Interval-Valued Neutrosophic Ideals of Hilbert Algebras

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Abstract

The concept of interval-valued neutrosophic sets (IVNSs) was first introduced by Wang et al. (Wang, H.; Smarandache, F.; Zhang, Y. Q.; Sunderraman, R. Interval neutrosophic sets and logic: Theory and applications in computing. Hexis, Phoenix, Ariz, USA, 2005.). In this paper, the concept of IVNSs to ideals of Hilbert algebras is introduced. The homomorphic inverse image of interval-valued neutrosophic ideals (IVN ideals) in Hilbert algebras is also studied and some related properties are investigated.

Keywords: Hilbert algebra; ideal; interval-valued neutrosophic ideal; level cut

1 Introduction and preliminaries

The concept of fuzzy sets was proposed by Zadeh.[21] The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in.[1,3,6] The idea of intuitionistic fuzzy sets suggested by Atanassov[2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multi-criteria decision-making.[11–13] The notion of neutrosophic sets was introduced by Smarandache[17] in 1999 which is a more general platform that extends the notions of classic sets, (intuitionistic) fuzzy sets and interval valued (intuitionistic) fuzzy sets (see[17,18]). Neutrosophic set theory is applied to various part which is referred to the site

http://fs.unm.edu/neutrosophy.htm.

The concept of Hilbert algebras was introduced in early 50-ties by Henkin and Skolem for some investigations of implication in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by Horn and Diego from algebraic point of view. Diego proved (cf.[8]) that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by Busneag (cf.,[45]) and Jun (cf.[15]) and some of their filters forming deductive systems were recognized. Dudek (cf.[9]) considered the fuzzification of subalgebras/ideals and deductive systems in Hilbert algebras.

In this paper, the concept of IVNSs to ideals of Hilbert algebras is introduced. The homomorphic inverse image of IVN ideals in Hilbert algebras is also studied and some related properties are investigated.

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Before we begin the study, let's review the definition of Hilbert algebras, which was defined by Diego[8] in 1966.

Definition 1.1. [8] A *Hilbert algebra* is a triplet $X = (X, \cdot, 1)$, where H is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed element of X such that the following axioms hold:

1. $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,
2. $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
3. $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

The following result was proved in.[9]

Lemma 1.2. *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. Then*

1. $(\forall x \in X)(x \cdot x = 1)$,
2. $(\forall x \in X)(1 \cdot x = x)$,
3. $(\forall x \in X)(x \cdot 1 = 1)$,
4. $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 1.3. [7] A nonempty subset I of a Hilbert algebra $X = (X, \cdot, 1)$ is called an *ideal* of X if the following conditions hold:

1. $1 \in I$,
2. $(\forall x \in X, \forall y \in I)(x \cdot y \in I)$,
3. $(\forall x \in X, \forall y_1, y_2 \in I)((y_1 \cdot (y_2 \cdot x)) \cdot x \in I)$.

A *fuzzy set*[21] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 1.4. [10] A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy ideal* of X if the following conditions hold:

1. $(\forall x \in X)(\mu(1) \geq \mu(x))$,
2. $(\forall x, y \in X)(\mu(x \cdot y) \geq \mu(y))$,
3. $(\forall x, y_1, y_2 \in X)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\mu(y_1), \mu(y_2)\})$

and an *anti fuzzy ideal* of X if the following conditions hold:

1. $(\forall x \in X)(\mu(1) \leq \mu(x))$,
2. $(\forall x, y \in X)(\mu(x \cdot y) \leq \mu(y))$,
3. $(\forall x, y_1, y_2 \in X)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{\mu(y_1), \mu(y_2)\})$.

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An *interval number* we mean a close subinterval $\tilde{a} = [a^l, a^u]$ of $[0, 1]$, where $0 \leq a^l \leq a^u \leq 1$. The interval number $\tilde{a} = [a^l, a^u]$ with $a^l = a^u$ is denoted by **a**. Denote by $D[0, 1]$ the set of all interval numbers. In particular, if $\tilde{a_1}$ and $\tilde{a_2}$ are interval numbers, we define the *refined minimum* and the *refined maximum* of $\tilde{a_1}$ and $\tilde{a_2}$, denoted by $\text{rmin}\{\tilde{a_1}, \tilde{a_2}\}$ and $\text{rmax}\{\tilde{a_1}, \tilde{a_2}\}$, respectively, as follows:

$$\text{rmin}\{\tilde{a_1}, \tilde{a_2}\} = [\min\{a_1^l, a_2^l\}, \min\{a_1^u, a_2^u\}],$$

$$\text{rmax}\{\tilde{a_1}, \tilde{a_2}\} = [\max\{a_1^l, a_2^l\}, \max\{a_1^u, a_2^u\}].$$

Definition 1.5. [16] Let $\tilde{a_1}$ and $\tilde{a_2}$ be interval numbers. We define the symbols $\succeq, \preceq$, and $=$ in case of $\tilde{a_1}$ and $\tilde{a_2}$ as follows:

$$\tilde{a_1} \succeq \tilde{a_2} \Leftrightarrow a_1^l \geq a_2^l \text{ and } a_1^u \geq a_2^u,$$

and similarly we may have $\tilde{a_1} \preceq \tilde{a_2}$ and $\tilde{a_1} = \tilde{a_2}$.

In $D[0, 1]$, the following assertions are valid (see[19]).

$$(\forall \tilde{a} \in D[0,1]) \left(\begin{array}{c} \text{rmax}\{\tilde{a}, \tilde{a}\} = \tilde{a} \\ \text{rmin}\{\tilde{a}, \tilde{a}\} = \tilde{a} \end{array} \right). \tag{1.1}$$

$$(\forall \tilde{a_1}, \tilde{a_2} \in D[0,1]) \left(\begin{array}{c} \text{rmax}\{\tilde{a_1}, \tilde{a_2}\} \succeq \tilde{a_1} \\ \tilde{a_1} \succeq \text{rmin}\{\tilde{a_1}, \tilde{a_2}\} \end{array} \right). \tag{1.2}$$

$$(\forall \tilde{a_1}, \tilde{a_2}, \tilde{a_3} \in D[0,1]) \left(\tilde{a_1} \succeq \tilde{a_2}, \tilde{a_3} \succeq \tilde{a_2} \Leftrightarrow \text{rmin}\{\tilde{a_1}, \tilde{a_3}\} \succeq \tilde{a_2} \right). \tag{1.3}$$

Definition 1.6. [20] An interval-valued neutrosophic set (IVNS) A in a nonempty set X is defined to be a structure

$$A = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\}, \tag{1.4}$$

where $T_A : X \to D[0, 1], I_A : X \to D[0, 1]$, and $F_A : X \to D[0, 1]$, which are called a truth membership function, an indeterminacy membership function, and a falsity membership function, respectively. The intervals $T_A(x), I_A(x)$, and $F_A(x)$ denote the intervals of the degree of membership, indeterminacy, and non-membership of the element x to the set $D[0, 1]$, respectively, where $T_A(x) = [T_A^l(x), T_A^u(x)], I_A(x) = [I_A^l(x), I_A^u(x)]$, and $F_A(x) = [F_A^l(x), F_A^u(x)]$ for all $x \in X$. Also note that $\overline{T_A}(x) = \mathbf{1} - T_A(x) = [1 - T_A^u(x), 1 - T_A^l(x)], \overline{I_A}(x) = \mathbf{1} - I_A(x) = [1 - I_A^u(x), 1 - I_A^l(x)]$, and $\overline{F_A}(x) = \mathbf{1} - F_A(x) = [1 - F_A^u(x), 1 - F_A^l(x)]$ for all $x \in X$, where $(x, \overline{T_A}(x), \overline{I_A}(x), \overline{F_A}(x))$ represents the complement of x in A. We define $\overline{A} = (\overline{T_A}, \overline{I_A}, \overline{F_A})$ as the complement of $A = (T_A, I_A, F_A)$. For the sake of simplicity, we shall use the symbol $A = (T_A, I_A, F_A)$ for the IVNS set $A = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\}$.

2 Interval-valued neutrosophic ideals of Hilbert algebras

In this section, we introduce the concept of IVN ideals of Hilbert algebras and investigate some related properties.

