Heptapartitioned neutrosophic soft set

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Abstract

The aim of this paper is to introduce the concept of heptapartitioned neutrosophic soft set and to study some of its basic properties. This kind of set can be considered as a significant extension of the idea of neutrosophic set. In general, our research is based on the earlier study presented by Radha and Stanis Arul Mary in 2021. However, we combine their initial definition with the inspiring notion of soft set (which has been established by Molodtsov in 1999). We examine algebraic operations on our sets. Then we show that the particular choose of exactly seven neutrosophic logical values has an interesting interpretation in terms of ancient Jaina logic (which has been developed centuries ago in India to deal with imprecise knowledge and the matter of paraconsistency). Moreover, we exhibit some other potential practical applications of our theoretical concept. For example, it can be applied in widely understood social sciences. **Keywords:**heptapartitioned neutrosophic soft; soft set; Jaina logic.

1 Introduction

Classical logic relies on the assumption that there are only two logical values, namely *true* and *false*. Undoubtedly, this approach is very natural and useful. Moreover, almost the whole building of mathematics has been founded on classical *metalogic* (with the important but still relatively small exception of *constructivist* mathematics). However, it turned our quite early that there are many areas where it is both convenient and legitimate to extend the set of logical constants. Such a need occurs in philosophy, computer science, economics and even in physics. Hence, the whole task is about capturing and modelling the idea of uncertainty (in a sense different than in probability theory, even if those approaches can be successfully mixed together).

This led Łukasiewicz, Post, Belnap and others to the development of three-valued, four-valued (and, in general, many-valued) logics. An important step has been made by Zadeh who presented the idea of fuzzy sets in his paper published in 1965 (see [26]). Then we had some other concepts. Among them there are intuitionistic fuzzy sets of Atanassov (see[1]), rough sets of Pawlak (see[13]), Wang's grey sets (presented in[26]), shadowed sets of Pedrycz (see[14]) or soft sets introduced by Molodtsov in.[12]

An important contribution to this field has been made by Smarandache who invented neutrosophic sets (together with neutrosophic logic and the whole idea of neutrosophy as a branch of philosophy and a useful research tool). The reader can check[22] and[23] to get the general overview of his theory. It is necessary to mention that the initial concept of Smarandache has grown into the vast field of study with many applications

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and modifications. Clearly, our paper can be placed in this domain. We would like to extend the concept of heptapartitioned neutrosophic set which has been already presented by Radha and Stanis Arul Mary in.[16] We combine their initial definition with the idea of soft sets which have been already mentioned in this introduction. In this way we obtain parametrized families of the sets in question. We check and show their algebraic properties (together with some numerical examples). Moreover, we point out the possible relationship between heptapartitioned neutrosophic sets (be them soft or not) and traditional seven-valued jaina logic. This last observation may lead us (and especially the reader who is encouraged to deepen this research) to some non-trivial philosophical conclusions.

2 Preliminaries

In this chapter we show some basic notions which stand at the root of our results. We start from the general idea of fuzzy set. Then we show its basic extension, namely intuitionistic fuzzy set of Atanassov, just to compare it with the concept of neutrosophic set (in the form presented by Smarandache). Then we define more complex extensions of neutrosophic set. Finally, we obtain their seven-valued version. As for the algebraic operations, we shall omit them in this subsection. They will be presented later, already in the context of heptapartitioned neutrosophic sets (and still later with respect to their soft version).

2.1 From fuzzy sets to neutrosophic sets

Let us start from the very basic and necessary definition.

Definition 2.1. [27]

Let $\mathfrak{U}$ be a non-empty universe. A *fuzzy set* $\mathcal{A}$ on $\mathfrak{U}$ can be defined as follows:

$\mathcal{A} = \{\langle x, T_{\mathcal{A}}(x)\rangle : x \in \mathfrak{U}\}$

We assume that $T_{\mathcal{A}}(x) \in [0, 1]$. This is so-called *membership function*.

As for the *non-membership function*, it is identified with $1-T_{\mathcal{A}}(x)$. As we can see, fuzzy logic is non-classical in the sense that it allows for partial truth. However, it is somewhat classical in its approach to the relationship between *belonging* and *non-belonging*. Clearly, the grade of non-membership depends strictly on the grade of membership (being its complement to 1).

