On $(\beta_{\rho n})$-OS in Pythagorean Neutrosophic Topological Spaces

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Abstract

In this paper, we introduce a new set called Pythagorean neutrosophic beta-open set with this concept, and we introduce interior and closure of Pythagorean neutrosophic beta-open set in a Pythagorean neutrosophic topological spaces by utilizing beta-open set and we introduce the chii, ii, iiibeta-spaces and di, ii, iiibeta-spaces from the pair of distinct points and we have derived the necessary and sufficient conditions by utilizing beta-open sets. We also go through some containment relations for interiors and closures of beta-open sets and studied some of their characteristics.

Keywords: Pythagorean neutrosophic beta-open set; interior of beta-open; closure of beta-open; chii; ii; iiibeta-spaces; di; ii; iiibeta-spaces.

1 Introduction

Traditional mathematical methods are not always advantageous, because there are uncertainties and ambiguity in real-world problems. There are several approaches to dealing with such situations. Unfortunately, each of these models has its own limitations and flaws. Zadeh introduced the concept of fuzzy sets as an addition to the traditional crisp set in 1965 to address these shortcomings by associating the membership function. As a result, in this new outline, we are confronted with topological issues, which are the subjects of fuzzy topology research. Chang defined fuzzy topology as a branch merging ordered and topological structure on a fuzzy set in 1968.

Chang's[5] paper paved the way for the rapid development of a number of fuzzy topological notions that followed. Several mathematicians have continued to apply all of the essential concepts of general topology to fuzzy situations, culminating in the present fuzzy topology theory. Fuzzy topology is now widely recognised as one of the fundamental fields of fuzzy mathematics. The construction of a fuzzy-point neighbourhood was defined by Pao-Ming and Ying-Ming [[10]]. Atanassov ([1] and[2]) proposed the concept of intuitionistic fuzzy sets in 1983. Smarandache.F.S[8] also popularised the concept of a neutrosophic set.

In 2013, Yager.R and Abbasov.A[11] Pythagorean membership grades were introduced as a notion in multicriteria decision making. In 2020, Granados.C[7] Pythagorean Neutrosophic pre-open sets are defined and Sneha.T and Nirmala.F,[9] they defined the concepts of pythagorean neutrosophic b-open sets and pythagorean nutrosophic semi-open sets, as well as several features and notions related with them. They also defined some continuity versions.

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In 2021, P.Basker and Broumi Said[3] were introduced the concept of $N\psi_{\alpha}^{\#0}$ and $N\psi_{\alpha}^{\#1}$-spaces in neutrosophic topological spaces and characterized some of their properties, Granados.C[6] Pythagorean neutrosophic semi-open sets in Pythagorean neutrosophic topological spaces are defined and Carlos Granados and Alok Dhital[4] Pythagorean neutrosophic $^*b$-open function, Pythagorean neutrosophic $^*b$-continuous function, and Pythagorean neutrosophic $^*b$-homeomorphism are defined Pythagorean neutrosophic $^*b$-open set on Pythagorean neutrosophic topology.

The concepts of Pythagorean neutrosophic open set and the notions stated above were utilised to introduce and analyse the concept of Pythagorean neutrosophic β-open set in this work. We also demonstrate some of its characteristics. Furthermore, several of their features are demonstrated. $PNPCS$, $PNSCS$ and $PN\alpha CS$ are the collections of all Pythagorean neutrosophic pre-open sets, Pythagorean neutrosophic semi-open sets and Pythagorean neutrosophic α-open sets respectively. The pythagorean neutrosophic topological space ($PNTS$) is referred to as (J, τ_{PN}) and PNS stands for Pythagorean Neutrosophic Set throughout this research work.

2 Preliminaries

Before we begin our research, we should review and discuss definitions.

Definition 2.1. For any PNS, A in a $PNTS(J, \tau_{PN})$, A is said to be Pythagorean neutrosophic pre-open set (briefly PN-P-open)[7] if $A \subseteq PNInt((PNCl(A))$.

Definition 2.2. For a PNS, A in a $PNTS(J, \tau_{PN})$, A is said to be Pythagorean neutrosophic semi-open set (briefly PN-S-open)[6] if $A \subseteq PNCl((PNInt(A))$. The complement of a Pythagorean neutrosophic semi-open set is called Pythagorean neutrosophic semi-closed set.

Definition 2.3. A pythagorean neutrosophic set A in a pythagorean neutrosophic topological space (J, τ_{PN}) is called a pythagorean neutrosophic α-open set (briefly $PN\alpha OS$),[9] if $A \subseteq PNint(PNcl(PNint(A)))$. The complement of $PN\alpha OS$ is called $(PN\alpha CS$.

3 On $(\beta_{\rho n})$-OS

In this section, we presented the set known as $(\beta_{\rho n})$-OS, and we investigated the concepts of $\rho n_{\beta}^{\#I}(P_1)$ and $\rho n_{\beta}^{\#C}(P_1)$ and their features utilising this notion.

Definition 3.1. A PNS P_1 in a $PNTS(J, \tau_{PN})$ is said to be
(i) Pythagorean Neutrosophic semi-preopen [briefly. $PN\beta$-open or $(\beta_{\rho n})$-OS] if there exists $P_2 \in PN$-$PO(J)$ such that $P_2 \subseteq P_1 \subseteq PNcl(P_1)$.
(ii) Pythagorean Neutrosophic semi-preclosed [briefly. $PN\beta$-closed or $(\beta_{\rho n})$-CS] if there exists a PN-$PC(J)$, P_2 such that $PNint(P_2) \subseteq P_1 \subseteq P_2$.

