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Characterization of ideals in L-algebras by neutrosophic $\mathcal{N}-$structures

Tugce Katican[1] · Tahsin Oner[2] · Arsham Borumand Saeid[3]

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Abstract
The main objective of this study is to introduce a neutrosophic $\mathcal{N}-$ subalgebra (ideal) of L-algebras and to investigate some properties. It is shown that the level-set of a neutrosophic $\mathcal{N}-$subalgebra (ideal) of an L-algebra is its subalgebra (ideal), and the family of all neutrosophic $\mathcal{N}-$subalgebras of an L-algebra forms a complete distributive modular lattice. Additionally, it is proved that every neutrosophic $\mathcal{N}-$ideal of an L-algebra is the neutrosophic $\mathcal{N}-$subalgebra but the inverse of the statement may not be true in general. As the concluding part, some special cases are provided as ideals which are particular subsets of an L-algebra defined due to $\mathcal{N}-$functions.

Keywords L-algebra · Ideal · Neutrosophic $\mathcal{N}-$subalgebra · Neutrosophic $\mathcal{N}-$ideal

Mathematics Subject Classification 06F05 · 03G25 · 03G10

1 Introduction

L-algebras which are defined in the light of the quantum Yang-Baxter equation are introduced, and studied in details by Rump in [13, 14]. For the readership, we can list the popular examples of L-algebras as Hilbert algebras, locales, (left) hoops, (pseudo)

✉ Tahsin Oner
tahsin.oner@ege.edu.tr

Tugce Katican
tugce.katican@izmirekonomi.edu.tr

Arsham Borumand Saeid
arsham@uk.ac.ir

1 Department of Mathematics, Faculty of Arts and Sciences, İzmir University of Economics, İzmir, Turkey

2 Department of Mathematics, Faculty of Science, Ege University, İzmir, Turkey

3 Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran

MV-algebras and l-group cones. In the existing literature, it is shown that algebraic structures of Łucasiewicz's logic are MV-algebras (see [2, 4]), and also, the paper [9] illustrates that the category of MV-algebras is equivalent to abelian lattice ordered groups (briefly, l-groups) with a strong order unit. In the sequel, in [13], L-algebras with a natural embedding into the negative cone of an l-group are introduced, and this results in Dvurecenskij's non-commutative generalization and Mundici's equivalence between MV-algebras and unital Abelian l-groups. Furthermore, pseudo-MV algebras and Bosbach's noncommutative bricks are characterized as L-algebras in [16], and it is underlined that an L-algebra is purposed as an interval in a lattice-ordered group if and only if it is semiregular with a smallest element and a bijective negation. We refer to readers the recent paper [15] which provides examples of L-algebras in logic, geometry, measure theory, and topology to polish potential for application. Also, we cite the papers [3, 5, 20, 21] as related studies on L-algebras in this field. For example, authors of [20] focus on the relationships between basic algebras and L-algebras, and then Yang and Wu represents orthomodular lattices as L-algebras in [21]. Moreover, specific properties of L-algebras are studied in the brand new papers [3, 5]. For further reading, we suggest the references there in.

The fuzzy set theory is first introduced by Zadeh as a generalization of the set theory in [22]. The notions truth (t) (membership) function, and positive meaning of information in the fuzzy set theory make the investigation of negative meaning of information reasonable. Consequentially, Atanassov introduced the intuitionistic fuzzy set theory as a generalization of the fuzzy set theory together with truth (t) (membership) and the falsehood (f) (nonmembership) functions (see [1]). Subsequently, Smarandache introduced the neutrosophic set theory for generalizing the intuitionistic fuzzy set theory in [17, 18], and employ the indeterminacy/neutrality (i) function with truth and falsehood functions. Therefore, neutrosophic sets are constructed by three components (t, i, f) (see [23]). It should be pointed out that the neutrosophy notion has been applied to the algebraic structures such as BCK/BCI-algebras, BE-algebras, strong Sheffer stroke non-associative MV-algebras, and Sheffer stroke Hilbert algebras. We refer to [6–8, 10–12, 19] as the corresponding literature.

The organization of the paper is as follows: the next section is devoted to presentation of essentials on L-algebras. In the third section, the setup of the main results is provided, and outcomes of the paper are presented with illustrative examples. The results of the manuscript are new and novel, therefore, contribute the ongoing theory of pure mathematics regarding L-algebras.

2 Preliminaries

In this section, basic definitions and notions about L-algebras and neutrosophic $\mathcal{N}-$structures are presented.

Definition 1 [13] An L-algebra is an algebra $(L; \longrightarrow, 1)$ of type (2, 0) satisfying

(L1) $x \longrightarrow x = x \longrightarrow 1 = 1, 1 \longrightarrow x = x$,
(L2) $(x \longrightarrow y) \longrightarrow (x \longrightarrow z) = (y \longrightarrow x) \longrightarrow (y \longrightarrow z)$,
(L3) $x \longrightarrow y = y \longrightarrow x = 1$ implies $x = y$,

for all $x, y, z \in L$.

Lemma 1 [13] *Let* $(L; \longrightarrow, 1)$ *be an L-algebra. Then the relation* $\leq$ *defined by*

$$x \leq y :\Longleftrightarrow x \longrightarrow y = 1$$

is a partial order on L. Also, 1 is the greatest element of L.

Lemma 2 [16] *Let* $(L; \longrightarrow, 1)$ *be an L-algebra. Then the following statements are equivalent:*

1. $y \leq x \longrightarrow y$,
2. $x \leq y$ *implies* $y \longrightarrow z \leq x \longrightarrow z$ *and* $z \longrightarrow x \leq z \longrightarrow y$,

for all $x, y, z \in L$.

Definition 2 [13] Let $(L; \longrightarrow, 1)$ be an L-algebra. Then a subset K of L is called an L-subalgebra if $x \longrightarrow y, y \longrightarrow x \in K$, for all $x, y \in K$.

Definition 3 [13] Let $(L; \longrightarrow, 1)$ be an L-algebra. Then a subset I of L is called an ideal if the following hold for all $x, y \in L$:

(I1) $1 \in I$,
(I2) $x, x \longrightarrow y \in I$ implies $y \in I$,
(I3) $x \in I$ implies $(x \longrightarrow y) \longrightarrow y \in I$,
(I4) $x \in I$ implies $y \longrightarrow x, y \longrightarrow (x \longrightarrow y) \in I$.

Definition 4 [6] $\mathcal{F}(A, [-1, 0])$ denotes the collection of functions from a set A to $[-1, 0]$ and an element of $\mathcal{F}(A, [-1, 0])$ is called a negative-valued function from A to $[-1, 0]$ (briefly, $\mathcal{N}-$function on A). An $\mathcal{N}-$structure refers to an ordered pair (A, f) of A and $\mathcal{N}-$function f on A.

Definition 5 [8] A neutrosophic $\mathcal{N}-$structure over a nonempty universe A is defined by $A_N := \dfrac{A}{(T_N, I_N, F_N)} = \{\dfrac{A}{(T_N(a), I_N(a), F_N(a))} : a \in A\}$ where T_N, I_N and F_N are $\mathcal{N}-$function on A, called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function, respectively.

Every neutrosophic $\mathcal{N}-$structure A_N over X satisfies the condition

$$(\forall a \in A)(-3 \leq T_N(a) + I_N(a) + F_N(a) \leq 0).$$

3 Neutrosophic $\mathcal{N}-$structures

In this section, neutrosophic $\mathcal{N}-$subalgebras and neutrosophic $\mathcal{N}-$ideals of L-algebras are given. Unless otherwise specified, L states an L-algebra.

Definition 6 A neutrosophic $\mathcal{N}-$subalgebra L_N of an L-algebra L is a neutrosophic $\mathcal{N}-$structure on L satisfying the condition

Fig. 1 Hasse diagram of L in Example 1

Table 1 Cayley table of a binary operation $\longrightarrow$ on L in Example 1

$\longrightarrow$	x	y	1
x	1	y	1
y	x	1	1
1	x	y	1

$$\begin{array}{l} \min\{T_N(x), T_N(y)\} \leq T_N(x \longrightarrow y), \\ I_N(x \longrightarrow y) \leq \max\{I_N(x), I_N(y)\} \\ \qquad\qquad and \\ F_N(x \longrightarrow y) \leq \max\{F_N(x), F_N(y)\}, \end{array} \tag{1}$$

for all $x, y \in L$.