Definition 2.1. An IVNS $A = (T_A, I_A, F_A)$ in a Hilbert algebra $X = (X, \cdot, 1)$ is called an *interval-valued neutrosophic subalgebra* (IVN subalgebra) of X if

$$(\forall x, y \in X) \left(\begin{array}{c} T_A(x \cdot y) \succeq \text{rmin}\{T_A(x), T_A(y)\} \\ I_A(x \cdot y) \preceq \text{rmax}\{I_A(x), I_A(y)\} \\ F_A(x \cdot y) \succeq \text{rmin}\{F_A(x), F_A(y)\} \end{array} \right). \tag{2.1}$$

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Definition 2.2. An IVNS $A = (T_A, I_A, F_A)$ in a Hilbert algebra $X = (X, \cdot, 1)$ is called an *interval-valued neutrosophic ideal* (IVN ideal) of X if

$$(\forall x \in X)\left(\begin{array}{c} T_A(1) \succeq T_A(x) \\ I_A(1) \preceq I_A(x) \\ F_A(1) \succeq F_A(x) \end{array}\right), \tag{2.2}$$

$$(\forall x, y \in X)\left(\begin{array}{c} T_A(x \cdot y) \succeq T_A(y) \\ I_A(x \cdot y) \preceq I_A(y) \\ F_A(x \cdot y) \succeq F_A(y) \end{array}\right), \tag{2.3}$$

$$(\forall x, y_1, y_2 \in X)\left(\begin{array}{c} T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \mathrm{rmin}\{T_A(y_1), T_A(y_2)\} \\ I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \preceq \mathrm{rmax}\{I_A(y_1), I_A(y_2)\} \\ F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \mathrm{rmin}\{F_A(y_1), F_A(y_2)\} \end{array}\right). \tag{2.4}$$

Example 2.3. Let $X = \{1, x, y, z, 0\}$ with the following Cayley table:

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then X is a Hilbert algebra. We define an IVNS $A = (T_A, I_A, F_A)$ as follows:

$$T_A(x) = \begin{cases} [0.5, 0.6] & \text{if } x \in \{1, x, y, z\} \\ [0.1, 0.2] & \text{if } x = 0, \end{cases}$$

$$I_A(x) = \begin{cases} [0.3, 0.4] & \text{if } x \in \{1, x, y, z\} \\ [0.4, 0.5] & \text{if } x = 0, \end{cases}$$

$$F_A(x) = \begin{cases} [0.1, 0.2] & \text{if } x \in \{1, x, y, z\} \\ [0.2, 0.3] & \text{if } x = 0. \end{cases}$$

Hence, A is an IVN ideal of X.

Proposition 2.4. *If $A = (T_A, I_A, F_A)$ is an IVN ideal of a Hilbert algebra X, then*

$$(\forall x, y \in X)\left(\begin{array}{c} T_A((y \cdot x) \cdot x) \succeq T_A(y) \\ I_A((y \cdot x) \cdot x) \preceq I_A(y) \\ F_A((y \cdot x) \cdot x) \succeq F_A(y) \end{array}\right). \tag{2.5}$$

Proof. Putting $y_1 = y$ and $y_2 = 1$ in (2.4), we have

$$T_A((y \cdot x) \cdot x) \succeq \mathrm{rmin}\{T_A(y), T_A(1)\} = T_A(y),$$

$$I_A((y \cdot x) \cdot x) \preceq \mathrm{rmax}\{I_A(y), I_A(1)\} = I_A(y),$$

$$F_A((y \cdot x) \cdot x) \succeq \mathrm{rmin}\{F_A(y), F_A(1)\} = F_A(y).$$

□

Lemma 2.5. *If $A = (T_A, I_A, F_A)$ is an IVN ideal of a Hilbert algebra $X = (X, \cdot, 1)$, then*

$$(\forall x, y \in X)\left(x \leq y \Rightarrow \begin{cases} T_A(x) \preceq T_A(y) \\ I_A(x) \succeq I_A(y) \\ F_A(x) \preceq F_A(y) \end{cases} \right). \tag{2.6}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x \cdot y = 1$ and so

$$\begin{array}{rcl} T_A(y) & = & T_A(1 \cdot y) \\ & = & T_A(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ & \succeq & \mathrm{rmin}\{T_A(x \cdot y), T_A(x)\} \\ & \succeq & \mathrm{rmin}\{T_A(1), T_A(x)\} \\ & = & T_A(x), \end{array}$$

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$$\begin{aligned} I_A(y) &= I_A(1 \cdot y) \\ &= I_A(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ &\preceq \text{rmax}\{I_A(x \cdot y), I_A(x)\} \\ &\preceq \text{rmax}\{I_A(1), I_A(x)\} \\ &= I_A(x), \end{aligned}$$

$$\begin{aligned} F_A(y) &= F_A(1 \cdot y) \\ &= F_A(((x \cdot y) \cdot (x \cdot y)) \cdot y) \\ &\succeq \text{rmin}\{F_A(x \cdot y), F_A(x)\} \\ &\succeq \text{rmin}\{F_A(1), F_A(x)\} \\ &= F_A(x). \end{aligned}$$

□

Theorem 2.6. *Every IVN ideal of a Hilbert algebra $X = (X, \cdot, 1)$ is an IVN subalgebra of X.*

Proof. Let $A = (T_A, I_A, F_A)$ be an IVN ideal of X. Since $y \leq x \cdot y$ for all $x, y \in X$, then it follows from Lemma 2.5 that

$$T_A(y) \succeq T_A(x \cdot y), I_A(y) \preceq I_A(x \cdot y), F_A(y) \succeq F_A(x \cdot y).$$

It follows from (2.3) that

$$\begin{aligned} T_A(x \cdot y) &\succeq T_A(y) \\ &\succeq \text{rmin}\{T_A(x \cdot y), T_A(x)\} \\ &\succeq \text{rmin}\{T_A(x), T_A(y)\}, \end{aligned}$$

$$\begin{aligned} I_A(x \cdot y) &\preceq I_A(y) \\ &\preceq \text{rmax}\{I_A(x \cdot y), I_A(x)\} \\ &\preceq \text{rmax}\{I_A(x), I_A(y)\}, \end{aligned}$$

$$\begin{aligned} F_A(x \cdot y) &\succeq F_A(y) \\ &\succeq \text{rmin}\{F_A(x \cdot y), F_A(x)\} \\ &\succeq \text{rmin}\{F_A(x), F_A(y)\}. \end{aligned}$$

Hence, A is an IVN subalgebra of X. □

Theorem 2.7. *An IVNS $A = (T_A, I_A, F_A)$ in a Hilbert algebra $X = (X, \cdot, 1)$ is an IVN ideal of X if and only if T^l_A, T^u_A, F^l_A, and F^u_A are fuzzy ideals of X and I^l_A and I^u_A are anti fuzzy ideals of X.*

Proof. Since $T^l_A(1) \geq T^l_A(x)$, $T^u_A(1) \geq T^u_A(x)$, $I^l_A(1) \leq I^l_A(x)$, $I^u_A(1) \leq I^u_A(x)$, $F^l_A(1) \geq F^l_A(x)$, and $F^u_A(1) \geq F^u_A(x)$, we have $T_A(1) \succeq T_A(x)$, $I_A(1) \preceq I_A(x)$, and $F_A(1) \succeq F_A(x)$. Let $x, y \in X$. Then

$$T_A(x \cdot y) = [T^l_A(x \cdot y), T^u_A(x \cdot y)] \succeq [T^l_A(y), T^u_A(y)] = T_A(y),$$

$$I_A(x \cdot y) = [I^l_A(x \cdot y), I^u_A(x \cdot y)] \preceq [I^l_A(y), I^u_A(y)] = I_A(y),$$

$$F_A(x \cdot y) = [F^l_A(x \cdot y), F^u_A(x \cdot y)] \succeq [F^l_A(y), F^u_A(y)] = F_A(y).$$

Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned} T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) &= [T^l_A((y_1 \cdot (y_2 \cdot x)) \cdot x), T^u_A((y_1 \cdot (y_2 \cdot x)) \cdot x)] \\ &\succeq [\min\{T^l_A(y_1), T^l_A(y_2)\}, \min\{T^u_A(y_1), T^u_A(y_2)\}] \\ &= \text{rmin}\{[T^l_A(y_1), T^u_A(y_1)], [T^l_A(y_2), T^u_A(y_2)]\} \\ &= \text{rmin}\{T_A(y_1), T_A(y_2)\}, \end{aligned}$$

$$\begin{aligned} I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) &= [I^l_A((y_1 \cdot (y_2 \cdot x)) \cdot x), I^u_A((y_1 \cdot (y_2 \cdot x)) \cdot x)] \\ &\preceq [\max\{I^l_A(y_1), I^l_A(y_2)\}, \max\{I^u_A(y_1), I^u_A(y_2)\}] \\ &= \text{rmax}\{[I^l_A(y_1), I^u_A(y_1)], [I^l_A(y_2), I^u_A(y_2)]\} \\ &= \text{rmax}\{I_A(y_1), I_A(y_2)\}, \end{aligned}$$

$$\begin{aligned} F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) &= [F^l_A((y_1 \cdot (y_2 \cdot x)) \cdot x), F^u_A((y_1 \cdot (y_2 \cdot x)) \cdot x)] \\ &\succeq [\min\{F^l_A(y_1), F^l_A(y_2)\}, \min\{F^u_A(y_1), F^u_A(y_2)\}] \\ &= \text{rmin}\{[F^l_A(y_1), F^u_A(y_1)], [F^l_A(y_2), F^u_A(y_2)]\} \\ &= \text{rmin}\{F_A(y_1), F_A(y_2)\}. \end{aligned}$$

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Hence, A is an IVN ideal of X.

