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On Neutrosophic Generalized Semi Generalized Closed Sets

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Abstract

The article considers a new generalization of closed sets in neutrosophic topological space. This generalization is called neutrosophic $\mathcal{gsg}$-closed set. Moreover, we discuss its essential features in neutrosophic topological spaces. Furthermore, we extend the research by displaying new related definitions such as neutrosophic $\mathcal{gsg}$-closure and neutrosophic $\mathcal{gsg}$-interior and debating their powerful characterizations and relationships.

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1. Introduction

The neutrosophic set theory was contributed by Smarandache in [1,2]. The neutrosophic topological space (simply Neu^{TS}) was offered by Salama et al. in [3]. The definition of semi-α-open sets in neutrosophic topological spaces was displayed by Imran et al. in [4]. The neutrosophic generalized homeomorphism was submitted by PAGE et al. in [5]. The class of generalized neutrosophic closed sets was given by Dhavaseelan et al. in [6]. The concepts of neutrosophic generalized $\alpha\mathcal{g}$-closed sets and neutrosophic generalized $\alpha\mathcal{g}$-continuous functions were provided by Imran et al. in [7]. The objective of this article is to show the sense of neutrosophic $\mathcal{gsg}$-closed set (briefly $Neu^{\mathcal{gsg}}CS$) and investigate their main characteristics in Neu^{TS}. Moreover, we argue neutrosophic $\mathcal{gsg}$-closure (in word $Neu^{\mathcal{gsg}}$-closure) and neutrosophic $\mathcal{gsg}$-interior (fleetingly $Neu^{\mathcal{gsg}}$-interior) with revealing several of their vital spots.

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2. Preliminaries

In this work, $(\mathfrak{U},\zeta)$ (or simply $\mathfrak{U}$) always mean Neu^{TS}. Let $\mathfrak{P}$ be a neutrosophic set in a Neu^{TS} $(\mathfrak{U},\zeta)$, we denote the neutrosophic closure, the neutrosophic interior, and the neutrosophic complement of $\mathfrak{P}$ by $NeuCl(\mathfrak{P})$, $NeuInt(\mathfrak{P})$ and $\mathfrak{P}^c = 1_{Neu} - \mathfrak{P}$, respectively.

Definition 2.1: [3]
The family ζ of neutrosophic subsets of a non-empty neutrosophic set $\mathfrak{U} \neq \emptyset$ is called a neutrosophic topology (in short, Neu^{T}) on $\mathfrak{U}$ if it satisfies the below axioms:
(i) $0_{Neu}, 1_{Neu} \in \zeta$,
(ii) $\mathfrak{P}_1 \sqcap \mathfrak{P}_2 \in \zeta$ being $\mathfrak{P}_1, \mathfrak{P}_2 \in \zeta$,
(iii) $\sqcup \mathfrak{P}_i \in \zeta$ for arbitrary family $\{\mathfrak{P}_i | i \in \Lambda\} \sqsubseteq \zeta$.
In this case, we signified Neu^{TS} by $(\mathfrak{U},\zeta)$ or $\mathfrak{U}$. Moreover, the neutrosophic set in ζ is named neutrosophic open (in short, $NeuOS$). Furthermore, for any $NeuOS$ $\mathfrak{P}$, then $\mathfrak{P}^c$ is titled neutrosophic closed set (briefly, $NeuCS$) in $\mathfrak{U}$.

Definition 2.2:
Let $\mathfrak{P}$ be a neutrosophic subset of a $Neu^{TS}(\mathfrak{U},\zeta)$, then it is called to be:
(i) a neutrosophic semi-open set and denoted by $Neu^{s}OS$ if $\mathfrak{P} \sqsubseteq NeuCl(NeuInt(\mathfrak{P}))$. [8]
(ii) a neutrosophic semi-closed set and denoted by $Neu^{s}CS$ if $NeuInt(NeuCl(\mathfrak{P})) \sqsubseteq \mathfrak{P}$. The intersection of entire $Neu^{s}CSs$, including $\mathfrak{P}$ is named a neutrosophic semi-closure, and it is symbolized by $Neu^{s}Cl(\mathfrak{P})$.[8]
(iii) a neutrosophic α-open set and denoted by $Neu^{\alpha}OS$ if $\mathfrak{P} \sqsubseteq NeuInt(NeuCl(NeuInt(\mathfrak{P})))$. [9]
(iv) a neutrosophic α-closed set and denoted by $Neu^{\alpha}CS$ if $NeucCl(NeuInt(NeuCl(\mathfrak{P}))) \sqsubseteq \mathfrak{P}$. The intersection of the whole $Neu^{\alpha}CSs$ including $\mathfrak{P}$ is named neutrosophic α-closure, and it is symbolized by $Neu^{\alpha}Cl(\mathfrak{P})$.[9]

Definition 2.3:
Let $\mathfrak{P}$ be a neutrosophic subset of a $Neu^{TS}(\mathfrak{U},\zeta)$, and let $\mathfrak{M}$ be a a $NeuOS$ in $(\mathfrak{U},\zeta)$ such that $\mathfrak{P} \sqsubseteq \mathfrak{M}$ then $\mathfrak{P}$ is called to be:
(i) a neutrosophic generalized closed set, and it is denoted by $Neu^{g}CS$ if $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The complement of a $Neu^{g}CS$ is a $Neu^{g}OS$ in $(\mathfrak{U},\zeta)$. [10]
(ii) a neutrosophic αg-closed set, and it is denoted by $Neu^{\alpha g}CS$ if $Neu^{\alpha}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The complement of a $Neu^{\alpha g}CS$ is a $Neu^{\alpha g}OS$ in $(\mathfrak{U},\zeta)$. [11]
(iii) a neutrosophic $g\alpha$-closed set, and it is denoted by $Neu^{g\alpha}CS$ if $Neu^{\alpha}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The complement of a $Neu^{g\alpha}CS$ is a $Neu^{g\alpha}OS$ in $(\mathfrak{U},\zeta)$. [12]
(iv) a neutrosophic sg-closed set, and it is denoted by $Neu^{sg}CS$ if $Neu^{s}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The complement of a $Neu^{sg}CS$ is a $Neu^{sg}OS$ in $(\mathfrak{U},\zeta)$. [13]
(v) a neutrosophic gs-closed set, and it is denoted by $Neu^{gs}CS$ if $Neu^{g}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The complement of a $Neu^{gs}CS$ is a $Neu^{gs}OS$ in $(\mathfrak{U},\zeta)$. [14]

Proposition 2.4:[9,10]
In a $Neu^{TS}(\mathfrak{U},\zeta)$, then the next arguments stand, and the opposite of every argument is not valid:
(i) Each $NeuOS$ (resp. $NeuCS$) is a $Neu^{\alpha}OS$ (resp. $Neu^{\alpha}CS$).
(ii) Each $NeuOS$ (resp. $NeuCS$) is a $Neu^{g}OS$ (resp. $Neu^{g}CS$).

