OPSEARCH
https://doi.org/10.1007/s12597-022-00584-2

THEORETICAL ARTICLE

A fuzzy programming approach to neutrosophic complex nonlinear programming problem of real functions in complex variables via lexicographic order

Hamiden Abd El-Wahed Khalifa[1,2] · Pavan Kumar[3]

Accepted: 30 April 2022

Abstract

In this article, a complex nonlinear programming problem with objective function coefficients characterized by neutrosophic numbers and fuzzy inequalities constraints is considered. Using the score function definition, the model is converted into the corresponding crisp model with fuzzy inequalities, which can be further partitioned into two real sub-models based on the Lexicographic order. A fuzzy programming approach is applied to each sub-problem by introducing the membership functions. Linear membership function is used to obtain optimal compromise solution. A numerical experimentation is performed for the sake of the suggested approach for illustration.

Keywords Nonlinear programming · Fuzzy set · Neutrosophic number · Kuhn–Tucker's optimality conditions · Lexicographic order · Optimal compromise solution

Abbreviations

NLP
Nonlinear programming
CNLP
Complex nonlinear programming

✉ Pavan Kumar
pavankmaths@gmail.com

Hamiden Abd El-Wahed Khalifa
hamiden@cu.edu.eg; Ha.Ahmed@qu.edu.sa

1 Department of Operations Research, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt

2 Present Address: Department of Mathematics, College of Science and Arts, Qassim University, Al-Badaya, Qassim, Saudi Arabia

3 Division of Mathematics, SASL, VIT Bhopal University, Sehore, MP 466114, India

Published online: 22 June 2022

NCNLP
Neutrosophic complex nonlinear programming
SVTN
Single-valued trapezoidal neutrosophic

1 Introduction

Applications of complex programming may be found in Mathematics, engineering, and in many other areas. In earlier works in the field of complex programming problem, all the researchers have been considered only the real part of the objective function of the problem as the objective function of the problem neglecting the imaginary part of the objective function, and the constraints of the problem have been considered as a cone in space $\mathbb{C}^n$. While, in many applications of the real world problem, especially in Mathematics, Physics, and Engineering, the imaginary part plays an important and vital role. Mathematical programming in complex space originated with Levinson [13], where he has generalized Farkas' theorem to the complex space, and has gave the duality theorems for a particular case of the complex linear optimization problem. Ferrero [7] considered the finite dimensional spaces for using the separation arguments and the one-to-one correspondence between $\mathbb{C}^n$, and $\mathbb{R}^{2n}$. In addition, the optimality conditions have established in both the real as well as imaginary parts. In 1967, Mond and Hanson have generalized Walfe's duality from the optimization in real space to the optimization in complex space. They have proved duality theorems for a particular case of a quadratic optimization problem in complex space. Ben-Israel [4] introduced two theorems with proofs for equalities and inequalities of finite dimensional real or complex vector spaces. Abrams [1] has established sufficient conditions for optimal points of the real part of the objective function neglecting the imaginary part. Duca [6] formulated the vectorial optimization problem in complex space and obtained some necessary and sufficient conditions for a point to be the efficient solution of a problem. From 1966 to 2004, hundreds of papers on optimization in complex space have written. In many of these papers, the authors considerd the real part of complex function as the objective function. Smart and Mond [17] have shown that the necessary conditions for optimality in polyhedral-cone constrained nonlinear programming problems are sufficient owing the hypothesis of special form of convexity. In addition, they extended the duality results for a Wolfe-type dual. Youness and Elborolosy [19] formulated the problem in complex space taking into account the two parts of complex objective function (real and imaginary together) and introduced an extension to necessary optimality conditions in complex programming. Malakooti [14] developed complex method with interior search directions to solve linear and nonlinear programming problems.

One of the difficulties occurring in the application of mathematical programming is that the model parameters are not deterministic. Nevertheless, they are uncertain. Zadeh [20] originally investigated fuzzy set theory. Fuzzy numbers represent the fuzzy numerical data. Dubois and Prade [5] have enlarged the applications of the algebraic operations on real numbers to fuzzy numbers. The concerned process is referred as fuzzification. The fuzzy nature, in a goal programming problem, firstly has been discussed by Zimmermann [23], and later followed by Narasimhan [16], Hanan [8], and lots of others authors working in that field. The decision maker cannot always articulate the goal precisely in a spite of having his / her decision making experience. Fuzzy decision making has been an improvement and a great help in the management decision problems [3]. Dubois and Prade [5] extended the use of algebraic operations on real numbers to fuzzy numbers by use of a fuzzification principle. In spite of having a vast decision making experience, the decision maker cannot always articulate the goals precisely. Zhang and Xia [21] developed two fast complex-valued algorithms for solving complex quadratic programming problem. Zhang and Xia [21] proposed two efficient complex-valued optimization methods for solving programming problems. They considered the real functions in complex variables to cope the uncertainty in the model parameters.

In recent future, several researchers studied the neutrosophic sets to deal the uncertainty in optimization problems. Khalifa et al. [10] introduced the optimization of neutrosophic complex programming using lexicographic order. They converted one neutrosophic complex programming model into two linear sub-models. Khalifa and Kumar [11] studied the interval-valued trapezoidal neutrosophic sets and incorporated the same into assignment problem. Khalifa and Kumar [11] proposed a fully neutrosophic linear programming problem using the neutrosophic sets concepts. They presented an application for the same to stock portfolio selection.

In this paper, we studied the complex nonlinear programming model in neutrosophic fuzzy environments. After converting the problem into an equivalent problem with fuzzy inequalities, a fuzzy programming approach is applied for the real and imaginary parts of the problem individually to determine the optimal compromise solution. The solution of the problem is being obtained based on the linear membership function with the Lexicographic order.