Let us make one step further.

Definition 2.2. [1]

Let $\mathfrak{U}$ be a non-empty universe. An *intuitionistic fuzzy set* $\mathcal{A}$ on $\mathfrak{U}$ can be defined as follows:

$\mathcal{A} = \{\langle x, T_{\mathcal{A}}(x), F_{\mathcal{A}}(x)\rangle : x \in \mathfrak{U}\}$

We assume that $T_{\mathcal{A}}(x)$, $F_{\mathcal{A}}(x) : \mathfrak{U} \to [0, 1]$ and $T_{\mathcal{A}}(x) + F_{\mathcal{A}}(x) \leq 1$.

Now membership and non-membership functions seem to be mutually independent. However, this feature is limited by the fact that their values sum up to 1. For example, if $T_{\mathcal{A}}(x) = 1$, then there is no place for non-zero degree of non-membership. In general, if $T_{\mathcal{A}} = \alpha$, then $F_{\mathcal{A}} \leq 1 - \alpha$.

Now we can deal with another concept.

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Definition 2.3. [22]

Let $\mathfrak{U}$ be a non-empty universe. A *neutrosophic set* $\mathcal{A}$ on $\mathfrak{U}$ can be defined as follows:

$\mathcal{A} = \{\langle x, T_{\mathcal{A}}(x), I_{\mathcal{A}}(x), F_{\mathcal{A}}(x)\rangle : x \in \mathfrak{U}\}$

We assume that $T_{\mathcal{A}}(x), I_{\mathcal{A}}(x), F_{\mathcal{A}}(x) : \mathfrak{U} \to [0,1]$ and $0 \leq T_{\mathcal{A}}(x)+I_{\mathcal{A}}(x)+F_{\mathcal{A}}(x) \leq 3$. We say that $T_{\mathcal{A}}(x)$ is the degree of *membership*, $I_{\mathcal{A}}(x)$ is the degree of *indeterminacy* and $F_{\mathcal{A}}(x)$ is the degree of *non-membership*.

At first glance, neutrosophic sets appear to be a simple, three-valued extension of intuitionistic fuzzy set. However, things are more complex. Note that those three values do not necessarily sum up to 1. In fact, they can sum up to any number greater than or equal to 0 and not bigger than 3. This feature is not accidental. Smarandache recognized and described (in[22]) three possible situations:

1. When the sum is lesser than 1 which means that we are dealing with incomplete information;
2. When the sum is 1 which means that our information is complete[1].
3. When the sum is strictly greater than 1 which refers to the idea of contradiction and paraconsistency.

As for the third option, we can also identify it with the situation of confusion and embarrassment. Imagine that we have listened to someone's speech on, let us say, certain political or philosophical matters. Some of the presented arguments seem to be convincing. However, we know that it is possible to disprove them (or, at least, to attack them). Besides, we can even believe that the orator said things that are not important for us or that are not verifiable. Hence, we are deeply in the state of disorientation. If there is some "universe $\mathfrak{U}$ of orators" and the one in question is denoted by x, then we may form neutrosophic set $\mathcal{A}$ to describe credibility of each orator. Now $T_{\mathcal{A}}(x) = I_{\mathcal{A}}(x) = F_{\mathcal{A}}(x) = 1$. This does not necessarily mean that the objective truth about the matters disputed by orator x does not exist or that it is not cognizable. Moreover, it does not mean that we are simultaneously voting for the orator, voting against him and not voting at all (which would not be possible). It just means that we would like to describe our own uncertainty and the fact that we cannot (at least not now) make the final decision. However, the reader is free to look for other interpretations.

One can interpret the concept of degree of truth (resp. ignorance and falsity) in terms of betting or gambling. It is like estimation of probability that an object x has (or does not have) certain property φ. Now, our decision maker says that he is ready to bet, say, 20% of his "capital" (not necessarily money) on the possibility that x satisfies φ, 30% on the possibility that φ does not apply at all to this object and 50% on the third option: that x does not have property φ. Thus, we obtain neutrosophic set $\langle x, 0.20, 0.30, 0.50\rangle$ Clearly, this "gamble interpretation" fits best to the case of complete information. In case of incomplete information we may think that our decision maker invested only some part of his "capital" and not the whole amount. As for the paraconsistency, here the situation is less clear. However, we may assume that decision maker *borrows* some capital. Then he may invest this "extra money" to get the sum (of logical values) greater than 1.