Example 1 Consider an L-algebra L where $L = \{x, y, 1\}$ with the Hasse diagram in Fig. 1 and a binary operation $\longrightarrow$ on L has the Cayley table in Table 1.

Then a neutrosophic $\mathcal{N}-$structure $L_N = \left\{\frac{x}{(-0.5, -0.4, -0.8)}, \frac{y}{(-0.6, -0.5, -0.8)}, \frac{1}{(0, -1, -1)}\right\}$ on L is a neutrosophic $\mathcal{N}-$subalgebra of L.

Definition 7 Let L_N be a neutrosophic $\mathcal{N}-$structure on an L-algebra L and π, ρ, σ be all elements of $[-1, 0]$ such that $-3 \leq \pi + \rho + \rho \leq 0$. For the sets

$$T_N^{\pi} := \{x \in L : \pi \leq T_N(x)\},$$
$$I_N^{\rho} := \{x \in L : I_N(x) \leq \rho\}$$

and

$$F_N^{\sigma} := \{x \in L : F_N(x) \leq \sigma\},$$

the set $L_N(\pi, \rho, \sigma) := \{x \in L : \pi \leq T_N(x), I_N(x) \leq \rho \; and \; F_N(x) \leq \sigma\}$ is called the $(\pi, \rho, \sigma)-$level set of L_N. Also, $L_N(\pi, \rho, \sigma) = T_N^{\pi} \cap I_N^{\rho} \cap F_N^{\sigma}$.

Theorem 3 *Let L_N be a neutrosophic $\mathcal{N}-$structure on an L-algebra L and π, ρ, σ be any elements of $[-1, 0]$ which implies $-3 \leq \pi + \rho + \sigma \leq 0$. If L_N is a neutrosophic $\mathcal{N}-$subalgebra of L, then the nonempty level set $L_N(\pi, \rho, \sigma)$ of L_N is an L-subalgebra of L.*

Proof Let L_N be a neutrosophic $\mathcal{N}-$subalgebra of L and x, y be any elements of $L_N(\pi, \rho, \sigma)$, for $\pi, \rho, \sigma \in [-1, 0]$ which implies $-3 \leq \pi + \rho + \sigma \leq 0$. Then $\pi \leq T_N(x), T_N(y)$; $I_N(x), I_N(y) \leq \rho$ and $F_N(x), F_N(y) \leq \sigma$. Since

$$\pi \leq \min\{T_N(y), T_N(x)\} \leq T_N(y \longrightarrow x),$$
$$I_N(y \longrightarrow x) \leq \max\{I_N(y), I_N(x)\} \leq \rho$$

and

$$F_N(y \longrightarrow x) \leq \max\{F_N(y), F_N(x)\} \leq \sigma,$$

for all $x, y \in L$, it is obtained that $y \longrightarrow x \in T_N^{\pi}, I_N^{\rho}, F_N^{\sigma}$. Thus, $y \longrightarrow x \in T_N^{\pi} \cap I_N^{\rho} \cap F_N^{\sigma} = L_N(\pi, \rho, \sigma)$. Hence, $L_N(\pi, \rho, \sigma)$ is an L-subalgebra of L. □

Theorem 4 *Let L_N be a neutrosophic $\mathcal{N}-$structure on an L-algebra L and T_N^{π}, I_N^{ρ} and F_N^{σ} be L-subalgebras of L, for all $\pi, \rho, \sigma \in [-1, 0]$ which implies $-3 \leq \pi + \rho + \sigma \leq 0$. Then L_N is a neutrosophic $\mathcal{N}-$subalgebra of L.*

Proof Let T_N^{π}, I_N^{ρ} and F_N^{σ} be L-subalgebras of L, for all $\pi, \rho, \sigma \in [-1, 0]$ which implies $-3 \leq \pi + \rho + \sigma \leq 0$. Suppose that

$$\pi_1 = T_N(y \longrightarrow x) < \min\{T_N(y), T_N(x)\} = \pi_2,$$
$$\rho_1 = \max\{I_N(y), I_N(x)\} < I(y \longrightarrow x) = \rho_2$$

and

$$\sigma_1 = \max\{F_N(y), F_N(x)\} < F(y \longrightarrow x) = \sigma_2.$$

If $\pi = \frac{1}{2}(\pi_1 + \pi_2)$, $\rho = \frac{1}{2}(\rho_1 + \rho_2)$, $\sigma = \frac{1}{2}(\sigma_1 + \sigma_2) \in [-1, 0)$, then $\pi_1 < \pi < \pi_2$, $\rho_1 < \rho < \rho_2$ and $\sigma_1 < \sigma < \sigma_2$. Thus, $x, y \in T_N^{\pi}, I_N^{\rho}, F_N^{\sigma}$ but $y \longrightarrow x \notin T_N^{\pi}, I_N^{\rho}, F_N^{\sigma}$ which is a contradiction. So, $\min\{T_N(x), T_N(y)\} \leq T_N(x \longrightarrow y)$, $I_N(x \longrightarrow y) \leq \max\{I_N(x), I_N(y)\}$ and $F_N(x \longrightarrow y) \leq \max\{F_N(x), F_N(y)\}$, for all $x, y \in L$. Hence, L_N is a neutrosophic $\mathcal{N}-$subalgebra of L. □

Lemma 5 *Let L_N be a neutrosophic $\mathcal{N}-$subalgebra of an L-algebra L. Then*

$$T_N(x) \leq T_N(1), I_N(1) \leq I_N(x) \text{ and } F_N(1) \leq F_N(x), \tag{2}$$

for any $x \in L$.

Proof Let L_N be a neutrosophic $\mathcal{N}-$subalgebra of L. Then it follows from (L1) that $T_N(x) = \min\{T_N(x), T_N(x)\} \leq T_N(x \longrightarrow x) = T_N(1)$, $I_N(1) = I_N(x \longrightarrow x) \leq \max\{I_N(x), I_N(x)\} = I_N(x)$ and $F_N(1) = F_N(x \longrightarrow x) \leq \max\{F_N(x), F_N(x)\} = F_N(x)$, for all $x \in L$. □

The inverse of Lemma 5 is generally not true.

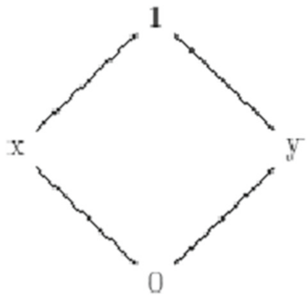

Fig. 2 Hasse diagram of L in Example 2

Table 2 Cayley table of a binary operation $\longrightarrow$ on L in Example 2

$\longrightarrow$	0	x	y	1
0	1	1	1	1
x	y	1	y	1
y	x	x	1	1
1	0	x	y	1

Example 2 Consider an L-algebra L where $L = \{x, y, z, 1\}$ with the Hasse diagram in Fig. 2 and a binary operation $\longrightarrow$ on L has the Cayley table in Table 2 [5].

Then a neutrosophic $\mathcal{N}-$structure $L_N = \left\{\frac{y}{(-0.7, -0.51, -0.1)}\right\} \cup \left\{\frac{u}{(-0.07, -0.83, -0.77)} : u \in L - \{y\}\right\}$ on L satisfies the condition (2) but it is not a neutrosophic $\mathcal{N}-$subalgebra of L since $\max\{F_N(x), F_N(0)\} = -0.77 < -0.1 = F_N(y) = F_N(x \longrightarrow 0)$.

Lemma 6 *Let L_N be a neutrosophic $\mathcal{N}-$subalgebra of an L-algebra L. If there exists a sequence $\{u_n\}$ in L such that* $\lim_{n\longrightarrow\infty} T_N(u_n) = 0$ *and* $\lim_{n\longrightarrow\infty} I_N(u_n) = -1 = \lim_{n\longrightarrow\infty} F_N(u_n)$, *then* $T_N(1) = 0$ *and* $I_N(1) = -1 = F_N(1)$.