Conversely, assume that A is an IVN ideal of X. Let $x \in X$. Then $[T_A^l(1), T_A^u(1)] = T_A(1) \succeq T_A(x) = [T_A^l(x), T_A^u(x)]$; hence $T_A^l(1) \succeq T_A^l(x)$, $[I_A^l(1), I_A^u(1)] = I_A(1) \preceq I_A(x) = [I_A^l(x), I_A^u(x)]$; hence $I_A^l(1) \preceq I_A^l(x)$, and $[F_A^l(1), F_A^u(1)] = F_A(1) \succeq F_A(x) = [F_A^l(x), F_A^u(x)]$; hence $F_A^l(1) \succeq F_A^l(x)$. Let $x, y \in X$. Then $[T_A^l(x \cdot y), T_A^u(x \cdot y)] = T_A(x \cdot y) \succeq T_A(y) = [T_A^l(y), T_A^u(y)]$; hence $T_A^l(x \cdot y) \succeq T_A^l(y)$ and $T_A^u(x \cdot y) \succeq T_A^u(y)$, $[I_A^l(x \cdot y), I_A^u(x \cdot y)] = I_A(x \cdot y) \preceq I_A(y) = [I_A^l(y), I_A^u(y)]$; hence $I_A^l(x \cdot y) \preceq I_A^l(y)$ and $I_A^u(x \cdot y) \preceq I_A^u(y)$, and $[F_A^l(x \cdot y), F_A^u(x \cdot y)] = F_A(x \cdot y) \succeq F_A(y) = [F_A^l(y), F_A^u(y)]$; hence $F_A^l(x \cdot y) \succeq F_A^l(y)$ and $F_A^u(x \cdot y) \succeq F_A^u(y)$. Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}
[T_A^l((y_1 \cdot (y_2 \cdot x)) \cdot x), T_A^u((y_1 \cdot (y_2 \cdot x)) \cdot x)] &= T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \\
&\succeq \text{rmin}\{T_A(y_1), T_A(y_2)\} \\
&= \text{rmin}\{[T_A^l(y_1), T_A^u(y_1)], [T_A^l(y_2), T_A^u(y_2)]\} \\
&= [\min\{T_A^l(y_1), T_A^l(y_2)\}, \min\{T_A^u(y_1), T_A^u(y_2)\}].
\end{aligned}$$

Hence, $T_A^l((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{T_A^l(y_1), T_A^l(y_2)\}$ and $T_A^u((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{T_A^u(y_1), T_A^u(y_2)\}$. Now,

$$\begin{aligned}
[I_A^l((y_1 \cdot (y_2 \cdot x)) \cdot x), I_A^u((y_1 \cdot (y_2 \cdot x)) \cdot x)] &= I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \\
&\preceq \text{rmax}\{I_A(y_1), I_A(y_2)\} \\
&= \text{rmax}\{[I_A^l(y_1), I_A^u(y_1)], [I_A^l(y_2), I_A^u(y_2)]\} \\
&= [\max\{I_A^l(y_1), I_A^l(y_2)\}, \max\{I_A^u(y_1), I_A^u(y_2)\}].
\end{aligned}$$

Hence, $I_A^l((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{I_A^l(y_1), I_A^l(y_2)\}$ and $I_A^u((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{I_A^u(y_1), I_A^u(y_2)\}$. Also,

$$\begin{aligned}
[F_A^l((y_1 \cdot (y_2 \cdot x)) \cdot x), F_A^u((y_1 \cdot (y_2 \cdot x)) \cdot x)] &= F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \\
&\succeq \text{rmin}\{F_A(y_1), F_A(y_2)\} \\
&= \text{rmin}\{[F_A^l(y_1), F_A^u(y_1)], [F_A^l(y_2), F_A^u(y_2)]\} \\
&= [\min\{F_A^l(y_1), F_A^l(y_2)\}, \min\{F_A^u(y_1), F_A^u(y_2)\}].
\end{aligned}$$

Hence, $F_A^l((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{F_A^l(y_1), F_A^l(y_2)\}$ and $F_A^u((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{F_A^u(y_1), F_A^u(y_2)\}$. Therefore, T_A^l, T_A^u, F_A^l, and F_A^u are fuzzy ideals of X and I_A^l and I_A^u are anti fuzzy ideals of X. □

Theorem 2.8. *If $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ are two IVN ideals of a Hilbert algebra $X = (X, \cdot, 1)$, then $A \cap B = (T_{A\cap B}, I_{A\cup B}, F_{A\cap B})$ is an IVN ideal of X, where*

$$(\forall x \in X) \left(\begin{array}{l} T_{A\cap B}(x) = [T_{A\cap B}^l(x), T_{A\cap B}^u(x)] \\ I_{A\cup B}(x) = [I_{A\cup B}^l(x), I_{A\cup B}^u(x)] \\ F_{A\cap B}(x) = [F_{A\cap B}^l(x), F_{A\cap B}^u(x)] \end{array} \right). \tag{2.7}$$

Proof. Let $x \in X$. Then

$$\begin{aligned}
T_{A\cap B}(1) &= [T_{A\cap B}^l(1), T_{A\cap B}^u(1)] \\
&= [\min\{T_A^l(1), T_B^l(1)\}, \min\{T_A^u(1), T_B^u(1)\}] \\
&\succeq [\min\{T_A^l(x), T_B^l(x)\}, \min\{T_A^u(x), T_B^u(x)\}] \\
&= [T_{A\cap B}^l(x), T_{A\cap B}^u(x)] \\
&= T_{A\cap B}(x),
\end{aligned}$$

$$\begin{aligned}
I_{A\cup B}(1) &= [I_{A\cup B}^l(1), I_{A\cup B}^u(1)] \\
&= [\max\{I_A^l(1), I_B^l(1)\}, \max\{I_A^u(1), I_B^u(1)\}] \\
&\preceq [\max\{I_A^l(x), I_B^l(x)\}, \max\{I_A^u(x), I_B^u(x)\}] \\
&= [I_{A\cup B}^l(x), I_{A\cup B}^u(x)] \\
&= I_{A\cup B}(x),
\end{aligned}$$

$$\begin{aligned}
F_{A\cap B}(1) &= [F_{A\cap B}^l(1), F_{A\cap B}^u(1)] \\
&= [\min\{F_A^l(1), F_B^l(1)\}, \min\{F_A^u(1), F_B^u(1)\}] \\
&\succeq [\min\{F_A^l(x), F_B^l(x)\}, \min\{F_A^u(x), F_B^u(x)\}] \\
&= [F_{A\cap B}^l(x), F_{A\cap B}^u(x)] \\
&= F_{A\cap B}(x).
\end{aligned}$$

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Let $x, y \in X$. Then

$$\begin{array}{rcl} T_{A\cap B}(x\cdot y) & = & [T^l_{A\cap B}(x\cdot y), T^u_{A\cap B}(x\cdot y)] \\ & = & [\min\{T^l_A(x\cdot y), T^l_B(x\cdot y)\}, \min\{T^u_A(x\cdot y), T^u_B(x\cdot y)\}] \\ & \succeq & [\min\{T^l_A(y), T^l_B(y)\}, \min\{T^u_A(y), T^u_B(y)\}] \\ & = & [T^l_{A\cap B}(y), T^u_{A\cap B}(y)] \\ & = & T_{A\cap B}(y), \end{array}$$

$$\begin{array}{rcl} I_{A\cup B}(x\cdot y) & = & [I^l_{A\cup B}(x\cdot y), I^u_{A\cup B}(x\cdot y)] \\ & = & [\max\{I^l_A(x\cdot y), I^l_B(x\cdot y)\}, \max\{I^u_A(x\cdot y), I^u_B(x\cdot y)\}] \\ & \preceq & [\max\{I^l_A(y), I^l_B(y)\}, \max\{I^u_A(y), I^u_B(y)\}] \\ & = & [I^l_{A\cup B}(y), I^u_{A\cup B}(y)] \\ & = & I_{A\cup B}(y), \end{array}$$

$$\begin{array}{rcl} F_{A\cap B}(x\cdot y) & = & [F^l_{A\cap B}(x\cdot y), F^u_{A\cap B}(x\cdot y)] \\ & = & [\min\{F^l_A(x\cdot y), F^l_B(x\cdot y)\}, \min\{F^u_A(x\cdot y), F^u_B(x\cdot y)\}] \\ & \succeq & [\min\{F^l_A(y), F^l_B(y)\}, \min\{F^u_A(y), F^u_B(y)\}] \\ & = & [F^l_{A\cap B}(y), F^u_{A\cap B}(y)] \\ & = & F_{A\cap B}(y). \end{array}$$