2.2 Extensions and generalizations of neutrosophic sets

We see that the whole theory of neutrosophic sets has great potential because of its generality. Of course, if someone does not need the very concept of paraconsistency in his research, then he can stay on the level of complete or incomplete information.

It is natural that even this capacious concept has been generalized. One way of generalization is to replace crisp logical values with intervals (that is, to *fuzzify* membership, non-membership and indeterminacy). Another method (on which we shall concentrate) is to extend the set of values[2]. Some examples are below:

[1] Note that this is not at odds with the fact that we determined degrees of membership, non-membership and indeterminacy. Completeness means (in this context) that our knowledge about fuzzy values is in some sense *complete*.

[2] Clearly, these two approaches can be combined. In fact, they are mixed by various authors. Consider the idea of interval pentapartitioned neutrosophic sets analyzed by Pramanik in[15]

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Definition 2.4. [17,19] Let $\mathfrak{U}$ be a non-empty universe. A *quadripartitioned neutrosophic set* $\mathcal{A}$ on $\mathfrak{U}$ is an object of the form:

$$\mathcal{A} = \{\langle x, T_{\mathcal{A}}(x), C_{\mathcal{A}}(x), U_{\mathcal{A}}(x), F_{\mathcal{A}}(x)\rangle : x \in \mathfrak{U}\}$$

We assume that $T_{\mathcal{A}}(x)$, $C_{\mathcal{A}}(x)$, $U_{\mathcal{A}}(x)$, $F_{\mathcal{A}}(x) : \mathfrak{U} \to [0,1]$. Moreover, $T_{\mathcal{A}}(x)+C_{\mathcal{A}}(x)+U_{\mathcal{A}}(x)+F_{\mathcal{A}}(x) \leq 4$. Here $T_{\mathcal{A}}(x)$ is truth membership, $C_{\mathcal{A}}(x)$ is contradiction, $U_{\mathcal{A}}(x)$ is ignorance and $F_{\mathcal{A}}(x)$ is false membership. As we can see, indeterminacy has been split into two parts.

Definition 2.5. [11] Let $\mathfrak{U}$ be a non-empty universe. A *pentapartitioned neutrosophic set* $\mathcal{A}$ on $\mathfrak{U}$ is an object of the form:

$$\mathcal{A} = \{\langle x, T_{\mathcal{A}}(x), C_{\mathcal{A}}(x), G_{\mathcal{A}}(x), U_{\mathcal{A}}(x), F_{\mathcal{A}}(x)\rangle : x \in \mathfrak{U}\}$$

We assume that $T_{\mathcal{A}}(x)$, $C_{\mathcal{A}}(x)$, $G_{\mathcal{A}}(x)$, $U_{\mathcal{A}}(x)$, $F_{\mathcal{A}}(x) : \mathfrak{U} \to [0,1]$. Moreover, $T_{\mathcal{A}}(x) + C_{\mathcal{A}}(x) + G_{\mathcal{A}}(x) + U_{\mathcal{A}}(x) + F_{\mathcal{A}}(x) \leq 5$. Here $T_{\mathcal{A}}(x)$ is truth membership, $C_{\mathcal{A}}(x)$ is uncertainty, $G_{\mathcal{A}}(x)$ is contradiction, $U_{\mathcal{A}}(x)$ is unknown membership and $F_{\mathcal{A}}(x)$ is false membership. As we can see, indeterminacy has been split into three parts.

In 2013 the concept of neutrosophic set was extended by Smarandache to the refined (that is, n-valued) neutrosophic set (see[25]). On this basis he defined refined neutrosophic logic and refined neutrosophic probability too. It means that the truth value T is refined (split) into several types of sub-truths such as T_1, T_2, ..., T_p. Similarly indeterminacy I is split into sub-indeterminacies I_1, I_2, ..., I_r and the falsehood F is split into sub-falsehood F_1, F_2, ..., F_s. We should assume that $p, r, s \in \mathbb{N}^+$ and at least one of p, r, s is ≥ 2.