For every PNS P_1 in (J, τ_{PN}), we have $P_1 \in (\beta_{\rho n})$-$CS(J) \iff \bar{P}_1 \in (\beta_{\rho n})$-$OS(J)$.

Theorem 3.2. *Every PN-S-open set is $(\beta_{\rho n})$-OS.*

Proof. Let P_1 be PN-S-open set in (J, τ_{PN}). Then, it follows that
$P_1 \subseteq PNcl(PNint(P_1)) \subseteq PNcl(PNint(PNcl(P_1)))$. Hence P_1 is an $(\beta_{\rho n})$-OS.

□

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For every PNS, P_1 in (J, τ_{PN}), we have $P_1 \in (\beta_{\rho n})$-$CS(J) \Longleftrightarrow \bar{P}_1 \in (\beta_{\rho n})$-$OS(X)$.

Theorem 3.3. *Let (J, τ_{PN}) be a $PNTS$ Then, (i) Any union of $(\beta_{\rho n})$-OS is $(\beta_{\rho n})$-OS and (ii) Any intersection of $(\beta_{\rho n})$-CS is $(\beta_{\rho n})$-CS.*

Proof. (i) Let $\{M_i\}_{i\in I}$ be a collection of $(\beta_{\rho n})$-OS of (J, τ_{PN}). Then there exists $N_i \in PNPO(J)$ such that $N_i \subseteq M_i \subseteq PNcl(N_i)$ for each $i \in I$. It follows that $\bigcup N_i \subseteq \bigcup M_i \subseteq \bigcup PNcl(\bigcup N_i)$ and $\bigcup N_i \in PNPO(J)$. Hence $\bigcup A_i \in I^{(T)}\beta O(X)$, (ii) is from (i) by taking compliments.

□

Theorem 3.4. *For any PNS P_1 in a $PNTS(J, \tau_{PN})$, $P_1 \in (\beta_{\rho n})$-$OS(J)$ if and only if $(\forall p(\alpha_1,\ \alpha_2) \in P_1)(\exists P_2 \in (\beta_{\rho n})$-$OS(J))(p(\alpha_1,\ \alpha_2) \in P_2 \subseteq P_1)$.*

Proof. If $P_1 \in (\beta_{\rho n})$-$OS(J)$, then we can take $P_2 = P_1$ so that $p(\alpha_1,\ \alpha_2) \in P_2 \subseteq P_1$ for every $p(\alpha_1,\ \alpha_2) \in P_1$. Let P_1 be a PNS in (J, τ_{PN}) and assume that there exists $P_2 \in (\beta_{\rho n})$-$OS(X)$ such that $p(\alpha_1,\ \alpha_2) \in P_2 \subseteq P_1$. Then $P_1 = \bigcup_{p(\alpha_1,\ \alpha_2)\in P_1} \{p(\alpha_1,\ \alpha_2)\} \subseteq \bigcup_{p(\alpha_1,\ \alpha_2)\in P_1} P_2 \subseteq P_1$, and so $P_1 = \bigcup_{p(\alpha_1,\ \alpha_2)\in P_1} P_2$ which is a $(\beta_{\rho n})$-OS.

□

Theorem 3.5. *Let $PNTS(J, \tau_{PN})$. Then (i) $(\forall P_1 \in (\beta_{\rho n})$-$OS(J))(\forall P_2 \in PN$-$SO(J))(P_1 \subseteq P_2 \subseteq PNcl(P_1)) \Longrightarrow P_2 \in (\beta_{\rho n})$-$OS(J)$ and (ii) $(\forall P_1 \in (\beta_{\rho n})$-$CS(J))(\forall P_2 \in PN$-$SC(J))(PNint(P_1) \subseteq P_2 \subseteq P_1) \Longrightarrow P_2 \in (\beta_{\rho n})$-$CS(J)$.*

Proof. (i) Assume that $P_1 \subseteq P_2 \subseteq PNcl(P_1)$ for every $P_1 \in (\beta_{\rho n})$-$OS(J)$ and $P_2 \in I^{(T)}S(J)$. Let $P_3 \in PN$-$PO(X)$ be such that $P_3 \subseteq P_1 \subseteq PNcl(P_3)$. Obviously, $P_3 \subseteq P_2$. From $P_1 \subseteq PNcl(P_3)$ it follows that $PNcl(P_1) \subseteq PNcl(P_3)$ so that $P_3 \subseteq P_2 \subseteq PNcl(P_1) \subseteq PNcl(P_3)$. Hence $P_2 \in (\beta_{\rho n})$-$OS(J)$, (ii) follows from (i).

□

Definition 3.6. Let (J, τ_{PN}) be a $PNTS$ and P_1 be a subset J. Then $(\beta_{\rho n})$-interior of P_1 is the union of all $(\beta_{\rho n})$-OS contained in P_1 and it is denoted by $\rho n_{\beta}^{\#I}(P_1)$.

Definition 3.7. Let (J, τ_{PN}) be a $PNTS$ and P_1 be a subset J. Then $(\beta_{\rho n})$-closure of P_1 is the intersection of all $(\beta_{\rho n})$-CS containing P_1 and it is denoted by $\rho n_{\beta}^{\#C}(P_1)$.