The rest of the article is organized as follows: Sect. 2 formulates complex nonlinear programming model in all of neutrosophic and fuzzy environments; respectively. Section 3 applied fuzzy programming approach for solving the CNLP problem. Section 4 proposed a solution method for obtaining the optimal compromise solution to the NCNLP problem Sect. 5 introduced a numerical experimentation to illustrate the efficiency of the suggested approach. In the last, some concluding remarks and further research directions are reported in Sect. 6s.

2 Model formulation and solution concepts

In this section, consider a complex programming problem as follows:

$$
\begin{aligned}
&(\mathrm{NP_C})\ \text{Min}\ \tilde{\mathrm{f}}^{N}(\mathrm{x}) = \mathrm{u}\left(\mathrm{x}, \tilde{\mathrm{a}}_{\mathrm{k}}^{\mathrm{N}}\right) + i\,\mathrm{v}\left(\mathrm{x}, \tilde{\mathrm{b}}_{\mathrm{k}}^{\mathrm{N}}\right), \quad \mathrm{k} = \overline{1, n} \\
&\text{Subject to} \\
&x \in \tilde{X}^{N} = \left\{ x \in \Re^{n} : g_r(x) = l_r(x) + i h_r(x) \tilde{\leq} \tilde{d}_r + i\tilde{e}_r, r = \overline{1, m}; i = \sqrt{-1} \right\}.
\end{aligned}
$$

Here, u, v $: \mathfrak{R}^n \rightarrow \mathfrak{R}$; $l_r, h_r : \mathfrak{R}^n \rightarrow \mathfrak{R}$, $r = 1, 2, \ldots, m$ are convex functions on X. Parameters $\tilde{a}_k^N$ and $\tilde{b}_k^N$, are characterized by single-valued trapezoidal neutrosophic numbers [18].

The ability of comparing any two single-valued trapezoidal neutrosophic (SVTN) numbers is related to score and accuracy functions. To fill this gap in the proposed study, the following definition is suggested.

Definition 1 Score and Accuracy functions of single valued trapezoidal neutrosophic number [18].

Let $\tilde{c}^N = (a_1, a_2, a_2, a_2); w_c, \varpi_c, y_c$ be a SVTN. Then, we define the following:

i. Score function $S(\tilde{c}^N) = \frac{1}{16}[a_1 + a_2 + a_2 + a_2] \times [\nu_{\tilde{c}^N} + (1 - \pi_{\tilde{c}^N}) + (1 - \rho_{\tilde{c}^N})]$,
ii. Accuracy function $A(\tilde{c}^N) = \frac{1}{16}[a_1 + a_2 + a_2 + a_2] \times [\nu_{\tilde{c}^N} + (1 - \pi_{\tilde{c}^N}) + (1 + \rho_{\tilde{c}^N})]$
.

Based on the Definition 1, the problem (P_C) can be revised to the following non-neutrosophic model as:

$$(\tilde{P}_C)\ \min \tilde{f}(x) = u(x, a_k) + iv(x, b_k), \quad k = 1, 2, \ldots, n$$
Subject to
$$x \in X = \left\{x \in \mathfrak{R}^n : g_r(x) = l_r(x) + i\, h_r(x) \tilde{\leq} d_r + ie_r,\ r = \overline{1, m}\right\}.$$

Definition 2 Lexicographic order of two complex numbers, $z_1 = a + ib$, and $z_2 = c + id$, is

$$z_1 \leq z_2 \Leftrightarrow a \leq c, \text{ and } b \leq d.$$

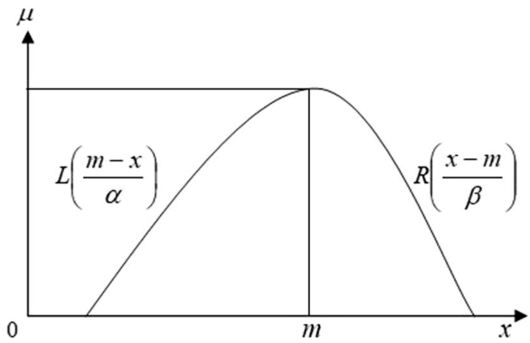
Definition 3 A fuzzy number $\tilde{A} = (x, \alpha, \beta)_{LR}$ is called a $L - R$ fuzzy number when

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right), & \alpha > 0, x \leq m, \\ R\left(\frac{x-m}{\beta}\right), & \beta > 0, x \geq m, \end{cases} \tag{1}$$

where α and β are respectively, the left and right spreads. m is the mean value of $\tilde{A}$. In addition, a function $L(.)$ is a left shape function that satisfies the following properties:

- $L(0) = 1$,
- $L(x) = L(-x)$,
- $L(x)$ is a non-increasing function on the interval $[0, \infty[$.

Fig. 1 *L–R* fuzzy number



Afterwards, the function R(.) is the right shape function that is similarly defined as L(.) as in the following Fig. 1.

For the characterization of the solution of Model $(\tilde{P}_C)$, let us partition it into the following two sub-models:

$$(\tilde{P}_u)\ \text{Min}\ u(x, a_k), \quad k = 1, 2, \ldots, n$$
$$\text{Subject to}$$
$$l_r(x) \tilde{\leq} d_r; h_r(x) \tilde{\leq} e_r, \quad r = 1, 2, \ldots, m, \quad \text{and}$$

$$(\tilde{P}_v)\ \text{Min}\ v(x, b_k), \quad k = 1, 2, \ldots., n$$
$$\text{Subject to}$$
$$l_r(x) \tilde{\leq} d_r; \quad h_r(x) \tilde{\leq} e_r, \quad r = 1, 2, \ldots, m.$$

The sign $\tilde{\leq}$ denotes a fuzzy satisfaction of the constraints.