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(iii) Each $Neu^{\alpha}OS$ (resp. $Neu^{\alpha}CS$) is a $Neu^{s}OS$ (resp. $Neu^{s}CS$).

Proposition 2.5:[11,12]
In a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$, then the next arguments stand, and the opposite of every argument is not valid:
(i) Each $Neu^{g}OS$ (resp. $Neu^{g}CS$) is a $Neu^{\alpha g}OS$ (resp. $Neu^{\alpha g}CS$).
(ii) Each $Neu^{\alpha}OS$ (resp. $Neu^{\alpha}CS$) is a $Neu^{g\alpha}OS$ (resp. $Neu^{g\alpha}CS$).
(iii) Each $Neu^{g\alpha}OS$ (resp. $Neu^{g\alpha}CS$) is a $Neu^{\alpha g}OS$ (resp. $Neu^{\alpha g}CS$).

Proposition 2.6:[13-15]
In a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$, then the next arguments stand, and the opposite of every argument is not valid:
(i) Each $Neu^{g}OS$ (resp. $Neu^{g}CS$) is a $Neu^{gs}OS$ (resp. $Neu^{gs}CS$).
(ii) Each $Neu^{s}OS$ (resp. $Neu^{s}CS$) is a $Neu^{sg}OS$ (resp. $Neu^{sg}CS$).
(iii) Each $Neu^{sg}OS$ (resp. $Neu^{sg}CS$) is a $Neu^{gs}OS$ (resp. $Neu^{gs}CS$).
(iv) Each $Neu^{g\alpha}OS$ (resp. $Neu^{g\alpha}CS$) is a $Neu^{gs}OS$ (resp. $Neu^{gs}CS$).

3. Neutrosophic Generalized sg-Closed Sets

In this sector, we present and analyse the neutrosophic generalized sg-closed sets and some of their features.

Definition 3.1:
Suppose that $\mathfrak{P}$ is a neutrosophic set in a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ and assume that $\mathfrak{M}$ is a $Neu^{sg}OS$ in $(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. The set $\mathfrak{P}$ is termed as a neutrosophic generalized sg-closed set, and it is signified by $Neu^{gsg}CS$ if $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The collection of all $Neu^{gsg}CSs$ in a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ is signified by $Neu^{gsg}C(\mathfrak{U})$.

Theorem 3.2:
In a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$, the subsequent arguments are valid:
(i) Each $NeuCS$ is a $Neu^{gsg}CS$.
(ii) Each $Neu^{gsg}CS$ is a $Neu^{g}CS$.
Proof:
(i) Let $NeuCS$ $\mathfrak{P}$ and $Neu^{sg}OS$ $\mathfrak{M}$ be in a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Then $NeuCl(\mathfrak{P}) = \mathfrak{P} \sqsubseteq \mathfrak{M}$. Therefore $\mathfrak{P}$ is a $Neu^{gsg}CS$.
(ii) Let $Neu^{gsg}CS$ $\mathfrak{P}$ and $NeuOS$ $\mathfrak{M}$ be in a $Neu^{\mathcal{TS}}(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $NeuOS$ is a $Neu^{sg}OS$, we get $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, $\mathfrak{P}$ is a $Neu^{g}CS$. ▪
The reverse of the above theorem is inaccurate, as displayed in the subsequent instances.

Example 3.3:
Suppose that $\mathfrak{U} = \{u_1, u_2\}$ is a set and assume that $\zeta = \{0_{Neu}, \mathfrak{P}_1, \mathfrak{P}_2, 1_{Neu}\}$ is a $Neu^{\mathcal{T}}$ defined on $\mathfrak{U}$. Suppose that we have the sets $\mathfrak{P}_1 = \langle u, (0.6,0.7), (0.1,0.1), (0.4,0.2) \rangle$ and $\mathfrak{P}_2 = \langle u, (0.1,0.2), (0.1,0.1), (0.8,0.8) \rangle$ are given. Then the neutrosophic set $\mathfrak{P}_3 = \langle u, (0.2,0.2), (0.1,0.1), (0.6,0.7) \rangle$ is a $Neu^{gsg}CS$. However, this latter set is not a $NeuCS$.

Example 3.4:
Suppose that $\mathfrak{U} = \{u_1, u_2, u_3\}$ is a set and assume that $\zeta = \{0_{Neu}, \mathfrak{P}_1, \mathfrak{P}_2, 1_{Neu}\}$ is a $Neu^{\mathcal{T}}$ defined on $\mathfrak{U}$. Suppose that we have the following sets $\mathfrak{P}_1 = \langle u, (0.5,0.5,0.4), (0.7,0.5,0.5), (0.4,0.5,0.5) \rangle$ and $\mathfrak{P}_2 =$

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$\langle u, (0.3,0.4,0.4), (0.4,0.5,0.5), (0.3,0.4,0.6)\rangle$ are given. Then the neutrosophic set $\mathfrak{P}_3 = \langle u, (0.4,0.6,0.5), (0.4,0.3,0.5), (0.5,0.6,0.4)\rangle$ is a $Neu^{g}CS$. However, this latter set is not a $Neu^{gsg}CS$.