Let $x, y_1, y_2 \in X$. Then

$$\begin{array}{rcl} T_{A\cap B}((y_1\cdot(y_2\cdot x))\cdot x) & = & [T^l_{A\cap B}((y_1\cdot(y_2\cdot x))\cdot x), T^u_{A\cap B}((y_1\cdot(y_2\cdot x))\cdot x)] \\ & = & \left[\begin{array}{l} \min\{T^l_A((y_1\cdot(y_2\cdot x))\cdot x), T^l_B((y_1\cdot(y_2\cdot x))\cdot x)\}, \\ \min\{T^u_A((y_1\cdot(y_2\cdot x))\cdot x), T^u_B((y_1\cdot(y_2\cdot x))\cdot x)\} \end{array}\right] \\ & \succeq & \left[\begin{array}{l} \min\{\min\{T^l_A(y_1), T^l_A(y_2)\}, \min\{T^l_B(y_1), T^l_B(y_2)\}\}, \\ \min\{\min\{T^u_A(y_1), T^u_A(y_2)\}, \min\{T^u_B(y_1), T^u_B(y_2)\}\} \end{array}\right] \\ & = & \left[\begin{array}{l} \min\{\min\{T^l_A(y_1), T^l_B(y_1)\}, \min\{T^l_A(y_2), T^l_B(y_2)\}\}, \\ \min\{\min\{T^u_A(y_1), T^u_B(y_2)\}, \min\{T^u_A(y_2), T^u_B(y_2)\}\} \end{array}\right] \\ & = & [\min\{T^l_{A\cap B}(y_1), T^l_{A\cap B}(y_2)\}, \min\{T^l_{A\cap B}(y_1), T^l_{A\cap B}(y_2)\}] \\ & = & \text{rmin}\{T_{A\cap B}(y_1), T_{A\cap B}(y_2)\}, \end{array}$$

$$\begin{array}{rcl} I_{A\cup B}((y_1\cdot(y_2\cdot x))\cdot x) & = & [I^l_{A\cup B}((y_1\cdot(y_2\cdot x))\cdot x), I^u_{A\cup B}((y_1\cdot(y_2\cdot x))\cdot x)] \\ & = & \left[\begin{array}{l} \max\{I^l_A((y_1\cdot(y_2\cdot x))\cdot x), I^l_B((y_1\cdot(y_2\cdot x))\cdot x)\}, \\ \max\{I^u_A((y_1\cdot(y_2\cdot x))\cdot x), I^u_B((y_1\cdot(y_2\cdot x))\cdot x)\} \end{array}\right] \\ & \preceq & \left[\begin{array}{l} \max\{\min\{I^l_A(y_1), I^l_A(y_2)\}, \min\{I^l_B(y_1), I^l_B(y_2)\}\}, \\ \max\{\min\{I^u_A(y_1), I^u_A(y_2)\}, \min\{I^u_B(y_1), I^u_B(y_2)\}\} \end{array}\right] \\ & = & \left[\begin{array}{l} \max\{\min\{I^l_A(y_1), I^l_B(y_1)\}, \min\{I^l_A(y_2), I^l_B(y_2)\}\}, \\ \max\{\min\{I^u_A(y_1), I^u_B(y_2)\}, \min\{I^u_A(y_2), I^u_B(y_2)\}\} \end{array}\right] \\ & = & [\max\{I^l_{A\cap B}(y_1), I^l_{A\cap B}(y_2)\}, \max\{I^l_{A\cap B}(y_1), I^l_{A\cap B}(y_2)\}] \\ & = & \text{rmax}\{I_{A\cap B}(y_1), I_{A\cap B}(y_2)\}, \end{array}$$

$$\begin{array}{rcl} F_{A\cap B}((y_1\cdot(y_2\cdot x))\cdot x) & = & [F^l_{A\cap B}((y_1\cdot(y_2\cdot x))\cdot x), F^u_{A\cap B}((y_1\cdot(y_2\cdot x))\cdot x)] \\ & = & \left[\begin{array}{l} \min\{F^l_A((y_1\cdot(y_2\cdot x))\cdot x), F^l_B((y_1\cdot(y_2\cdot x))\cdot x)\}, \\ \min\{F^u_A((y_1\cdot(y_2\cdot x))\cdot x), F^u_B((y_1\cdot(y_2\cdot x))\cdot x)\} \end{array}\right] \\ & \succeq & \left[\begin{array}{l} \min\{\min\{F^l_A(y_1), F^l_A(y_2)\}, \min\{F^l_B(y_1), F^l_B(y_2)\}\}, \\ \min\{\min\{F^u_A(y_1), F^u_A(y_2)\}, \min\{F^u_B(y_1), F^u_B(y_2)\}\} \end{array}\right] \\ & = & \left[\begin{array}{l} \min\{\min\{F^l_A(y_1), F^l_B(y_1)\}, \min\{F^l_A(y_2), F^l_B(y_2)\}\}, \\ \min\{\min\{F^u_A(y_1), F^u_B(y_2)\}, \min\{F^u_A(y_2), F^u_B(y_2)\}\} \end{array}\right] \\ & = & [\min\{F^l_{A\cap B}(y_1), F^l_{A\cap B}(y_2)\}, \min\{F^l_{A\cap B}(y_1), F^l_{A\cap B}(y_2)\}] \\ & = & \text{rmin}\{F_{A\cap B}(y_1), F_{A\cap B}(y_2)\}. \end{array}$$

Hence, $A \cap B$ is an IVN ideal of X. □

Corollary 2.9. *If $A = (T_A, I_A, F_A)$ is an IVN ideal of a Hilbert algebra $X = (X, \cdot, 1)$, then $\overline{A}$ is also an IVN ideal of X.*

Definition 2.10. Let $A = (T_A, I_A, F_A)$ be an IVNS in a Hilbert algebra $X = (X, \cdot, 1)$. The IVNSs $\oplus A, \otimes A$, and $\odot A$ in X are defined as follows: $\oplus A = (T_A, \overline{T_A}, F_A), \otimes A = (\overline{I_A}, I_A, F_A)$, and $\odot A = (\overline{I_A}, I_A, \overline{I_A})$.

Theorem 2.11. *If $A = (T_A, I_A, F_A)$ is an IVN ideal of a Hilbert algebra $X = (X, \cdot, 1)$, then $\oplus A, \otimes A$, and $\odot A$ are IVN ideals of X.*

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Proof. Let $x \in X$. Then $\overline{T_A}(1) = \mathbf{1} - T_A(1) \preceq \mathbf{1} - T_A(x) \preceq \overline{T_A}(x)$. Let $x, y \in X$. Then $\overline{T_A}(x \cdot y) = \mathbf{1} - T_A(x \cdot y) \preceq \mathbf{1} - T_A(y) \preceq \overline{T_A}(y)$. Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}\overline{T_A}((y_1 \cdot (y_2 \cdot x)) \cdot x) &= \mathbf{1} - T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \\ &\preceq \mathbf{1} - \text{rmin}\{T_A(y_1), T_A(y_2)\} \\ &= \text{rmax}\{\mathbf{1} - T_A(y_1), \mathbf{1} - T_A(y_2)\} \\ &= \text{rmax}\{\overline{T_A}(y_1), \overline{T_A}(y_2)\}.\end{aligned}$$

Hence, $\oplus A$ is an IVN ideal of X.

Let $x \in X$. Then $\overline{I_A}(1) = \mathbf{1} - I_A(1) \succeq \mathbf{1} - I_A(x) \succeq \overline{I_A}(x)$. Let $x, y \in X$. Then $\overline{I_A}(x \cdot y) = \mathbf{1} - I_A(x \cdot y) \succeq \mathbf{1} - I_A(y) \succeq \overline{I_A}(y)$. Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}\overline{I_A}((y_1 \cdot (y_2 \cdot x)) \cdot x) &= \mathbf{1} - I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \\ &\succeq \mathbf{1} - \text{rmax}\{I_A(y_1), I_A(y_2)\} \\ &= \text{rmin}\{\mathbf{1} - I_A(y_1), \mathbf{1} - I_A(y_2)\} \\ &= \text{rmin}\{\overline{I_A}(y_1), \overline{I_A}(y_2)\}.\end{aligned}$$

Hence, $\otimes A$ is an IVN ideal of X. The proof of $\odot A$ is similar. □

Theorem 2.12. *An IVNS $A = (T_A, I_A, F_A)$ in a Hilbert algebra $X = (X, \cdot, 1)$ is an IVN ideal of X if and only if for every $[s_1, s_2], [t_1, t_2], [u_1, u_2] \in D[0, 1]$, the sets $U(T_A : [t_1, t_2])$, $L(I_A : [s_1, s_2])$, and $U(F_A : [u_1, u_2])$ are either empty or ideals of X.*

Proof. Let $A = (T_A, I_A, F_A)$ be an IVN ideal of X and let $[s_1, s_2], [t_1, t_2], [u_1, u_2] \in D[0, 1]$ be such that $U(T_A : [t_1, t_2])$, $L(I_A : [s_1, s_2])$, and $U(F_A : [u_1, u_2])$ are nonempty sets of X. It is clear that $1 \in U(T_A : [t_1, t_2]) \cap L(I_A : [s_1, s_2]) \cap U(F_A : [u_1, u_2])$ since $T_A(1) \succeq [t_1, t_2]$, $I_A(1) \preceq [s_1, s_2]$, and $F_A(1) \succeq [u_1, u_2]$.

Let $x \in X$ and $y \in U(T_A : [t_1, t_2])$. Then $T_A(y) \succeq [t_1, t_2]$. It follows that $T_A(x \cdot y) \succeq T_A(y) \succeq [t_1, t_2]$ so that $x \cdot y \in U(T_A : [t_1, t_2])$. Let $x \in X$ and $y_1, y_2 \in U(T_A : [t_1, t_2])$. Then $T_A(y_1) \succeq [t_1, t_2]$ and $T_A(y_2) \succeq [t_1, t_2]$. Hence, $T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \text{rmin}\{T_A(y_1), T_A(y_2)\} \succeq [t_1, t_2]$ so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(T_A : [t_1, t_2])$. Hence, $U(T_A : [t_1, t_2])$ is an ideal of X.