Definition 4 A point $x^* \in X$ is called an optimal solution for the Model $(\tilde{P}_C)$, if the following condition is satisfied:

$$u(x^*, a_k) \leq u(x, a_k), \text{ and } v(x^*, b_k) \leq v(x, b_k); \quad \forall x \in X.$$

3 Fuzzy programming approach for $(\tilde{P}_u)$, and $(\tilde{P}_v)$ models

Bellman and Zadeh [3] have developed three fundamental terms viz., fuzzy constraints (*C*), fuzzy decision (*D*), and fuzzy goal (*G*). These terms are applied explorer to the mathematical models related to the decision-making models in fuzziness. The fuzzy decision for them is defined in the following Eq. (2):

$$D = G \cap C \tag{2}$$

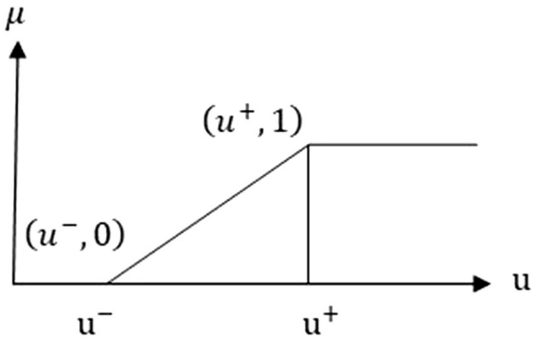


Fig. 2 Membership functions for $u(x, a_k)$

The membership function for this problem is characterized as

$$\mu_D(x) = \mu_{G\cap C}(x) = \min\left(\mu_{C(x)}, \mu_G(x)\right), \tag{3}$$

Fuzzify the objective function $u\left(x, a_k\right)$, let u^-, and u^+ be respectively the lower and upper bounds of $u\left(x, a_k\right)$, which can be determined as in the following Models (4) and (5):

$$\begin{aligned} u_1 = {} & \text{Minimize } u\left(x, a_k\right) \\ & \text{Subject to} \\ & l_r(x) \le d_r; h_r(x) \le e_r, \quad r = 1, 2, \dots, m; \quad x \in \Re^n, \end{aligned} \tag{4}$$

$$\begin{aligned} u_2 = {} & \text{Minimize } u\left(x, a_k\right) \\ & \text{Subject to} \\ & l_r(x) \le d_r + d'_r; h_r(x) \le e_r + e'_r, \quad r = 1, 2, \dots, m; \quad x \in \Re^n, \end{aligned} \tag{5}$$

Let $u^- = \text{Minimize}\left(u_1, u_2\right)$, and $u^+ = \text{Maximize}\left(u_1, u_2\right)$.

Suppose that $\tilde{S}_u$ be the fuzzy set represented by the fuzzy values $u\left(x, a_k\right)$, which is defined as in Eq. (6):

$$\tilde{S}_u = \left\{\left(x, \mu_{\tilde{S}_u}(x)\right) : x \in \Re^n\right\}, \tag{6}$$

where $\mu_{\tilde{S}_u}$ is a linear membership functions defined as follows [23] in Eq. (7), and illustrated in Fig. 2:

$$\mu_{\tilde{S}_u}(x) = \begin{cases} 0, & \text{if } u\left(x, a_k\right) \le u^-, \\ \frac{u\left(x, a_k\right) - u^-}{u^+ - u^-}, & \text{if } u^- \le u\left(x, a_k\right) \le u^+, \\ 1, & \text{if } u\left(x, a_k\right) \ge u^+. \end{cases} \tag{7}$$

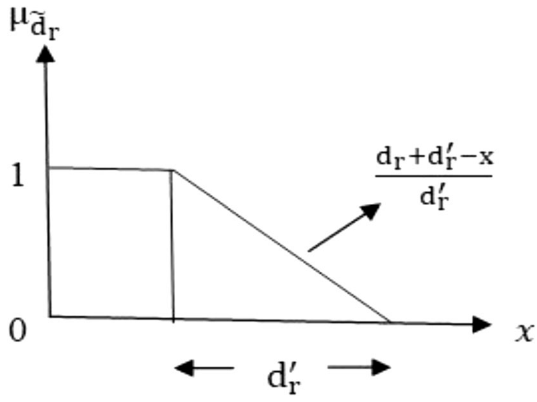


Fig. 3 Membership functions for $\tilde{d}_r$

Remark 1 d_r', and e_r' are vectors of relaxation, which can be determined by fuzzifying $d_r\left(i.e., \tilde{d}_r\right)$, and $e_r\left(i.e., \tilde{e}_r\right)$ as follows

$$\tilde{d}_r = \left\{\left(x, \mu_{\tilde{d}_r}(x) : x \in \mathfrak{R}^n\right)\right\}, \tag{8}$$

where

$$\mu_{\tilde{d}_r}(x) = \begin{cases} 1, & if\ x \le d_r; \\ \frac{d_r+d_r'-x}{d_r'}, & if\ d_r \le x \le d_r + d_r'; \\ 0, & if\ d_r + d_r' \le x \end{cases} \tag{9}$$

Proceeding in the same way,

$$\tilde{e}_r = \left\{\left(x, \mu_{\tilde{e}_r}(x)\right) : x \in \mathbb{R}^n\right\}, \tag{10}$$

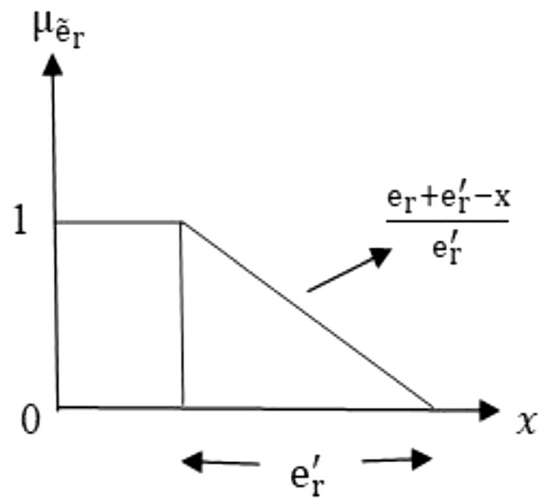
where

$$\mu_{\tilde{e}_r}(x) = \begin{cases} 1, & if\ x \le e_r; \\ \frac{e_r+e_r'-x}{d_r'}, & if\ e_r \le x \le e_r + e_r'; \\ 0, & if\ e_r + e_r' \le x \end{cases} \tag{11}$$

The membership functions for $\tilde{d}_r$ and $\tilde{e}_r$ are demonstrated in Figs. 3 and 4.