Theorem 3.8. *$\bigcup PN\alpha OS$ is invariably a $PN\alpha OS$.*

Proof. Let P_1 and P_2 be 2 $PN\alpha OS$, $P_1 \subseteq PNint(PNcl(PNint(P_1)))$ and $P_2 \subseteq PNint(PNcl(PNint(P_1)))$ $\Longrightarrow P_1 \cup P_2 \subseteq PNint(PNcl(PNint(P_1 \cup P_2)))$. Therefore $P_1 \cup P_2$ is a $PN\alpha OS$.

□

Proposition 3.9. *Let (J, τ_{PN}) be a $PNTS$ and let $P_1 \in PNS(J)$. Then $P_1 \in PNPOS(X) \Longleftrightarrow (\exists P_2 \in T)(P_1 \subseteq P_2 \subseteq PNcl(P_1))$.*

Proof. If $P_1 \in PNPOS(X)$, then $P_1 \subseteq PNint(PNcl(P_1))$ Take $P_2 = PNint(PNcl(P_1))$. Then $P_2 \in T$ and $P_1 \subseteq P_2 \subseteq PNcl(P_1)$.

Conversely, let $P_2 \in T$ be such that $P_1 \subseteq P_2 \subseteq PNcl(P_1)$. Then $P_1 \subseteq PNint(P_2) \subseteq PNint(PNcl(P_1))$, and so $P_1 \in PNPOS(J)$.

□

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Theorem 3.10. *Let (J, τ_{PN}) be a $PNTS$. A subset P_1 of J is $PN\alpha OS \iff$ it is PN-S-open set and PN-P-open.*

Proof. Necessity: Let P_1 be a $PN\alpha OS$. Then, we have $P_1 \subseteq PNint(PNcl(PNint(P_1)))$. This implies that $P_1 \subseteq PNcl(PNint(P_1))$ and $P_1 \subseteq PNint(PNcl(P_1))$. Hence, P_1 is PN-S-open and PN-P-open.
Sufficiency: Let P_1 is PN-S-open and PN-P-open.
Then, we have $P_1 \subseteq PNint(PNcl(A)) \subseteq PNint(PNcl(PNint(P_1)))$. This shows that P_1 is $PN\alpha OS$. □

Definition 3.11. A subset P_1 of J is said to be $PN\alpha CS \iff X - A$ is $PN\alpha OS$, which is equivalently.
Let (J, τ_{PN}) be a $PNTS$ and P_1 be a subset J. Then, P_1 is $PN\alpha OS \iff P_1 \supseteq PNint(PNcl(PNint(P_1)))$.

Definition 3.12. Let (J, τ_{PN}) be a $PNTS$ and P_1 be a subset J. Then $PN\alpha$-interior of P_1 is the union of all $PN\alpha OS$ contained in P_1 and it is denoted by $\alpha^{*I}_{\rho n}(P_1)$.

Definition 3.13. Let (J, τ_{PN}) be a $PNTS$ and P_1 be a subset J. Then $PN\alpha$-closure of P_1 is the intersection of all $PN\alpha CS$ containing P_1 and it is denoted by $\alpha^{*C}_{\rho n}(P_1)$.

Theorem 3.14. *Let (J, τ_{PN}) be a $PNTS$ and P_1 be a subset of J.*
(a) If $Y \in PNSO(J)$ and $P_1 \in T_{PN\alpha}$ then $Y \cap P_1 \in PNSO(X)$.
(b) If $Y \in PNPO(J)$ and $P_1 \in T_{PN\alpha}$ then $Y \cap P_1 \in PNPO(X)$.

Proof. (a) Let $Y \in PNSO(J)$ and $P_1 \in T_{PN\alpha}, Y \cap P_1 \subseteq PNcl(PNint(Y)) \cap PNint(PNcl(PNint(P_1))) \subseteq PNcl(PNint(Y)) \cap PNcl(PNint(P_1)) \subseteq PNint(PNcl$
$(PNint(P_1) \cap PNcl(Y))) \subseteq PNcl(PNint(Y \cap P_1))$. Therefore, $Y \cap P_1 \in PNSO(J)$.

(b) Let $Y \in PNPO(J)$ and $P_1 \in T_{PN\alpha}$, $Y \cap P_1 \subseteq PNint(PNcl(Y)) \cap PNint(PNcl$
$(PNint(P_1))) \subseteq PNint(PNcl(Y)) \cap PNcl(PNint(P_1)) \subseteq PNint(PNcl(PNint(P_1) \cap PNcl(Y))) \subseteq PNint(PNcl(Y \cap P_1))$. Therefore, $Y \cap P_1 \in PNPO(J)$. □

Theorem 3.15. *Let (J, τ_{PN}) be a $PNTS$ and $P_1,\ P_2$ be subset of J.*

- *(a) If $P_1,\ P_2 \in T_{PN\alpha}$ then $P_1 \cap P_2 \in T_{PN\alpha}$.*
- *(b) If $\{P_1^\alpha : \alpha \in H\}$ be the family of $PN\alpha OS$ in (J, τ_{PN}). Then, $\bigcup_{\alpha \in H} P_1^\alpha$ is also an $PN\alpha OS$.*

Proof. (a) Let $P_1,\ P_2 \in T_{PN\alpha}$, $P_1,\ P_2$ is PN-S-open and PN-P-open and $P_1 \cap P_2$ is PN-S-open and PN-P-open. Therefore, $P_1 \cap P_2 \in T(\alpha_{\rho n})$.