Theorem 3.5:
In a $Neu^{TS}(\mathfrak{U},\zeta)$, the subsequent arguments are valid:
(i) Each $Neu^{gsg}CS$ is a $Neu^{\alpha g}CS$.
(ii) Each $Neu^{gsg}CS$ is a $Neu^{g\alpha}CS$.
(iii) Each $Neu^{gsg}CS$ is a $Neu^{sg}CS$.
(iv) Each $Neu^{gsg}CS$ is a $Neu^{gs}CS$.
Proof:
(i) Let $Neu^{gsg}CS$ $\mathfrak{P}$ and $NeuOS$ $\mathfrak{M}$ be in a $Neu^{TS}(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $NeuOS$ is a $Neu^{\alpha g}CS$, we get $Neu^{\alpha}Cl(\mathfrak{P}) \sqsubseteq NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The latter implies $Neu^{\alpha}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, $\mathfrak{P}$ is a $Neu^{\alpha g}CS$.
(ii) Let $Neu^{gsg}CS$ $\mathfrak{P}$ and $Neu^{\alpha}OS$ $\mathfrak{M}$ be in a $Neu^{TS}(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $Neu^{\alpha}OS$ is a $Neu^{s}OS$, which is a $Neu^{sg}OS$, we get $Neu^{\alpha}Cl(\mathfrak{P}) \sqsubseteq NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The latter implies $Neu^{\alpha}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, $\mathfrak{P}$ is a $Neu^{g\alpha}CS$.
(iii) Let $Neu^{gsg}CS$ $\mathfrak{P}$ and $Neu^{s}OS$ $\mathfrak{M}$ be in a $Neu^{TS}(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $Neu^{s}OS$ is a $Neu^{sg}OS$, we get $Neu^{s}Cl(\mathfrak{P}) \sqsubseteq NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The latter implies $Neu^{s}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, $\mathfrak{P}$ is a $Neu^{sg}CS$.
(iv) Let $Neu^{gsg}CS$ $\mathfrak{P}$ and $NeuOS$ $\mathfrak{M}$ be in a $Neu^{TS}(\mathfrak{U},\zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $NeuOS$ is a $Neu^{sg}OS$, we get $Neu^{s}Cl(\mathfrak{P}) \sqsubseteq NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. That implies $Neu^{s}Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, $\mathfrak{P}$ is a $Neu^{gs}CS$. ▪
The reverse of the above theorem is inaccurate, as displayed in the subsequent instances.

Example 3.6:
Let $\mathfrak{U} = \{u_1, u_2\}$ be a set and assume that $\zeta = \{0_{Neu}, \mathfrak{P}_1, \mathfrak{P}_2, 1_{Neu}\}$ is a $Neu^{\mathcal{T}}$ defined on $\mathfrak{U}$. Suppose that we have the following sets $\mathfrak{P}_1 = \langle u, (0.5,0.6), (0.3,0.2), (0.4,0.1)\rangle$ and $\mathfrak{P}_2 = \langle u, (0.4,0.4), (0.4,0.3), (0.5,0.4)\rangle$ are given. Then the neutrosophic set $\mathfrak{P}_3 = \langle u, (0.5,0.4), (0.4,0.4), (0.4,0.5)\rangle$ is a $Neu^{\alpha g}CS$ and hence $Neu^{g\alpha}CS$ but not a $Neu^{gsg}CS$.

Example 3.7:
Let $\mathfrak{U} = \{u_1, u_2\}$ and let $\zeta = \{0_{Neu}, \mathfrak{P}_1, 1_{Neu}\}$ be a $Neu^{\mathcal{T}}$ on $\mathfrak{U}$. Take $\mathfrak{P}_1 = \langle u, (0.3,0.4,0.6), (0.6,0.6,0.4)\rangle$. Then the neutrosophic set $\mathfrak{P}_2 = \langle u, (0.3,0.2,0.5), (0.6,0.6,0.8)\rangle$ is a $Neu^{sg}CS$ but not a $Neu^{gsg}CS$.

Example 3.8:
Let $\mathfrak{U} = \{u_1, u_2\}$ and let $\zeta = \{0_{Neu}, \mathfrak{P}_1, 1_{Neu}\}$ be a $Neu^{\mathcal{T}}$ on $\mathfrak{U}$. Where $\mathfrak{P}_1 = \langle u, (0.3,0.2,0.3), (0.8,0.6,0.7)\rangle$. Then the neutrosophic set $\mathfrak{P}_2 = \langle u, (0.3,0.2,0.6), (0.8,0.9,0.8)\rangle$ is a $Neu^{gs}CS$. However, this latter set is not a $Neu^{gsg}CS$.

Remark 3.9:
The $Neu^{gsg}CS$ are independent of $Neu^{\alpha}CS$ and $Neu^{s}CS$.

Definition 3.10:
A neutrosophic subset $\mathfrak{P}$ of a $Neu^{TS}(\mathfrak{U},\zeta)$ is called a neutrosophic generalized sg-open set (in short, $Neu^{gsg}OS$) iff $1_{Neu} - \mathfrak{P}$ is a $Neu^{gsg}CS$. The collection of entire $Neu^{gsg}OS$s of a $Neu^{TS}(\mathfrak{U},\zeta)$ is signified by $Neu^{gsg}O(\mathfrak{U})$.

Proposition 3.11:
Let $\mathfrak{P}$ be a $NeuOS$ in $Neu^{TS}(\mathfrak{U},\zeta)$, then this set $\mathfrak{P}$ is $Neu^{gsg}OS$ in the space $(\mathfrak{U},\zeta)$.

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Proof:
Let $\mathfrak{P}$ be a $NeuOS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $1_{Neu} - \mathfrak{P}$ is a $NeuCS$ in $(\mathfrak{U},\zeta)$. According to theorem (3.2), point (i), $1_{Neu} - \mathfrak{P}$ is a $Neu^{gsg}CS$. Therefore, $\mathfrak{P}$ is a $Neu^{gsg}OS$ in $(\mathfrak{U},\zeta)$. ▪

Proposition 3.12:
Let $\mathfrak{P}$ be a $Neu^{gsg}OS$ in $Neu^{TS}(\mathfrak{U},\zeta)$, then this set $\mathfrak{P}$ is $Neu^{g}OS$ in the space $(\mathfrak{U},\zeta)$.
Proof:
Let $\mathfrak{P}$ be a $Neu^{gsg}OS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $1_{Neu} - \mathfrak{P}$ is a $Neu^{gsg}CS$ in $(\mathfrak{U},\zeta)$. According to theorem (3.2), point (ii), $1_{Neu} - \mathfrak{P}$ is a $Neu^{g}CS$. Therefore, $\mathfrak{P}$ is a $Neu^{g}OS$ in $(\mathfrak{U},\zeta)$. ▪

