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Abstract

According to its definition, a topological space could be a highly unexpected object. There are spaces (indiscreet space) which have only two open sets: the empty set and the entire space. In a discrete space, on the other hand, each set is open. These two artificial extremes are very rarely seen in actual practice. Most spaces in geometry and analysis fall somewhere between these two types of spaces. Accordingly, the separation axioms allow us to say with confidence whether a topological space contains a sufficient number of open sets to meet our needs. To this end, we use bipolar hypersoft (BHS) sets (one of the efficient tools to deal with ambiguity and vagueness) to define a new kind of separation axioms called BHS $\widetilde{T}_i$-space ($i = 0, 1, 2, 3, 4$). We show that BHS $\widetilde{T}_i$-space ($i = 1, 2$) implies BHS $\widetilde{T}_{i-1}$-space; however, the converse is false, as shown by an example. For $i = 0, 1, 2, 3, 4$, we prove that BHS $\widetilde{T}_i$-space is hypersoft (HS) T_i-space and we present a condition so that HS T_i-space is BHS $\widetilde{T}_i$-space. Moreover, we study that a BHS subspace of a BHS $\widetilde{T}_i$-space is a BHS $\widetilde{T}_i$-space for $i = 0, 1, 2, 3$.

Keywords: bipolar hypersoft separation axioms; hypersoft separation axioms; bipolar hypersoft topology; hypersoft topology; bipolar hypersoft sets

1 Introduction

Several mathematicians have proposed new mathematical methodologies such as probability theory, fuzzy set theory[1] and rough set theory[2] to approach and describe complex problems containing uncertainty, vagueness and ambiguity in fields such as economics, engineering, medical sciences and social sciences. Molodtsov[3] demonstrated that these theories have their own set of problems. As a result, he developed a new mathematical technique known as soft set theory and examined its applications in various fields, including game theory, operations research, and probability theory. Later, many types of soft set operators were defined in the literature for use in theoretical and applied research of soft set theory (see, for example,[4–7]). In 2011, the concept of soft topologies[8] was created based on soft set theory. Many studies have been conducted to compare and contrast the characteristics of soft topologies with those of classical topologies.[9–14] The study of the axioms of soft separation, in particular, has aroused the interest of researchers who have approached the subject from several angles (see, for example,[15,16]).

Smarandache[17] extended the concept of the soft set to the HS set in 2018. The fundamental motive for utilizing HS set is that the soft set environment cannot manage cases where the attributes are more than one and further bisected. Some works on HS set and its extensions can be seen in.[18–30] In,[31] Musa and Asaad employed HS sets to define the concept of hypersoft topological spaces (HSTSs). They defined the essential concepts of HSTSs such as HS open (closed) sets, HS closure, HS interior, and HS boundary. Later, some fundamental

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notions via HSTSs such as HS connectedness,[32] HS continuity and HS compactness,[33] and HS separation axioms[34] were studied and investigated.

The concept of BHS sets (a hybridization of the structure of HS set and bipolarity) with its application in decision-making was first introduced by Musa and Asaad.[35,36] They also defined mappings between BH classes and investigated some of its properties.[37] Based on BHS sets, the authors of[38] defined the concept of bipolar hypersoft topological spaces (BHSTSs) and its some basics operators such as BHS closure, BHS interior, and BH boundary. The concepts of BHS connected space,[39] BH homeomorphism maps and BHS compact spaces[40] were presented and investigated.

The rest of the study is carried out as follows: in section 2 basic definitions and results are provided for the use in this article. Then, in section 3, we introduce the notion of BHS $\widetilde{T_i}$-spaces $(i = 0, 1, 2, 3, 4)$ in terms of different ordinary points. Also, we give some results and discuss the relationship between them. In section 4, the concepts of BHS regular and BHS normal spaces are studied in details. In the end, in section 5, we summarize the key findings and make recommendations for further research.

2 Preliminaries

In this part, we'll go through the essential definitions and outcomes that are required to make this work self-contained. Suppose that Ω is the universal set and 2^Ω is the power set of Ω. Let $\sigma_i \cap \sigma_j = \phi$ with $i \neq j$ and let $\lambda_i, \beta_i \subseteq \Sigma_i$ for $i = 1, 2, ..., n$. We denote $\lambda_1 \times \lambda_2 \times ... \times \lambda_n$, $\beta_1 \times \beta_2 \times ... \times \beta_n$, and $\sigma_1 \times \sigma_2 \times ... \times \sigma_n$ as Λ, B, and Σ respectively. Let $\Lambda, B \subseteq \Sigma$.

[35] A BHS set over Ω is defined as a triple $(g, \hat{g}, \Sigma)$ provided that g and $\hat{g}$ are mappings given by $g : \Sigma \to 2^\Omega$ and $\hat{g} : \neg\Sigma \to 2^\Omega$ with $g(s) \cap \hat{g}(\neg s) = \phi$ for all $s \in \Sigma$.

The representation of a BHS set $(g, \hat{g}, \Sigma)$ is as follows:

$$(g, \hat{g}, \Sigma) = \{(s, g(s), \hat{g}(\neg s)) : s \in \Sigma \text{ and } g(s) \cap \hat{g}(\neg s) = \phi\}.$$

[35] A BHS set $(g_1, \hat{g}_1, \Lambda)$ is a BHS subset of BHS set $(g_2, \hat{g}_2, B)$, denoted by $(g_1, \hat{g}_1, \Lambda) \,\widetilde{\widetilde{\sqsubseteq}}\, (g_2, \hat{g}_2, B)$, if $\Lambda \subseteq B$ and for all $s \in \Lambda$, $g_1(s) \subseteq g_2(s)$ and $\hat{g}_2(\neg s) \subseteq \hat{g}_1(\neg s)$. If $(g_2, \hat{g}_2, B)$ is a BHS subset of $(g_1, \hat{g}_1, \Lambda)$, then $(g_1, \hat{g}_1, \Lambda)$ is a BHS superset of $(g_2, \hat{g}_2, B)$ and is denoted by $(g_1, \hat{g}_1, \Lambda) \,\widetilde{\widetilde{\sqsupseteq}}\, (g_2, \hat{g}_2, B)$.

[35] The BHS sets $(g_1, \hat{g}_1, \Lambda)$ and $(g_2, \hat{g}_2, B)$ over Ω are said to be BHS equal if they are both BHS subsets of each other.

[35] The complement of $(g, \hat{g}, \Sigma)$, denoted by $(g, \hat{g}, \Sigma)^c = (g^c, \hat{g}^c, \Sigma)$, is defined by $g^c(s) = \hat{g}(\neg s)$ and $\hat{g}^c(\neg s) = g(s)$ for all $s \in \Sigma$.

[35] A BHS set $(g, \hat{g}, \Lambda)$ over Ω is said to be a relative null BHS set, denoted by $(\widetilde{\Phi}, \widetilde{\Omega}, \Lambda)$, if $\widetilde{\Phi}(s) = \phi$ and $\widetilde{\Omega}(\neg s) = \Omega$ for all $s \in \Lambda$. The absolute null BHS set over Ω is denoted by $(\widetilde{\Phi}, \widetilde{\Omega}, \Sigma)$.

[35] A BHS set $(g, \hat{g}, \Lambda)$ over Ω is said to be a relative whole BHS set, denoted by $(\widetilde{\Omega}, \widetilde{\Phi}, \Lambda)$, if $\widetilde{\Omega}(s) = \Omega$ and $\widetilde{\Phi}(\neg s) = \phi$ for all $s \in \Lambda$. The absolute whole BHS set over Ω is denoted by $(\widetilde{\Omega}, \widetilde{\Phi}, \Sigma)$.

[35] The union of BHS sets $(g_1, \hat{g}_1, \Lambda)$ and $(g_2, \hat{g}_2, B)$, denoted by $(g_1, \hat{g}_1, \Lambda) \,\widetilde{\widetilde{\sqcup}}\, (g_2, \hat{g}_2, B)$, is a BHS set $(f, \hat{f}, \Delta)$ where $\Delta = \Lambda \cap B$ and the mappings f and $\hat{f}$ are defined by $f(s) = g_1(s) \cup g_2(s)$ and $\hat{f}(\neg s) = \hat{g}_1(\neg s) \cap \hat{g}_2(\neg s)$ for all $s \in \Delta$.

[35] The intersection of BHS sets $(g_1, \hat{g}_1, \Lambda)$ and $(g_2, \hat{g}_2, B)$, denoted by $(g_1, \hat{g}_1, \Lambda) \,\widetilde{\widetilde{\sqcap}}\, (g_2, \hat{g}_2, B)$, is a BHS set $(f, \hat{f}, \Delta)$ where $\Delta = \Lambda \cap B$ and the mappings f and $\hat{f}$ are defined by $f(s) = g_1(s) \cap g_2(s)$ and $\hat{f}(\neg s) = \hat{g}_1(\neg s) \cup \hat{g}_2(\neg s)$ for all $s \in \Delta$.

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[35] The difference of BHS sets $(g_1, \hat{g}_1, \Lambda)$ and $(g_2, \hat{g}_2, B)$, denoted by $(g_1, \hat{g}_1, \Lambda) \setminus (g_2, \hat{g}_2, B)$, is a BHS set $(f, \hat{f}, \Delta)$ where $\Delta = \Lambda \cap B$ and the mappings f and $\hat{f}$ are defined by $f(s) = g_1(s) \cap \hat{g}_2(\neg s)$ and $\hat{f}(\neg s) = \hat{g}_1(\neg s) \cup g_2(s)$ for all $s \in \Delta$.