(b) Let $P_1 \in T_{PN\alpha}$ for each $\alpha \in H$. Then, $P_1^\alpha \subseteq PNint(PNcl(PNint(A))) \subseteq PNint(PNcl(PNint(\cup P_1^\alpha)))$ and hence $\cup P_1^\alpha \subseteq PNint(PNcl(PNint(\cup_{\alpha \in H} P_1^\alpha)))$. This shows that $\cup_{\alpha \in H} P_1^\alpha$ is also a $PN\alpha OS$. □

$\forall PNS\ P_1$ in (J, τ_{PN}), we've $P_1 \in (\beta_{\rho n})$-$OS(J) \iff \overline{P_1} \in (\beta_{\rho n})$-$CS(J)$.

$\forall PNS\ P_1$ in (J, τ_{PN}), we've $P_1 \in (\beta_{\rho n})$-$OS(J) \iff \overline{P_1} \in (\beta_{\rho n})$-$CS(J)$.

Theorem 3.16. *For any $PNS\ P_1$ in $PNTS(J, \tau_{PN})$, $P_1 \in (\beta_{\rho n})$-$OS(X) \iff (\forall q(\alpha_1,$
$\alpha_2) \in P_1)(\exists P_2 \in (\beta_{\rho n})$-$OS(X))(q(\alpha_1,\ \alpha_2) \in P_2 \subseteq P_1)$.*

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Proof. If $P_1 \in (\beta_{\rho n})\text{-}OS(X)$, take $P_2 = P_1$ so that $q(\alpha_1,\ \alpha_2) \in P_2 \subseteq P_1\ \forall q(\alpha_1,\ \alpha_2) \in P_1$. Let P_1 be an PNS in (J, τ_{PN}) and assume that $\exists P_2 \in (\beta_{\rho n})\text{-}OS(X)$ such that $q(\alpha_1,\ \alpha_2) \in P_2 \subseteq P_1$. Then $P_1 = \bigcup_{q(\alpha_1,\ \alpha_2)\in P_1} q(\alpha_1,\ \alpha_2) = \bigcup_{q(\alpha_1,\ \alpha_2)\in P_1} P_2 \subseteq P_1$, and so $P_1 = \bigcup_{q(\alpha_1,\ \alpha_2)\in P_1} P_2$ which is an $(\beta_{\rho n})$-$OS(X)$.

□

For subsets P_1 and P_2 of an $PNTS(J, \tau_{PN})$, the following statements hold:

(a) $\alpha^{*I}_{\rho n}(P_1)$ is the largest $(\alpha_{\rho n})$-OS contained in P_1.

(b) P_1 is $(\alpha_{\rho n})$-$OS \Longleftrightarrow P_1 = \alpha^{*I}_{\rho n}(P_1)$.

(c) $\alpha^{*I}_{\rho n}(\alpha^{*I}_{\rho n}(P_1)) = \alpha^{*I}_{\rho n}(P_1)$.

(d) $J - \alpha^{*I}_{\rho n}(P_1) = \alpha^{*C}_{\rho n}(J - P_1)$.

(e) $J - \alpha^{*C}_{\rho n}(P_1) = \alpha^{*I}_{\rho n}(J - P_1)$.

(f) $P_1 \subset P_2$, then $\alpha^{*I}_{\rho n}(P_1) \subset \alpha^{*I}_{\rho n}(P_2)$.

(g) $\alpha^{*I}_{\rho n}(P_1) \cup \alpha^{*I}_{\rho n}(P_2) \subset \alpha^{*I}_{\rho n}(P_1 \cup P_2)$

Theorem 3.17. *If P_1 is a subset of a $PNTS(J, \tau_{PN})$, then*

(a) $PNint(A) \subset \alpha^{*I}_{\rho n}(P_1)$

(b) $\alpha^{*I}_{\rho n}(P_1) \subset PNPINT(P_1)$

(c) $\alpha^{*I}_{\rho n}(P_1) \subset PNSINT(P_1)$

(d) $\alpha^{*I}_{\rho n}(P_1) \subset \rho n^{\#I}_{\beta}(P_1)$

Proof. (a) Let P_1 be a subset of a $PNTS(J, \tau_{PN})$.
Let $j \in PNint(P_1) \Longrightarrow j \in \bigcup\{H \subset J : H\ is\ a\ PN\ open,\ H \subset P_1$.
$\Longrightarrow \exists$ a $PNOS$, H such that $j \in H \subset P_1$.
$\Longrightarrow \exists$ a $PN\alpha OS$, H such that $j \in H \subset P_1, \forall\ PNOS \Longrightarrow PN\alpha OS$.
$\Longrightarrow j \in \bigcup\{H \subset J : H\ is\ a\ PN\alpha OS,\ H \subset P_1$.
$\Longrightarrow j \in \alpha^{*I}_{\rho n}(P_1)$.
$\Longrightarrow j \in PNint(A) \Longrightarrow j \in \alpha^{*I}_{\rho n}(P_1)$.
Hence $PNint(A) \subset \alpha^{*I}_{\rho n}(P_1)$.