Fuzzify the constraints: $l_r(x) \le d_r;h_r(x) \le e_r, r = 1, 2, \dots, m$. Assume that $\tilde{c}_r$ and $\tilde{c}_r'$ are the fuzzy set for r^{th} constraints such that:
$\tilde{c}_r = \left\{\left(x, \mu_{\tilde{c}_r}(x)\right) : x \in \mathfrak{R}^n\right\}$, and $\tilde{c}_r' = \left\{\left(x, \mu_{\tilde{c}_r'}(x)\right) : x \in \mathfrak{R}^n\right\}$, where

Fig. 4 Membership function for $\tilde{e}_r$



$$\mu_{\tilde{c}_r}(x)=\begin{cases}1, & \text{if } l_r(x)\le d_r,\\ \frac{d_r+d_r'-l_r(x)}{d_r'}, & \text{if } d_r\le l_r(x)\le d_r+d_r',\\ 0, & \text{if } d_r+d_r'\le l_r(x).\end{cases}\quad \text{and} \tag{12}$$

$$\mu_{\tilde{c}_r'}(x)=\begin{cases}1, & \text{if } h_r(x)\le e_r,\\ \frac{e_r+e_r'-h_r(x)}{e_r'}, & \text{if } d_r\le h_r(x)\le e_r+e_r',\\ 0, & \text{if } e_r+e_r'\le h_r(x).\end{cases} \tag{13}$$

Let $\tilde{D}_u$ be the fuzzy decision set of problem $\tilde{P}_u$, where

$$\tilde{D}_u=\tilde{S}_u\cap\tilde{c}_r\cap\tilde{c}_r',\quad r=1,2,\ldots,m, \tag{14}$$

Therefore,

$$\tilde{D}_u=\tilde{S}_u\cap\tilde{c}_1\cap\tilde{c}_2\cap\cdots\cap\tilde{c}_m\cap\tilde{c}_1'\cap\tilde{c}_2'\cap\cdots\cap\tilde{c}_m,\ \text{and}\ \tilde{D}_u=\left\{\left(x,\mu_{\tilde{D}_u}(x)\right)\,:\,x\in\Re^n\right\}.$$

Then,

$$\mu_{\tilde{D}_u}(x)=\left\{\min\mu_{\tilde{S}_u}(x),\min\left\{\mu_{\tilde{c}_1}(x),\ldots,\mu_{\tilde{c}_m}(x)\right\},\min\left\{\mu_{\tilde{c}_1}(x),\ldots,\mu_{\tilde{c}_m'}(x)\right\},\right\}.$$

Applying the fuzzy decision of Bellman and Zadeh (1970) together with (7), (12), (13), and with the help of the auxiliary variable ϑ, the Model ($\tilde{P}_u$) is changed to the corresponding nonlinear programming problem (23) as in Model (15).

$$\begin{aligned}&\text{Maximize } \vartheta\\&\text{Subject to}\\&\quad \vartheta \le \mu_{\tilde{S}_u}(x);\\&\quad \vartheta \le \mu_{\tilde{c}_r}(x), r = \overline{1,m};\\&\quad \vartheta \le \mu_{\tilde{c}'_r}(x), r = \overline{1,m};\\&\quad 0 \le \vartheta \le 1; x \in \mathfrak{R}^n.\end{aligned} \tag{15}$$

Or equivalently,

$$\begin{aligned}&\text{Maximize } \vartheta\\&\text{Subject to}\\&\quad \frac{u(x, a_k) - u^-}{u^+ - u^-} \ge \vartheta,\\&\quad \frac{d_r + d'_r - l_r(x)}{d'_r} \ge \vartheta, r = \overline{1,m};\\&\quad \frac{e_r + e'_r - h_r(x)}{e'_r} \ge \vartheta, r = \overline{1,m};\\&\quad 0 \le \vartheta \le 1; x \in \mathfrak{R}^n.\end{aligned} \tag{16}$$

Let $x_u^* \in \mathfrak{R}^n$ be the optimal solution of problem (16) with the fuzzy decision set

$$\tilde{D}_u = \tilde{S}_u \cap \tilde{c}_r \cap \tilde{c}'_r, \quad r = 1, 2, \ldots, m,$$
$$\mu_{\tilde{D}_u}(x) = \min\left\{\mu_{\tilde{S}_u}(x); \mu_{\tilde{c}_1}(x), \ldots, \mu_{\tilde{c}_m}(x); \mu_{\tilde{c}_1}(x), \ldots, \mu_{\tilde{c}'_m}(x)\right\}.$$

The problem is to find $x_u^* \in \mathfrak{R}^n$ such that:

$$\mu_{\tilde{D}_u}\left(x_u^*\right) = \text{Maximize(Minimize}\left\{\begin{array}{c}\mu_{\tilde{S}_u}\left(x_u^*\right); \mu_{\tilde{c}_1}\left(x_u^*\right), \ldots, \mu_{\tilde{c}_m}\left(x_u^*\right);\\ \mu_{\tilde{c}_1}\left(x_u^*\right), \ldots, \mu_{\tilde{c}'_m}\left(x_u^*\right)\end{array}\right\}, \tag{17}$$

where $\mu_{\tilde{S}_u}(x)$, $\mu_{\tilde{c}_r}(x)$, and $\mu_{\tilde{c}'_r}(x)$ are defined in (7), (12), and (13); respectively.