Theorem 3.13:
In a $Neu^{TS}(\mathfrak{U},\zeta)$, the subsequent arguments are valid:
(i) Each $Neu^{gsg}OS$ is a $Neu^{\alpha g}OS$ and $Neu^{g\alpha}OS$.
(ii) Each $Neu^{gsg}OS$ is a $Neu^{sg}OS$ and $Neu^{gs}OS$.
Proof:
Similar to above proposition. ▪

Proposition 3.14:
If $\mathfrak{P}$ and $\mathfrak{Q}$ are $Neu^{gsg}CSs$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $\mathfrak{P}\sqcup\mathfrak{Q}$ is a $Neu^{gsg}CS$.
Proof:
Let $\mathfrak{P}$ and $\mathfrak{Q}$ be two $Neu^{gsg}CSs$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ and let $\mathfrak{M}$ be any $Neu^{sg}OS$ in $\mathfrak{U}$ such that $\mathfrak{P} \sqsubseteq \mathfrak{M}$ and $\mathfrak{Q} \sqsubseteq \mathfrak{M}$. Then we have $\mathfrak{P}\sqcup\mathfrak{Q} \sqsubseteq \mathfrak{M}$. Since $\mathfrak{P}$ and $\mathfrak{Q}$ are $Neu^{gsg}CSs$ in $\mathfrak{U}$, $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$ and $NeuCl(\mathfrak{Q}) \sqsubseteq \mathfrak{M}$. Now, $NeuCl(\mathfrak{P}\sqcup\mathfrak{Q}) = NeuCl(\mathfrak{P})\sqcup NeuCl(\mathfrak{Q}) \sqsubseteq \mathfrak{M}$ and so $NeuCl(\mathfrak{P}\sqcup\mathfrak{Q}) \sqsubseteq \mathfrak{M}$. Hence $\mathfrak{P}\sqcup\mathfrak{Q}$ is a $Neu^{gsg}CS$ in $\mathfrak{U}$. ▪

Proposition 3.15:
If $\mathfrak{P}$ is a $Neu^{gsg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $NeuCl(\mathfrak{P}) - \mathfrak{P}$ does not include non-empty $NeuCS$ in $(\mathfrak{U},\zeta)$.
Proof:
Let $\mathfrak{P}$ be a $Neu^{gsg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ and let $\mathfrak{F}$ be any $NeuCS$ in $(\mathfrak{U},\zeta)$ such that $\mathfrak{F} \sqsubseteq NeuCl(\mathfrak{P}) - \mathfrak{P}$. Since $\mathfrak{P}$ is a $Neu^{gsg}CS$, we have $NeuCl(\mathfrak{P}) \sqsubseteq 1_{Neu} - \mathfrak{F}$. This implies $\mathfrak{F} \sqsubseteq 1_{Neu} - NeuCl(\mathfrak{P})$. Then $\mathfrak{F} \sqsubseteq NeuCl(\mathfrak{P})\sqcap(1_{Neu} - NeuCl(\mathfrak{P})) = 0_{Neu}$. Thus, $\mathfrak{F} = 0_{Neu}$. Hence $NeuCl(\mathfrak{P}) - \mathfrak{P}$ does not include non-empty $NeuCS$ in $(\mathfrak{U},\zeta)$. ▪

Proposition 3.16:
A neutrosophic set $\mathfrak{P}$ is $Neu^{gsg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ iff $NeuCl(\mathfrak{P}) - \mathfrak{P}$ does not include non-empty $Neu^{sg}CS$ in $(\mathfrak{U},\zeta)$.
Proof:
Let $\mathfrak{P}$ be a $Neu^{gsg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ and let $\mathfrak{R}$ be any $Neu^{sg}CS$ in $(\mathfrak{U},\zeta)$ such that $\mathfrak{R} \sqsubseteq NeuCl(\mathfrak{P}) - \mathfrak{P}$. Since $\mathfrak{P}$ is a $Neu^{gsg}CS$, we have $NeuCl(\mathfrak{P}) \sqsubseteq 1_{Neu} - \mathfrak{R}$. This implies $\mathfrak{R} \sqsubseteq 1_{Neu} - NeuCl(\mathfrak{P})$. Then $\mathfrak{R} \sqsubseteq NeuCl(\mathfrak{P})\sqcap(1_{Neu} - NeuCl(\mathfrak{P})) = 0_{Neu}$. Thus, $\mathfrak{R}$ is empty.
Conversely, suppose that $NeuCl(\mathfrak{P}) - \mathfrak{P}$ does not include non-empty $Neu^{sg}CS$ in $(\mathfrak{U},\zeta)$. Let $\mathfrak{P} \sqsubseteq \mathfrak{M}$ and $\mathfrak{M}$ is $Neu^{sg}OS$. If $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$ then $NeuCl(\mathfrak{P})\sqcap(1_{Neu} - \mathfrak{M})$ is non-empty. Since $NeuCl(\mathfrak{P})$ is $NeuCS$ and $1_{Neu} - \mathfrak{M}$ is $Neu^{sg}CS$, we have $NeuCl(\mathfrak{P})\sqcap(1_{Neu} - \mathfrak{M})$ is not empty $Neu^{sg}CS$ of $NeuCl(\mathfrak{P}) - \mathfrak{P}$, which is a contradiction. Therefore $NeuCl(\mathfrak{P}) \not\sqsubseteq \mathfrak{M}$. Hence $\mathfrak{P}$ is a $Neu^{gsg}CS$. ▪

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Proposition 3.17:
If $\mathfrak{P}$ is a $Neu^{gsg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ and $\mathfrak{P} \sqsubseteq \mathfrak{Q} \sqsubseteq NeuCl(\mathfrak{P})$, then $\mathfrak{Q}$ is a $Neu^{gsg}CS$ in $(\mathfrak{U},\zeta)$.
Proof:
Assume the set $\mathfrak{P}$ is a $Neu^{gsg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$. Suppose the set $\mathfrak{M}$ is a $Neu^{sg}OS$ in $(\mathfrak{U},\zeta)$ where $\mathfrak{Q} \sqsubseteq \mathfrak{M}$. So, $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because $\mathfrak{P}$ is a $Neu^{gsg}CS$, it observes that $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Currently, $\mathfrak{Q} \sqsubseteq NeuCl(\mathfrak{P})$ suggests $NeuCl(\mathfrak{Q}) \sqsubseteq NeuCl(NeuCl(\mathfrak{P})) = NeuCl(\mathfrak{P})$. Thus, $NeuCl(\mathfrak{Q}) \sqsubseteq \mathfrak{M}$. Hence $\mathfrak{Q}$ is a $Neu^{gsg}CS$. ▪