Proof Let L_N be a neutrosophic $\mathcal{N}-$subalgebra of L. Suppose that there exists a sequence $\{u_n\}$ in L such that $\lim_{n\longrightarrow\infty} T_N(u_n) = 0$ *and* $\lim_{n\longrightarrow\infty} I_N(u_n) = -1 = \lim_{n\longrightarrow\infty} F_N(u_n)$. Since $T_N(u_n) \leq T_N(1)$, $I_N(1) \leq I_N(u_n)$ and $F_N(1) \leq F_N(u_n)$ from Lemma 5, we have that

$$0 = \lim_{n\longrightarrow\infty} T_N(u_n) \leq \lim_{n\longrightarrow\infty} T_N(1) = T_N(1) \leq 0,$$
$$-1 \leq I_N(1) = \lim_{n\longrightarrow\infty} I_N(1) \leq \lim_{n\longrightarrow\infty} I_N(u_n) = -1$$

and

$$-1 \leq F_N(1) = \lim_{n\longrightarrow\infty} F_N(1) \leq \lim_{n\longrightarrow\infty} F_N(u_n) = -1.$$

Therefore, $T_N(1) = 0$ and $I_N(1) = -1 = F_N(1)$. □

Lemma 7 *A neutrosophic $\mathcal{N}-$subalgebra L_N of an L-algebra L satisfies $T_N(x \longrightarrow y) \leq T_N(y)$, $I_N(y) \leq I_N(x \longrightarrow y)$ and $F_N(y) \leq F_N(x \longrightarrow y)$, for all $x, y \in L$ if and only if T_N, I_N and F_N are constant.*

Proof Let L_N be a neutrosophic $\mathcal{N}-$subalgebra of L such that $T_N(x \longrightarrow y) \leq T_N(y)$, $I_N(y) \leq I_N(x \longrightarrow y)$ and $F_N(y) \leq F_N(x \longrightarrow y)$, for any $x, y \in L$. Since $T_N(1) = T_N(x \longrightarrow x) \leq T_N(x)$, $I_N(x) \leq I_N(x \longrightarrow x) = I_N(1)$ and $F_N(x) \leq F_N(x \longrightarrow x) = F_N(1)$ from (L1), we get from Lemma 5 that $T_N(x) = T_N(1)$, $I_N(x) = I_N(1)$ and $F_N(x) = F_N(1)$, for all $x \in L$. Hence, T_N, I_N and F_N are constant. Conversely, it is obvious since T_N, I_N and F_N are constant. □

Definition 8 A neutrosophic $\mathcal{N}-$structure L_N on an L-algebra L is called a neutrosophic $\mathcal{N}-$ideal of L if it satisfies the following conditions for all $x, y \in L$:

(NI1) $T_N(x) \leq T_N(1)$, $I_N(1) \leq I_N(x)$ *and* $F_N(1) \leq F_N(x)$,

(NI2) $\min\{T_N(x), T_N(x \longrightarrow y)\} \leq T_N(y)$, $I_N(y) \leq \max\{I_N(x), I_N(x \longrightarrow y)\}$ and $F_N(y) \leq \max\{F_N(x), F_N(x \longrightarrow y)\}$

(NI3) $T_N(x) \leq T_N((x \longrightarrow y) \longrightarrow y)$, $I_N((x \longrightarrow y) \longrightarrow y) \leq I_N(x)$ and $F_N((x \longrightarrow y) \longrightarrow y) \leq F_N(x)$,

(NI4) $T_N(x) \leq \min\{T_N(y \longrightarrow x), T_N(y \longrightarrow (x \longrightarrow y))\}$, $\max\{I_N(y \longrightarrow x), I_N(y \longrightarrow (x \longrightarrow y))\} \leq I_N(x)$ and $\max\{F_N(y \longrightarrow x), F_N(y \longrightarrow (x \longrightarrow y))\} \leq F_N(x)$.

Example 3 Consider the L-algebra L in Example 1. Then a neutrosophic $\mathcal{N}-$structure

$$L_N = \left\{ \frac{x}{(0, -1, -0.4)}, \frac{y}{(-1, -0.17, 0)}, \frac{1}{(0, -1, -0.4)} \right\}$$

on L is a neutrosophic $\mathcal{N}-$ideal of L.

Lemma 8 *Let L_N be a neutrosophic $\mathcal{N}-$structure on an L-algebra L. If L_N is a neutrosophic $\mathcal{N}-$ideal of L, then*

$$\begin{aligned} x \leq y \ implies \ T_N(x) \leq T_N(y),\ I_N(y) \leq I_N(x) \\ and \ F_N(y) \leq F_N(x), \end{aligned} \tag{3}$$

for any $x, y \in L$.

Proof Let L_N be a neutrosophic $\mathcal{N}-$ideal of L and $x \leq y$. Since $x \longrightarrow y = 1$ from Lemma 1, it follows from (NI1) and (NI2) that $T_N(x) = \min\{T_N(x), T_N(1)\} = \min\{T_N(x), T_N(x \longrightarrow y)\} \leq T_N(y)$, $I_N(y) \leq \max\{I_N(x), I_N(x \longrightarrow y)\} = \max\{I_N(x), I_N(1)\} = I_N(x)$ and $F_N(y) \leq \max\{F_N(x), F_N(x \longrightarrow y)\} = \max\{F_N(x), F_N(1)\} = F_N(x)$, for any $x, y \in L$. □

However, the inverse of Lemma 8 does not usually hold.

Example 4 Consider the L-algebra L in Example 2. Then a neutrosophic $\mathcal{N}-$structure $L_N = \{\frac{a}{(0, -1, -0.7)} : a \in L - \{0\}\} \cup \{\frac{0}{(-0.03, -0.1, -0.32)}\}$ on L satisfies the condition (3) but it is not a neutrosophic $\mathcal{N}-$ideal of L since $\min\{T_N(x), T_N(x \longrightarrow 0)\} = \min\{T_N(x), T_N(y)\} = 0 > -0.03 = T_N(0)$.

Lemma 9 *Let L_N be a neutrosophic $\mathcal{N}-$ideal of an L-algebra L satisfying $y \leq x \longrightarrow y$, for all $x, y \in L$. Then*

$$\left(\begin{array}{c} T_N(x \longrightarrow y) \leq T_N((x \longrightarrow y) \longrightarrow y), \\ I_N((x \longrightarrow y) \longrightarrow y) \leq I_N(x \longrightarrow y) \\ \text{and} \\ F_N((x \longrightarrow y) \longrightarrow y) \leq F_N(x \longrightarrow y) \end{array}\right)$$

if and only if

$$\left(\begin{array}{c} T_N((x \longrightarrow y) \longrightarrow z) \leq T_N((y \longrightarrow z) \longrightarrow (x \longrightarrow z)), \\ I_N((y \longrightarrow z) \longrightarrow (x \longrightarrow z)) \leq I_N((x \longrightarrow y) \longrightarrow z) \\ \text{and} \\ F_N((y \longrightarrow z) \longrightarrow (x \longrightarrow z)) \leq F_N((x \longrightarrow y) \longrightarrow z), \end{array}\right)$$

for all $x, y, z \in L$.

Proof Let L_N be a neutrosophic $\mathcal{N}-$ideal of L. Suppose that

$$\left(\begin{array}{c} T_N(x \longrightarrow y) \leq T_N((x \longrightarrow y) \longrightarrow y), \\ I_N((x \longrightarrow y) \longrightarrow y) \leq I_N(x \longrightarrow y) \text{ and} \\ F_N((x \longrightarrow y) \longrightarrow y) \leq F_N(x \longrightarrow y) \end{array}\right)$$

for any $x, y, z \in L$. Since $y \leq x \longrightarrow y$ and $z \leq x \longrightarrow z$ from Lemma 2 (1), we have from Lemma 2 (2) that $(x \longrightarrow y) \longrightarrow z \leq y \longrightarrow z \leq y \longrightarrow (x \longrightarrow z)$ and $(y \longrightarrow (x \longrightarrow z)) \longrightarrow (x \longrightarrow z) \leq (y \longrightarrow z) \longrightarrow (x \longrightarrow z)$. Then it follows from Lemma 8 that

$$\begin{aligned} T_N((x \longrightarrow y) \longrightarrow z) &\leq T_N(y \longrightarrow (x \longrightarrow z)) \\ &\leq T_N((y \longrightarrow (x \longrightarrow z)) \\ &\longrightarrow (x \longrightarrow z)) \\ &\leq T_N((y \longrightarrow z) \longrightarrow (x \longrightarrow z)), \\ I_N((y \longrightarrow z) \longrightarrow (x \longrightarrow z)) &\leq I_N((y \longrightarrow (x \longrightarrow z)) \\ &\longrightarrow (x \longrightarrow z)) \\ &\leq I_N(y \longrightarrow (x \longrightarrow z)) \\ &\leq I_N((x \longrightarrow y) \longrightarrow z) \end{aligned}$$

and similarly,

$$\begin{aligned} F_N((y \longrightarrow z) \longrightarrow (x \longrightarrow z)) &\leq F_N((y \longrightarrow (x \longrightarrow z)) \\ &\longrightarrow (x \longrightarrow z)) \\ &\leq F_N(y \longrightarrow (x \longrightarrow z)) \\ &\leq F_N((x \longrightarrow y) \longrightarrow z), \end{aligned}$$

for all $x, y, z \in L$.