Let $x \in X$ and $y \in L(I_A : [s_1, s_2])$. Then $I_A(y) \preceq [s_1, s_2]$. It follows that $I_A(x \cdot y) \preceq I_A(y) \preceq [s_1, s_2]$ so that $x \cdot y \in L(I_A : [s_1, s_2])$. Let $x \in X$ and $y_1, y_2 \in L(I_A : [s_1, s_2])$. Then $I_A(y_1) \preceq [s_1, s_2]$ and $I_A(y_2) \preceq [s_1, s_2]$. Hence, $I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \preceq \text{rmax}\{I_A(y_1), I_A(y_2)\} \preceq [s_1, s_2]$ so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L(I_A : [s_1, s_2])$. Hence, $L(I_A : [s_1, s_2])$ is an ideal of X.

Let $x \in X$ and $y \in U(F_A : [u_1, u_2])$. Then $F_A(y) \succeq [u_1, u_2]$. It follows that $F_A(x \cdot y) \succeq F_A(y) \succeq [u_1, u_2]$ so that $x \cdot y \in U(F_A : [u_1, u_2])$. Let $x \in X$ and $y_1, y_2 \in U(F_A : [u_1, u_2])$. Then $F_A(y_1) \succeq [u_1, u_2]$ and $F_A(y_2) \succeq [u_1, u_2]$. Hence, $F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \text{rmin}\{F_A(y_1), F_A(y_2)\} \succeq [u_1, u_2]$ so that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(F_A : [u_1, u_2])$. Hence, $U(F_A : [u_1, u_2])$ is an ideal of X.

Assume now that every nonempty sets $U(T_A : [t_1, t_2])$, $L(I_A : [s_1, s_2])$, and $U(F_A : [u_1, u_2])$ are ideals of X. If $T_A(1) \succeq T_A(x)$ is not true for all $x \in X$, then there exists $x_0 \in X$ such that $T_A(1) \prec T_A(x_0)$. But in this case for $[s_1, s_2] = \frac{1}{2}(T_A(1) + T_A(x_0))$. Then $x_0 \in U(T_A : [s_1, s_2])$, that is, $U(T_A : [s_1, s_2]) \neq \emptyset$. By the assumption, $U(T_A : [s_1, s_2])$ is an ideal of X and so $T_A(1) \succeq [s_1, s_2]$, which is impossible. Hence, $T_A(1) \succeq T_A(x)$. If $I_A(1) \preceq I_A(x)$ is not true, then there exists $y_0 \in X$ such that $I_A(1) \prec I_A(y_0)$. But in this case for $[s'_0, s''_0] = \frac{1}{2}(I_A(1) + I_A(y_0))$. Then $y_0 \in L(I_A : [s'_0, s''_0])$, that is, $L(I_A : [s'_0, s''_0]) \neq \emptyset$. By the assumption, $L(I_A : [s'_0, s''_0])$ is an ideal of X and so $I_A(1) \preceq [s'_0, s''_0]$, which is impossible. Hence, $I_A(1) \preceq I_A(x)$. If $F_A(1) \succeq F_A(x)$ is not true for all $x \in X$, then there exists $x_0 \in X$ such that $F_A(1) \prec F_A(x_0)$. But in this case for $[u_1, u_2] = \frac{1}{2}(F_A(1) + F_A(x_0))$. Then $x_0 \in U(F_A : [u_1, u_2])$, that is, $U(F_A : [u_1, u_2]) \neq \emptyset$. By the assumption, $U(F_A : [u_1, u_2])$ is an ideal of X, then $F_A(1) \succeq [u_1, u_2]$, which is impossible. Hence, $F_A(1) \succeq F_A(x)$.

If $T_A(x \cdot y) \succeq T_A(y)$ is not true for all $x, y \in X$, then there exist $x_0, y_0 \in X$ such that $T_A(x_0 \cdot y_0) \prec T_A(y_0)$. Let $[t_1, t_2] = \frac{1}{2}(T_A(x_0 \cdot y_0) + T_A(y_0))$. Then $[t_1, t_2] \in D[0, 1]$ and $T_A(x_0 \cdot y_0) \prec [t_1, t_2] \prec T_A(y_0)$, which prove that $y_0 \in U(T_A : [t_1, t_2])$. Since $U(T_A : [t_1, t_2])$ is an ideal of X, $x_0 \cdot y_0 \in U(T_A : [t_1, t_2])$. Hence, $T_A(x_0 \cdot y_0) \succeq [t_1, t_2]$, a contradiction. Thus $T_A(x \cdot y) \succeq T_A(y)$ is true for all $x, y \in X$. If $I_A(x \cdot y) \preceq I_A(y)$

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is not true for all $x, y \in X$, then there exist $x_0, y_0 \in X$ such that $I_A(x_0 \cdot y_0) \succ I_A(y_0)$. Let $[t_0', t_0''] = \frac{1}{2}(I_A(x_0 \cdot y_0) + I_A(y_0))$. Then $[t_0', t_0''] \in D[0,1]$ and $I_A(x_0 \cdot y_0) \succ [t_0', t_0''] \succ I_A(y_0)$, which prove that $y_0 \in L(I_A : [t_0', t_0''])$. Since $L(I_A : [t_0', t_0''])$ is an ideal of X, $x_0 \cdot y_0 \in L(I_A : [t_0', t_0''])$. Hence, $I_A(x_0 \cdot y_0) \preceq [t_0', t_0'']$, a contradiction. Thus $I_A(x \cdot y) \preceq I_A(y)$ is true for all $x, y \in X$. If $F_A(x \cdot y) \succeq F_A(y)$ is not true for all $x, y \in X$, then there exist $x_0, y_0 \in X$ such that $F_A(x_0 \cdot y_0) \prec F_A(y_0)$. Let $[u_1, u_2] = \frac{1}{2}(F_A(x_0 \cdot y_0) + F_A(y_0))$. Then $[u_1, u_2] \in D[0,1]$ and $F_A(x_0 \cdot y_0) \prec [u_1, u_2] \prec F_A(y_0)$, which prove that $y_0 \in U(F_A : [u_1, u_2])$. Since $U(F_A : [u_1, u_2])$ is an ideal of X, $x_0 \cdot y_0 \in U(F_A : [u_1, u_2])$. Hence, $F_A(x_0 \cdot y_0) \succeq [u_1, u_2]$, a contradiction. Thus $F_A(x \cdot y) \succeq F_A(y)$ is true for all $x, y \in X$.

Suppose that $T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \text{rmin}\{T_A(y_1), T_A(y_2)\}$ is not true for all $x, y_1, y_2 \in X$. Then there exist $u_0, v_0, x_0 \in X$ such that $T_A((u_0 \cdot (v_0 \cdot x_0))) \cdot x_0) \prec \text{rmin}\{T_A(u_0), T_A(v_0)\}$. Taking $[p', p''] = \frac{1}{2}(T_A((u_0 \cdot (v_0 \cdot x_0))) \cdot x_0) + \text{rmin}\{T_A(u_0), T_A(v_0)\})$. Then $T_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) \prec [p', p''] \prec \text{rmin}\{T_A(u_0), T_A(v_0)\}$, which prove that $u_0, v_0 \in U(T_A : [p', p''])$. Since $U(T_A : [p', p''])$ is an ideal of X, $(u_0 \cdot (v_0 \cdot x_0)) \cdot x_0 \in U(T_A : [p', p''])$, a contradiction. Thus $T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \text{rmin}\{T_A(y_1), T_A(y_2)\}$ is true for all $x, y_1, y_2 \in X$. Suppose that $I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \preceq \text{rmax}\{I_A(y_1), I_A(y_2)\}$ is not true for all $x, y_1, y_2 \in X$. Then there exist $u_0, v_0, x_0 \in X$ such that $I_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) \succ \text{rmax}\{I_A(u_0), I_A(v_0)\}$. Taking $[p_0', p_0''] = \frac{1}{2}(I_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) + \text{rmax}\{I_A(u_0), I_A(v_0)\})$. Then $I_A((u_0 \cdot (v_0 \cdot x_0)) \cdot x_0) \succ [p_0', p_0''] \succ \text{rmax}\{I_A(u_0), I_A(v_0)\}$, which prove that $u_0, v_0 \in L(I_A : [p_0', p_0''])$. Since $L(I_A : [p_0', p_0''])$ is an ideal of X, $(u_0 \cdot (v_0 \cdot x_0)) \cdot x_0 \in L(I_A : [p_0', p_0''])$, a contradiction. Thus $I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \preceq \text{rmax}\{I_A(y_1), I_A(y_2)\}$ is true for all $x, y_1, y_2 \in X$. Suppose that $F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \text{rmin}\{F_A(y_1), F_A(y_2)\}$ is not true for all $x, y_1, y_2 \in X$. Then there exist $f_0, g_0, h_0 \in X$ such that $F_A((f_0 \cdot (g_0 \cdot h_0))) \cdot h_0) \prec \text{rmin}\{F_A(f_0), T_A(g_0)\}$. Taking $[q', q''] = \frac{1}{2}(F_A((f_0 \cdot (g_0 \cdot h_0))) \cdot h_0) + \text{rmin}\{F_A(f_0), F_A(g_0)\})$. Then $F_A((f_0 \cdot (g_0 \cdot h_0)) \cdot h_0) \prec [q', q''] \prec \text{rmin}\{F_A(f_0), F_A(g_0)\}$, which prove that $f_0, g_0 \in U(F_A : [q', q''])$. Since $U(F_A : [q', q''])$ is an ideal of X, $(f_0 \cdot (g_0 \cdot h_0)) \cdot h_0 \in U(F_A : [q', q''])$, a contradiction. Thus $F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \text{rmin}\{F_A(y_1), F_A(y_2)\}$ is true for all $x, y_1, y_2 \in X$. Hence, A is an IVN ideal of X. □