[38] Suppose that $(g, \hat{g}, \Sigma)$ is a BHS set over Ω and $\omega \in \Omega$. We write $\omega \in (g, \hat{g}, \Sigma)$ if $\omega \in g(s)$ for all $s \in \Sigma$. Also for any $\omega \in \Omega$, $\omega \notin (g, \hat{g}, \Sigma)$, if $\omega \notin g(s)$ for some $s \in \Sigma$.

[38] Two BHS sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ are said to be disjoint BHS sets if $g_1(s) \cap g_2(s) = \phi$ for all $s \in \Sigma$. We denote it by $(g_1, \hat{g}_1, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$ where $\widetilde{\Phi}(s) = \phi$ and $\hat{g}(\neg s) \subseteq \Omega$ for all $s \in \Sigma$.

[38] Suppose that $(g, \hat{g}, \Sigma)$ is a BHS set over Ω and $\Upsilon \subseteq \Omega$. The sub BHS set of $(g, \hat{g}, \Sigma)$ over Υ, denoted by $(g_\Upsilon, \hat{g}_\Upsilon, \Sigma)$, is defined by $g_\Upsilon(s) = \Upsilon \cap g(s)$ and $\hat{g}_\Upsilon(\neg s) = \Upsilon \cap \hat{g}(\neg s)$, for each $s \in \Sigma$.

[38] If $\mathcal{T}_{B\mathcal{H}}$ be a collection of BHS sets over Ω, then $\mathcal{T}_{B\mathcal{H}}$ is a bipolar hypersoft topology (BHST) on Ω if:

1. $(\widetilde{\Phi}, \widetilde{\Omega}, \Sigma)$, $(\widetilde{\Omega}, \widetilde{\Phi}, \Sigma)$ belong to $\mathcal{T}_{B\mathcal{H}}$.
2. the intersection of any two BHS sets in $\mathcal{T}_{B\mathcal{H}}$ belongs to $\mathcal{T}_{B\mathcal{H}}$.
3. the union of any number of BHS sets in $\mathcal{T}_{B\mathcal{H}}$ belongs to $\mathcal{T}_{B\mathcal{H}}$.

We called $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ a BHSTS. Every member of $\mathcal{T}_{B\mathcal{H}}$ is called a BHS open set and its complement is called a BHS closed set.

[38] Suppose that $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHST and $\Upsilon \subseteq \Omega$. Then the relative BHST on Υ is defined by $\mathcal{T}_{B\mathcal{H}_\Upsilon} = \{(g_\Upsilon, \hat{g}_\Upsilon, \Sigma) \mid (g, \hat{g}, \Sigma) \,\widetilde{\widetilde{\in}}\, \mathcal{T}_{B\mathcal{H}}\}$. We called $(\Upsilon, \mathcal{T}_{B\mathcal{H}_\Upsilon}, \Sigma, \neg\Sigma)$ a BHS subspace of $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$.

[39] The property $\mathcal{P}$ is said to be BHS hereditary if every BHS subspace of a BHSTS has the same property.

[31] If $\mathcal{T}_{\mathcal{H}}$ be a collection of HS sets over Ω, then $\mathcal{T}_{\mathcal{H}}$ is a hypersoft topology (HST) on Ω if:

1. $(\widetilde{\Phi}, \Sigma)$, $(\widetilde{\Omega}, \Sigma)$ belong to $\mathcal{T}_{\mathcal{H}}$.
2. the intersection of any two HS sets in $\mathcal{T}_{\mathcal{H}}$ belongs to $\mathcal{T}_{\mathcal{H}}$.
3. the union of any number of HS sets in $\mathcal{T}_{\mathcal{H}}$ belongs to $\mathcal{T}_{\mathcal{H}}$.

We called $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ an HSTS over Ω. Every member of $\mathcal{T}_{\mathcal{H}}$ is called an HS open set and its complement is called an HS closed set.

[34] An HSTS $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is said to be:

i. an HS T_0-space if for every $\omega_1 \neq \omega_2 \in \Omega$, there exists an HS open set (g, Σ) such that $\omega_1 \in (g, \Sigma)$, $\omega_2 \notin (g, \Sigma)$ or $\omega_2 \in (g, \Sigma)$, $\omega_1 \notin (g, \Sigma)$.

ii. an HS T_1-space if for every $\omega_1 \neq \omega_2 \in \Omega$, there exist HS open sets (g_1, Σ) and (g_2, Σ) such that $\omega_1 \in (g_1, \Sigma)$, $\omega_2 \notin (g_1, \Sigma)$ and $\omega_2 \in (g_2, \Sigma)$, $\omega_1 \notin (g_2, \Sigma)$.

iii. an HS T_2-space if for every $\omega_1 \neq \omega_2 \in \Omega$, there exist HS open sets (g_1, Σ) and (g_2, Σ) such that $\omega_1 \in (g_1, \Sigma)$, $\omega_2 \in (g_2, \Sigma)$ and $(g_1, \Sigma) \,\widetilde{\sqcap}\, (g_2, \Sigma) = (\widetilde{\Phi}, \Sigma)$.

iv. an HS regular space if for every HS closed set (f, Σ) with $\omega \notin (f, \Sigma)$, there are HS open sets (g_1, Σ) and (g_2, Σ) such that $\omega \in (g_1, \Sigma)$ and $(f, \Sigma) \,\widetilde{\sqsubseteq}\, (g_2, \Sigma)$ with $(g_1, \Sigma) \,\widetilde{\sqcap}\, (g_2, \Sigma) = (\widetilde{\Phi}, \Sigma)$.

v. an HS T_3-space if it is both HS regular and HS T_1-space.

vi. an HS normal space if for every HS closed sets (f_1, Σ) and (f_2, Σ) with $(f_1, \Sigma) \,\widetilde{\sqcap}\, (f_2, \Sigma) = (\widetilde{\Phi}, \Sigma)$, there exist HS open sets (g_1, Σ) and (g_2, Σ) such that $(f_1, \Sigma) \,\widetilde{\sqsubseteq}\, (g_1, \Sigma)$ and $(f_2, \Sigma) \,\widetilde{\sqsubseteq}\, (g_2, \Sigma)$ with $(g_1, \Sigma) \,\widetilde{\sqcap}\, (g_2, \Sigma) = (\widetilde{\Phi}, \Sigma)$.

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vii. an HS T_4-space if it is both HS normal and HS T_1-space.

[38] Suppose that $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHSTS. Then the following collections defined HST on Ω.

1. $\mathcal{T}_{\mathcal{H}} = \{(g, \Sigma) \mid (g, \hat{g}, \Sigma) \,\widetilde{\widetilde{\in}}\, \mathcal{T}_{B\mathcal{H}}\}$.
2. $\neg\mathcal{T}_{\mathcal{H}} = \{(\hat{g}, \neg\Sigma) \mid (g, \hat{g}, \Sigma) \,\widetilde{\widetilde{\in}}\, \mathcal{T}_{B\mathcal{H}}\}$. (Provided that Ω is a finite set)

[38] Suppose that $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is an HSTS. Then $\mathcal{T}_{B\mathcal{H}}$ is said to be BHST if it consists BHS sets $(g, \hat{g}, \Sigma)$ such that $(g, \Sigma) \,\widetilde{\in}\, \mathcal{T}_{\mathcal{H}}$ and $\hat{g}(\neg s) = \Omega \setminus g(s)$ for all $s \in \Sigma$.

Let $\omega \in \Omega$, then $(g_\omega, \hat{g}_\omega, \Sigma)$ denotes the BHS set defined by $g_\omega(s) = \{\omega\}$ and $\hat{g}_\omega(\neg s) = \Omega \setminus \{\omega\}$ for all $s \in \Sigma$.

Suppose that $(g, \hat{g}, \Sigma)$ is a BHS set and $\omega \in \Omega$. Then:

i. $\omega \in (g, \hat{g}, \Sigma)$ if and only if $(g_\omega, \hat{g}_\omega, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, (g, \hat{g}, \Sigma)$.

ii. if $(g_\omega, \hat{g}_\omega, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g, \hat{g}, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$, then $\omega \notin (g, \hat{g}, \Sigma)$.

Proof. Straightforward. □

The opposite of Proposition 2 (ii.) is incorrect.

Let $\Omega = \{\omega_1, \omega_2\}$, $\sigma_1 = \{\ell_1, \ell_2, \ell_3\}$, $\sigma_2 = \{\ell_4\}$, and $\sigma_3 = \{\ell_5\}$. Let $(g, \hat{g}, \Sigma) = \{((\ell_1, \ell_4, \ell_5), \{\omega_1\}, \{\omega_2\}), ((\ell_2, \ell_4, \ell_5), \{\omega_1\},$
Then $\omega_2 \notin (g, \hat{g}, \Sigma)$ but $(g_{\omega_2}, \hat{g}_{\omega_2}, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g, \hat{g}, \Sigma) \neq (\widetilde{\Phi}, \hat{g}, \Sigma)$ since $g_{\omega_2}(\ell_3, \ell_4, \ell_5) \cap g(\ell_3, \ell_4, \ell_5) \neq \phi$.