Proposition 3.18:
Let $\mathfrak{P} \sqsubseteq \mathfrak{D} \sqsubseteq \mathfrak{U}$ and if $\mathfrak{P}$ is a $Neu^{gsg}CS$ in $\mathfrak{U}$ then $\mathfrak{P}$ is a $Neu^{gsg}CS$ relative to $\mathfrak{D}$.
Proof:
$\mathfrak{P} \sqsubseteq \mathfrak{D} \sqcap \mathfrak{M}$ where $\mathfrak{M}$ is a $Neu^{sg}OS$ in $\mathfrak{U}$. Then $\mathfrak{P} \sqsubseteq \mathfrak{M}$ and hence $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$.
This implies that $\mathfrak{D} \sqcap NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{D} \sqcap \mathfrak{M}$. Thus $\mathfrak{P}$ is a $Neu^{gsg}CS$ relative to $\mathfrak{D}$. ▪

Proposition 3.19:
If $\mathfrak{P}$ is a $Neu^{sg}OS$ and a $Neu^{gsg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $\mathfrak{P}$ is a $NeuCS$ in $(\mathfrak{U},\zeta)$.
Proof:
Suppose that $\mathfrak{P}$ is a $Neu^{sg}OS$ and a $Neu^{gsg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{P}$ and since $\mathfrak{P} \sqsubseteq NeuCl(\mathfrak{P})$. Thus, $NeuCl(\mathfrak{P}) = \mathfrak{P}$. Hence $\mathfrak{P}$ is a $NeuCS$. ▪

Theorem 3.20:
For each $u \in \mathfrak{U}$ either $\langle u,(0.1,0.1)\rangle$ is a $Neu^{sg}CS$ or $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is a $Neu^{gsg}CS$ in $\mathfrak{U}$.
Proof:
If $\langle u,(0.1,0.1)\rangle$ is not a $Neu^{sg}CS$ in $\mathfrak{U}$ then $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is not a $Neu^{sg}OS$ and the only $Neu^{sg}OS$ containing $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is the space $\mathfrak{U}$ itself. Therefore $NeuCl(1_{Neu} - \langle u,(0.1,0.1)\rangle) \sqsubseteq 1_{Neu}$ and so $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is a $Neu^{gsg}CS$ in $\mathfrak{U}$. ▪

Proposition 3.21:
If $\mathfrak{P}$ and $\mathfrak{Q}$ are $Neu^{gsg}OSs$ in a $Neu^{TS}(\mathfrak{U},\zeta)$, then $\mathfrak{P} \sqcap \mathfrak{Q}$ is a $Neu^{gsg}OS$.
Proof:
Let $\mathfrak{P}$ and $\mathfrak{Q}$ be $Neu^{gsg}OSs$ in a $Neu^{TS}(\mathfrak{U},\zeta)$. Then $1_{Neu} - \mathfrak{P}$ and $1_{Neu} - \mathfrak{Q}$ are $Neu^{gsg}CSs$. By proposition (3.14), $(1_{Neu} - \mathfrak{P}) \sqcup (1_{Neu} - \mathfrak{Q})$ is a $Neu^{gsg}CS$. Since $(1_{Neu} - \mathfrak{P}) \sqcup (1_{Neu} - \mathfrak{Q}) = 1_{Neu} - (\mathfrak{P} \sqcap \mathfrak{Q})$. Hence $\mathfrak{P} \sqcap \mathfrak{Q}$ is a $Neu^{gsg}OS$. ▪

Theorem 3.22:
A neutrosophic set $\mathfrak{P}$ is $Neu^{gsg}OS$ iff $\mathfrak{S} \sqsubseteq NeuInt(\mathfrak{P})$ where $\mathfrak{S}$ is a $Neu^{gsg}CS$ and $\mathfrak{S} \sqsubseteq \mathfrak{P}$.
Proof:
Suppose that $\mathfrak{S} \sqsubseteq NeuInt(\mathfrak{P})$ where $\mathfrak{S}$ is a $Neu^{gsg}CS$ and $\mathfrak{S} \sqsubseteq \mathfrak{P}$. Then $1_{Neu} - \mathfrak{P} \sqsubseteq 1_{Neu} - \mathfrak{S}$ and $1_{Neu} - \mathfrak{S}$ is a $Neu^{sg}OS$ by theorem (3.13) part (ii). Now, $NeuCl(1_{Neu} - \mathfrak{P}) = 1_{Neu} - NeuInt(\mathfrak{P}) \sqsubseteq 1_{Neu} - \mathfrak{S}$. Then $1_{Neu} - \mathfrak{P}$ is a $Neu^{gsg}CS$. Hence $\mathfrak{P}$ is a $Neu^{gsg}OS$.
Conversely, let $\mathfrak{P}$ be a $Neu^{gsg}OS$ and $\mathfrak{S}$ be a $Neu^{gsg}CS$ and $\mathfrak{S} \sqsubseteq \mathfrak{P}$. Then $1_{Neu} - \mathfrak{P} \sqsubseteq 1_{Neu} - \mathfrak{S}$. Since $1_{Neu} - \mathfrak{P}$ is a $Neu^{gsg}CS$ and $1_{Neu} - \mathfrak{S}$ is a $Neu^{sg}OS$, we have $NeuCl(1_{Neu} - \mathfrak{P}) \sqsubseteq 1_{Neu} - \mathfrak{S}$. Then $\mathfrak{S} \sqsubseteq NeuInt(\mathfrak{P})$. ▪

Theorem 3.23:
If $\mathfrak{P} \sqsubseteq \mathfrak{Q} \sqsubseteq \mathfrak{U}$ where $\mathfrak{P}$ is a $Neu^{gsg}OS$ relative to $\mathfrak{Q}$ and $\mathfrak{Q}$ is a $Neu^{gsg}OS$ in $\mathfrak{U}$, then $\mathfrak{P}$ is a $Neu^{gsg}OS$ in $\mathfrak{U}$.