By substituting in the objective function of problem $(\tilde{P}_u)$, we find the accurate value u^{AV} is

$$u^- \le u^{AV} \le u^+. \tag{18}$$

Similarly, the optimal solution $x_v^* \in \mathfrak{R}^n$ of problem $(\tilde{P}_v)$ with fuzzy decision set

$$\tilde{D}_v = \tilde{S}_v \cap \tilde{c}_r \cap \tilde{c}'_r, \quad r = \overline{1,m}, \tag{19}$$

$$\mu_{\tilde{D}_v}(x) = \text{Minimize}\left\{\mu_{\tilde{S}_v}(x), \mu_{\tilde{c}_r}(x), \mu_{\tilde{c}'_r}(x)\right\}. \tag{20}$$

Also, the problem is to find x_v^* such that

$$\mu_{\tilde{D}_v}\left(x_v^*\right) = \text{Maximize}(\text{Minimize}\left\{ \begin{array}{c} \mu_{\tilde{S}_v}\left(x_v^*\right);\mu_{\tilde{S}_v}(x);\mu_{\tilde{c}_1}(x),\dots,\mu_{\tilde{c}_m}(x); \\ \mu_{\tilde{c}_1}(x),\dots,\mu_{\tilde{c}'_m}(x) \end{array} \right\}, \tag{21}$$

where

$$\mu_{\tilde{S}_v}(x) = \begin{cases} 1, & if\ v(x) \geq v^+, \\ \frac{v(x)-v^-}{v^+-v^-}, & if\ v^- \leq v(x) < v^+, \\ 0, & if\ v(x) \leq v^-. \end{cases} \tag{22}$$

where $\mu_{\tilde{c}_r}(x)$ and $\mu_{\tilde{c}'_r}(x)$ are defined in (12) and (13); respectively.

Our aim is to find $x_v^* \in \mathfrak{R}^{\mathbf{n}}$ such that

$$\mu_{\tilde{D}_v}\left(x_v^*\right) = \text{Maximize}(\text{Minimize}\left\{ \begin{array}{c} \mu_{\tilde{S}_v}\left(x_v^*\right);\mu_{\tilde{c}_1}\left(x_v^*\right),\dots,\mu_{\tilde{c}_m}\left(x_v^*\right); \\ \mu_{\tilde{c}_1}\left(x_v^*\right),\dots,\mu_{\tilde{c}'_m}\left(x_v^*\right) \end{array} \right\}. \tag{23}$$

By substituting in the objective function of problem $\tilde{P}_v$, we have

$$v^- \leq v^{AV} \leq v^+ \tag{24}$$

The optimal solution of ($\tilde{P}_C$) problem is to determine the variable $x^* \in \mathfrak{R}$ with the fuzzy decision set such that:

$$\tilde{D} = \tilde{D}_u \cap \tilde{D}_v, \tag{25}$$

$$\tilde{D} = \tilde{S}_u \cap \tilde{S}_v \cap \tilde{c}_r \cap \tilde{c}'_r, \tag{26}$$

$$\mu_{\tilde{D}} = \min\left\{ \mu_{\tilde{S}_u}(x);\mu_{\tilde{S}_v}(x);\mu_{\tilde{c}_r}(x);\mu_{\tilde{c}'_r}\left(x_v^*\right) \right\}, \tag{27}$$

Now, the problem is to find $x^* \in \mathfrak{R}$ such that:

$$\begin{aligned} \mu_D(x^*) &= \max\min\left(\mu_{\tilde{D}_u}(x^*), \mu_{\tilde{D}_v}(x^*)\right) \\ &= \max\min\left(\begin{array}{c} \mu_{\tilde{S}_u}\left(x_u^*\right);\mu_{\tilde{c}_r}\left(x_u^*\right),\dots,\mu_{\tilde{c}_m}\left(x_u^*\right); \\ \mu_{\tilde{S}_v}\left(x_v^*\right);\mu_{\tilde{c}_1}\left(x_v^*\right),\dots,\mu_{\tilde{c}'_m}\left(x_v^*\right) \end{array} \right), \quad r = \overline{1,m} \end{aligned} \tag{28}$$

4 Solution procedure

In this Section, the following steps are considered for the suggested to solve Model (NP_C):

Step 0: Start.