Let $(\Upsilon, \mathcal{T}_{B\mathcal{H}_\Upsilon}, \Sigma, \neg\Sigma)$ be a BHS subspace of $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$. Then:

i. $(g_\Upsilon, \hat{g}_\Upsilon, \Sigma)$ is BHS open set in Υ if and only if $g_\Upsilon(s) = \Upsilon \cap g(s)$ and $\hat{g}_\Upsilon(\neg s) = \Upsilon \cap \hat{g}(\neg s)$ for some BHS open set $(g, \hat{g}, \Sigma)$ in Ω.

ii. $(f_\Upsilon, \hat{f}_\Upsilon, \Sigma)$ is BHS closed set in Υ if and only if $f_\Upsilon(s) = \Upsilon \cap f(s)$ and $\hat{f}_\Upsilon(\neg s) = \Upsilon \cap \hat{f}(\neg s)$ for some BHS closed set $(f, \hat{f}, \Sigma)$ in Ω.

Proof. i. Follows from Definition 2.

ii. $(f_\Upsilon, \hat{f}_\Upsilon, \Sigma)$ is BHS closed set in Υ *iff* $(f_\Upsilon, \hat{f}_\Upsilon, \Sigma)^c$ is BHS open set in Υ *iff* ${f_\Upsilon}^c(s) = \Upsilon \cap g(s)$ and $\hat{f}^c_\Upsilon(\neg s) = \Upsilon \cap \hat{g}(\neg s)$ for some BHS open set $(g, \hat{g}, \Sigma)$ in Ω *iff* $\Upsilon \setminus {f_\Upsilon}^c(s) = \Upsilon \setminus [\Upsilon \cap g(s)]$ and $\Upsilon \setminus \hat{f}^c_\Upsilon(\neg s) = \Upsilon \setminus [\Upsilon \cap \hat{g}(\neg s)]$ *iff* $f_\Upsilon(s) = \Upsilon \cap g^c(s)$ and $\hat{f}_\Upsilon(\neg s) = \Upsilon \cap \hat{g}^c(\neg s)$ *iff* $f_\Upsilon(s) = \Upsilon \cap f(s)$ and $\hat{f}_\Upsilon(\neg s) = \Upsilon \cap \hat{f}(\neg s)$ where $(f, \hat{f}, \Sigma) = (g, \hat{g}, \Sigma)^c$ is BHS closed set in Ω since $(g, \hat{g}, \Sigma)$ is BHS open set in Ω.

□

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3 Bipolar Hypersoft Separation Axioms

In this section, the definitions of BHS $\widetilde{T}_i$-space for $i = 0, 1, 2$ are given. The properties of these spaces are studied and discussed.

A BHSTS $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is said to be:

i. a BHS $\widetilde{T}_0$-space if for every $\omega_1 \neq \omega_2 \in \Omega$, there exists a BHS open set $(g, \hat{g}, \Sigma)$ such that $\omega_1 \in (g, \hat{g}, \Sigma)$, $\omega_2 \notin (g, \hat{g}, \Sigma)$ or $\omega_2 \in (g, \hat{g}, \Sigma)$, $\omega_1 \notin (g, \hat{g}, \Sigma)$.

ii. a BHS $\widetilde{T}_1$-space if for every $\omega_1 \neq \omega_2 \in \Omega$, there exist BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \notin (g_1, \hat{g}_1, \Sigma)$ and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$, $\omega_1 \notin (g_2, \hat{g}_2, \Sigma)$.

iii. a BHS $\widetilde{T}_2$-space if for every $\omega_1 \neq \omega_2 \in \Omega$, there exist BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$ and $(g_1, \hat{g}_1, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$.

Below we examine some results related to BHS $\widetilde{T}_0$-space.

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega_1 \neq \omega_2 \in \Omega$. If there is a BHS open set $(g, \hat{g}, \Sigma)$ with $\omega_1 \in (g, \hat{g}, \Sigma)$, $\omega_2 \in (g, \hat{g}, \Sigma)^c$ or $\omega_2 \in (g, \hat{g}, \Sigma)$, $\omega_1 \in (g, \hat{g}, \Sigma)^c$, then $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_0$-space.

Proof. Let $\omega_1 \neq \omega_2 \in \Omega$ and $(g, \hat{g}, \Sigma)$ be a BHS open set with $\omega_1 \in (g, \hat{g}, \Sigma)$ and $\omega_2 \in (g, \hat{g}, \Sigma)^c$. Since $\omega_2 \in (g, \hat{g}, \Sigma)^c$ then $\omega_2 \in g^c(s)$ for all $s \in \Sigma$. This means $\omega_2 \notin g(s)$ for all $s \in \Sigma$. Hence $\omega_2 \notin (g, \hat{g}, \Sigma)$. In the same way, we may verify that $\omega_1 \in (g, \hat{g}, \Sigma)^c$, then $\omega_1 \notin (g, \hat{g}, \Sigma)$. Therefore, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_0$-space. □

If $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_0$-space, then $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is an HS T_0-space.

Proof. Let $\omega_1 \neq \omega_2 \in \Omega$. Since $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_0$-space, then there is a BHS open set $(g, \hat{g}, \Sigma)$ with $\omega_1 \in (g, \hat{g}, \Sigma)$, $\omega_2 \notin (g, \hat{g}, \Sigma)$ or $\omega_2 \in (g, \hat{g}, \Sigma)$, $\omega_1 \notin (g, \hat{g}, \Sigma)$. Say, $\omega_1 \in (g, \hat{g}, \Sigma)$ and $\omega_2 \notin (g, \hat{g}, \Sigma)$. This means $\omega_1 \in g(s)$ for all $s \in \Sigma$ and $\omega_2 \notin g(s)$ for some $s \in \Sigma$. Therefore $\omega_1 \in (g, \Sigma)$ and $\omega_2 \notin (g, \Sigma)$. In the same way, we may verify that $\omega_2 \in (g, \Sigma)$ and $\omega_1 \notin (g, \Sigma)$. Hence, $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is an HS T_0-space. □

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega_1 \neq \omega_2 \in \Omega$. If there is a BHS open set $(g, \hat{g}, \Sigma)$ with $\omega_1 \in (g, \hat{g}, \Sigma)$, $\omega_2 \in (g, \hat{g}, \Sigma)^c$ or $\omega_2 \in (g, \hat{g}, \Sigma)$, $\omega_1 \in (g, \hat{g}, \Sigma)^c$, then $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is HS T_0-space.

Proof. Follows from Proposition 3 and Proposition 3. □

Let Ω be a finite set. Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega_1 \neq \omega_2 \in \Omega$. If there is a BHS open set $(g, \hat{g}, \Sigma)$ with $\omega_1 \in (g, \hat{g}, \Sigma)$, $\omega_2 \in (g, \hat{g}, \Sigma)^c$ or $\omega_2 \in (g, \hat{g}, \Sigma)$, $\omega_1 \in (g, \hat{g}, \Sigma)^c$, then $(\Omega, \neg\mathcal{T}_{\mathcal{H}}, \neg\Sigma)$ is HS T_0-space.

Proof. Let $\omega_1 \in (g, \hat{g}, \Sigma)$ and $\omega_2 \in (g, \hat{g}, \Sigma)^c$. This means $\omega_1 \in g(s)$ and $\omega_2 \in g^c(s)$ for all $s \in \Sigma$. Then we have $\omega_1 \notin \hat{g}(\neg s)$ and $\omega_2 \in \hat{g}(\neg s)$ for all $s \in \Sigma$. So, $\omega_2 \in (\hat{g}, \neg\Sigma)$ and $\omega_1 \notin (\hat{g}, \neg\Sigma)$. In the same way, we may verify that $\omega_1 \in (\hat{g}, \neg\Sigma)$ and $\omega_2 \notin (\hat{g}, \neg\Sigma)$. Hence, $(\Omega, \neg\mathcal{T}_{\mathcal{H}}, \neg\Sigma)$ is an HS T_0-space. □

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS constructed from HSTS $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ as in Proposition 2. If $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ be an HS T_0-space, then $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_0$-space.

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Proof. Let $\omega_1 \neq \omega_2 \in \Omega$. Since $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is an HS T_0-space, then there is an HS open set (g, Σ) with $\omega_1 \in (g, \Sigma)$, $\omega_2 \notin (g, \Sigma)$ or $\omega_2 \in (g, \Sigma)$, $\omega_1 \notin (g, \Sigma)$. Say, $\omega_1 \in (g, \Sigma)$ and $\omega_2 \notin (g, \Sigma)$. This means $\omega_1 \in g(s)$ for all $s \in \Sigma$ and $\omega_2 \notin g(s)$ for some $s \in \Sigma$. Since $\hat{g}(\neg s) = \Omega \setminus g(s)$ for all $s \in \Sigma$, then $\omega_1 \notin \hat{g}(\neg s)$ for all $s \in \Sigma$ and $\omega_2 \in \hat{g}(\neg s)$ for some $s \in \Sigma$. Therefore $\omega_1 \in (g, \hat{g}, \Sigma)$ and $\omega_2 \notin (g, \hat{g}, \Sigma)$. In the same way, we may verify that $\omega_2 \in (g, \hat{g}, \Sigma)$ and $\omega_1 \notin (g, \hat{g}, \Sigma)$. Hence, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_0$-space. □

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\Upsilon \subseteq \Omega$. If $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_0$-space, then $(\Upsilon, \mathcal{T}_{B\mathcal{H}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_0$-space.