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Proof:

Let $\mathfrak{F}$ be a $Neu^{sg}CS$ in $\mathfrak{U}$ and suppose that $\mathfrak{F} \sqsubseteq \mathfrak{P}$. Then $\mathfrak{F} = \mathfrak{F} \sqcap \mathfrak{Q}$ is a $Neu^{sg}CS$ in $\mathfrak{Q}$. But $\mathfrak{P}$ is a $Neu^{gsg}OS$ relative to $\mathfrak{Q}$. Therefore $\mathfrak{F} \sqsubseteq NeuInt_{\mathfrak{Q}}(\mathfrak{P})$. Since $NeuInt_{\mathfrak{Q}}(\mathfrak{P})$ is a $NeuOS$ relative to $\mathfrak{Q}$. We have $\mathfrak{F} \sqsubseteq \mathfrak{M} \sqcap \mathfrak{Q} \sqsubseteq \mathfrak{P}$, for some $NeuOS$ $\mathfrak{M}$ in $\mathfrak{U}$. Since $\mathfrak{Q}$ is a $Neu^{gsg}OS$ in $\mathfrak{U}$, we have $\mathfrak{F} \sqsubseteq NeuInt(\mathfrak{Q}) \sqsubseteq \mathfrak{Q}$. Therefore $\mathfrak{F} \sqsubseteq NeuInt(\mathfrak{Q}) \sqcap \mathfrak{M} \sqsubseteq \mathfrak{Q} \sqcap \mathfrak{M} \sqsubseteq \mathfrak{P}$. It follows that $\mathfrak{F} \sqsubseteq NeuInt(\mathfrak{P})$. Thus, $\mathfrak{P}$ is a $Neu^{gsg}OS$ in $\mathfrak{U}$. ▪

Theorem 3.24:

If $\mathfrak{P}$ is a $Neu^{gsg}OS$ in a $Neu^{TS}(\mathfrak{U}, \zeta)$ and $NeuInt(\mathfrak{P}) \sqsubseteq \mathfrak{Q} \sqsubseteq \mathfrak{P}$, then $\mathfrak{Q}$ is a $Neu^{gsg}OS$ in $(\mathfrak{U}, \zeta)$.

Proof:

Suppose that $\mathfrak{P}$ is a $Neu^{gsg}OS$ in a $Neu^{TS}(\mathfrak{U}, \zeta)$ and $NeuInt(\mathfrak{P}) \sqsubseteq \mathfrak{Q} \sqsubseteq \mathfrak{P}$. Then $1_{Neu} - \mathfrak{P}$ is a $Neu^{gsg}CS$ and $1_{Neu} - \mathfrak{P} \sqsubseteq 1_{Neu} - \mathfrak{Q} \sqsubseteq NeuCl(1_{Neu} - \mathfrak{P})$. Then $1_{Neu} - \mathfrak{Q}$ is a $Neu^{gsg}CS$ by proposition (3.17). Hence, $\mathfrak{Q}$ is a $Neu^{gsg}OS$. ▪

Theorem 3.25:

For a neutrosophic subset $\mathfrak{P}$ of a $Neu^{TS}(\mathfrak{U}, \zeta)$, the following statements are equivalent:

(i) $\mathfrak{P}$ is a $Neu^{gsg}CS$.

(ii) $NeuCl(\mathfrak{P}) - \mathfrak{P}$ contains no non-empty $Neu^{sg}CS$.

(iii) $NeuCl(\mathfrak{P}) - \mathfrak{P}$ is a $Neu^{gsg}OS$.

Proof:

Follows from proposition (3.16) and proposition (3.18). ▪

Remark 3.26:

The subsequent illustration reveals the relative among the diverse kinds of $NeuCS$:

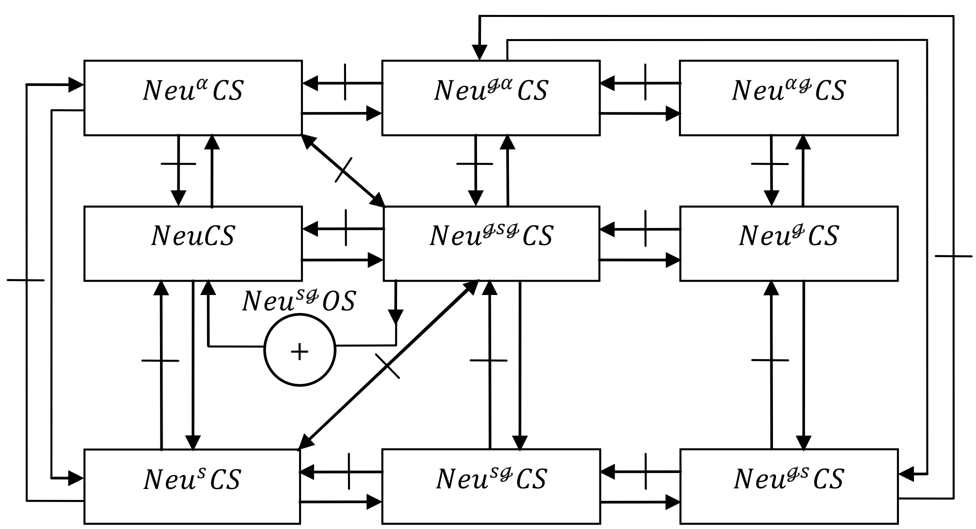

Fig. 3.1

4. Neutrosophic *gsg*-Closure and Neutrosophic *gsg*-Interior

We present neutrosophic *gsg*-closure and neutrosophic *gsg*-interior and obtain some of its properties in this section.

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Definition 4.1:
The intersection of all $Neu^{gsg}CSs$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ containing $\mathfrak{P}$ is called neutrosophic gsg-closure of $\mathfrak{P}$ and is denoted by $Neu^{gsg}Cl(\mathfrak{P})$.

Definition 4.2:
The union of all $Neu^{gsg}OSs$ in a $Neu^{TS}(\mathfrak{U},\zeta)$ contained in $\mathfrak{P}$ is called neutrosophic gsg-interior of $\mathfrak{P}$ and is denoted by $Neu^{gsg}Int(\mathfrak{P})$.