(b) Let P_1 be a subset of a $PNTS(J, \tau_{PN})$.
Let $j \in \alpha^{*I}_{\rho n}(P_1) \Longrightarrow j \in \bigcup\{H \subset J : H\ is\ a\ PN\alpha OS,\ H \subset P_1$.
$\Longrightarrow \exists$ a $PN\alpha OS$, H such that $j \in H \subset P_1$.
$\Longrightarrow \exists$ a $PNPOS$, H such that $j \in H \subset P_1, \forall\ PN\alpha OS \Longrightarrow PNPOS$.
$\Longrightarrow j \in \bigcup\{H \subset J : H\ is\ a\ PNPOS,\ H \subset P_1$.
$\Longrightarrow j \in PNPINT(P_1)$.
$\Longrightarrow j \in \alpha^{*I}_{\rho n}(P_1) \Longrightarrow j \in PNPINT(P_1)$.
Hence $\alpha^{*I}_{\rho n}(P_1) \subset PNPINT(P_1)$.

(c) Let P_1 be a subset of a $PNTS(J, \tau_{PN})$.
Let $j \in \alpha^{*I}_{\rho n}(P_1) \Longrightarrow j \in \bigcup\{H \subset J : H\ is\ an\ PN\alpha OS,\ H \subset P_1$.
$\Longrightarrow \exists$ a $PN\alpha OS$, H such that $j \in H \subset P_1$.
$\Longrightarrow \exists$ a $PNSOS$, H such that $j \in H \subset P_1, \forall\ PN\alpha OS \Longrightarrow PNSOS$.
$\Longrightarrow j \in \bigcup\{H \subset J : H\ is\ a\ PNSOS,\ H \subset P_1\}$.
$\Longrightarrow j \in PNSINT(P_1)$.
$\Longrightarrow j \in \alpha^{*I}_{\rho n}(P_1) \Longrightarrow j \in PNSINT(P_1)$.
Hence $\alpha^{*I}_{\rho n}(P_1) \subset PNSINT(P_1)$.

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(d) Let P_1 be a subset of a $PNTS(J, \tau_{PN})$.
Let $j \in \alpha^{*I}_{\rho n}(P_1) \implies j \in \bigcup\{H \subset J : H\ is\ a\ PN\alpha OS,\ H \subset P_1$.
$\implies \exists$ a $PN\alpha OS$, H such that $j \in H \subset P_1$.
$\implies \exists$ a $(\beta_{\rho n})$-OS, H such that $j \in H \subset P_1, \forall\ PN\alpha OS \implies (\beta_{\rho n})$-$OS$.
$\implies j \in \bigcup\{H \subset J : H\ is\ a\ (\beta_{\rho n})\text{-}OS,\ H \subset P_1$.
$\implies j \in \rho n^{\#I}_{\beta}(P_1)$.
$\implies x \in \alpha^{*I}_{\rho n}(P_1) \implies j \in \rho n^{\#I}_{\beta}(P_1)$.
Hence $\alpha^{*I}_{\rho n}(P_1) \subset \rho n^{\#I}_{\beta}(P_1)$. □

4 On $\Psi^{(\beta_{\rho n})}_{\mapsto I}{}_{I=i,ii,iii}$-space

We proposed the space named $\Psi^{(\beta_{\rho n})}_{\mapsto I}{}_{I=i,ii,iii}$-space utilising the notion of $(\beta_{\rho n})$-OS, and their attributes are being researched.

Definition 4.1. A pythagorean neutrosophic topological space (J, τ_{PN}) for each pair of distinct points is said to be

(a) $\Psi^{(\beta_{\rho n})}_{\mapsto i}$, $j_1 \neq j_2 \in J$, $\exists$ $(\beta_{\rho n})$-OS, F such that either $j_1 \in F$ and $j_2 \notin F$ or $j_1 \notin F$ and $j_2 \in F$.

(b) $\Psi^{(\beta_{\rho n})}_{\mapsto ii}$, $j_1 \neq j_2 \in J$, $\exists$ two $(\beta_{\rho n})$-OS, F and G such that $j_1 \in F$ but $j_2 \notin F$ and $j_2 \in G$ but $j_1 \notin G$.

(c) $\Psi^{(\beta_{\rho n})}_{\mapsto iii}$, $j_1 \neq j_2 \in J$, $\exists$ two disjoint $(\beta_{\rho n})$-OS, F and G containing j_1 and j_2 respectively.

Proposition 4.2. *A pythagorean neutrosophic topological space (J, τ_{PN}) is $\Psi^{(\beta_{\rho n})}_{\mapsto i}$ $\iff$ for each pair of distinct points $j_1,\ j_2$ of J, $\rho n^{\#C}_{\beta}(\{j_1\}) \neq \rho n^{\#C}_{\beta}(\{j_2\})$.*

Proof. Necessity. Let (J, τ_{PN}) be a $\Psi^{(\beta_{\rho n})}_{\mapsto i}$-space and $j_1,\ j_2$ be any two distinct points of J, $\exists$ a $(\beta_{\rho n})$-CS, L containing j_1 or j_2, say j_1 but not j_2. Then J/L is a $\Psi^{(\beta_{\rho n})}_{\mapsto i}$ which does not contain j_1 but contains j_2. Since $\rho n^{\#C}_{\beta}(\{j_2\})$ is the smallest $(\beta_{\rho n})$-CS containing j_2, $\rho n^{\#C}_{\beta}(\{j_2\}) \subseteq J/L$ and therefore $j_1 \notin \rho n^{\#C}_{\beta}(\{j_2\})$. Consequently $\rho n^{\#C}_{\beta}(\{j_1\}) \neq \rho n^{\#C}_{\beta}(\{j_2\})$.

