113.710
	101.331	101.504	101.261
	107.170	108.132	106.778
Dec	100.087	100.100	100.082
	101.580	101.807	101.489
	100.643	100.726	100.609

Table 9: Relative efficiencies of proposed product estimators using OLS method with proposed estimators using Huber M method

$d=6$	$M\left(\bar{a}_{S(R1+j)}\right)/M\left(\bar{a}_{P(R1+j)}\right)$	$M\left(\bar{a}_{S(R2+j)}\right)/M\left(\bar{a}_{P(R2+j)}\right)$	$M\left(\bar{a}_{S(R3+j)}\right)/M\left(\bar{a}_{P(R3+j)}\right)$
Jan	107.340	107.324	107.346
	114.423	114.503	114.389
	108.404	108.499	108.364
Feb	104.148	104.207	104.124
	100.921	100.939	100.914
	102.921	102.970	102.901
Mar	110.368	110.328	110.383
	110.451	110.527	110.419
	103.851	103.908	103.827
April	115.760	115.566	115.838
	116.785	116.578	116.869
	127.176	126.896	127.289
May	100.364	100.372	100.361
	101.137	101.159	101.127
	100.295	100.301	100.292
June	108.885	108.987	108.843
	111.297	111.425	111.244
	101.484	101.515	101.472
July	129.334	128.959	129.485
	134.882	134.501	135.034
	129.328	129.127	129.406
Aug	136.135	135.683	136.317
	143.932	143.376	144.156
	136.226	135.893	136.358
Sep	101.333	101.359	101.322
	100.606	100.619	100.601
	100.019	100.019	100.018
Oct	106.865	106.958	106.827
	119.378	119.379	119.374
	122.613	122.611	122.610
Nov	131.222	130.899	131.352
	106.747	106.838	106.709
	123.060	123.005	123.079
Dec	100.588	100.600	100.583
	108.620	108.734	108.572

	103.547	103.609	103.522

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Declaration of interests

☐ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☒ The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Mohammad A. Alqudah reports a relationship with German Jordanian University, Amman, 11180 Jordan that includes: employment. If there are other authors, they declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.