Proposition 4.3:
Let $\mathfrak{P}$ be any neutrosophic set in a $Neu^{TS}(\mathfrak{U},\zeta)$. Then the following properties hold:
(i) $Neu^{gsg}Int(\mathfrak{P}) = \mathfrak{P}$ iff $\mathfrak{P}$ is a $Neu^{gsg}OS$.
(ii) $Neu^{gsg}Cl(\mathfrak{P}) = \mathfrak{P}$ iff $\mathfrak{P}$ is a $Neu^{gsg}CS$.
(iii) $Neu^{gsg}Int(\mathfrak{P})$ is the largest $Neu^{gsg}OS$ contained in $\mathfrak{P}$.
(iv) $Neu^{gsg}Cl(\mathfrak{P})$ is the smallest $Neu^{gsg}CS$ containing $\mathfrak{P}$.
Proof:
(i), (ii), (iii) and (iv) are obvious. ▪

Proposition 4.4:
Let $\mathfrak{P}$ be any neutrosophic set in a $Neu^{TS}(\mathfrak{U},\zeta)$. Then the following properties hold:
(i) $Neu^{gsg}Int(1_{Neu} - \mathfrak{P}) = 1_{Neu} - (Neu^{gsg}Cl(\mathfrak{P}))$,
(ii) $Neu^{gsg}Cl(1_{Neu} - \mathfrak{P}) = 1_{Neu} - (Neu^{gsg}Int(\mathfrak{P}))$.
Proof:
(i) By definition, $Neu^{gsg}Cl(\mathfrak{P}) = \sqcap\{\mathfrak{Q}: \mathfrak{P} \sqsubseteq \mathfrak{Q}, \mathfrak{Q} \text{ is a } Neu^{gsg}CS\}$

$$\begin{aligned} 1_{Neu} - (Neu^{gsg}Cl(\mathfrak{P})) &= 1_{Neu} - \sqcap\{\mathfrak{Q}: \mathfrak{P} \sqsubseteq \mathfrak{Q}, \mathfrak{Q} \text{ is a } Neu^{gsg}CS\} \\ &= \sqcup\{1_{Neu} - \mathfrak{Q}: \mathfrak{P} \sqsubseteq \mathfrak{Q}, \mathfrak{Q} \text{ is a } Neu^{gsg}CS\} \\ &= \sqcup\{\mathfrak{M}: \mathfrak{M} \sqsubseteq 1_{Neu} - \mathfrak{P}, \mathfrak{M} \text{ is a } Neu^{gsg}OS\} \\ &= Neu^{gsg}Int(1_{Neu} - \mathfrak{P}). \end{aligned}$$

(ii) The evidence is analogous to (i). ▪

Theorem 4.5:
Let $\mathfrak{P}$ and $\mathfrak{Q}$ be two neutrosophic sets in a $Neu^{TS}(\mathfrak{U},\zeta)$. Then the following properties hold:
(i) $Neu^{gsg}Cl(0_{Neu}) = 0_{Neu}$, $Neu^{gsg}Cl(1_{Neu}) = 1_{Neu}$.
(ii) $\mathfrak{P} \sqsubseteq Neu^{gsg}Cl(\mathfrak{P})$.
(iii) $\mathfrak{P} \sqsubseteq \mathfrak{Q} \Longrightarrow Neu^{gsg}Cl(\mathfrak{P}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{Q})$.
(iv) $Neu^{gsg}Cl(\mathfrak{P}\sqcap\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P}) \sqcap Neu^{gsg}Cl(\mathfrak{Q})$.
(v) $Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q}) = Neu^{gsg}Cl(\mathfrak{P}) \sqcup Neu^{gsg}Cl(\mathfrak{Q})$.
(vi) $Neu^{gsg}Cl(Neu^{gsg}Cl(\mathfrak{P})) = Neu^{gsg}Cl(\mathfrak{P})$.
Proof:
(i) and (ii) are obvious.
(iii) By part (ii), $\mathfrak{Q} \sqsubseteq Neu^{gsg}Cl(\mathfrak{Q})$. Since $\mathfrak{P} \sqsubseteq \mathfrak{Q}$, we have $\mathfrak{P} \sqsubseteq Neu^{gsg}Cl(\mathfrak{Q})$. But $Neu^{gsg}Cl(\mathfrak{Q})$ is a $Neu^{gsg}CS$. Thus $Neu^{gsg}Cl(\mathfrak{Q})$ is a $Neu^{gsg}CS$ containing $\mathfrak{P}$. Since $Neu^{gsg}Cl(\mathfrak{P})$ is the smallest $Neu^{gsg}CS$ containing $\mathfrak{P}$, we have $Neu^{gsg}Cl(\mathfrak{P}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{Q})$.
(iv) We know that $\mathfrak{P}\sqcap\mathfrak{Q} \sqsubseteq \mathfrak{P}$ and $\mathfrak{P}\sqcap\mathfrak{Q} \sqsubseteq \mathfrak{Q}$. Therefore, by part (iii), $Neu^{gsg}Cl(\mathfrak{P}\sqcap\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P})$ and $Neu^{gsg}Cl(\mathfrak{P}\sqcap\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{Q})$. Hence $Neu^{gsg}Cl(\mathfrak{P}\sqcap\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P}) \sqcap Neu^{gsg}Cl(\mathfrak{Q})$.
(v) Since $\mathfrak{P} \sqsubseteq \mathfrak{P}\sqcup\mathfrak{Q}$ and $\mathfrak{Q} \sqsubseteq \mathfrak{P}\sqcup\mathfrak{Q}$, it follows from part (iii) that $Neu^{gsg}Cl(\mathfrak{P}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q})$ and $Neu^{gsg}Cl(\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q})$. Hence $Neu^{gsg}Cl(\mathfrak{P}) \sqcup Neu^{gsg}Cl(\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q})$………… (1)

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Since $Neu^{gsg}Cl(\mathfrak{P})$ and $Neu^{gsg}Cl(\mathfrak{Q})$ are $Neu^{gsg}CSs$, $Neu^{gsg}Cl(\mathfrak{P})\sqcup Neu^{gsg}Cl(\mathfrak{Q})$ is also $Neu^{gsg}CS$ by proposition (3.14). Also $\mathfrak{P} \sqsubseteq Neu^{gsg}Cl(\mathfrak{P})$ and $\mathfrak{Q} \sqsubseteq Neu^{gsg}Cl(\mathfrak{Q})$ implies that $\mathfrak{P}\sqcup\mathfrak{Q} \sqsubseteq Neu^{gsg}Cl(\mathfrak{P})\sqcup Neu^{gsg}Cl(\mathfrak{Q})$. Thus $Neu^{gsg}Cl(\mathfrak{P})\sqcup Neu^{gsg}Cl(\mathfrak{Q})$ is a $Neu^{gsg}CS$ containing $\mathfrak{P}\sqcup\mathfrak{Q}$. Since $Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q})$ is the smallest $Neu^{gsg}CS$ containing $\mathfrak{P}\sqcup\mathfrak{Q}$, we have $Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q}) \sqsubseteq Neu^{gsg}Cl(\mathfrak{P})\sqcup Neu^{gsg}Cl(\mathfrak{Q})$………… (2)
From (1) and (2), we have $Neu^{gsg}Cl(\mathfrak{P}\sqcup\mathfrak{Q}) = Neu^{gsg}Cl(\mathfrak{P})\sqcup Neu^{gsg}Cl(\mathfrak{Q})$.
(vi) Since $Neu^{gsg}Cl(\mathfrak{P})$ is a $Neu^{gsg}CS$, we have by proposition (4.3) part (ii), $Neu^{gsg}Cl(Neu^{gsg}Cl(\mathfrak{P})) = Neu^{gsg}Cl(\mathfrak{P})$. ▪