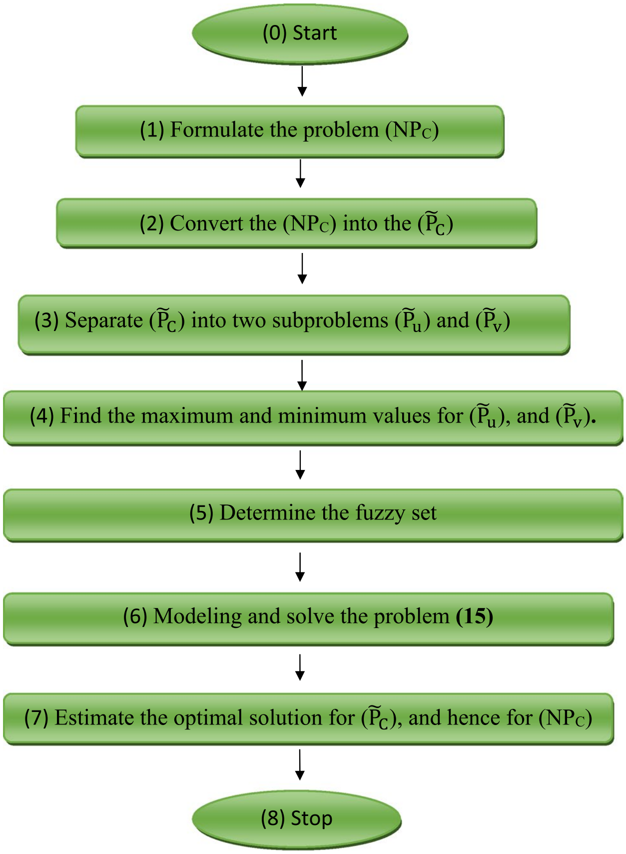


Fig. 5 Algorithms of the proposed approach

Step 1: Formulate the model (NP_C),
Step 2: Convert the (NP_C) into the ($\tilde{P}_C$),
Step 3: Separate ($\tilde{P}_C$) into two sub-models ($\tilde{P}_u$), and ($\tilde{P}_v$),
Step 4: Find the maximum and minimum values for the sub-models ($\tilde{P}_u$), and ($\tilde{P}_v$). Afterwards, construct the linear memberships $\mu_{\tilde{S}_u}(x)$, $\mu_{\tilde{S}_v}(x)$,
Step 5: Determine the fuzzy set for r^{th} constraints, $\tilde{c}_r$ and $\tilde{c}'_r$.
Step 6: Modeling and solve the problem (15) to obtain x^*_u. In the same way, we obtain x^*_v.
Step 7: Estimate the optimal solution for ($\tilde{P}_C$), and hence for (NP_C).
Step 8: Stop.

A flowchart of the suggested approach is depicted in Fig. 5.

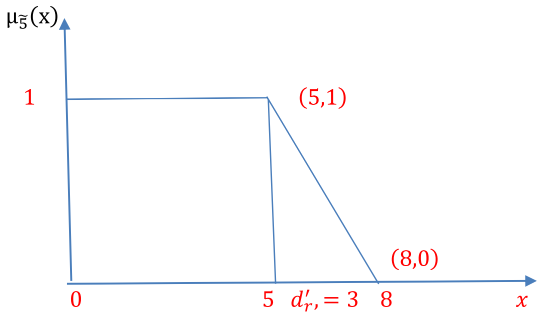


Fig. 6 Membership function of $\mu_{\tilde{5}}(x)$

5 Numerical experimentation

Consider the following (NP_C)

$$\min\ \tilde{f}^N(x) = \left(\begin{array}{c} (5, 7, 9, 11; 0.9, 0.7, 0.5x_1(+) \\ (+)(4, 8, 11, 15; 0.6, 0.3, 0.2x_1(-)14, 19, 25, 30; 0.8, 0.2, 0.6x_2) \end{array} \right)$$

Subject to

$$x_1^2 + x_2^2 + i(x_1 - x_2) \tilde{\leq} 5 + i. \tag{29}$$

Applying the score function as in Definition 1, problem (29) becomes

$$\min\ \tilde{f} = (3x_1 + x_2) + i(5x_1 - 11x_2)$$

$$x_1^2 + x_2^2 + i(x_1 - x_2) \tilde{\leq} 5 + i \tag{30}$$

Based on the Lexicographic order in Definition 2, problem (30) can be divided into the following two sub-models

$$\begin{array}{ll} (\tilde{P}_u) & \min\ \tilde{f} = 3x_1 + x_2 \\ & \text{Subject to} \\ & \quad x_1^2 + x_2^2 \tilde{\leq} 5, \\ & \quad x_1 - x_2 \tilde{\leq} 1, \text{ and} \end{array} \tag{31}$$

$$\begin{array}{ll} (\tilde{P}_v) & \min\ \tilde{f} = 5x_1 - 11x_2 \\ & \text{Subject to} \\ & \quad x_1^2 + x_2^2 \tilde{\leq} 5, \\ & \quad x_1 - x_2 \tilde{\leq} 1. \end{array} \tag{32}$$

According to the Kuhn–Tucker' optimality conditions [2], the optimal solutions of $\tilde{P}_u$, and $\tilde{P}_v$ are $(x_1^*, x_2^*) = (-2, -1)$ with $u_1 = -7$, and $(x_1^*, x_2^*) = (-2, 1)$, with $v_1 = -21$; respectively.

To calculate the maximum values for $\tilde{P}_u$, and $\tilde{P}_v$, let us consider

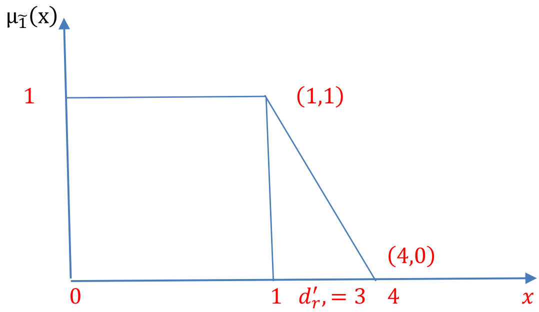


Fig. 7 Membership function of $\mu_{\tilde{1}}(x)$

$$\begin{aligned} u_2 &= \min\left(3x_1 + x_2\right) \\ &\text{Subject to} \\ &x_1^2 + x_2^2 \le 8, \\ &x_1 - x_2 \le 4, \ \text{and} \end{aligned} \tag{33}$$

$$\begin{aligned} v_2 &= \min\left(5x_1 - 11\right) \\ &\text{Subject to} \\ &x_1^2 + x_2^2 \le 8, \\ &x_1 - x_2 \le 4. \end{aligned} \tag{34}$$

It is clear that (Fig. 6)

$$\mu_{\tilde{5}}(x) = \begin{cases} 1, & if\ x \le 5; \\ \frac{8-x}{3}, & if\ 5 \le x < 8; \text{ with membership function} \\ 0, & if\ x \ge 8. \end{cases}$$

Similarly (Fig. 7),

$$\mu_{\tilde{1}}(x) = \begin{cases} 1, & if\ x \le 1; \\ \frac{4-x}{3}, & if\ 1 \le x < 4; \quad \text{with membership function} \\ 0, & if\ x \ge 4. \end{cases}$$