Theorem 2.13. *Let B be a nonempty subset of a Hilbert algebra $X = (X, \cdot, 1)$ and $A = (T_A, I_A, F_A)$ be an IVNS in X defined by*

$$T_A(x) = \begin{cases} \alpha_0 & \text{if } x \in B \\ \alpha_1 & \text{otherwise}, \end{cases}$$

$$I_A(x) = \begin{cases} \beta_0 & \text{if } x \in B \\ \beta_1 & \text{otherwise}, \end{cases}$$

$$F_A(x) = \begin{cases} \theta_0 & \text{if } x \in B \\ \theta_1 & \text{otherwise} \end{cases}$$

for all $x \in X$ and $\alpha_i, \beta_i, \theta_i \in D[0,1]$ such that $\alpha_0 \succ \alpha_1$, $\beta_0 \prec \beta_1$, $\theta_0 \succ \theta_1$ for $i = 1, 2$. Then A is an IVN ideal of X if and only if B is an ideal of X.

Proof. Assume that A is an IVN ideal of X. Since $T_A(1) \succeq T_A(x)$, $I_A(1) \preceq I_A(x)$, and $F_A(1) \succeq F_A(x)$ for all $x \in X$, we have $T_A(1) = \alpha_1$, $I_A(1) = \beta_1$, and $F_A(1) = \theta_1$ and so $1 \in B$. Let $x \in X$ and $y \in B$. Then $T_A(x \cdot y) \succeq T_A(y) = \alpha_1$ and then $T_A(x \cdot y) = \alpha_1$. Also, $I_A(x \cdot y) \preceq I_A(y) = \beta_1$ and then $I_A(x \cdot y) = \beta_1$ and $F_A(x \cdot y) \succeq F_A(y) = \theta_1$; hence, $F_A(x \cdot y) = \theta_1$. Hence, $x \cdot y \in B$. For any $y_1, y_2 \in B$ and $x \in X$, we get $T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \text{rmin}\{T_A(y_1), T_A(y_2)\} = \alpha_1$, $I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \preceq \text{rmax}\{I_A(y_1), I_A(y_2)\} = \beta_1$, and $F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \text{rmin}\{F_A(y_1), F_A(y_2)\} = \theta_1$, which implies that $T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) = \alpha_1$, $I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) = \beta_1$, and $F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) = \theta_1$. It follows that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in B$. Therefore, B is an ideal of X.

Conversely, suppose that B is an ideal of X. Since $1 \in B$, it follows that $T_A(1) = \alpha_1 \succeq T_A(x)$, $I_A(1) = \beta_1 \preceq I_A(x)$, and $F_A(1) = \theta_1 \succeq F_A(x)$ for all $x \in X$. Let $x, y \in X$. If $y \in B$, then $x \cdot y \in B$ and so $T_A(x \cdot y) = \alpha_1 = T_A(y)$, $I_A(x \cdot y) = \beta_1 = I_A(y)$, and $F_A(x \cdot y) = \theta_1 = F_A(y)$. If $y \in X \backslash B$, then $T_A(y) = \alpha_2$, $I_A(y) = \beta_2$, and $F_A(y) = \theta_2$ and so $T_A(x \cdot y) \succeq \alpha_2 = T_A(y)$, $I_A(x \cdot y) \preceq \beta_2 = I_A(y)$, and $F_A(x \cdot y) \succeq \theta_2 = F_A(y)$. Finally, let $y_1, y_2 \in X$. If $y_1 \in X \backslash B$ or $y_2 \in X \backslash B$, then $T_A(y_1) = \alpha_2$ or $T_A(y_2) = \alpha_2$. It follows that $T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \alpha_2 = \min\{T_A(y_1), T_A(y_2)\}$. Also, if $y_1 \in X \backslash B$ or $y_2 \in X \backslash B$, then $I_A(y_1) = \beta_2$ or $I_A(y_2) = \beta_2$. It follows that $I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \preceq \beta_2 = \text{rmax}\{I_A(y_1), I_A(y_2)\}$. If $y_1 \in X \backslash B$ or $y_2 \in X \backslash B$, then $F_A(y_1) = \theta_2$ or $F_A(y_2) = \theta_2$. It follows that $F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \theta_2 = \min\{F_A(y_1), F_A(y_2)\}$. Assume that $y_1, y_2 \in B$. Then $(y_1 \cdot (y_2 \cdot x)) \cdot x \in B$ and thus $T_A((y_1 \cdot (y_2 \cdot x)) \cdot x) = \alpha_1 = \text{rmin}\{T_A(y_1), T_A(y_2)\}$, $I_A((y_1 \cdot (y_2 \cdot x)) \cdot x) = \beta_1 = \text{rmax}\{I_A(y_1), I_A(y_2)\}$, and $F_A((y_1 \cdot (y_2 \cdot x)) \cdot x) = \theta_1 = \text{rmin}\{F_A(y_1), F_A(y_2)\}$. Hence, A is an IVN ideal of X. □

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Definition 2.14. Let $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be IVNSs in Hilbert algebras X and Y, respectively. The cartesian product $A \times B = \{((x,y),(T_A \times T_B)(x,y),(I_A \times I_B)(x,y),(F_A \times F_B)(x,y)) \mid x \in X, y \in Y\}$ is defined by

$$(\forall (x,y) \in X \times Y)\left(\begin{array}{l}(T_A \times T_B)(x,y) = \text{rmin}\{T_A(x), T_B(y)\}\\ (I_A \times I_B)(x,y) = \text{rmax}\{I_A(x), I_B(y)\}\\ (F_A \times F_B)(x,y) = \text{rmin}\{F_A(x), F_B(y)\}\end{array}\right),$$

where $T_A \times T_B : X \times Y \to D[0,1]$, $I_A \times I_B : X \times Y \to D[0,1]$, and $F_A \times F_B : X \times Y \to D[0,1]$.

Remark 2.15. Let X and Y be Hilbert algebras. We define the binary operation $\cdot$ on $X \times Y$ by $(x,y)\cdot(u,v) = (x \cdot u, y \cdot v)$ for every $(x,y),(u,v) \in X \times Y$. Then clearly $(X \times Y, \cdot, (1,1))$ is a Hilbert algebra.

Proposition 2.16. *If $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ are IVN ideals of Hilbert algebras X and Y, respectively, then the cartesian product $A \times B$ is also an IVN ideal of $X \times Y$.*

Proof. Let $(x,y) \in X \times Y$. Then

$$\begin{array}{rcl}(T_A \times T_B)(1,1) & = & \text{rmin}\{T_A(1), T_B(1)\}\\ & \succeq & \text{rmin}\{T_A(x), T_B(y)\}\\ & = & (T_A \times T_B)(x,y),\end{array}$$

$$\begin{array}{rcl}(I_A \times I_B)(1,1) & = & \text{rmax}\{I_A(1), I_B(1)\}\\ & \preceq & \text{rmax}\{I_A(x), I_B(y)\}\}\\ & = & (I_A \times I_B)(x,y),\end{array}$$

$$\begin{array}{rcl}(F_A \times F_B)(1,1) & = & \text{rmin}\{F_A(1), F_B(1)\}\\ & \succeq & \text{rmin}\{F_A(x), F_B(y)\}\\ & = & (F_A \times F_B)(x,y).\end{array}$$

Let $(x_1,x_2),(y_1,y_2) \in X \times Y$. Then

$$\begin{array}{rcl}(T_A \times T_B)((x_1,x_2)\cdot(y_1,y_2)) & = & (T_A \times T_B)((x_1 \cdot y_1),(x_2 \cdot y_2))\\ & = & \text{rmin}\{T_A(x_1 \cdot y_1), T_B(x_2 \cdot y_2)\}\\ & \succeq & \text{rmin}\{T_A(y_1), T_B(y_2)\}\\ & = & (T_A \times T_B)(y_1,y_2),\end{array}$$

$$\begin{array}{rcl}(I_A \times I_B)((x_1,x_2)\cdot(y_1,y_2)) & = & (I_A \times I_B)((x_1 \cdot y_1),(x_2 \cdot y_2))\\ & = & \text{rmax}\{I_A(x_1 \cdot y_1), I_B(x_2 \cdot y_2)\}\\ & \preceq & \text{rmax}\{I_A(y_1), I_B(y_2)\}\\ & = & (I_A \times I_B)(y_1,y_2),\end{array}$$

$$\begin{array}{rcl}(F_A \times F_B)((x_1,x_2)\cdot(y_1,y_2)) & = & (F_A \times F_B)((x_1 \cdot y_1),(x_2 \cdot y_2))\\ & = & \text{rmin}\{F_A(x_1 \cdot y_1), F_B(x_2 \cdot y_2)\}\\ & \succeq & \text{rmin}\{F_A(y_1), F_B(y_2)\}\\ & = & (F_A \times F_B)(y_1,y_2).\end{array}$$