Conversely, assume that

$$\begin{pmatrix} T_N((x \longrightarrow y) \longrightarrow z) \le T_N((y \longrightarrow z) \longrightarrow (x \longrightarrow z)), \\ I_N((y \longrightarrow z) \longrightarrow (x \longrightarrow z)) \le I_N((x \longrightarrow y) \longrightarrow z) \\ and \\ F_N((y \longrightarrow z) \longrightarrow (x \longrightarrow z)) \le F_N((x \longrightarrow y) \longrightarrow z), \end{pmatrix}$$

for any $x, y, z \in L$. By substituting $[x := 1]$, $[y := x]$ and $[z := y]$ in the assumption, simultaneously, it is obtained from (L1) that

$$\begin{aligned} T_N(x \longrightarrow y) &= T_N((1 \longrightarrow x) \longrightarrow y) \\ &\le T_N((x \longrightarrow y) \longrightarrow (1 \longrightarrow y)) \\ &= T_N((x \longrightarrow y) \longrightarrow y), \\ I_N((x \longrightarrow y) \longrightarrow y) &= I_N((x \longrightarrow y) \longrightarrow (1 \longrightarrow y)) \\ &\le I_N((1 \longrightarrow x) \longrightarrow y) \\ &= I_N(x \longrightarrow y) \end{aligned}$$

and

$$\begin{aligned} F_N((x \longrightarrow y) \longrightarrow y) &= F_N((x \longrightarrow y) \longrightarrow (1 \longrightarrow y)) \\ &\le F_N((1 \longrightarrow x) \longrightarrow y) \\ &= F_N(x \longrightarrow y), \end{aligned}$$

for all $x, y \in L$. □

Theorem 10 *Let L_N be a neutrosophic $\mathcal{N}$−structure on an L-algebra L and π, ρ, σ be any elements of $[-1, 0]$ which implies $-3 \le \pi + \rho + \sigma \le 0$. If L_N is a neutrosophic $\mathcal{N}$−ideal of L, then the nonempty (π, ρ, σ)-level set $L_N(\pi, \rho, \sigma)$ of L_N is an ideal of L.*

Proof Let L_N be a neutrosophic $\mathcal{N}$−ideal of L and $L_N(\pi, \rho, \sigma) \neq \emptyset$, for $\pi, \rho, \sigma \in [-1, 0]$ which implies $-3 \le \pi + \rho + \sigma \le 0$. Assume that $x \in L_N(\pi, \rho, \sigma)$. Since $\pi \le T_N(x) \le T_N(1)$, $I_N(1) \le I_N(x) \le \rho$ and $F_N(1) \le F_N(x) \le \sigma$ from (NI1), it follows that $1 \in L_N(\pi, \rho, \sigma)$. Suppose that $x, x \longrightarrow y \in L_N(\pi, \rho, \sigma)$. Since $\pi \le T_N(x), \pi \le T_N(x \longrightarrow y)$, $I_N(x) \le \rho$, $I_N(x \longrightarrow y) \le \rho$, $F_N(x) \le \sigma$, $F_N(x \longrightarrow y) \le \sigma$, it is obtained from (NI2) that $\pi \le \min\{T_N(x), T_N(x \longrightarrow y)\} \le T_N(y)$, $I_N(y) \le \max\{I_N(x), I_N(x \longrightarrow y)\} \le \rho$ and $F_N(y) \le \max\{F_N(x), F_N(x \longrightarrow y)\} \le \sigma$. Thus, $y \in L_N(\pi, \rho, \sigma)$. Let $x \in L_N(\pi, \rho, \sigma)$. Since $\pi \le T_N(x) \le T_N((x \longrightarrow y) \longrightarrow y)$, $I_N((x \longrightarrow y) \longrightarrow y) \le I_N(x) \le \rho$ and $F_N((x \longrightarrow y) \longrightarrow y) \le F_N(x) \le \sigma$ from (NI3), we get that $(x \longrightarrow y) \longrightarrow y \in L_N(\pi, \rho, \sigma)$. Also, $y \longrightarrow x, y \longrightarrow (x \longrightarrow y) \in L_N(\pi, \rho, \sigma)$ since $\pi \le T_N(x) \le \min\{T_N(y \longrightarrow x), T_N(y \longrightarrow (x \longrightarrow y))\}$, $\max\{I_N(y \longrightarrow x), I_N(y \longrightarrow (x \longrightarrow y))\} \le I_N(x) \le \rho$ and $\max\{F_N(y \longrightarrow x), F_N(y \longrightarrow (x \longrightarrow y))\} \le F_N(x) \le \sigma$. Hence, $L_N(\pi, \rho, \sigma)$ is an ideal of L. □

Theorem 11 *Let L_N be a neutrosophic $\mathcal{N}-$structure on an L-algebra L and $T_N^{\pi}, I_N^{\rho}, F_N^{\sigma}$ be ideals of L, for all $\pi, \rho, \sigma \in [-1, 0]$ which implies $-3 \leq \pi+\rho+\sigma \leq 0$. Then L_N is a neutrosophic $\mathcal{N}-$ideal of L.*

Proof Let L_N be a neutrosophic $\mathcal{N}-$structure on L and $T_N^{\pi}, I_N^{\rho}, F_N^{\sigma}$ be ideals of L, for all $\pi, \rho, \sigma \in [-1, 0]$ which implies $-3 \leq \pi + \rho + \sigma \leq 0$. Suppose that $T_N(1) < T_N(x)$, $I_N(x) < I_N(1)$ and $F_N(x) < F_N(1)$, for some $x \in L$. If $\pi = \frac{1}{2}(T_N(x)+T_N(1))$, $\rho = \frac{1}{2}(I_N(x)+I_N(1))$ and $\sigma = \frac{1}{2}(F_N(x)+F_N(1))$ are elements in $[-1, 0)$, then $T_N(1) < \pi < T_N(x)$, $I_N(x) < \rho < I_N(1)$ and $F_N(x) < \sigma < F_N(1)$. Hence, $1 \notin T_N^{\pi}, I_N^{\rho}, F_N^{\sigma}$ which is a contradiction with (I1). So, $T_N(x) \leq T_N(1)$, $I_N(1) \leq I_N(x)$ and $F_N(1) \leq F_N(x)$, for all $x \in L$. Assume that

$$\pi_1 = T_N(y) < \min\{T_N(x), T_N(x \longrightarrow y)\} = \pi_2,$$
$$\rho_1 = \max\{I_N(x), I_N(x \longrightarrow y)\} < I_N(y) = \rho_2,$$

and

$$\sigma_1 = \max\{F_N(x), F_N(x \longrightarrow y)\} < F_N(y) = \sigma_2.$$

If $\pi^{'} = \frac{1}{2}(\pi_1+\pi_2)$, $\rho^{'} = \frac{1}{2}(\rho_1+\rho_2)$ and $\sigma^{'} = \frac{1}{2}(\sigma_1+\sigma_2)$ are elements in $[-1, 0)$, then $\pi_1 < \pi^{'} < \pi_2$, $\rho_1 < \rho^{'} < \rho_2$ and $\sigma_1 < \sigma^{'} < \sigma_2$. Thus, $x, x \longrightarrow y \in T_N^{\pi^{'}}, I_N^{\rho^{'}}, F_N^{\sigma^{'}}$ but $y \notin T_N^{\pi^{'}}, I_N^{\rho^{'}}, F_N^{\sigma^{'}}$, which is a contradiction with (I2). Thereby,

$$\min\{T_N(x), T_N(x \longrightarrow y)\} \leq T_N(y),$$
$$I_N(y) \leq \max\{I_N(x), I_N(x \longrightarrow y)\}$$