Theorem 4.6:
Let $\mathfrak{P}$ and $\mathfrak{Q}$ be two neutrosophic sets in a $Neu^{TS}(\mathfrak{U},\zeta)$. Then the following properties hold:
(i) $Neu^{gsg}Int(0_{Neu}) = 0_{Neu}$, $Neu^{gsg}Int(1_{Neu}) = 1_{Neu}$.
(ii) $Neu^{gsg}Int(\mathfrak{P}) \sqsubseteq \mathfrak{P}$.
(iii) $\mathfrak{P} \sqsubseteq \mathfrak{Q} \Longrightarrow Neu^{gsg}Int(\mathfrak{P}) \sqsubseteq Neu^{gsg}Int(\mathfrak{Q})$.
(iv) $Neu^{gsg}Int(\mathfrak{P}\sqcap\mathfrak{Q}) = Neu^{gsg}Int(\mathfrak{P})\sqcap Neu^{gsg}Int(\mathfrak{Q})$.
(v) $Neu^{gsg}Int(\mathfrak{P}\sqcup\mathfrak{Q}) \sqsupseteq Neu^{gsg}Int(\mathfrak{P})\sqcup Neu^{gsg}Int(\mathfrak{Q})$.
(vi) $Neu^{gsg}Int(Neu^{gsg}Int(\mathfrak{P})) = Neu^{gsg}Int(\mathfrak{P})$.
Proof:
(i), (ii), (iii), (iv), (v) and (vi) are obvious. ▪

Definition 4.7:
A $Neu^{TS}(\mathfrak{U},\zeta)$ is called a neutrosophic $T_{\frac{1}{2}}$-space (in short, $NeuT_{\frac{1}{2}}$-space) if each $Neu^{g}CS$ in this space is a $NeuCS$.

Definition 4.8:
A $Neu^{TS}(\mathfrak{U},\zeta)$ is called a neutrosophic T_{gsg}-space (in short, $NeuT_{gsg}$-space) if each $Neu^{gsg}CS$ in this space is a $NeuCS$.

Proposition 4.9:
Every $NeuT_{\frac{1}{2}}$-space is a $NeuT_{gsg}$-space.
Proof:
Let $(\mathfrak{U},\zeta)$ be a $NeuT_{\frac{1}{2}}$-space and let $\mathfrak{P}$ be a $Neu^{gsg}CS$ in $\mathfrak{U}$. Then $\mathfrak{P}$ is a $Neu^{g}CS$, by theorem (3.2) part (ii). Since $(\mathfrak{U},\zeta)$ is a $NeuT_{\frac{1}{2}}$-space, then $\mathfrak{P}$ is a $NeuCS$ in $\mathfrak{U}$. Hence $(\mathfrak{U},\zeta)$ is a $NeuT_{gsg}$-space. ▪

Theorem 4.10:
For a $Neu^{TS}(\mathfrak{U},\zeta)$, the following statements are equivalent:
(i) $(\mathfrak{U},\zeta)$ is a $NeuT_{gsg}$-space.
(ii) Every singleton of a $Neu^{TS}(\mathfrak{U},\zeta)$ is either $Neu^{sg}CS$ or $NeuOS$.
Proof:
(i) ⟹(ii) Assume that for some $u \in \mathfrak{U}$ the neutrosophic set $\langle u,(0.1,0.1)\rangle$ is not a $Neu^{sg}CS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$. Then the only $Neu^{sg}OS$ containing $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is the space $\mathfrak{U}$ itself and $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is a $Neu^{gsg}CS$ in $(\mathfrak{U},\zeta)$. By assumption $1_{Neu} - \langle u,(0.1,0.1)\rangle$ is a $NeuCS$ in $(\mathfrak{U},\zeta)$ or equivalently $\langle u,(0.1,0.1)\rangle$ is a $NeuOS$.

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(ii) ⟹(i) Let $\mathfrak{P}$ be a $Neu^{gsg}CS$ in $(\mathfrak{U},\zeta)$ and let $u \in NeuCl(\mathfrak{P})$. By assumption $\langle u,(0.1,0.1)\rangle$ is either $Neu^{sg}CS$ or $NeuOS$.

Case(1). Suppose $\langle u,(0.1,0.1)\rangle$ is a $Neu^{sg}CS$. If $u \notin \mathfrak{P}$ then $NeuCl(\mathfrak{P}) - \mathfrak{P}$ contains a non-empty $Neu^{sg}CS$ $\langle u,(0.1,0.1)\rangle$ which is a contradiction to proposition (3.18). Therefore $u \in \mathfrak{P}$.

Case(2). Suppose $\langle u,(0.1,0.1)\rangle$ is a $NeuOS$. Since $u \in NeuCl(\mathfrak{P})$, $\langle u,(0.1,0.1)\rangle \sqcap \mathfrak{P} \neq 0_{Neu}$ and therefore $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{P}$ or equivalently $\mathfrak{P}$ is a $NeuCS$ in a $Neu^{TS}(\mathfrak{U},\zeta)$. ▪

5. Conclusion

The concept of $Neu^{gsg}CS$ identified utilizing $Neu^{sg}CS$ constructs a neutrosophic topology and sits between the concept of $NeuCS$ and the concept of $Neu^{g}CS$. The $Neu^{gsg}CS$ can be used to derive a new decomposition of Neu^{gsg}-continuity and new Neu^{gsg}-separation axioms.

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