Proof. Let $\omega_1 \neq \omega_2 \in \Upsilon$. Since $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_0$-space, then there is a BHS open set $(g, \hat{g}, \Sigma)$ with $\omega_1 \in (g, \hat{g}, \Sigma)$, $\omega_2 \notin (g, \hat{g}, \Sigma)$ or $\omega_2 \in (g, \hat{g}, \Sigma)$, $\omega_1 \notin (g, \hat{g}, \Sigma)$. Say, $\omega_1 \in (g, \hat{g}, \Sigma)$ and $\omega_2 \notin (g, \hat{g}, \Sigma)$. As, $\omega_1 \in (g, \hat{g}, \Sigma)$ then $\omega_1 \in g(s)$ and $\omega_1 \notin \hat{g}(\neg s)$ for all $s \in \Sigma$. Since $\omega_1 \in \Upsilon$, then $\omega_1 \in \Upsilon \cap g(s) = g_\Upsilon(s)$ and $\omega_2 \notin \Upsilon \cap \hat{g}(\neg s) = \hat{g}_\Upsilon(\neg s)$ for all $s \in \Sigma$. Hence, $\omega_1 \in (g_\Upsilon, \hat{g}_\Upsilon, \Sigma)$. Consider $\omega_2 \notin (g, \hat{g}, \Sigma)$. Then $\omega_2 \notin g(s)$ for some $s \in \Sigma$. This implies $\omega_2 \notin \Upsilon \cap g(s) = g_\Upsilon(s)$ for some $s \in \Sigma$. Hence, $\omega_2 \notin (g_\Upsilon, \hat{g}_\Upsilon, \Sigma)$. In the same way, we may verify that $\omega_2 \in (g_\Upsilon, \hat{g}_\Upsilon, \Sigma)$ and $\omega_1 \notin (g_\Upsilon, \hat{g}_\Upsilon, \Sigma)$. Hence, $(\Upsilon, \mathcal{T}_{B\mathcal{H}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_0$-space. □

In the following result, we present a complete description of a BHS $\widetilde{T}_1$-space and then establish several characteristics of this space.

If $(g_\omega, \hat{g}_\omega, \Sigma)$ be a BHS closed set of $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ for each $\omega \in \Omega$, then $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_1$-space.

Proof. Suppose for each $\omega \in \Omega$, $(g_\omega, \hat{g}_\omega, \Sigma)$ is a BHS closed set of $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$. Then $(g_\omega, \hat{g}_\omega, \Sigma)^c$ is a BHS open set in $\mathcal{T}_{B\mathcal{H}}$. Let $\omega \neq \varpi \in \Omega$. For $\omega \in \Omega$, $(g_\omega, \hat{g}_\omega, \Sigma)^c$ is a BHS open set with $\varpi \in (g_\omega, \hat{g}_\omega, \Sigma)^c$ and $\omega \notin (g_\omega, \hat{g}_\omega, \Sigma)^c$. Similarly, $(g_\varpi, \hat{g}_\varpi, \Sigma)^c \,\widetilde{\widetilde{\in}}\, \mathcal{T}_{B\mathcal{H}}$ with $\omega \in (g_\varpi, \hat{g}_\varpi, \Sigma)^c$ and $\varpi \notin (g_\varpi, \hat{g}_\varpi, \Sigma)^c$. Thus, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_1$-space. □

The converse of Proposition 3 is not true in general as shown below.

Let $\Omega = \{\omega_1, \omega_2\}$, $\sigma_1 = \{\ell_1, \ell_2\}$, $\sigma_2 = \{\ell_3\}$, and $\sigma_3 = \{\ell_4\}$. Let $\mathcal{T}_{B\mathcal{H}} = \{(\widetilde{\Phi}, \widetilde{\Omega}, \Sigma), (\widetilde{\Omega}, \widetilde{\Phi}, \Sigma), (g_1, \hat{g}_1, \Sigma), (g_2, \hat{g}_2, \Sigma), (g_3, \hat{g}_3, \Sigma)\}$ be a BHST defined on Ω, where

$(g_1, \hat{g}_1, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{\omega_1\}, \{\omega_2\}), ((\ell_2, \ell_3, \ell_4), \{\omega_2\}, \{\omega_1\})\}$.

$(g_2, \hat{g}_2, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{\omega_1\}, \{\omega_2\}), ((\ell_2, \ell_3, \ell_4), \Omega, \phi)\}$.

$(g_3, \hat{g}_3, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \Omega, \phi), ((\ell_2, \ell_3, \ell_4), \{\omega_2\}, \{\omega_1\})\}$.

Then, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_1$-space.

We note that for $(g_{\omega_1}, \hat{g}_{\omega_1}, \Sigma)$, $(g_{\omega_2}, \hat{g}_{\omega_2}, \Sigma)$ over Ω, where

$(g_{\omega_1}, \hat{g}_{\omega_1}, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{\omega_1\}, \{\omega_2\}), ((\ell_2, \ell_3, \ell_4), \{\omega_1\}, \{\omega_2\})\}$.

$(g_{\omega_2}, \hat{g}_{\omega_2}, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{\omega_2\}, \{\omega_1\}), ((\ell_2, \ell_3, \ell_4), \{\omega_2\}, \{\omega_1\})\}$.

The BHS complement $(g_{\omega_1}, \hat{g}_{\omega_1}, \Sigma)^c$, $(g_{\omega_2}, \hat{g}_{\omega_2}, \Sigma)^c$ over Ω are defined by

$(g_{\omega_1}, \hat{g}_{\omega_1}, \Sigma)^c = \{((\ell_1, \ell_3, \ell_4), \{\omega_2\}, \{\omega_1\}), ((\ell_2, \ell_3, \ell_4), \{\omega_2\}, \{\omega_1\})\}$.

$(g_{\omega_2}, \hat{g}_{\omega_2}, \Sigma)^c = \{((\ell_1, \ell_3, \ell_4), \{\omega_1\}, \{\omega_2\}), ((\ell_2, \ell_3, \ell_4), \{\omega_1\}, \{\omega_2\})\}$.

Then, $(g_{\omega_1}, \hat{g}_{\omega_1}, \Sigma)^c$, $(g_{\omega_2}, \hat{g}_{\omega_2}, \Sigma)^c \,\widetilde{\widetilde{\notin}}\, \mathcal{T}_{B\mathcal{H}}$. Thus, $(g_{\omega_1}, \hat{g}_{\omega_1}, \Sigma)$ and $(g_{\omega_2}, \hat{g}_{\omega_2}, \Sigma)$ are not BHS closed sets of $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$.

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega \in \Omega$. If Ω be a BHS $\widetilde{T}_1$-space, then for each BHS open sets $(g, \hat{g}, \Sigma)$ such that $\omega \in (g, \hat{g}, \Sigma)$:

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i. $(g_\omega, \hat{g}_\omega, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, [\widetilde{\widetilde{\sqcap}}\,(g, \hat{g}, \Sigma)]$.

ii. $\varpi \notin \widetilde{\widetilde{\sqcap}}\,(g, \hat{g}, \Sigma)$ for all $\varpi \neq \omega$.

Proof. i. As $\omega \in \widetilde{\widetilde{\sqcap}}\,(g, \hat{g}, \Sigma)$, then by Proposition 2, $(g_\omega, \hat{g}_\omega, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, [\widetilde{\widetilde{\sqcap}}\,(g, \hat{g}, \Sigma)]$.

ii. Let $\varpi \neq \omega \in \Omega$, then there are BHS open sets $(f, \hat{f}, \Sigma)$ with $\omega \in (f, \hat{f}, \Sigma)$ and $\varpi \notin (f, \hat{f}, \Sigma)$. Then $\varpi \notin f(s)$ for some $s \in \Sigma$ and hence $\varpi \notin \cap_{s\in\Sigma}\, g(s)$. Thus, $\varpi \notin \widetilde{\widetilde{\sqcap}}\,(g, \hat{g}, \Sigma)$.

□

The equality of Proposition 3 (i.) is false as the next example shows.

Let $\Omega = \{\omega_1, \omega_2\}$, $\sigma_1 = \{\ell_1, \ell_2\}$, $\sigma_2 = \{\ell_3, \ell_4\}$, and $\sigma_3 = \{\ell_5\}$. Let $\mathcal{T}_{B\mathcal{H}} = \{(\widetilde{\Phi}, \widetilde{\Omega}, \Sigma)$, $(\widetilde{\Omega}, \widetilde{\Phi}, \Sigma)$, $(g_1, \hat{g}_1, \Sigma)$, $(g_2, \hat{g}_2, \Sigma)$, $(g_3, \hat{g}_3, \Sigma)$, $(g_4, \hat{g}_4, \Sigma)$, $(g_5, \hat{g}_5, \Sigma)\}$ be a BHST defined on Ω, where

$$(g_1, \hat{g}_1, \Sigma) = \{((\ell_1, \ell_3, \ell_5), \Omega, \phi), ((\ell_1, \ell_4, \ell_5), \{\omega_1\}, \{\omega_2\}), ((\ell_2, \ell_3, \ell_5), \{\omega_1\}, \{\omega_2\}), ((\ell_2, \ell_4, \ell_5), \{\omega_1\}, \{\omega_2\})\}.$$

$$(g_2, \hat{g}_2, \Sigma) = \{((\ell_1, \ell_3, \ell_5), \{\omega_2\}, \{\omega_1\}), ((\ell_1, \ell_4, \ell_5), \{\omega_2\}, \{\omega_1\}), ((\ell_2, \ell_3, \ell_5), \{\omega_2\}, \{\omega_1\}), ((\ell_2, \ell_4, \ell_5), \Omega, \phi)\}.$$

$$(g_3, \hat{g}_3, \Sigma) = \{((\ell_1, \ell_3, \ell_5), \Omega, \phi), ((\ell_1, \ell_4, \ell_5), \Omega, \phi), ((\ell_2, \ell_3, \ell_5), \{\omega_1\}, \{\omega_2\}), ((\ell_2, \ell_4, \ell_5), \{\omega_1\}, \{\omega_2\})\}.$$

$$(g_4, \hat{g}_4, \Sigma) = \{((\ell_1, \ell_3, \ell_5), \{\omega_2\}, \{\omega_1\}), ((\ell_1, \ell_4, \ell_5), \phi, \Omega), ((\ell_2, \ell_3, \ell_5), \phi, \Omega), ((\ell_2, \ell_4, \ell_5), \{\omega_1\}, \{\omega_2\})\}.$$

$$(g_5, \hat{g}_5, \Sigma) = \{((\ell_1, \ell_3, \ell_5), \{\omega_2\}, \{\omega_1\}), ((\ell_1, \ell_4, \ell_5), \{\omega_2\}, \{\omega_1\}), ((\ell_2, \ell_3, \ell_5), \phi, \Omega), ((\ell_2, \ell_4, \ell_5), \{\omega_1\}, \{\omega_2\})\}.$$