and

$$F_N(y) \leq \max\{F_N(x), F_N(x \longrightarrow y)\}$$

for all $x, y \in L$. Suppose that $\pi_a = T_N((x \longrightarrow y) \longrightarrow y) < T_N(x) = \pi_b$, $\rho_a = I_N(x) < I_N((x \longrightarrow y) \longrightarrow y) = \rho_b$ and $\sigma_a = F_N(x) < F_N((x \longrightarrow y) \longrightarrow y) = \sigma_b$. If $\pi^{''} = \frac{1}{2}(\pi_a + \pi_b)$, $\rho^{''} = \frac{1}{2}(\rho_a + \rho_b)$ and $\sigma^{''} = \frac{1}{2}(\sigma_a + \sigma_b)$ are elements in $[-1, 0)$, then $\pi_a < \pi^{''} < \pi_b$, $\rho_a < \rho^{''} < \rho_b$ and $\sigma_a < \sigma^{''} < \sigma_b$. Hence, $x \in T_N^{\pi^{''}}, I_N^{\rho^{''}}, F_N^{\sigma^{''}}$ but $(x \longrightarrow y) \longrightarrow y \notin T_N^{\pi^{''}}, I_N^{\rho^{''}}, F_N^{\sigma^{''}}$, which is a contradiction with (I3), and so, $T_N(x) \leq T_N((x \longrightarrow y) \longrightarrow y)$, $I_N((x \longrightarrow y) \longrightarrow y) \leq I_N(x)$ and $F_N((x \longrightarrow y) \longrightarrow y) \leq F_N(x)$, for all $x, y \in L$. Assume that $\pi_u = \min\{T_N(y \longrightarrow x), T_N(y \longrightarrow (x \longrightarrow y))\} < T_N(x) = \pi_v$, $\rho_u = I_N(x) < \max\{I_N(y \longrightarrow x), I_N(y \longrightarrow (x \longrightarrow y))\} = \rho_v$ and $\sigma_u = F_N(x) < \max\{F_N(y \longrightarrow x), F_N(y \longrightarrow (x \longrightarrow y))\} = \sigma_v$. If $\pi^* = \frac{1}{2}(\pi_u + \pi_v)$, $\rho^* = \frac{1}{2}(\rho_u + \rho_v)$ and $\sigma^* = \frac{1}{2}(\sigma_u + \sigma_v)$ are elements in $[-1, 0)$, then $\pi_u < \pi^* < \pi_v$, $\rho_u < \rho^* < \rho_v$

and $\sigma_u < \sigma^* < \sigma_v$. Thus, $x \in T_N^{\pi^*}, I_N^{\rho^*}, F_N^{\sigma^*}$ but $y \longrightarrow x, y \longrightarrow (x \longrightarrow y) \notin T_N^{\pi^*}, I_N^{\rho^*}, F_N^{\sigma^*}$, which is a contradiction with (I4), and so, $T_N(x) \leq \min\{T_N(y \longrightarrow x), T_N(y \longrightarrow (x \longrightarrow y))\}$, $\max\{I_N(y \longrightarrow x), I_N(y \longrightarrow (x \longrightarrow y))\} \leq I_N(x)$ and $\max\{F_N(y \longrightarrow x), F_N(y \longrightarrow (x \longrightarrow y))\} \leq F_N(x)$, for all $x, y \in L$. Therefore, L_N is a neutrosophic $\mathcal{N}-$ideal of L. □

Theorem 12 *Let L_N be a neutrosophic $\mathcal{N}-$ideal of an L-algebra L. Then*

$$\begin{aligned} x \leq y \longrightarrow z \textit{ implies } &\min\{T_N(x), T_N(y)\} \leq T_N(z), \\ I_N(z) &\leq \max\{I_N(x), I_N(y)\} \\ \textit{and } F_N(z) &\leq \max\{F_N(x), F_N(y)\}, \end{aligned} \tag{4}$$

for all $x, y, z \in L$.

Proof Let L_N be a neutrosophic $\mathcal{N}-$ideal of L and $x \leq y \longrightarrow z$. Then $\min\{T_N(x), T_N(y)\} \leq \min\{T_N(y), T_N(y \longrightarrow z)\} \leq T_N(z)$, $I_N(z) \leq \max\{I_N(y), I_N(y \longrightarrow z)\} \leq \max\{I_N(x), I_N(y)\}$ and $F_N(z) \leq \max\{F_N(y), F_N(y \longrightarrow z)\} \leq \max\{F_N(x), I_N(y)\}$ from Lemma 8 and (NI2). □

Theorem 13 *Let L be an L-algebra such that $x \leq (x \longrightarrow y) \longrightarrow y$ and $x \leq y \longrightarrow x$, for all $x, y \in L$, and let L_N be a neutrosophic $\mathcal{N}-$structure of L satisfying the condition (4). Then L_N is a neutrosophic $\mathcal{N}-$ideal of L.*

Proof Let L be an L-algebra such that $x \leq (x \longrightarrow y) \longrightarrow y$ and $x \leq y \longrightarrow x$, for any $x, y \in L$, and let L_N be a neutrosophic $\mathcal{N}-$structure of L satisfying the condition (4). Then

(NI1): Since $x \leq 1 = x \longrightarrow 1$ from (L1), we have from the condition (4) that $T_N(x) = \min\{T_N(x), T_N(x)\} \leq T_N(1)$, $I_N(1) \leq \max\{I_N(x), I_N(x)\} = I_N(x)$ and $F_N(1) \leq \max\{F_N(x), F_N(x)\} = F_N(x)$, for all $x \in L$.

(NI2): Since $x \leq (x \longrightarrow y) \longrightarrow y$, it follows from the condition (4) that $\min\{T_N(x), T_N(x \longrightarrow y)\} \leq T_N(y)$, $I_N(y) \leq \max\{I_N(x), I_N(x \longrightarrow y)\}$ and $F_N(y) \leq \max\{F_N(x), F_N(x \longrightarrow y)\}$, for all $x, y \in L$.

(NI3): Since $x \leq (x \longrightarrow y) \longrightarrow y = 1 \longrightarrow ((x \longrightarrow y) \longrightarrow y)$ from (L1), it is obtained from the condition (4) and (NI1) that $T_N(x) = \min\{T_N(x), T_N(1)\} \leq T_N((x \longrightarrow y) \longrightarrow y)$, $I_N((x \longrightarrow y) \longrightarrow y) \leq \max\{I_N(x), I_N(1)\} = I_N(x)$ and $F_N((x \longrightarrow y) \longrightarrow y) \leq \max\{F_N(x), F_N(1)\} = F_N(x)$,

(NI4) Since $x \leq y \longrightarrow x = 1 \longrightarrow (y \longrightarrow x)$ from (L1), we get from (NI1) and the condition (4) that

$$\begin{aligned} T_N(x) &= \min\{T_N(x), T_N(1)\} \\ &\leq T_N(y \longrightarrow x) \\ &= \min\{T_N(y \longrightarrow x), T_N(1)\} \\ &= \min\{T_N(y \longrightarrow x), T_N(y \longrightarrow (x \longrightarrow y))\}, \\ \max\{I_N(y \longrightarrow x), &I_N(y \longrightarrow (x \longrightarrow y))\} \\ &= \max\{I_N(y \longrightarrow x), I_N(1)\} \end{aligned}$$

$$\begin{aligned} &= I_N(y \longrightarrow x) \\ &\leq \max\{I_N(x), I_N(1)\} \\ &= I_N(x) \end{aligned}$$

and

$$\begin{aligned} &\max\{F_N(y \longrightarrow x), F_N(y \longrightarrow (x \longrightarrow y))\} \\ &\quad = \max\{F_N(y \longrightarrow x), F_N(1)\} \\ &\quad = F_N(y \longrightarrow x) \\ &\quad \leq \max\{F_N(x), F_N(1)\} \\ &\quad = F_N(x), \end{aligned}$$

for all $xiy \in L$.