Let $(x_1,y_1),(x_2,y_2),(x_3,y_3) \in X \times Y$. Then

$$\begin{array}{l}(T_A \times T_B)(((x_2,y_2)\cdot((x_3,y_3)\cdot(x_1,y_1)))\cdot(x_1,y_1))\\ = (T_A \times T_B)(((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1)),(y_2 \cdot (y_3 \cdot y_1)) \cdot y_1))\\ = \text{rmin}\{T_A((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1)), T_B((y_2 \cdot (y_3 \cdot y_1)) \cdot y_1))\}\\ \succeq \text{rmin}\{\text{rmin}\{T_A(x_2), T_A(x_3)\}, \text{rmin}\{T_B(y_2), T_B(y_3)\}\}\\ = \text{rmin}\{\text{rmin}\{T_A(x_2), T_B(y_2)\}, \text{rmin}\{T_A(x_3), T_B(y_3)\}\}\\ = \text{rmin}\{(T_A \times T_B)(x_2,y_2),(T_A \times T_B)(x_3,y_3)\},\end{array}$$

$$\begin{array}{l}(I_A \times I_B)(((x_2,y_2)\cdot((x_3,y_3)\cdot(x_1,y_1)))\cdot(x_1,y_1))\\ = (I_A \times I_B)(((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1)),(y_2 \cdot (y_3 \cdot y_1)) \cdot y_1))\\ = \text{rmax}\{I_A((x_2 \cdot (x_3 \cdot x_1)) \cdot x_1)), I_B((y_2 \cdot (y_3 \cdot y_1)) \cdot y_1))\}\\ \preceq \text{rmax}\{\text{rmax}\{I_A(x_2), I_A(x_3)\}, \text{rmax}\{I_B(y_2), I_B(y_3)\}\}\\ = \text{rmax}\{\text{rmax}\{I_A(x_2), I_B(y_2)\}, \text{rmax}\{I_A(x_3), I_B(y_3)\}\}\\ = \text{rmax}\{(I_A \times I_B)(x_2,y_2),(I_A \times I_B)(x_3,y_3)\},\end{array}$$

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$$\begin{aligned}
&(F_A \times F_B)(((x_2, y_2)\cdot((x_3, y_3)\cdot(x_1, y_1)))\cdot(x_1, y_1)) \\
&= (F_A \times F_B)(((x_2\cdot(x_3\cdot x_1))\cdot x_1)), (y_2\cdot(y_3\cdot y_1))\cdot y_1)) \\
&= \text{rmin}\{F_A((x_2\cdot(x_3\cdot x_1))\cdot x_1)), F_B((y_2\cdot(y_3\cdot y_1))\cdot y_1))\} \\
&\succeq \text{rmin}\{\text{rmin}\{F_A(x_2), F_A(x_3)\}, \text{rmin}\{F_B(y_2), F_B(y_3)\}\} \\
&= \text{rmin}\{\text{rmin}\{F_A(x_2), F_B(y_2)\}, \text{rmin}\{F_A(x_3), F_B(y_3)\}\} \\
&= \text{rmin}\{(F_A \times F_B)(x_2, y_2), (F_A \times F_B)(x_3, y_3)\}.
\end{aligned}$$

Hence, $A \times B$ is an IVN ideal of $X \times Y$. □

Theorem 2.17. *If $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ are IVN ideals of Hilbert algebras X and Y, respectively, then $\oplus(A \times B), \otimes(A \times B)$, and $\odot(A \times B)$ are IVN ideals of $X \times Y$.*

Proof. It follows from Theorem 2.11 and Proposition 2.16. □

Theorem 2.18. *Let $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be any two IVNSs in Hilbert algebras X and Y, respectively. Then $A \times B$ is an IVN ideal of $X \times Y$ if and only if the nonempty upper $[s_1, s_2]$-level cut $U(T_A \times T_B : [s_1, s_2])$, the nonempty lower $[t_1, t_2]$-level cut $L(I_A \times I_B : [t_1, t_2])$, and the nonempty upper $[u_1, u_2]$-level cut $U(F_A \times F_B : [u_1, u_2])$ are ideals of $X \times Y$ for all $[s_1, s_2], [t_1, t_2], [u_1, u_2] \in D[0, 1]$.*

Proof. It follows from Theorem 2.12. □

For any fixed interval numbers $\tilde{a}^+, \tilde{a}^-, \tilde{b}^+, \tilde{b}^-, \tilde{c}^+, \tilde{c}^- \in D[0, 1]$ such that $\tilde{a}^+ \succ \tilde{a}^-, \tilde{b}^+ \succ \tilde{b}^-, \tilde{c}^+ \succ \tilde{c}^-$ and a nonempty subset G of a Hilbert algebra X, the IVNS

$$A^G \begin{bmatrix} \tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+ \\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^- \end{bmatrix} = \left(T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}, I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix}, F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} \right)$$

in X is defined by for all $x \in X$,

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(x) = \begin{cases} \tilde{a}^+ & \text{if } x \in G \\ \tilde{a}^- & \text{otherwise,} \end{cases}$$

$$I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix}(x) = \begin{cases} \tilde{b}^- & \text{if } x \in G \\ \tilde{b}^+ & \text{otherwise,} \end{cases}$$

$$F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix}(x) = \begin{cases} \tilde{c}^+ & \text{if } x \in G \\ \tilde{c}^- & \text{otherwise.} \end{cases}$$

Lemma 2.19. [14] *If the constant 1 of a Hilbert algebra $X = (X, \cdot, 1)$ is in a nonempty subset G of X, then the IVNS $A^G \begin{bmatrix} \tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+ \\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^- \end{bmatrix}$ in X satisfies the condition* (2.2).

Lemma 2.20. [14] *If the IVNS $A^G \begin{bmatrix} \tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+ \\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^- \end{bmatrix}$ in a Hilbert algebra $X = (X, \cdot, 1)$ satisfies the condition* (2.2)*, then the constant 1 of X is in a nonempty subset G of X.*

Theorem 2.21. *The IVNS $A^G \begin{bmatrix} \tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+ \\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^- \end{bmatrix}$ in a Hilbert algebra $X = (X, \cdot, 1)$ is an IVN ideal of X if and only if a nonempty subset G of X is a ideal of X.*

Proof. Assume that $A^G \begin{bmatrix} \tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+ \\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^- \end{bmatrix}$ is an IVN ideal of X. Since $A^G \begin{bmatrix} \tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+ \\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^- \end{bmatrix}$ satisfies the condition (2.2), it follows from Lemma 2.20 that $1 \in G$. Next, let $x \in X$ and $y \in G$. Then $T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(y) = \tilde{a}^+$. Hence,

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(x \cdot y) \succeq T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(y) = \tilde{a}^+ \succeq T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(x \cdot y)$$

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and so $T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(x \cdot y) = \tilde{a}^+$. Thus $x \cdot y \in G$. Next, let $x \in X$ and $y_1, y_2 \in G$. Then $T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(y_1) = \tilde{a}^+ = T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(y_2)$. Now,

$$\begin{aligned} T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}((y_1 \cdot (y_2 \cdot x)) \cdot x) &\succeq \operatorname{rmin}\left\{T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(y_1), T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(y_2)\right\} \\ &= \operatorname{rmin}\{\tilde{a}^+, \tilde{a}^+\} \\ &= \tilde{a}^+ \\ &\succeq T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}((y_1 \cdot (y_2 \cdot x)) \cdot x) \end{aligned}$$

and so $T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}((y_1 \cdot (y_2 \cdot x)) \cdot x) = \tilde{a}^+$. Thus $(y_1 \cdot (y_2 \cdot x)) \cdot x \in G$. Hence, G is an ideal of X.