If we take $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ where $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$ and $\omega_2 \notin (g_1, \hat{g}_1, \Sigma)$; and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$ and $\omega_1 \notin (g_2, \hat{g}_2, \Sigma)$. Then, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_1$-space. But for all BHS open sets $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$ and $\omega_1 \in (g_3, \hat{g}_3, \Sigma)$ we have $(g_1, \hat{g}_1, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g_3, \hat{g}_3, \Sigma) = (g_1, \hat{g}_1, \Sigma) \neq (g_{\omega_1}, \hat{g}_{\omega_1}, \Sigma)$.

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega_1 \neq \omega_2 \in \Omega$. If there are BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ with $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \in (g_1, \hat{g}_1, \Sigma)^c$ and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$, $\omega_1 \in (g_2, \hat{g}_2, \Sigma)^c$, then $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_1$-space.

Proof. Similar to the proof of Proposition 3. □

If $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_1$-space, then $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is HS T_1-space.

Proof. Similar to the proof of Proposition 3. □

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega_1 \neq \omega_2 \in \Omega$. If there is a BHS open set $(g, \hat{g}, \Sigma)$ with $\omega_1 \in (g, \hat{g}, \Sigma)$, $\omega_2 \in (g, \hat{g}, \Sigma)^c$ and $\omega_2 \in (g, \hat{g}, \Sigma)$, $\omega_1 \in (g, \hat{g}, \Sigma)^c$, then $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is HS T_1-space.

Proof. Follows from Proposition 3 and Proposition 3. □

Let Ω be a finite set. Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega_1 \neq \omega_2 \in \Omega$. If there is a BHS open set $(g, \hat{g}, \Sigma)$ with $\omega_1 \in (g, \hat{g}, \Sigma)$, $\omega_2 \in (g, \hat{g}, \Sigma)^c$ and $\omega_2 \in (g, \hat{g}, \Sigma)$, $\omega_1 \in (g, \hat{g}, \Sigma)^c$, then $(\Omega, \neg\mathcal{T}_{\mathcal{H}}, \neg\Sigma)$ is HS T_1-space.

Proof. Similar to the proof of Proposition 3. □

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS constructed from HSTS $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ as in Proposition 2. If $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ be an HS T_1-space, then $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_1$-space.

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Proof. Similar to the proof of Proposition 3. □

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\Upsilon \subseteq \Omega$. If $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_1$-space, then $(\Upsilon, \mathcal{T}_{B\mathcal{H}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_1$-space.

Proof. Similar to the proof of Proposition 3. □

In the following results, we characterize a BHS $\widetilde{T}_2$-space and investigate some of its properties.

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega \in \Omega$. If Ω be a BHS $\widetilde{T}_2$-space, then for each BHS open sets $(g, \hat{g}, \Sigma)$ such that $\omega \in (g, \hat{g}, \Sigma)$:

i. $(g_\omega, \hat{g}_\omega, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, [\widetilde{\widetilde{\sqcap}}\,(g, \hat{g}, \Sigma)]$;

ii. $\varpi \notin \widetilde{\widetilde{\sqcap}}\,(g, \hat{g}, \Sigma)$ for all $\varpi \neq \omega$.

Proof. Similar to the proof of Proposition 3. □

The equality of Proposition 3 (i.) is not true in general.

Let $\Omega = \{\omega_1, \omega_2\}$, $\sigma_1 = \{\ell_1, \ell_2\}$, $\sigma_2 = \{\ell_3\}$, and $\sigma_3 = \{\ell_4\}$. Let $\mathcal{T}_{B\mathcal{H}} = \{(\widetilde{\Phi}, \widetilde{\Omega}, \Sigma), (\widetilde{\Omega}, \widetilde{\Phi}, \Sigma), (g_1, \hat{g}_1, \Sigma), (g_2, \hat{g}_2, \Sigma), (g_3, \hat{g}_3, \Sigma)\}$ be a BHST defined on Ω, where

$(g_1, \hat{g}_1, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{\omega_1\}, \{\omega_2\}), ((\ell_2, \ell_3, \ell_4), \{\omega_1\}, \phi)\}$.

$(g_2, \hat{g}_2, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{\omega_2\}, \{\omega_1\}), ((\ell_2, \ell_3, \ell_4), \{\omega_2\}, \phi)\}$.

$(g_3, \hat{g}_3, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \phi, \Omega), ((\ell_2, \ell_3, \ell_4), \phi, \phi)\}$.

Then, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_2$-space. We note that $(g_1, \hat{g}_1, \Sigma)$ is the only BHS open set with $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, but $(g_{\omega_1}, \hat{g}_{\omega_1}, \Sigma) \neq (g_1, \hat{g}_1, \Sigma)$.

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega_1 \neq \omega_2 \in \Omega$. If there are BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \in (g_1, \hat{g}_1, \Sigma)^c$ and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$, $\omega_1 \in (g_2, \hat{g}_2, \Sigma)^c$, then $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_2$-space.

Proof. Let $\omega_1 \neq \omega_2 \in \Omega$ and $(g_1, \hat{g}_1, \Sigma)$, $(g_2, \hat{g}_2, \Sigma)$ be two BHS open sets with $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \in (g_1, \hat{g}_1, \Sigma)^c$ and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$, $\omega_1 \in (g_2, \hat{g}_2, \Sigma)^c$. This means $\omega_1 \in g_1(s)$, $\omega_2 \in g_1^c(s)$ and $\omega_2 \in g_2(s)$, $\omega_1 \in g_2^c(s)$ for all $s \in \Sigma$. Then, $\omega_1 \in g_1(s)$, $\omega_2 \notin g_1(s)$ and $\omega_2 \in g_2(s)$, $\omega_1 \notin g_2(s)$ with $g_1(s) \cap g_2(s) = \phi$. Also, $\omega_1 \notin \hat{g}_1(\neg s)$, $\omega_2 \in \hat{g}_1(\neg s)$ and $\omega_2 \notin \hat{g}_2(\neg s)$, $\omega_1 \in \hat{g}_2(\neg s)$ with $\hat{g}_1(\neg s) \cup \hat{g}_2(\neg s) = \Omega$. Then we have $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$ and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_1, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \widetilde{\Omega}, \Sigma)$. Hence, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_2$-space. □

If $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_2$-space, then $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is an HS T_2-space.

Proof. Let $\omega_1 \neq \omega_2 \in \Omega$. Since $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_2$-space, then there are BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$ and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_1, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$. This means $\omega_1 \in g_1(s)$ and $\omega_2 \in g_2(s)$ with $g_1(s) \cap g_2(s) = \phi$ for all $s \in \Sigma$. Consequently, $\omega_1 \in (g_1, \Sigma)$ and $\omega_2 \in (g_2, \Sigma)$ with $(g_1, \Sigma) \,\widetilde{\sqcap}\, (g_2, \Sigma) = (\widetilde{\Phi}, \Sigma)$. Hence, $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is an HS T_2-space. □

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Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega_1 \neq \omega_2 \in \Omega$. If there are BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \in (g_1, \hat{g}_1, \Sigma)^c$ and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$, $\omega_1 \in (g_2, \hat{g}_2, \Sigma)^c$, then $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is HS T_2-space.

Proof. Follows from Proposition 3 and Proposition 3. □

Let Ω be a finite set. Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega_1 \neq \omega_2 \in \Omega$. If there are BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \in (g_1, \hat{g}_1, \Sigma)^c$ and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$, $\omega_1 \in (g_2, \hat{g}_2, \Sigma)^c$, then $(\Omega, \neg\mathcal{T}_{\mathcal{H}}, \neg\Sigma)$ is HS T_2-space.

Proof. Let $\omega_1 \neq \omega_2 \in \Omega$ and $(g_1, \hat{g}_1, \Sigma)$, $(g_2, \hat{g}_2, \Sigma)$ be two BHS open sets such that $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \in (g_1, \hat{g}_1, \Sigma)^c$ and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$, $\omega_1 \in (g_2, \hat{g}_2, \Sigma)^c$. This means $\omega_1 \notin \hat{g}_1(\neg s)$, $\omega_2 \in \hat{g}_1(\neg s)$ and $\omega_2 \notin \hat{g}_2(\neg s)$, $\omega_1 \in \hat{g}_2(\neg s)$ with $\hat{g}_1(\neg s) \cap \hat{g}_2(\neg s) = \phi$ for all $s \in \Sigma$. Then we have $\omega_2 \in (\hat{g}_1, \neg\Sigma)$ and $\omega_1 \in (\hat{g}_2, \neg\Sigma)$ with $(\hat{g}_1, \neg\Sigma) \widetilde{\sqcap} (\hat{g}_2, \neg\Sigma) = (\widetilde{\Phi}, \neg\Sigma)$. Thus, $(\Omega, \neg\mathcal{T}_{\mathcal{H}}, \neg\Sigma)$ is an HS T_2-space. □

Let $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ be an HSTS and let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS constructed from $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ as in Proposition 2. If $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ be an HS T_2-space, then $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_2$-space.