Therefore, L_N is a neutrosophic $\mathcal{N}-$ideal of L. □

Theorem 14 *Let $(L; \longrightarrow_L, 1_L)$ and $(K, \longrightarrow_K, 1_K)$ be L-algebras, $f : L \longrightarrow K$ be a surjective homomorphism and $K_N = \dfrac{K}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}-$structure on K. Then K_N is a neutrosophic $\mathcal{N}-$ideal of K if and only if $K_N^f = \dfrac{L}{(T_N^f, I_N^f, F_N^f)}$ is a neutrosophic $\mathcal{N}-$ideal of L where the $\mathcal{N}-$functions $T_N^f, I_N^f, F_N^f : L \longrightarrow [-1, 0]$ on L are defined by $T_N^f(x) = T_N(f(x))$, $I_N^f(x) = I_N(f(x))$ and $F_N^f(x) = F_N(f(x))$, for all $x \in L$, respectively.*

Proof Let $(L; \longrightarrow_L, 1_L)$ and $(K, \longrightarrow_K, 1_K)$ be L-algebras, $f : L \longrightarrow K$ be a surjective homomorphism and $K_N = \dfrac{K}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}-$ideal of K. Then $T_N^f(x) = T_N(f(x)) = T_N(a) \leq T_N(1_K) = T_N(f(1_L)) = T_N^f(1_L)$, $I_N^f(1_L) = I_N(f(1_L)) = I_N(1_K) \leq I_N(a) = I_N(f(x)) = I_N^f(x)$ and $F_N^f(1_L) = F_N(f(1_L)) = F_N(1_K) \leq F_N(a) = F_N(f(x)) = F_N^f(x)$, for all $x \in L$. Also,

$$\begin{aligned} &\min\{T_N^f(x), T_N^f(x \longrightarrow_L y)\} \\ &\quad = \min\{T_N(f(x)), T_N(f(x \longrightarrow_L y))\} \\ &\quad = \min\{T_N(f(x)), T_N(f(x) \longrightarrow_K f(y))\} \\ &\quad = \min\{T_N(a), T_N(a \longrightarrow_K b)\} \\ &\quad \leq T_N(b) \\ &\quad = T_N(f(y)) \\ &\quad = T_N^f(y), \\ &I_N^f(y) = I_N(f(y)) \\ &\quad = I_N(b) \\ &\quad \leq \max\{I_N(a), I_N(a \longrightarrow_K b)\} \end{aligned}$$

$$
\begin{aligned}
&= \max\{I_N(f(x)), I_N(f(x) \longrightarrow_K f(y))\} \\
&= \max\{I_N(f(x)), I_N(f(x \longrightarrow_L y))\} \\
&= \max\{I_N^f(x), I_N^f(x \longrightarrow_L y)\},
\end{aligned}
$$

and similarly, $F_N^f(y) \leq \max\{F_N^f(x), F_N^f(x \longrightarrow_L y)\}$, for all $x, y \in L$. Moreover,

$$
\begin{aligned}
T_N^f(x) &= T_N(f(x)) \\
&= T_N(a) \\
&\leq T_N((a \longrightarrow_K b) \longrightarrow_K b) \\
&= T_N((f(x) \longrightarrow_K f(y)) \longrightarrow_K f(y)) \\
&= T_N(f((x \longrightarrow_L y) \longrightarrow_L y)) \\
&= T_N^f((x \longrightarrow_L y) \longrightarrow_L y), \\
I_N^f((x \longrightarrow_L y) \longrightarrow_L y) &= I_N(f((x \longrightarrow_L y) \longrightarrow_L y)) \\
&= I_N((f(x) \longrightarrow_K \\
&\quad f(y)) \longrightarrow_K f(y)) \\
&= I_N((a \longrightarrow_K b) \longrightarrow_K b) \\
&\leq I_N(a) \\
&= I_N(f(x)) \\
&= I_N^f(x),
\end{aligned}
$$

and similarly, $F_N^f((x \longrightarrow_L y) \longrightarrow_L y) \leq F_N^f(x)$, for all $x, y \in L$. Besides,

$$
\begin{aligned}
&T_N^f(x) = T_N(f(x)) \\
&\quad = T_N(a) \\
&\quad \leq \min\{T_N(b \longrightarrow_K a), \\
&\qquad T_N(b \longrightarrow_K (a \longrightarrow_K b))\} \\
&\quad = \min\{T_N(f(y) \longrightarrow_K f(x)), \\
&\qquad T_N(f(y) \longrightarrow_K (f(x) \longrightarrow_K f(y)))\} \\
&\quad = \min\{T_N(f(y \longrightarrow_L x)), \\
&\qquad T_N(f(y \longrightarrow_L (x \longrightarrow_L y)))\} \\
&\quad = \min\{T_N^f(y \longrightarrow_L x), \\
&\qquad T_N^f(y \longrightarrow_L (x \longrightarrow_L y))\}, \\
&\max\{I_N^f(y \longrightarrow_L x), I_N^f(y \longrightarrow_L (x \longrightarrow_L y))\} \\
&\quad = \max\{I_N(f(y \longrightarrow_L x)), \\
&\qquad I_N(f(y \longrightarrow_L (x \longrightarrow_L y)))\} \\
&\quad = \max\{I_N(f(y) \longrightarrow_K f(x)), \\
&\qquad I_N(f(y) \longrightarrow_K (f(x) \longrightarrow_K f(y)))\}
\end{aligned}
$$

$$
\begin{aligned}
&= \max\{I_N(b \longrightarrow_K a), \\
&\quad I_N(b \longrightarrow_K (a \longrightarrow_K b))\} \\
&\leq I_N(a) \\
&= I_N(f(x)) \\
&= I_N^f(x),
\end{aligned}
$$

and similarly, $\max\{F_N^f(y \longrightarrow_L x), F_N^f(y \longrightarrow_L (x \longrightarrow_L y))\} \leq F_N^f(x)$, for all $x, y \in L$. Therefore, $K_N^f = \dfrac{L}{(T_N^f, I_N^f, F_N^f)}$ is a neutrosophic $\mathcal{N}-$ideal of L.

Conversely, let K_N^f be a neutrosophic $\mathcal{N}-$ideal of L. So, $T_N(a) = T_N(f(x)) = T_N^f(x) \leq T_N^f(1_L) = T_N(f(1_L)) = T_N(1_K)$, $I_N(1_K) = I_N(f(1_L)) = I_N^f(1_L) \leq I_N^f(x) = I_N(f(x)) = I_N(a)$ and $F_N(1_K) = F_N(f(1_L)) = F_N^f(1_L) \leq F_N^f(x) = F_N(f(x)) = F_N(a)$, for all $a \in K$. Moreover,

$$
\begin{aligned}
&\min\{T_N(a), T_N(a \longrightarrow_K b)\} \\
&\quad = \min\{T_N(f(x)), T_N(f(x) \longrightarrow_K f(y))\} \\
&\quad = \min\{T_N^f(x), T_N^f(x \longrightarrow_L y)\} \\
&\quad \leq T_N^f(y) \\
&\quad = T_N(f(y)) \\
&\quad = T_N(b), \\
&I_N(b) = I_N(f(y)) \\
&\quad = I_N^f(y) \\
&\quad \leq \max\{I_N^f(x), I_N^f(x \longrightarrow_L y)\} \\
&\quad = \max\{I_N(f(x)), I_N(f(x) \longrightarrow_K f(y))\} \\
&\quad = \max\{I_N(a), I_N(a \longrightarrow_K b)\},
\end{aligned}
$$

and similarly, $F_N(b) \leq \max\{F_N(a), F_N(a \longrightarrow_K b)\}$, for all $a, b \in K$. Besides,

$$
\begin{aligned}
T_N(a) &= T_N(f(x)) \\
&= T_N^f(x) \\
&\leq T_N^f((x \longrightarrow_L y) \longrightarrow_L y) \\
&= T_N(f((x \longrightarrow_L y) \longrightarrow_L y)) \\
&= T_N((f(x) \longrightarrow_K f(y)) \longrightarrow_K f(y)) \\
&= T_N((a \longrightarrow_K b) \longrightarrow_K b), \\
I_N((a \longrightarrow_K b) \longrightarrow_K b) &= I_N((f(x) \longrightarrow_K \\
&\quad f(y)) \longrightarrow_K f(y)) \\
&= I_N(f((x \longrightarrow_L y) \longrightarrow_L y)) \\
&= I_N^f((x \longrightarrow_L y) \longrightarrow_L y)
\end{aligned}
$$

$$\begin{aligned} &\leq I_N^f(x) \\ &= I_N(f(x)) \\ &= I_N(a), \end{aligned}$$

and similarly, $F_N((a \longrightarrow_K b) \longrightarrow_K b) \leq F_N(a)$, for all $a, b \in K$. Also,

$$\begin{aligned} T_N(a) &= T_N(f(x)) \\ &= T_N^f(x) \\ &\leq \min\{T_N^f(y \longrightarrow_L x), \\ &\quad T_N^f(y \longrightarrow_L (x \longrightarrow_L y))\} \\ &= \min\{T_N(f(y \longrightarrow_L x)), \\ &\quad T_N(f(y \longrightarrow_L (x \longrightarrow_L y)))\} \\ &= \min\{T_N(f(y) \longrightarrow_K f(x)), \\ &\quad T_N(f(y) \longrightarrow_K (f(x) \longrightarrow_K f(y)))\} \\ &= \min\{T_N(b \longrightarrow_K a), \\ &\quad T_N(b \longrightarrow_K (a \longrightarrow_K b))\}, \end{aligned}$$

$$\begin{aligned} &\max\{I_N(b \longrightarrow_K a), I_N(b \longrightarrow_K (a \longrightarrow_K b))\} \\ &\quad = \max\{I_N(f(y) \longrightarrow_K f(x)), \\ &\qquad I_N(f(y) \longrightarrow_K (f(x) \longrightarrow_K f(y)))\} \\ &\quad = \max\{I_N(f(y \longrightarrow_L x)), \\ &\qquad I_N(f(y \longrightarrow_L (x \longrightarrow_L y)))\} \\ &\quad = \max\{I_N^f(y \longrightarrow_L x), \\ &\qquad I_N^f(y \longrightarrow_L (x \longrightarrow_L y))\} \\ &\quad \leq I_N^f(x) \\ &\quad = I_N(f(x)) \\ &\quad = I_N(a), \end{aligned}$$

and similarly, $\max\{F_N(b \longrightarrow_K a), F_N(b \longrightarrow_K (a \longrightarrow_K b))\} \leq F_N(a)$, for all $a, b \in K$. Thereby, K_N is a neutrosophic $\mathcal{N}-$ideal of K. □