Now, we can find the lower and upper bound of the optimal value u denoted by u^- and u^+ respectively, by solving the two crisp nonlinear programming problems

(1) $u_1 = u$: Since the problem is the same first problem $\tilde{P}_u$, and have the same solution, therefore

$$u^- = u = -7 \text{ occurs at } \left(x_1^*, x_2^*\right) = (-2, 1)$$

$$(2)\quad \begin{aligned} & u_2 = \min\left(3x_1 + x_2\right) \\ & \quad \text{Subject to} \\ & \quad x_1^2 + x_2^2 \le 8, \\ & \quad x_1 - x_2 \le 4 \end{aligned}$$

The optimal solution is $\left(x_1^*, x_2^*\right) = ()$ which is satisfies the constraints and the optimal value $u_2 = 4\sqrt{5} = 8.9442719$. Then,

$$u^- = \min\left(u_1, u_2\right) = \min(-7, 8.9443) = -7,$$
$$u^+ = \max\left(u_1, u_2\right) = \max(-7, 8.9443) = 8.9443,$$

The optimal solutions of problems (33) and (34) are $\left(x_1^*, x_2^*\right) = (2.6833, 0.8944)$ with $u_2 = 8.9443$, and $\left(x_1^*, x_2^*\right) = (1.1704, -2.5749)$ with $v_2 = 34.1670$, respectively. Therefore,

$$u^- = \min\left(u_1, u_2\right) = \min(-7, 8.9443) = -7,$$
$$u^+ = \max\left(u_1, u_2\right) = \max(-7, 8.9443) = 8.9443,$$
$$v^- = \min\left(v, v_2\right) = \min(-21, 8.9449) = -21, \text{ and}$$
$$v^+ = \max\left(v_1, v_2\right) = \max(-21, 8.9449) = 8.9449$$

Let the S be the set of all objective function such that:

$$\tilde{S}_u = \left\{\left(x, \mu_{\tilde{S}_u}\right) : x \in \mathfrak{R}^n\right\}, \text{ and}$$

$$\mu_{\tilde{S}_u}(x) = \begin{cases} 0, & \textit{if } 3x_1 + x_2 \le -7; \\ \frac{3x_1 + x_2 + 7}{15.9443}, & \textit{if } u^- \le 3x_1 + x_2 \le 8.9443; \\ 1, & \textit{if } 8.9443 \le 3x_1 + x_2. \end{cases}$$

Also,

$$\mu_{\tilde{c}_1}(x) = \begin{cases} 1, & \textit{if } x_1^2 + x_2^2 \le 1; \\ \frac{8 - x_1^2 - x_2^2}{3}, & \textit{if } 5 \le x_1^2 + x_2^2 < 8; \\ 0, & \textit{if } x_1^2 + x_2^2 \ge 8, \end{cases} \quad \text{and}$$

$$\mu_{\tilde{c}_1'}(x) = \begin{cases} 1, & \textit{if } x_1 - x_2 \le 1; \\ \frac{4 - x_1 + x_2}{3}, & \textit{if } 1 \le x_1 - x_2 < 4; \\ 0, & \textit{if } x_1 - x_2 \ge 4. \end{cases}$$

Hence, problem (15) corresponding to problem (33) is as follows:

$$\begin{aligned}
&\text{Max } \vartheta \\
&\text{s.t} \\
&\vartheta \le \frac{3\mathrm{x}_1 + \mathrm{x}_2 + 7}{15.9443}; \\
&\vartheta \le \frac{8 - x_1^2 - x_2^2}{3}; \\
&\vartheta \le \frac{4 - x_1 + x_2}{3}; \\
&0 \le \vartheta \le 1; x \in \mathfrak{R}^2.
\end{aligned} \tag{35}$$

The solution of problem (35) is $x_u^* = (1.76233, 0.0914)$, and $\vartheta^* = 0.7764$. So, the accurate value for u is $u^{AF} = u\left(x_1^*, x_2^*\right) = 5.3784$, and the neutrosophic value is as follows:

$$\tilde{u}^N = \langle 8.8115, 12.4275, 16.1349, 19.9337; 0.7, 0.7, 0.5 \rangle$$

Similarly, problem (15) corresponding to problem (34) is

$$\begin{aligned}
&\text{Max } \vartheta \\
&\text{Subject to} \\
&\vartheta \le \frac{5x_1 - 11x_2 + 21}{55.1761}, \\
&\vartheta \le \frac{8 - x_1^2 - x_2^2}{3}, \\
&\vartheta \le \frac{4 - x_1 + x_2}{3}, \\
&0 \le \vartheta \le 1; \quad x \in \mathbb{R}^2.
\end{aligned} \tag{36}$$

Then, $x_v^* = (0.9585, -1.0256)$, and $\vartheta^* = 0.6719$, with the accurate value for v is $v^{AF} = v(0.9585, -1.0256) = 16.0748$, neutrosophic value is $\tilde{v}^N = 18.1924, 27.1544, 36.1835, 45.1455; 0.6, 0.3, 0.6$.Thus, the optimal solution of Model ($\tilde{\mathrm{P}}_{\mathrm{C}}$) is determined such that the variable x^* satisfies the following condition in Eq. (30):

$$\begin{aligned}
\mu_{\tilde{D}}(x) &= \min\left\{ \mu_{\tilde{D}_u}(x^*), \mu_{\tilde{D}_v}(x^*) \right\} \\
&= \max\min\{0.7764, 0.6719, 1.6286, 0.7764, 2.0098, 0.6719\} = 0.6719
\end{aligned} \tag{37}$$

In addition, the optimal solution for Model (30) is (0.9585, −1.0256), with accurate value is 0.6719, which is the same as the solution of Model ($\tilde{\mathrm{P}}_{\mathrm{v}}$), and its neutrosophic solution is as follows:

$$\langle 18.1924, 27.1544, 36.1835, 45.1455; 0.6, 0.3, 0.6. \rangle$$

It is observed that the results obtained in a fuzzy environment are less than, more applicable and satisfactory than the introduced by Khalifa et al. [10].