Proof. Let $\omega_1 \neq \omega_2 \in \Omega$. Since $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is an HS T_2-space, then there are two HS open sets (g_1, Σ) and (g_2, Σ) such that $\omega_1 \in (g_1, \Sigma)$, $\omega_2 \in (g_2, \Sigma)$ with $(g_1, \Sigma) \widetilde{\sqcap} (g_2, \Sigma) = (\widetilde{\Phi}, \Sigma)$. This means $\omega_1 \in g_1(s)$ and $\omega_2 \in g_2(s)$ with $g_1(s) \cap g_2(s) = \phi$ for all $s \in \Sigma$. Then we have $\omega_1 \notin \hat{g}_1(\neg s)$ and $\omega_2 \notin \hat{g}_2(\neg s)$ with $\hat{g}_1(\neg s)$ $\cup\, \hat{g}_2(\neg s) = \Omega$ for all $s \in \Sigma$. Therefore, $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$ and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_1, \Sigma) \widetilde{\widetilde{\sqcap}} (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \widetilde{\Omega}, \Sigma)$. Thus, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_2$-space. □

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\Upsilon \subseteq \Omega$. If $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_2$-space, then $(\Upsilon, \mathcal{T}_{B\mathcal{H}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_2$-space.

Proof. Let $\omega_1 \neq \omega_2 \in \Upsilon$. Since $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_2$-space, then there are two BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_1, \Sigma) \widetilde{\widetilde{\sqcap}} (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$. This means $\omega_1 \in g_1(s)$ and $\omega_2 \in g_2(s)$ with $g_1(s) \cap g_2(s) = \phi$ for all $s \in \Sigma$. Since $\omega_1, \omega_2 \in \Upsilon$, then $\omega_1 \in \Upsilon \cap g_1(s) = g_{1_\Upsilon}(s)$ and $\omega_2 \in \Upsilon \cap g_2(s) = g_{2_\Upsilon}(s)$ with $g_{1_\Upsilon}(s) \cap g_{2_\Upsilon}(s) = \phi$ for all $s \in \Sigma$. Again, $\omega_1 \notin \Upsilon \cap \hat{g}_1(\neg s) = \hat{g}_{1_\Upsilon}(\neg s)$ and $\omega_2 \notin \Upsilon \cap \hat{g}_2(\neg s) = \hat{g}_{2_\Upsilon}(\neg s)$ with $\hat{g}_{1_\Upsilon}(\neg s) \cup \hat{g}_{2_\Upsilon}(\neg s) = \Omega$ for all $s \in \Sigma$. Then, $\omega_1 \in (g_{1_\Upsilon}, \hat{g}_{1_\Upsilon}, \Sigma)$ and $\omega_2 \in (g_{2_\Upsilon}, \hat{g}_{2_\Upsilon}, \Sigma)$ with $(g_{1_\Upsilon}, \hat{g}_{1_\Upsilon}, \Sigma) \widetilde{\widetilde{\sqcap}} (g_{2_\Upsilon}, \hat{g}_{2_\Upsilon}, \Sigma) = (\widetilde{\Phi}, \widetilde{\Omega}, \Sigma)$. Hence, $(\Upsilon, \mathcal{T}_{B\mathcal{H}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_2$-space. □

Every BHS $\widetilde{T}_i$-space is BHS $\widetilde{T}_{i-1}$-space, for $i = 1, 2$.

Proof. Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega_1 \neq \omega_2 \in \Omega$. For the case $i = 1$, let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_1$-space, then there are BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ with $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \notin (g_1, \hat{g}_1, \Sigma)$ and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$, $\omega_1 \notin (g_2, \hat{g}_2, \Sigma)$. Obviously, we have $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \notin (g_1, \hat{g}_1, \Sigma)$ or $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$, $\omega_1 \notin (g_2, \hat{g}_2, \Sigma)$. Thus, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_0$-space. Now, for the case $i = 2$, let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_2$-space, then there are BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$ and $(g_1, \hat{g}_1, \Sigma) \widetilde{\widetilde{\sqcap}} (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$. This means $\omega_1 \in g_1(s)$, $\omega_2 \in g_2(s)$ and $g_1(s) \cap g_2(s)$ $= \phi$ for all $s \in \Sigma$. Then we have $\omega_1 \in g_1(s)$, $\omega_2 \notin g_1(s)$ and $\omega_2 \in g_2(s)$, $\omega_1 \notin g_2(s)$ for all $s \in \Sigma$. Thus, $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \notin (g_1, \hat{g}_1, \Sigma)$ and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$, $\omega_1 \notin (g_2, \hat{g}_2, \Sigma)$. Therefore, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_1$-space. □

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The converse of Proposition 3 is false.

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be the same as in Example 3. Then $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_1$-space but it is not a BHS $\widetilde{T}_2$-space since for $\omega_1, \omega_2 \in \Omega$ there do not exist any two BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ with $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$ and $(g_1, \hat{g}_1, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$.

Now let $\mathcal{T}_{B\mathcal{H}} = \{(\widetilde{\Phi}, \widetilde{\Omega}, \Sigma), (\widetilde{\Omega}, \widetilde{\Phi}, \Sigma), (g, \hat{g}, \Sigma)\}$ where

$(g, \hat{g}, \Sigma) = \{((\ell_1, \ell_3, \ell_4), \{\omega_1\}, \{\omega_2\}), ((\ell_2, \ell_3, \ell_4), \Omega, \phi)\}$.

Then, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_0$-space but it is not a BHS $\widetilde{T}_1$-space since for $\omega_1, \omega_2 \in \Omega$ there do not exist BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ with $\omega_1 \in (g_1, \hat{g}_1, \Sigma)$, $\omega_2 \notin (g_1, \hat{g}_1, \Sigma)$ and $\omega_2 \in (g_2, \hat{g}_2, \Sigma)$, $\omega_1 \notin (g_2, \hat{g}_2, \Sigma)$.

4 Bipolar Hypersoft Regular and Bipolar Hypersoft Normal Spaces

This section dedicates for studying in details BHS regular and BHS normal spaces.

A BHSTS $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is said to be BHS regular space if for every BHS closed set $(f, \hat{f}, \Sigma)$ with $\omega \notin (f, \hat{f}, \Sigma)$, there are BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $\omega \in (g_1, \hat{g}_1, \Sigma)$ and $(f, \hat{f}, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_1, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$.

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and let $(f, \hat{f}, \Sigma)$ be a BHS closed set with $\omega \notin (f, \hat{f}, \Sigma)$. If $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS regular space, then there is a BHS open set $(g, \hat{g}, \Sigma)$ with $\omega \in (g, \hat{g}, \Sigma)$ and $(g, \hat{g}, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (f, \hat{f}, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$.

Proof. Straightforward. □

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega \in \Omega$. If Ω be a BHS regular space, then:

i. for a BHS closed set $(f, \hat{f}, \Sigma)$, $\omega \notin (f, \hat{f}, \Sigma)$ if and only if $(g_\omega, \hat{g}_\omega, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (f, \hat{f}, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$.

ii. for a BHS open set $(g, \hat{g}, \Sigma)$, $\omega \notin (g, \hat{g}, \Sigma)$ if and only if $(g_\omega, \hat{g}_\omega, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g, \hat{g}, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$.

Proof. i. Let $\omega \notin (f, \hat{f}, \Sigma)$, then there is a BHS open set $(g, \hat{g}, \Sigma)$ with $\omega \in (g, \hat{g}, \Sigma)$ and $(g, \hat{g}, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (f, \hat{f}, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$ by Proposition 4. Since $\omega \in (g, \hat{g}, \Sigma)$, then $(g_\omega, \hat{g}_\omega, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, (g, \hat{g}, \Sigma)$ by Proposition 2 (i.). Hence, $(g_\omega, \hat{g}_\omega, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (f, \hat{f}, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$.
The converse is obtained by Proposition 2 (ii.).

ii. Let $\omega \notin (g, \hat{g}, \Sigma)$. Then we have two cases: (1) for all $s \in \Sigma$, $\omega \notin g(s)$ and (2) for some $s, t \in \Sigma$, $\omega \notin g(s)$ and $\omega \in g(t)$. In case (1) it is clear that $(g_\omega, \hat{g}_\omega, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g, \hat{g}, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$. In case (2), $\omega \in g(t)$ implies that $\omega \notin g^c(t)$ for some $t \in \Sigma$. Hence, $(g, \hat{g}, \Sigma)^c$ is a BHS closed set with $\omega \notin (g, \hat{g}, \Sigma)^c$, by (i.), $(g_\omega, \hat{g}_\omega, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g, \hat{g}, \Sigma)^c = (\widetilde{\Phi}, \hat{g}, \Sigma)$. So $(g_\omega, \hat{g}_\omega, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, (g, \hat{g}, \Sigma)$ but this is inconsistent with $\omega \notin g(s)$ for some $s \in \Sigma$. Consequently, $(g_\omega, \hat{g}_\omega, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g, \hat{g}, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$.
The converse is obvious.