Sufficiency. Suppose that $j_1,\ j_2 \in J$, $j_1 \neq j_2$ and $\rho n^{\#C}_{\beta}(\{j_1\}) \neq \rho n^{\#C}_{\beta}(\{j_2\})$. Let j_3 be a point of J such that $j_3 \in \rho n^{\#C}_{\beta}(\{j_1\})$ but $j_3 \notin \rho n^{\#C}_{\beta}(\{j_2\})$. We claim that $j_1 \notin \rho n^{\#C}_{\beta}(\{j_2\})$. For, if $j_1 \in \rho n^{\#C}_{\beta}(\{j_2\})$ then $\rho n^{\#C}_{\beta}(\{j_1\}) \subseteq \rho n^{\#C}_{\beta}(\{j_2\})$. This contradicts the fact that $j_3 \notin \rho n^{\#C}_{\beta}(\{j_2\})$. Consequently j_1 belongs to the $(\beta_{\rho n})$-CS, $J/\rho n^{\#C}_{\beta}(\{j_2\})$ to which j_2 does not belong. □

Proposition 4.3. *A pythagorean neutrosophic topological space (J, τ_{PN}) is $\Psi^{(\beta_{\rho n})}_{\mapsto i}$ $\iff$ the singletons are $(\beta_{\rho n})$-CS.*

Proof. Let pythagorean neutrosophic topological space (J, τ_{PN}) be $\Psi^{(\beta_{\rho n})}_{\mapsto i}$ and j_1 any point of J. Suppose $j_2 \in J/\{j_1\}$, then $j_1 \neq j_2$ and so $\exists$ $(\beta_{\rho n})$-OS, K such that $j_2 \in K$ but $j_1 \notin K$. Consequently $j_2 \in K \subseteq K/\{j_1\}$, i.e., $J/\{j_1\} = \cup\{K : j_2 \in J/\{j_1\}\}$ which is $(\beta_{\rho n})$-OS.

Conversely, suppose $\{j_3\}$ is $(\beta_{\rho n})$-CS, $\forall\ j_3 \in J$. Let $j_1,\ j_2 \in J$ with $j_1 \neq j_2$. Now $j_1 \neq j_2 \implies j_2 \in J/\{j_1\}$. Hence $J/\{j_1\}$ is a $(\beta_{\rho n})$-OS contains j_2 but not j_1. Similarly $J/\{j_2\}$ is a $(\beta_{\rho n})$-OS contains j_1 but not j_2. Accordingly J is a $\Psi^{(\beta_{\rho n})}_{\mapsto i}$-space. □

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Proposition 4.4. *The following statements are equivalent for a pythagorean neutrosophic topological space* (J, τ_{PN})*:*

(a) *J is* $\Psi^{(\beta_{\rho n})}_{\mapsto ii}$.

(b) *Let* $j_1 \in J$*. For each* $j_2 \neq j_1$, $\exists$ *a* $(\beta_{\rho n})$-OS, K *containing* j_1 *such that* $j_2 \notin \rho n^{\#C}_{\beta}(K)$.

(c) *For each* $j_1 \in J$, $\cap\{\rho n^{\#C}_{\beta}(K) : K \in (\beta_{\rho n})$-$O(J)$ *and* $j_1 \in K\} = \{j_1\}$.

Proof. $(a) \implies (b)$ Since J is $\Psi^{(\beta_{\rho n})}_{\mapsto ii}$, $\exists$ disjoint $(\beta_{\rho n})$-OS, K and L containing j_1 and j_2 respectively. So, $K \subseteq J/L$. Therefore, $\rho n^{\#C}_{\beta}(K) \subseteq J/L$. So $j_2 \notin \rho n^{\#C}_{\beta}(K)$.

$(b) \implies (c)$ If possible for some $j_2 \neq j_1$, we have $j_2 \in \rho n^{\#C}_{\beta}(K)$, $\forall$ $(\beta_{\rho n})$-OS, K containing j_1, which then contradicts (b).

$(c) \implies (a)$ Let $j_1,\ j_2 \in J$ and $j_1 \neq j_2$. Then $\exists$ a $(\beta_{\rho n})$-OS, K containing j_1 such that $j_2 \notin \rho n^{\#C}_{\beta}(K)$. Let $L = J/\rho n^{\#C}_{\beta}(K)$, then $j_2 \in L$ and $j_1 \in K$ and also $K \cap L = \varphi$

□

Theorem 4.5. *If* J_1 *and* J_2 *are subsets of J, then* $\rho n^{\#I}_{\beta}(J_1) \cup \rho n^{\#I}_{\beta}(J_2) \subset \rho n^{\#I}_{\beta}(J_1 \cup J_2)$.

Proof. We know that $J_1 \subset J_1 \cup J_2$ and $J_2 \subset J_1 \cup J_2$, $\rho n^{\#I}_{\beta}(J_1) \subset \rho n^{\#I}_{\beta}(J_1 \cup J_2)$ and $\rho n^{\#I}_{\beta}(J_2) \subset \rho n^{\#I}_{\beta}(J_1 \cup J_2) \implies \rho n^{\#I}_{\beta}(J_1) \cup \rho n^{\#I}_{\beta}(J_2) \subset \rho n^{\#I}_{\beta}(J_1 \cup J_1)$.