Theorem 15 *Every neutrosophic $\mathcal{N}-$ideal of an L-algebra L is a neutrosophic $\mathcal{N}-$subalgebra of L.*

Proof Let L_N be a neutrosophic $\mathcal{N}-$ideal of L. Since

$$\begin{aligned} &\min\{T_N(x), T_N(y)\} \\ &\quad \leq \min\{\min\{T_N(y \longrightarrow x), T_N(y \longrightarrow (x \longrightarrow y))\}, T_N(y)\} \\ &\quad \leq \min\{T_N(y), T_N(y \longrightarrow (x \longrightarrow y))\} \\ &\quad \leq T_N(x \longrightarrow y), \end{aligned}$$

$$\begin{aligned} I_N(x \longrightarrow y) &\leq \max\{I_N(y), I_N(y \longrightarrow (x \longrightarrow y))\} \\ &\leq \max\{\max\{I_N(y \longrightarrow x), \\ &\quad I_N(y \longrightarrow (x \longrightarrow y))\}, I_N(y)\} \\ &\leq \max\{I_N(x), I_N(y)\}, \end{aligned}$$

and similarly, $F_N(x \longrightarrow y) \leq \max\{F_N(x), F_N(y)\}$ from (NI2) and (NI4), it is obtained that L_N is a neutrosophic $\mathcal{N}-$subalgebra of L. □

The inverse of Theorem 15 is mostly not true.

Example 5 Consider the L-algebra L in Example 2. Then a neutrosophic $\mathcal{N}-$structure $L_N = \{\frac{0}{(-0.79, 0, -0.23)}\} \cup \{\frac{u}{(-0.1, -1, -0.8)} : u \in L - \{0\}\}$ on L is a neutrosophic $\mathcal{N}-$subalgebra of L but it is not a neutrosophic $\mathcal{N}-$ideal of L since $T_N(0) = -0.79 < -0.1 = \min\{T_N(x), T_N(y)\} = \min\{T_N(x), T_N(x \longrightarrow 0)\}$.

Lemma 16 *Let L_N be a neutrosophic $\mathcal{N}-$ideal of an L-algebra L. Then the subsets $L_{T_N} = \{x \in L : T_N(x) = T_N(1)\}$, $L_{I_N} = \{x \in L : I_N(x) = I_N(1)\}$ and $L_{F_N} = \{x \in L : F_N(x) = F_N(1)\}$ of L are ideals of L.*

Proof Let L_N be a neutrosophic $\mathcal{N}-$ideal of L. Then $1 \in L_{T_N}, L_{I_N}, L_{F_N}$. Assume that $x, x \longrightarrow y \in L_{T_N}, L_{I_N}, L_{F_N}$. Then $T_N(x) = T_N(1) = T_N(x \longrightarrow y)$, $I_N(x) = I_N(1) = I_N(x \longrightarrow y)$ and $F_N(x) = F_N(1) = F_N(x \longrightarrow y)$. Since

$$\begin{aligned} T_N(1) = \min\{T_N(x), T_N(x \longrightarrow y)\} \leq T_N(y), \\ I_N(y) \leq \max\{I_N(x), I_N(x \longrightarrow y)\} = I_N(1) \end{aligned}$$

and

$$F_N(y) \leq \max\{F_N(x), F_N(x \longrightarrow y)\} = F_N(1)$$

from (NI2), it is obtained from (NI1) that $T_N(y) = T_N(1)$, $I_N(y) = I_N(1)$ and $F_N(y) = F_N(1)$. So, $y \in L_{T_N}, L_{I_N}, L_{F_N}$. Suppose that $x \in L_{T_N}, L_{I_N}, L_{F_N}$. Since $T_N(1) = T_N(x) \leq T_N((x \longrightarrow y) \longrightarrow y)$, $I_N((x \longrightarrow y) \longrightarrow y) \leq I_N(x) = I_N(1)$ and $F_N((x \longrightarrow y) \longrightarrow y) \leq F_N(x) = F_N(1)$ from (NI3), it follows from (NI1) that $T_N((x \longrightarrow y) \longrightarrow y) = T_N(1)$, $I_N((x \longrightarrow y) \longrightarrow y) = I_N(1)$ and $F_N((x \longrightarrow y) \longrightarrow y) F_N(1)$, which imply that $(x \longrightarrow y) \longrightarrow y \in L_{T_N}, L_{I_N}, L_{F_N}$. Also, $T_N(1) = T_N(x) \leq \min\{T_N(y \longrightarrow x), T_N(y \longrightarrow (x \longrightarrow y))\}$, $\max\{I_N(y \longrightarrow x), I_N(y \longrightarrow (x \longrightarrow y))\} \leq I_N(x) = I_N(1)$ and $\max\{F_N(y \longrightarrow x), F_N(y \longrightarrow (x \longrightarrow y))\} \leq F_N(x) = F_N(1)$ from (NI4). We get from (NI1) that $\min\{T_N(y \longrightarrow x), T_N(y \longrightarrow (x \longrightarrow y))\} = T_N(1)$, $\max\{I_N(y \longrightarrow x), I_N(y \longrightarrow (x \longrightarrow y))\} = I_N(1)$ and $\max\{F_N(y \longrightarrow x), F_N(y \longrightarrow (x \longrightarrow y))\} = F_N(1)$, and so, $T_N(y \longrightarrow x) = T_N(y \longrightarrow (x \longrightarrow y)) = T_N(1)$, $I_N(y \longrightarrow x) = I_N(y \longrightarrow (x \longrightarrow y)) = I_N(1)$ and $F_N(y \longrightarrow x) = F_N(y \longrightarrow (x \longrightarrow y)) = F_N(1)$. Hence, $y \longrightarrow x, y \longrightarrow (x \longrightarrow y) \in L_{T_N}, L_{I_N}, L_{F_N}$. Therefore, L_{T_N}, L_{I_N} and L_{F_N} are ideals of L. □

Definition 9 Let L be an L-algebra. Define the subsets

$$L_N^{u_t} := \{x \in L : T_N(u_t) \leq T_N(x)\},$$
$$L_N^{u_i} := \{x \in L : I_N(x) \leq I_N(u_i)\}$$

and

$$L_N^{u_f} := \{x \in L : F_N(x) \leq F_N(u_f)\}$$

of L, for all $u_t, u_i, u_f \in L$. Moreover, $u_t \in L_N^{u_t}$, $u_i \in L_N^{u_i}$ and $u_f \in L_N^{u_f}$.