□

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\omega \in \Omega$. Then these are equivalent:

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i. $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS regular space.

ii. for each BHS closed set $(f, \hat{f}, \Sigma)$ with $(g_\omega, \hat{g}_\omega, \Sigma) \mathrel{\widetilde{\widetilde{\sqcap}}} (f, \hat{f}, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$, there are BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ with $(g_\omega, \hat{g}_\omega, \Sigma) \mathrel{\widetilde{\widetilde{\sqsubseteq}}} (g_1, \hat{g}_1, \Sigma)$, $(f, \hat{f}, \Sigma) \mathrel{\widetilde{\widetilde{\sqsubseteq}}} (g_2, \hat{g}_2, \Sigma)$ and $(g_1, \hat{g}_1, \Sigma) \mathrel{\widetilde{\widetilde{\sqcap}}} (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$.

Proof. Follows from Proposition 4 (i.) and Proposition 2 (i.). □

Let $(g, \hat{g}, \Sigma)$ be a BHS open subset of $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ and $\omega \in \Omega$. If Ω be a BHS regular space, then $\omega \in (g, \hat{g}, \Sigma)$ if and only if $\omega \in g(s)$ for some $s \in \Sigma$.

Proof. Suppose that for some $s \in \Sigma$, $\omega \in g(s)$ and $\omega \notin (g, \hat{g}, \Sigma)$. Then, $(g_\omega, \hat{g}_\omega, \Sigma) \mathrel{\widetilde{\widetilde{\sqcap}}} (g, \hat{g}, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$ by Proposition 4 (ii.). But this contradicts our assumption and so $\omega \in (g, \hat{g}, \Sigma)$.
The converse is obvious. □

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS. If $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS regular space, then these are equivalent:

i. $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_1$-space.

ii. for $\omega_1 \neq \omega_2 \in \Omega$, there are BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ with $(g_{\omega_1}, \hat{g}_{\omega_1}, \Sigma) \mathrel{\widetilde{\widetilde{\sqsubseteq}}} (g_1, \hat{g}_1, \Sigma)$ and $(g_{\omega_2}, \hat{g}_{\omega_2}, \Sigma) \mathrel{\widetilde{\widetilde{\sqcap}}} (g_1, \hat{g}_1, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$, and $(g_{\omega_2}, \hat{g}_{\omega_2}, \Sigma) \mathrel{\widetilde{\widetilde{\sqsubseteq}}} (g_2, \hat{g}_2, \Sigma)$ and $(g_{\omega_1}, \hat{g}_{\omega_1}, \Sigma) \mathrel{\widetilde{\widetilde{\sqcap}}} (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$.

Proof. It should be clear that $\omega \in (g, \hat{g}, \Sigma)$ if and only if $(g_\omega, \hat{g}_\omega, \Sigma) \mathrel{\widetilde{\widetilde{\sqsubseteq}}} (g, \hat{g}, \Sigma)$, and by Proposition 4 (ii.), $\omega \notin (g, \hat{g}, \Sigma)$ if and only if $(g_\omega, \hat{g}_\omega, \Sigma) \mathrel{\widetilde{\widetilde{\sqcap}}} (g, \hat{g}, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$. As a result, the above assertions are equivalent. □

A BHSTS $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is said to be BHS $\widetilde{T}_3$-space if it is BHS regular and BHS $\widetilde{T}_1$-space.

If $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_3$-space, then $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is an HS T_3-space.

Proof. Since $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_3$-space, then it is BHS $\widetilde{T}_1$-space. By Proposition 3, $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is HS T_1-space. Now, let $(f, \hat{f}, \Sigma)$ be a BHS closed set with $\omega \notin (f, \hat{f}, \Sigma)$. This implies that $\omega \notin f(s)$ for some $s \in \Sigma$ and hence $\omega \notin (f, \Sigma)$ as an HS closed set. As $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is BHS regular, then there exist BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $\omega \in (g_1, \hat{g}_1, \Sigma)$ and $(f, \hat{f}, \Sigma) \mathrel{\widetilde{\widetilde{\sqsubseteq}}} (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_1, \Sigma) \mathrel{\widetilde{\widetilde{\sqcap}}} (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$. Then we have $\omega \in g_1(s)$ and $f(s) \subseteq g_2(s)$ with $g_1(s) \cap g_2(s) = \phi$ for all $s \in \Sigma$. It follows that for an HS closed set (f, Σ) with $\omega \notin (f, \Sigma)$, we have $\omega \in (g_1, \Sigma)$ and $(f, \Sigma) \mathrel{\widetilde{\sqsubseteq}} (g_2, \Sigma)$ with $(g_1, \Sigma) \mathrel{\widetilde{\sqcap}} (g_2, \Sigma) = (\widetilde{\Phi}, \Sigma)$. Thus, $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is HS regular and hence $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is HS $\widetilde{T}_3$-space. □

Let $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ be an HSTS and let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS constructed from $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ as in Proposition 2. If $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ be an HS T_3-space, then $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_3$-space.

Proof. Since $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ be an HS T_3-space, then it is HS T_1-space. By Proposition 3, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is BHS $\widetilde{T}_1$-space. Now, let (f, Σ) be an HS closed set with $\omega \notin (f, \Sigma)$. This implies that $\omega \notin f(s)$ for some $s \in \Sigma$ and hence $\omega \notin (f, \hat{f}, \Sigma)$ as a BHS closed set. As $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is HS regular, then there exist HS open sets (g_1, Σ) and (g_2, Σ) such that $\omega \in (g_1, \Sigma)$ and $(f, \Sigma) \mathrel{\widetilde{\sqsubseteq}} (g_2, \Sigma)$ with $(g_1, \Sigma) \mathrel{\widetilde{\sqcap}} (g_2, \Sigma) = (\widetilde{\Phi}, \Sigma)$. Then $\omega \in g_1(s)$ and $f(s) \subseteq g_2(s)$ with $g_1(s) \cap g_2(s) = \phi$ for all $s \in \Sigma$. Also, we have $\omega \notin \hat{g}_1(\neg s)$ and $g_2(\neg s) \subseteq f(\neg s)$ with $\hat{g}_1(\neg s) \cup \hat{g}_2(\neg s) = \Omega$ for all $s \in \Sigma$. It follows that for a BHS closed set $(f, \hat{f}, \Sigma)$ with $\omega \notin (f, \hat{f}, \Sigma)$, we have $\omega \in (g_1, \hat{g}_1, \Sigma)$ and $(f, \hat{f}, \Sigma) \mathrel{\widetilde{\widetilde{\sqsubseteq}}} (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_1, \Sigma) \mathrel{\widetilde{\widetilde{\sqcap}}} (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \widetilde{\Omega}, \Sigma)$. Thus, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is BHS regular and hence $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is BHS $\widetilde{T}_3$-space. □

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Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS and $\Upsilon \subseteq \Omega$. If $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_3$-space, then $(\Upsilon, \mathcal{T}_{B\mathcal{H}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_3$-space.

Proof. Since $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_3$-space, then it is BHS $\widetilde{T}_1$-space. By Proposition 3, $(\Upsilon, \mathcal{T}_{B\mathcal{H}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_1$-space. Let $\omega \in \Upsilon$ and let $(f, \hat{f}, \Sigma)$ be a BHS closed set in Υ with $\omega \notin (f, \hat{f}, \Sigma)$. Then, $\omega \notin f(s)$ for some $s \in \Sigma$. Since $(f, \hat{f}, \Sigma)$ be a BHS closed set in Υ, then there exists a BHS closed set $(h, \hat{h}, \Sigma)$ in Ω such that $f(s) = h(s) \cap \Upsilon$ and $\hat{f}(\neg s) = \hat{h}(\neg s) \cap \Upsilon$. Since $\omega \notin f(s)$ for some $s \in \Sigma$, then $\omega \notin h(s) \cap \Upsilon = f(s)$ and hence $\omega \notin (h, \hat{h}, \Sigma)$. As $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is BHS regular space, then there exist BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $\omega \in (g_1, \hat{g}_1, \Sigma)$ and $(h, \hat{h}, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_1, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$. Now, if we take $(g_{1_\Upsilon}, \hat{g}_{1_\Upsilon}, \Sigma)$ and $(g_{2_\Upsilon}, \hat{g}_{2_\Upsilon}, \Sigma)$ as two BHS open sets in Υ, then $g_{1_\Upsilon}(s) = g_1(s) \cap \Upsilon$, $\hat{g}_{1_\Upsilon}(\neg s) = \hat{g}_1(\neg s) \cap \Upsilon$, and $g_{2_\Upsilon}(s) = g_2(s) \cap \Upsilon$, $\hat{g}_{2_\Upsilon}(\neg s) = \hat{g}_2(\neg s) \cap \Upsilon$. This implies that $\omega \in (g_{1_\Upsilon}, \hat{g}_{1_\Upsilon}, \Sigma)$ and $(f, \hat{f}, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, (g_{2_\Upsilon}, \hat{g}_{2_\Upsilon}, \Sigma)$ with $(g_{1_\Upsilon}, \hat{g}_{1_\Upsilon}, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g_{2_\Upsilon}, \hat{g}_{2_\Upsilon}, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$. Thus, $(\Upsilon, \mathcal{T}_{B\mathcal{H}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BHS regular space and hence $(\Upsilon, \mathcal{T}_{B\mathcal{H}_\Upsilon}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_3$-space. □

A BHSTS $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is said to be BHS normal space if for every BHS closed sets $(f_1, \hat{f}_1, \Sigma)$ and $(f_2, \hat{f}_2, \Sigma)$ with $(f_1, \hat{f}_1, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (f_2, \hat{f}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$, there exist BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $(f_1, \hat{f}_1, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, (g_1, \hat{g}_1, \Sigma)$ and $(f_2, \hat{f}_2, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_1, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$.