□

Let (J, τ_{PN}) be a pythagorean neutrosophic topological space, then every $\Psi^{(\beta_{\rho n})}_{\mapsto ii}$-space is $\Psi^{(\beta_{\rho n})}_{\mapsto i}$-space.

5 On $\left\langle \beta_{\rho n},\ \widetilde{d^I} \right\rangle_{I=i,ii,iii}$-space

We presented the set $\aleph^{\hbar}_{\lfloor \beta_{\rho n} \rfloor}$-set and defined the spaces $\left\langle \beta_{\rho n},\ \widetilde{d^I} \right\rangle_{I=i,ii,iii}$-space in this section, and their features are being investigated.

Definition 5.1. A pythagorean neutrosophic set J_1 in a (J, τ_{PN}) is called a $(\beta_{\rho n})$-Difference set (briefly, $\aleph^{\hbar}_{\lfloor \beta_{\rho n} \rfloor}$-set) if there are $K, L \in (\beta_{\rho n})O(J, \tau_{PN})$ such that $K \neq J$ and $J_1 = K/L$.

It is true that every $(\beta_{\rho n})$-OS, K different from 1_N is a $\aleph^{\hbar}_{\lfloor \beta_{\rho n} \rfloor}$-set if $J_1 = K$ and $L = 0_N$.

Remark 5.2. Every proper $(\beta_{\rho n})$-OS is a $\aleph^{\hbar}_{\lfloor \beta_{\rho n} \rfloor}$-set.

Now we define another set of separation axioms called $\left\langle \beta_{\rho n},\ \widetilde{d^I} \right\rangle$, for $I = i, ii, iii$ by using the $\aleph^{\hbar}_{\lfloor \beta_{\rho n} \rfloor}$-sets.

Definition 5.3. A $PNTS(J, \tau_{PN})$ is said to be

(a) $\left\langle \beta_{\rho n},\ \widetilde{d^0} \right\rangle$ if for any pair of distinct points j_1 and j_2 of J $\exists$ an $\aleph^{\hbar}_{\lfloor \beta_{\rho n} \rfloor}$-set of J containing j_1 but not j_2 or $\aleph^{\hbar}_{\lfloor \beta_{\rho n} \rfloor}$-set of J containing j_2 but not j_1.

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(b) $\left\langle \beta_{\rho n},\ \widetilde{d}^{1} \right\rangle$ if for any pair of distinct points j_1 and j_2 of J $\exists$ an $\aleph^{\hbar}_{\lfloor\beta_{\rho n}\rfloor}$-set of J containing j_1 but not j_2 and $\aleph^{\hbar}_{\lfloor\beta_{\rho n}\rfloor}$-set of J containing j_2 but not j_1.

(c) $\left\langle \beta_{\rho n},\ \widetilde{d}^{2} \right\rangle$ if for any pair of distinct points j_1 and j_2 of J disjoint $\aleph^{\hbar}_{\lfloor\beta_{\rho n}\rfloor}$-set J and K of J containing j_1 and k_1, respectively.

Remark 5.4. For a pythagorean neutrosophic topological space (J, τ_{PN}), the following properties hold:

(a) If (J, τ_{PN}) is $\Psi^{(\beta_{\rho n})}_{\mapsto I}$, then it is $\left\langle \beta_{\rho n},\ \widetilde{d}^{I} \right\rangle$, for $I = i, ii, iii$.

(b) If (J, τ_{PN}) is $\left\langle \beta_{\rho n},\ \widetilde{d}^{I} \right\rangle$, then it is $\left\langle \beta_{\rho n},\ \widetilde{d}^{I-1} \right\rangle$, for $I = i, ii$.

Proposition 5.5. *A pythagorean neutrosophic topological space (J, τ_{PN}) is $\left\langle \beta_{\rho n},\ \widetilde{d}^{i} \right\rangle \iff \Psi^{(\beta_{\rho n})}_{\mapsto i}$.*

Proof. Suppose that (J, τ_{PN}) is $\widetilde{d}^{i}$. Then for each distinct pair $j_1,\ j_2 \in J$, at least one of $j_1,\ j_2$, say j_1, belongs to a $\aleph^{\hbar}_{\lfloor\beta_{\rho n}\rfloor}$-set H but $j_2 \notin H$. Let $H = \varpi_1/\varpi_2$ where $\varpi_1 \neq J$ and $\varpi_1,\ \varpi_2 \in (\beta_{\rho n})\text{-}O(J, \tau_{PN})$. Then $j_1 \in \varpi_1$, and for $j_2 \notin H$ we have two cases:

(a) $j_2 \notin \varpi_1$, (b) $j_2 \in \varpi_1$ and $j_2 \in \varpi_2$.

In case (a), $j_1 \in \varpi_1$ but $j_2 \notin \varpi_1$.

In case (b), $j_2 \in \varpi_2$ but $j_1 \notin \varpi_2$.

Thus in both the cases, we obtain that J is $\Psi^{(\beta_{\rho n})}_{\mapsto i}$.

Conversely, if J is $\Psi^{(\beta_{\rho n})}_{\mapsto i}$, by the previous remark, J is $\left\langle \beta_{\rho n},\ \widetilde{d}^{I} \right\rangle$.