Conversely, assume that G is an ideal of X. Since $1 \in G$, it follows from Lemma 2.19 that $A^G \begin{bmatrix} \tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+ \\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^- \end{bmatrix}$ satisfies the condition (2.2). Next, let $x, y \in X$. Then

Case (1): Suppose $x, y \in G$. Then

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(x) = \tilde{a}^+ = T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(y),$$

$$I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix}(x) = \tilde{b}^- = I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix}(y),$$

$$F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix}(x) = \tilde{c}^+ = F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix}(y).$$

Since G is an ideal of X, we have $x \cdot y \in G$ and so

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(x \cdot y) = \tilde{a}^+, I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix}(x \cdot y) = \tilde{b}^-, F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix}(x \cdot y) = \tilde{c}^+.$$

It follows from (1.1) that

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(x \cdot y) = \tilde{a}^+ \succeq \tilde{a}^+ = T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(y),$$

$$I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix}(x \cdot y) = \tilde{b}^- \preceq \tilde{b}^- = I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix}(y),$$

$$F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix}(x \cdot y) = \tilde{c}^+ \succeq \tilde{c}^+ = F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix}(y).$$

Case (2): Suppose $x \notin G$ or $y \notin G$. Then

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(x) = \tilde{a}^- \text{ or } T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(y) = \tilde{a}^-,$$

$$I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix}(x) = \tilde{b}^+ \text{ or } I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix}(y) = \tilde{b}^+,$$

$$F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix}(x) = \tilde{c}^- \text{ or } F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix}(y) = \tilde{c}^-.$$

It follows from (1.1) that

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(x \cdot y) \succeq \tilde{a}^- = T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}(y),$$

$$I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix}(x \cdot y) \preceq \tilde{b}^+ = I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix}(y),$$

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$$F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (x \cdot y) \succeq \tilde{c}^- = F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y).$$

Let $x, y_1, y_2 \in X$. Then

Case(1): Suppose $y_1, y_2 \in G$. Then

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y_1) = \tilde{a}^+ = T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y_2),$$

$$I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (y_1) = \tilde{b}^- = I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (y_2),$$

$$F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y_1) = \tilde{c}^+ = F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y_2).$$

Since G is an ideal of X, we have $(y_1 \cdot (y_2 \cdot x)) \cdot x \in G$ and so

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} ((y_1 \cdot (y_2 \cdot x)) \cdot x) = \tilde{a}^+,$$

$$I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} ((y_1 \cdot (y_2 \cdot x)) \cdot x) = \tilde{b}^-,$$

$$F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} ((y_1 \cdot (y_2 \cdot x)) \cdot x) = \tilde{c}^+.$$

It follows from (1.1) that

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} ((y_1 \cdot (y_2 \cdot x)) \cdot x) = \tilde{a}^+ \succeq \tilde{a}^+ = \text{rmin}\{\tilde{a}^+, \tilde{a}^+\} = \text{rmin}\left\{ T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y_1), T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y_2) \right\},$$

$$I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} ((y_1 \cdot (y_2 \cdot x)) \cdot x) = \tilde{b}^- \preceq \tilde{b}^- = \text{rmax}\{\tilde{b}^-, \tilde{b}^-\} = \text{rmax}\left\{ I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (y_1), I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (y_2) \right\},$$

$$F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} ((y_1 \cdot (y_2 \cdot x)) \cdot x) = \tilde{c}^+ \succeq \tilde{c}^+ = \text{rmin}\{\tilde{c}^+, \tilde{c}^+\} = \text{rmin}\left\{ F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y_1), F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y_2) \right\}.$$

Case(2): Suppose $y_1 \notin G$ or $y_2 \notin G$. Then

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y_1) = \tilde{a}^- \text{ or } T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y_2) = \tilde{a}^-,$$

$$I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (y_1) = \tilde{b}^+ \text{ or } I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (y_2) = \tilde{b}^+,$$

$$F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y_1) = \tilde{c}^- \text{ or } F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y_2) = \tilde{c}^-.$$

It follows from (1.1) that

$$\text{rmin}\left\{ T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y_1), T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y_2) \right\} = \text{rmin}\{\tilde{a}^-, \tilde{a}^-\} = \tilde{a}^-,$$

$$\text{rmax}\left\{ I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (y_1), I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (y_2) \right\} = \text{rmax}\{\tilde{b}^-, \tilde{b}^-\} = \tilde{b}^-,$$

$$\text{rmin}\left\{ F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y_1), F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y_2) \right\} = \text{rmin}\{\tilde{c}^-, \tilde{c}^-\} = \tilde{c}^-.$$

Therefore,

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} ((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \tilde{a}^- = \text{rmin}\left\{ T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y_1), T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y_2) \right\},$$

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$$I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} ((y_1 \cdot (y_2 \cdot x)) \cdot x) \preceq \tilde{b}^- = \text{rmax}\left\{ I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (y_1), I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (y_2) \right\},$$

$$F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} ((y_1 \cdot (y_2 \cdot x)) \cdot x) \succeq \tilde{c}^- = \text{rmin}\left\{ F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y_1), F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y_2) \right\}.$$

Hence, $A^G \begin{bmatrix} \tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+ \\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^- \end{bmatrix}$ is an IVN ideal of X. □

A mapping $f : X \to Y$ of Hilbert algebras X and Y is called a *homomorphism* if $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of Hilbert algebras X and Y, then $f(1) = 1$. Let $f : X \to Y$ be a homomorphism of Hilbert algebras X and Y. For any IVNS $A = (T_A, I_A, F_A)$ in Y, we define the IVNS $f^{-1}(A) = (T_{f^{-1}(A)}, I_{f^{-1}(A)}, F_{f^{-1}(A)})$ in X by

$$(\forall x \in X) \begin{pmatrix} T_{f^{-1}(A)}(x) = T_A(f(x)) \\ I_{f^{-1}(A)}(x) = I_A(f(x)) \\ F_{f^{-1}(A)}(x) = F_A(f(x)) \end{pmatrix}.$$

Proposition 2.22. *Let $f : X \to Y$ be a homomorphism of a Hilbert algebra X into a Hilbert algebra Y. If $A = (T_A, I_A, F_A)$ is an IVN ideal of Y, then the inverse image $f^{-1}(A)$ of A is an IVN ideal of X.*

Proof. Since f is a homomorphism of X into Y, then $f(1) = 1 \in Y$. By the assumption, $T_A(f(1)) = T_A(1) \succeq T_A(y)$ for every $y \in Y$. In particular, $T_A(f(1)) \succeq T_A(f(x))$ for all $x \in X$. Hence, $T_{f^{-1}(A)}(1) \succeq T_{f^{-1}(A)}(x)$ for all $x \in X$. Also, $I_A(f(1)) = I_A(1) \preceq I_A(y)$ for every $y \in Y$. In particular, $I_A(f(1)) \preceq I_B(f(x))$ for all $x \in X$. Hence, $I_{f^{-1}(A)}(1) \preceq I_{f^{-1}(A)}(x)$ for all $x \in X$, and $F_A(f(1)) \succeq F_A(f(x))$ for all $x \in X$. Hence, $F_{f^{-1}(A)}(1) \succeq F_{f^{-1}(A)}(x)$, which proves (2.2).

Now, let $x, y \in X$. Then

$$T_{f^{-1}(A)}(x \cdot y) = T_A(f(x \cdot y)) = T_A(f(x) \cdot f(y)) \succeq T_A(f(y)) = T_{f^{-1}(A)}(y),$$
$$I_{f^{-1}(A)}(x \cdot y) = I_A(f(x \cdot y)) = I_A(f(x) \cdot f(y)) \preceq I_A(f(y)) = I_{f^{-1}(A)}(y),$$
$$F_{f^{-1}(A)}(x \cdot y) = F_A(f(x \cdot y)) = F_A(f(x) \cdot f(y)) \succeq F_A(f(y)) = F_{f^{-1}(A)}(y),$$

which proves (2.3).

Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}
T_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x) &= T_A(f(y_1 \cdot (y_2 \cdot x) \cdot x)) \\
&= T_A(f(y_1) \cdot (f(y_2 \cdot x)) \cdot f(x)) \\
&= T_A(f(y_1 \cdot (y_2 \cdot x)) \cdot f(x)) \\
&= T_A(f(y_1 \cdot (y_2 \cdot x)) \cdot x)) \\
&\succeq \text{rmin}\{T_A(f(y_1)), T_A(f(y_2))\} \\
&= \text{rmin}\{T_{f^{-1}(A)}(y_1), T_{f^{-1}(A)}(y_2)\}, \\
I_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x) &= I_A(f(y_1 \cdot (y_2 \cdot x) \cdot x)) \\
&= I_A(f(y_1) \cdot (f(y_2 \cdot x)) \cdot f(x)) \\
&= I_A(f(y_1 \cdot (y_2 \cdot x)) \cdot f(x)) \\
&= I_A(f(y_1 \cdot (y_2 \cdot x)) \cdot x)) \\
&\preceq \text{rmax}\{I_A(f(y_1)), I_A(f(y_2))\} \\
&= \text{rmax}\{I_{f^{-1}(A)}(y_1), I_{f^{-1}(A)}(y_2)\}, \\
F_{f^{-1}(A)}((y_1 \cdot (y_2 \cdot x)) \cdot x) &= F_A(f(y_1 \cdot (y_2 \cdot x) \cdot x)) \\
&= F_A(f(y_1) \cdot (f(y_2 \cdot x)) \cdot f(x)) \\
&= F_A(f(y_1 \cdot (y_2 \cdot x)) \cdot f(x)) \\
&= F_A(f(y_1 \cdot (y_2 \cdot x)) \cdot x)) \\
&\succeq \text{rmin}\{F_A(f(y_1)), F_A(f(y_2))\} \\
&= \text{rmin}\{F_{f^{-1}(A)}(y_1), F_{f^{-1}(A)}(y_2)\},
\end{aligned}$$

which proves (2.4). Hence, $f^{-1}(A)$ is an IVN ideal of X. □

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