Example 6 Consider the L-algebra L in Example 1. Let

$$T_N(a) = \begin{cases} -0.7, & \text{if } a = 1 \\ 0, & \text{otherwise} \end{cases} \quad I_N(a) = \begin{cases} -1, & \text{if } a = x \\ 0, & \text{otherwise,} \end{cases}$$
$$F_N(a) = \begin{cases} -0.21, & \text{if } a = y \\ -0.69, & \text{otherwise,} \end{cases} \quad u_t = y, \ u_i = 1$$

and $u_f = x$. Then

$$L_N^{u_t} = \{a \in L : T_N(y) \leq T_N(a)\} = \{x, y\},$$
$$L_N^{u_i} = \{a \in L : I_N(a) \leq I_N(1)\} = L$$

and

$$L_N^{u_f} = \{a \in L : F_N(a) \leq F_N(x)\} = \{x, 1\}.$$

Theorem 17 *Let u_t, u_i and u_f be any elements of an L-algebra L. If L_N is a neutrosophic $\mathcal{N}-$ideal of L, then $L_N^{u_t}, L_N^{u_i}$ and $L_N^{u_f}$ are ideals of L.*

Proof Let u_t, u_i and u_f be any elements of L and L_N be a neutrosophic $\mathcal{N}-$ideal of L. Since $T_N(u_t) \leq T_N(1)$, $I_N(1) \leq I_N(u_i)$ and $F_N(1) \leq F_N(u_f)$ from (NI1), it follows that $1 \in L_N^{u_t}, L_N^{u_i}, L_N^{u_f}$. Assume that $x, x \longrightarrow y \in L_N^{u_t}, L_N^{u_i}, L_N^{u_f}$. Since $T_N(u_t) \leq T_N(x)$, $T_N(u_t) \leq T_N(x \longrightarrow y)$; $I_N(x) \leq I_N(u_i)$, $I_N(x \longrightarrow y) \leq I_N(u_i)$ and $F_N(x) \leq F_N(u_f)$, $F_N(x \longrightarrow y) \leq F_N(u_f)$, it is obtained from (NI2) that $T_N(u_t) \leq \min\{T_N(x), T_N(x \longrightarrow y)\} \leq T_N(y)$, $I_N(y) \leq \max\{I_N(x), I_N(x \longrightarrow y)\} \leq I_N(u_i)$ and $F_N(y) \leq \max\{F_N(x), F_N(x \longrightarrow y)\} \leq F_N(u_i)$, which imply that $y \in L_N^{u_t}, L_N^{u_i}, L_N^{u_f}$. Suppose that $x \in L_N^{u_t}, L_N^{u_i}, L_N^{u_f}$. Since $T_N(u_t) \leq T_N(x)$, $I_N(x) \leq I_N(u_i)$ and $F_N(x) \leq F_N(u_f)$, we have from (NI3) that $T_N(u_t) \leq T_N(x) \leq T_N((x \longrightarrow y) \longrightarrow y)$, $I_N((x \longrightarrow y) \longrightarrow y) \leq I_N(x) \leq I_N(u_i)$ and $F_N((x \longrightarrow y) \longrightarrow y) \leq F_N(x) \leq F_N(u_i)$. Hence, $(x \longrightarrow y) \longrightarrow y \in L_N^{u_t}, L_N^{u_i}, L_N^{u_f}$. Moreover, $T_N(u_t) \leq T_N(x) \leq \min\{T_N(y \longrightarrow x), T_N(y \longrightarrow (x \longrightarrow y))\}$, $\max\{I_N(y \longrightarrow x), I_N(y \longrightarrow (x \longrightarrow y))\} \leq I_N(x) \leq I_N(u_i)$ and $\max\{F_N(y \longrightarrow x), F_N(y \longrightarrow (x \longrightarrow y))\} \leq F_N(x) \leq F_N(u_f)$ from (NI4). Thus, $y \longrightarrow x, y \longrightarrow (x \longrightarrow y) \in L_N^{u_t}, L_N^{u_i}, L_N^{u_f}$. Therefore, $L_N^{u_t}, L_N^{u_i}, L_N^{u_f}$ are ideals of L. □

Example 7 Consider the L-algebra L in Example 1. For a neutrosophic $\mathcal{N}-$ideal $L_N = \{\frac{x}{(-0.7, -0.2, 0)}, \frac{y}{(-0.3, -0.8, -1)}, \frac{1}{(-0.3, -0.8, -1)}\}$ of L and elements $u_t = 1, u_i = y, u_f = x \in L$, the subsets

$$L_N^{u_t} = \{a \in L : T_N(1) \leq T_N(a)\} = \{y, 1\},$$
$$L_N^{u_i} = \{a \in L : I_N(a) \leq I_N(y)\} = \{y, 1\}$$

and

$$L_N^{u_f} = \{a \in L : F_N(a) \leq F_N(x)\} = L$$

of L are ideals of L.

Theorem 18 *Let u_t, u_i and u_f be any elements of an L-algebra L and L_N be a neutrosophic $\mathcal{N}-$structure on L. If $L_N^{u_t}, L_N^{u_i}$ and $L_N^{u_f}$ are ideals of L, then*

$$\begin{aligned} &T_N(z) \leq \min\{T_N(x), T_N(x \longrightarrow y)\} \Rightarrow T_N(z) \leq T_N(y), \\ &\max\{I_N(x), I_N(x \longrightarrow y)\} \leq I_N(z) \Rightarrow I_N(y) \leq I_N(z) \\ &\qquad\qquad\qquad \textit{and} \\ &\max\{F_N(x), F_N(x \longrightarrow y)\} \leq F_N(z) \Rightarrow F_N(y) \leq F_N(z), \end{aligned} \tag{5}$$

for all $x, y, z \in L$.

Proof Let u_t, u_i and u_f be any elements of L and L_N be a neutrosophic $\mathcal{N}-$structure on L. Suppose that $L_N^{u_t}, L_N^{u_i}$ and $L_N^{u_f}$ are ideals of L, and $T_N(z) \leq \min\{T_N(x), T_N(x \longrightarrow y)\}$, $\max\{I_N(x), I_N(x \longrightarrow y)\} \leq I_N(z)$ and $\max\{F_N(x), F_N(x \longrightarrow y)\} \leq F_N(z)$, for any $x, y, z \in L$. Since $x, x \longrightarrow y \in L_N^{u_t}, L_N^{u_i}, L_N^{u_f}$ in which $u_t = u_t = u_f = z$, it follows from (I2) that $y \in L_N^{u_t}, L_N^{u_i}, L_N^{u_f}$ where $u_t = u_t = u_f = z$. Then $T_N(z) \leq T_N(y)$, $I_N(y) \leq I_N(z)$ and $F_N(y) \leq F_N(z)$, for all $x, y, z \in L$. □

Example 8 Consider the L-algebra L in Example 2. Let

$$T_N(a) = \begin{cases} -0.92, & \text{if } a = 0, x \\ -0.1, & \text{otherwise,} \end{cases}$$
$$I_N(a) = \begin{cases} -0.53, & \text{if } a = 1 \\ -1, & \text{otherwise,} \end{cases}$$
$$F_N(a) = \begin{cases} -0.3, & \text{if } a = 0, y \\ -0.4, & \text{otherwise,} \end{cases}$$

and $u_t = y, u_i = 1\ u_f = x \in L$. Then the ideals

$$L_N^{u_t} = \{y, 1\}, L_N^{u_i} = L \text{ and } L_N^{u_f} = \{x, 1\}$$

of L satisfy the condition (5). However, $L_N(T_N, I_N, F_N)$ is not a neutrosophic $\mathcal{N}-$ideal of L since $I_N(a) < I_N(1)$, for all $a \in L - \{1\}$.

4 Conclusion

In this research, neutrosophic $\mathcal{N}-$subalgebras, neutrosophic $\mathcal{N}-$ideals and level-sets of neutrosophic $\mathcal{N}-$structures on L-algebras are proposed. First, we observe that the level-set of a neutrosophic $\mathcal{N}-$subalgebra (ideal) of an L-algebra is an L-subalgebra (ideal), and the converse of the statement always holds. Next, we show that the family of all neutrosophic $\mathcal{N}-$subalgebras of an L-algebra forms a complete distributive modular lattice. The particular cases at which $\mathcal{N}-$functions are constant have been examined, and some properties of neutrosophic $\mathcal{N}-$ideals of an L-algebra are obtained. Furthermore, we get a neutrosophic $\mathcal{N}-$ideal of an L-algebra from that of another L-algebra by employing a surjective homomorphism. Also, we highlight that every neutrosophic $\mathcal{N}-$ideal of an L-algebra is the neutrosophic $\mathcal{N}-$subalgebra however the inverse may not hold. To finalize, we show that the subsets L_{T_N}, L_{I_N} and L_{F_N} of an L-algebra are ideals for its neutrosophic $\mathcal{N}-$ideal, and illustrate that the subsets $L_N^{u_t}$, $L_N^{u_i}$ and $L_N^{u_f}$ of an L-algebra are ideals for a neutrosophic $\mathcal{N}-$ideal and any elements u_t, u_i, u_f of this algebraic structure.

As the continuiation of this research, we would like to focus on plithogenic structures of L-algebras.

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