Let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS. If $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS normal space and if $(g_\omega, \hat{g}_\omega, \Sigma)$ is a BHS closed set for each $\omega \in \Omega$, then $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_3$-space.

Proof. Since for each $\omega \in \Omega$, $(g_\omega, \hat{g}_\omega, \Sigma)$ is a BHS closed set, then $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_1$-space by Proposition 3. Also by Proposition 4 and Proposition 4 it is BHS regular space. Hence, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_3$-space. □

A BHSTS $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is said to be BHS $\widetilde{T}_4$-space if it is BHS normal and BHS $\widetilde{T}_1$-space.

If $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_4$-space, then $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is an HS T_4-space.

Proof. Since $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS $\widetilde{T}_4$-space, then it is BHS $\widetilde{T}_1$-space and hence $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is HS T_1-space by Proposition 3. Also, since $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHS normal space, then for every two BHS closed sets $(f_1, \hat{f}_1, \Sigma)$ and $(f_2, \hat{f}_2, \Sigma)$ with $(f_1, \hat{f}_1, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (f_2, \hat{f}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$ there exist BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $(f_1, \hat{f}_1, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, (g_1, \hat{g}_1, \Sigma)$ and $(f_2, \hat{f}_2, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_2, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g_2, g_2, \Sigma)$ $= (\widetilde{\Phi}, \hat{g}, \Sigma)$. This implies that, for all $s \in \Sigma$, $f_1(s) \cap f_2(s) = \phi$ and $f_1(s) \subseteq g_1(s)$, $f_2(s) \subseteq g_2(s)$ with $g_1(s)$ $\cap\, g_2(s) = \phi$. Then for two HS closed sets (f_1, Σ) and (f_2, Σ) with $(f_1, \Sigma) \,\widetilde{\sqcap}\, (f_2, \Sigma) = (\widetilde{\Phi}, \Sigma)$ there exist two HS open sets (g_1, Σ) and (g_2, Σ) such that $(f_1, \Sigma) \,\widetilde{\sqsubseteq}\, (g_1, \Sigma)$ and $(f_2, \Sigma) \,\widetilde{\sqsubseteq}\, (g_2, \Sigma)$ with $(g_1, \Sigma) \,\widetilde{\sqcap}\, (g_2, \Sigma) = (\widetilde{\Phi}, \Sigma)$. Thus, $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is HS normal and hence $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is HS T_4-space. □

Let $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ be an HSTS and let $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ be a BHSTS constructed from $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ as in Proposition 2. If $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ be an HS T_4-space, then $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS $\widetilde{T}_4$-space.

Proof. Since $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ be an HS T_4-space, then it is HS T_1-space and hence $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is a BHS T_1-space by Proposition 3. Also, since $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ be an HS normal space, then for every two HS closed sets (f_1, Σ) and (f_2, Σ) with $(f_1, \Sigma) \,\widetilde{\sqcap}\, (f_2, \Sigma) = (\widetilde{\Phi}, \Sigma)$ there exist two HS open sets (g_1, Σ) and (g_2, Σ) such that $(f_1, \Sigma) \,\widetilde{\sqsubseteq}\, (g_1, \Sigma)$ and $(f_2, \Sigma) \,\widetilde{\sqsubseteq}\, (g_2, \Sigma)$ with $(g_1, \Sigma) \,\widetilde{\sqcap}\, (g_2, \Sigma) = (\widetilde{\Phi}, \Sigma)$. This implies that $f_1(s) \cap f_2(s) = \phi$ and $f_1(s) \subseteq g_1(s)$, $f_2(s) \subseteq g_2(s)$ with $g_1(s) \cap g_2(s) = \phi$. Then we have $\hat{f}_1(\neg s) \cup \hat{f}_2(\neg s) = \Omega$ and $\hat{g}_1(\neg s) \subseteq \hat{f}_1(\neg s)$, $\hat{g}_2(\neg s) \subseteq \hat{f}_2(\neg s)$ with $\hat{g}_1(\neg s) \cup \hat{g}_2(\neg s) = \Omega$. It follows that for two BHS closed sets $(f_1, \hat{f}_1, \Sigma)$ and $(f_2, \hat{f}_2, \Sigma)$ with $(f_1, \hat{f}_1, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (f_2, \hat{f}_2, \Sigma) = (\widetilde{\Phi}, \hat{g}, \Sigma)$ there exist two BHS open sets $(g_1, \hat{g}_1, \Sigma)$ and $(g_2, \hat{g}_2, \Sigma)$ such that $(f_1, \hat{f}_1, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, (g_1, \hat{g}_1, \Sigma)$ and $(f_2, \hat{f}_2, \Sigma) \,\widetilde{\widetilde{\sqsubseteq}}\, (g_2, \hat{g}_2, \Sigma)$ with $(g_1, \hat{g}_1, \Sigma) \,\widetilde{\widetilde{\sqcap}}\, (g_2, \hat{g}_2, \Sigma) = (\widetilde{\Phi}, \widetilde{\Omega}, \Sigma)$. Thus, $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is BHS normal and hence $(\Omega, \mathcal{T}_{\mathcal{H}}, \Sigma)$ is BHS $\widetilde{T}_4$-space. □

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A BHS $\widetilde{T}_4$-space need not be BHS $\widetilde{T}_3$-space.

Let $\Omega = \{\omega_1, \omega_2\}$, $\sigma_1 = \{\ell_1, \ell_2, \ell_3\}$, $\sigma_2 = \{\ell_4\}$, and $\sigma_3 = \{\ell_5\}$. Let $\mathcal{T}_{B\mathcal{H}} = \{(\widetilde{\Phi}, \widetilde{\Omega}, \Sigma), (\widetilde{\Omega}, \widetilde{\Phi}, \Sigma), (g_1, \hat{g}_1, \Sigma), (g_2, \hat{g}_2, \Sigma), (g_3, \hat{g}_3, \Sigma), (g_4, \hat{g}_4, \Sigma), (g_5, \hat{g}_5, \Sigma), (g_6, \hat{g}_6, \Sigma), (g_7, \hat{g}_7, \Sigma)\}$ be a BHST defined on Ω, where

$(g_1, \hat{g}_1, \Sigma) = \{((\ell_1, \ell_4, \ell_5), \{\omega_1\}, \{\omega_2\}), ((\ell_2, \ell_4, \ell_5), \{\omega_1\}, \{\omega_2\}), ((\ell_3, \ell_4, \ell_5), \{\omega_1\}, \{\omega_2\})\}$.

$(g_2, \hat{g}_2, \Sigma) = \{((\ell_1, \ell_4, \ell_5), \{\omega_2\}, \{\omega_1\}), ((\ell_2, \ell_4, \ell_5), \{\omega_2\}, \{\omega_1\}), ((\ell_3, \ell_4, \ell_5), \{\omega_2\}, \{\omega_1\})\}$.

$(g_3, \hat{g}_3, \Sigma) = \{((\ell_1, \ell_4, \ell_5), \phi, \Omega), ((\ell_2, \ell_4, \ell_5), \{\omega_1\}, \{\omega_2\}), ((\ell_3, \ell_4, \ell_5), \{\omega_1\}, \{\omega_2\})\}$.

$(g_4, \hat{g}_4, \Sigma) = \{((\ell_1, \ell_4, \ell_5), \phi, \Omega), ((\ell_2, \ell_4, \ell_5), \{\omega_2\}, \{\omega_1\}), ((\ell_3, \ell_4, \ell_5), \{\omega_2\}, \{\omega_1\})\}$.

$(g_5, \hat{g}_5, \Sigma) = \{((\ell_1, \ell_4, \ell_5), \{\omega_1\}, \{\omega_2\}), ((\ell_2, \ell_4, \ell_5), \Omega, \phi), ((\ell_3, \ell_4, \ell_5), \Omega, \phi)\}$.

$(g_6, \hat{g}_6, \Sigma) = \{((\ell_1, \ell_4, \ell_5), \{\omega_2\}, \{\omega_1\}), ((\ell_2, \ell_4, \ell_5), \Omega, \phi), ((\ell_3, \ell_4, \ell_5), \Omega, \phi)\}$.

$(g_7, \hat{g}_7, \Sigma) = \{((\ell_1, \ell_4, \ell_5), \phi, \Omega), ((\ell_2, \ell_4, \ell_5), \Omega, \phi), ((\ell_3, \ell_4, \ell_5), \Omega, \phi)\}$.

Then, it is easy to see that $(\Omega, \mathcal{T}_{B\mathcal{H}}, \Sigma, \neg\Sigma)$ is BHS $\widetilde{T}_4$-space but not BHS $\widetilde{T}_3$-space.

5 Conclusions

During this work we extended the study of BHSTS by defining BHS separation axioms, called BHS $\widetilde{T}_i$-space for $i = 0, 1, 2, 3, 4$. The relations between them and with HS $\widetilde{T}_i$-space are presented and discussed. In addition, we studied in detail the concepts of BHS regular and BHS normal spaces. Moreover, we investigated that the property of BHS $\widetilde{T}_i$-space ($i = 0, 1, 2, 3$) is BHS hereditary. Ultimately, we hope that our results and conclusions can be used to solve existing problems in a variety of disciplines that contain uncertainty.

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