□

Proposition 5.6. *A pythagorean neutrosophic topological space (J, τ_{PN}) is $\left\langle \beta_{\rho n},\ \widetilde{d}^{ii} \right\rangle \iff \left\langle \beta_{\rho n},\ \widetilde{d}^{iii} \right\rangle$.*

Proof. Necessity. Let $j_1,\ j_2 \in J$, $j_1 \neq j_2$. Then $\exists$ $\aleph^{\hbar}_{\lfloor\beta_{\rho n}\rfloor}$-sets $H_1,\ H_2$ in J such that $j_1 \in H_1$, $j_2 \notin H_1$ and $j_2 \in H_2$, $j_1 \notin H_2$. Let $H_1 = \varpi_1/\varpi_2$ and $G_2 = \varpi_3/\varpi_4$, where $\varpi_1,\ \varpi_2,\ \varpi_3$ and ϖ_4 are $(\beta_{\rho n})\text{-}OS$ in (J, τ_{PN}). From $j_1 \notin H_2$, it follows that either $j_1 \notin \varpi_3$ or $j_1 \in \varpi_3$ and $j_1 \in \varpi_4$.

We discuss the two cases separately.

(a) $j_1 \notin \varpi_3$. By $j_2 \notin H_1$ we have two subcases:

(i) $j_2 \notin \varpi_1$. Since $j_1 \in \varpi_1/\varpi_2$, it follows that $j_1 \in \varpi_1/(\varpi_2 \cup \varpi_3)$, and since $j_2 \in \varpi_3/\varpi_4$ we have $j_2 \in \varpi_3/(\varpi_1 \cup \varpi_4)$. Therefore $(\varpi_1/(\varpi_2 \cup \varpi_3)) \cap (\varpi_3/(\varpi_1 \cup \varpi_4)) = 0_N$.

(ii) $j_2 \in \varpi_1$ and $j_2 \in \varpi_2$. We have $j_1 \in \varpi_1/\varpi_2$, and $j_2 \in \varpi_2$. Therefore $(\varpi_1/\varpi_2) \cap \varpi_2 = 0_N$.

(b) $j_1 \in \varpi_3$ and $j_1 \in \varpi_4$. We've $j_2 \in \varpi_3/\varpi_4$ and $j_1 \in \varpi_4$. Hence $(\varpi_3/\varpi_4) \cap \varpi_4 = 0_N$. Therefore J is $\left\langle \beta_{\rho n},\ \widetilde{d}^{iii} \right\rangle$.

Sufficiency. Follows from the previous remark.

□

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If pythagorean neutrosophic topological space (J, τ_{PN}) is $\left\langle \beta_{\rho n},\ \widetilde{d}^{ii} \right\rangle \Longrightarrow \Psi_{\mapsto i}^{(\beta_{\rho n})}$.

Definition 5.7. A point $j \in J$ which has only J as the $(\beta_{\rho n})$-n^* is called a $N^{\beta_{\rho n}}$-point.

Proposition 5.8. *For a $\Psi_{\mapsto i}^{(\beta_{\rho n})}$ pythagorean neutrosophic topological space (J, τ_{PN}) the following are equivalent:*

(a) (J, τ_{PN}) *is* $\left\langle \beta_{\rho n},\ \widetilde{d}^{ii} \right\rangle$.

(b) (J, τ_{PN}) *has no* $N^{\beta_{\rho n}}$*-point.*

Proof. $(a) \Longrightarrow (b)$ Since (J, τ_{PN}) is $\left\langle \beta_{\rho n},\ \widetilde{d}^{ii} \right\rangle$, then each point j_1 of J is contained in a $\aleph^{\hbar}_{\lfloor \beta_{\rho n} \rfloor}$-set $J_1 = H_1/H_2$ and thus in H_1. By definition $H_1 \neq J$. This implies that j_1 is not a $N^{\beta_{\rho n}}$-point.

$(b) \Longrightarrow (a)$ If J is $\Psi_{\mapsto i}^{(\beta_{\rho n})}$, then for each distinct pair of points $j_1,\ j_2 \in J$, at least one of them, j_1 (say) has a $(\beta_{\rho n})$-n^*, h_1 containing J_1 and not J_2. Thus H_1 which is different from J is a $\aleph^{\hbar}_{\lfloor \beta_{\rho n} \rfloor}$-set. If J has no $N^{\beta_{\rho n}}$-point, then j_2 is not a $N^{\beta_{\rho n}}$-point. This means that $\exists$ $(\beta_{\rho n})$-n^*, H_2 of j_2 such that $H_2 \neq J$. Thus $j_2 \in H_2/H_1$ but not j_1 and H_2/H_1 is a $\aleph^{\hbar}_{\lfloor \beta_{\rho n} \rfloor}$-set. Hence J is $\left\langle \beta_{\rho n},\ \widetilde{d}^{ii} \right\rangle$.

□

Corollary 5.9. *A $\Psi_{\mapsto i}^{(\beta_{\rho n})}$-space J is not $\left\langle \beta_{\rho n},\ \widetilde{d}^{ii} \right\rangle \Longleftrightarrow$ there is a unique $N^{\beta_{\rho n}}$-point in J.*

Proof. We only prove the uniqueness of the $N^{\beta_{\rho n}}$-point. If j_1 and j_2 are two $N^{\beta_{\rho n}}$-points in J, then since J is $\Psi_{\mapsto i}^{(\beta_{\rho n})}$, at least one of j_1 and j_2, say j_1, has a $(\beta_{\rho n})$-n^*, H_1 containing j_1 but not j_2. Hence $H_1 \neq J$. Therefore j_1 is not a $N^{\beta_{\rho n}}$-point which is a contradiction.

□

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