6 Conclusions and future work

In this paper, a complex nonlinear programming problem with objective function coefficients represented as interval-valued trapezoidal neutrosophic parameters and fuzzy inequalities $\left(\tilde{\leq}\right)$ has studied. A fuzzy programming approach has been implemented to solve the suggested model by defining membership functions. For obtaining optimal compromise solution in neutrosophic environment, the linear membership function is incorporated. Kuhn-Tucker optimality conditions have applied for obtaining the optimal solution. For further research, there are several research directions. For instance, we can extend the current work to multi objective case. Secondly, we can extent the current work to some real life applications in decision science, production planning, financial mathematics, etc. In addition, there are some limitations of the proposed study. There are many open problems and research points to be investigated in the field of complex mathematical programming problems that include the two parts in the objective functions. Some of these points are:

a) Developing the duality theory in both single-objective and multi-objective optimization problems.
b) Deriving the optimality conditions under other types of generalized convexity such as E-convexity and invexity.
c) Extending the various concepts of proper efficiency to complex space and establishing the relationships between them.

Acknowledgements The authors would like to thank the Editor-in-Chief and anonymous referees for their suggestive and helpful comments that have led to an improvement both in the quality and clarity of the paper.

Data availability The data used to support the findings of this research are available from the corresponding author upon request.

Declarations

Conflict of interest Authors do not have any conflicts of interest.

References

1. Abrams, R.A.: Nonlinear programming in complex space: sufficient conditions and duality. J. Math. Anal. Appl. **38**, 619–632 (1972)
2. Bazaraa, M.S., Shetty, M.H.: Non-Linear Programming: Theory and Algorithms. Wiley, New York (1979)
3. Bellmann, R., Zadeh, L.: Decision making in a fuzzy environment. Manag. Sci. **17**, 141–164 (1970)
4. Ben-Israel, A.: Linear equations and inequalities on finite dimensional real or complex, vector spaces: a unified theory. J. Math. Anal. Appl. **27**, 367–389 (1969)

5. Dubois, D., Prade, H.: Fuzzy sets and systems: theory and applications. Academic Press, New York (1980)
6. Duca, D.I.: Constraint qualifications in nonlinear programming in complex space, Stadia Univ. Babes-Bolyai. Mathematics **23**(1), 61–65 (1978)
7. Ferrero, O.: On nonlinear programming in complex spaces. J. Math. Anal. Appl. **164**(2), 399–416 (1992)
8. Hanan, E.L.: On fuzzy goal programming. Decis. Sci. **12**, 522–531 (1981)
9. Hossein, K., Mohsen, S.D., Majid, J.: A fully fuzzy best–worst multi attribute decision making method with triangular fuzzy number: a case study of maintenance assessment in the hospitals. Appl. Soft Comput. **86**, 1–11 (2020). https://doi.org/10.1016/j.asoc.2019.105882
10. Khalifa, H.A., Kumar, P., Florentin, S.: On optimizing neutrosophic complex programming using lexicographic order. Neutrosophic Sets Syst. **32**, 330–343 (2020). https://doi.org/10.5281/zenodo.3723173
11. Khalifa, H.A., Kumar, P.: A novel method for neutrosophic assignment problem by using interval-valued trapezoidal neutrosophic number. Neutrosophic Sets Syst. **36**, 24–36 (2020). https://doi.org/10.5281/zenodo.4065363
12. Khalifa, H.A., Kumar, P.: Solving fully neutrosophic linear programming problem with application to stock portfolio selection. Croat. Oper. Res. Rev. **11**(2), 165–176 (2020). https://doi.org/10.17535/crorr.2020.0014
13. Levinson, N.: Linear programming in complex space. J. Math. Anal. Appl. **14**, 44–62 (1966)
14. Malakooti, B.: The complex interior-boundary method for linear and nonlinear programming with linear constraints. Appl. Math. Comput. **216**, 1903–1917 (2010)
15. Mond, B., Hanson, M.A.: Quadratic programming in complex space. J. Math. Anal. Appl. **20**, 507–514 (1967)
16. Narasimhan, R.: Goal programming in a fuzzy environment. Decis. Sci. **13**, 325–336 (1980)
17. Smart, I., Mond, B.: Complex nonlinear programming: duality with invexity and equivalent real program. J. Optim. Theory Appl. **69**, 469–488 (1991)
18. Thamaraiselvi, A., Santhi, R.: A new approach for optimization of real life transportation problem in neutrosophic environment. Mathematical Problems in Engineering, Volume 2016, Article ID 5950747/ 9 pages. Math. Comput. Simul. **180**, 152–177 (2016). https://doi.org/10.1016/j.matcom.2020.08.019
19. Youness, E.A., Elbrolosy, M.E.: Extension to necessary optimality conditions in complex programming. J. Appl. Math. Comput. **154**, 229–237 (2004)
20. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**(1), 338–353 (1965)
21. Zhang, S., Xia, Y.: Two fast complex-valued algorithms for solving complex quadratic programming problems. IEEE Trans. Cybern. **46**(12), 2837–2847 (2016)
22. Zhang, S., Xia, Y.: Solving nonlinear optimization problems of real functions in complex variables by complex-valued iterative methods. IEEE Trans. Cybern. **48**(1), 277–287 (2018). https://doi.org/10.1109/tcyb.2016.2632159
23. Zimmermann, H.-J.: Fuzzy programming and linear programming with several objective functions. Fuzzy Sets Syst. **1**(1), 45–55 (1978)