In 2018 Smarandache generalized the concept of soft set to the *hypersoft set* by transforming the classical uni-argument function F into a multi-argument function (see[24]).

2.3 Heptapartitioned neutrosophic sets

In this section we present some basic facts about neutrosophic sets with exactly seven logical values.

Definition 2.6. [16] Let $\mathfrak{U}$ be a non-empty universe. A *heptapartitioned neutrosophic set* $\mathcal{A}$ on $\mathfrak{U}$ is an object of the form:

$$\mathcal{A} = \{\langle x, T_{\mathcal{A}}(x), M_{\mathcal{A}}(x), C_{\mathcal{A}}(x), U_{\mathcal{A}}(x), I_{\mathcal{A}}(x), K_{\mathcal{A}}(x), F_{\mathcal{A}}(x)\rangle : x \in \mathfrak{U}\}$$

We assume that $T_{\mathcal{A}}(x)$, $M_{\mathcal{A}}(x)$, $C_{\mathcal{A}}(x)$, $U_{\mathcal{A}}(x)$, $I_{\mathcal{A}}(x), K_{\mathcal{A}}(x), F_{\mathcal{A}}(x) : \mathfrak{U} \to [0,1]$. Moreover, $T_{\mathcal{A}}(x) + M_{\mathcal{A}}(x) + C_{\mathcal{A}}(x) + U_{\mathcal{A}}(x) + I_{\mathcal{A}}(x) + K_{\mathcal{A}}(x) + F_{\mathcal{A}}(x) \leq 7$. Here $T_{\mathcal{A}}(x)$ is truth membership, $M_{\mathcal{A}}(x)$ is relative truth, $C_{\mathcal{A}}(x)$ is contradiction, $U_{\mathcal{A}}(x)$ is unknown membership, $I_{\mathcal{A}}(x)$ is ignorance, $K_{\mathcal{A}}(x)$ is relative falsity and $F_{\mathcal{A}}(x)$ is absolute falsity.

The last definition will be discussed briefly in the next section. We shall try to answer the question of whether it is possible and reasonable to interpret those seven values in a slightly different manner. However, before this let us introduce some algebraic operations.

Definition 2.7. [16] Let $\mathcal{A}$ and $\mathcal{B}$ be two HPNSs over the universe $\mathfrak{U}$. We say that $\mathcal{A}$ is *contained* in $\mathcal{B}$ (that is, $\mathcal{A} \subseteq \mathcal{B}$) if and only if for any $x \in \mathfrak{U}$ the following holds: $T_{\mathcal{A}}(x) \leq T_{\mathcal{B}}(x)$, $M_{\mathcal{A}}(x) \leq M_{\mathcal{B}}(x)$, $C_{\mathcal{A}}(x) \leq C_{\mathcal{B}}(x)$, $U_{\mathcal{A}}(x) \geq U_{\mathcal{B}}(x)$, $I_{\mathcal{A}}(x) \geq I_{\mathcal{B}}(x)$, $K_{\mathcal{A}}(x) \geq K_{\mathcal{B}}(x)$ and $F_{\mathcal{A}}(x) \geq F_{\mathcal{B}}(x)$.

Definition 2.8. Let $\mathcal{A}$ be a HPNS over the universe $\mathfrak{U}$. The *complement* of $\mathcal{A}$ is denoted by $\mathcal{A}^c$ and its membership functions are defined in the following way (for every $x \in \mathfrak{U}$):

$T_{\mathcal{A}^c}(x) = F_{\mathcal{A}}(x)$, $M_{\mathcal{A}^c}(x) = K_{\mathcal{A}}(x)$, $C_{\mathcal{A}^c}(x) = I_{\mathcal{A}}(x)$, $U_{\mathcal{A}^c}(x) = 1 - U_{\mathcal{A}}(x)$, $I_{\mathcal{A}^c}(x) = C_{\mathcal{A}}(x)$, $K_{\mathcal{A}^c}(x) = M_{\mathcal{A}}(x)$ and $F_{\mathcal{A}^c}(x) = T_{\mathcal{A}}(x)$.

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Definition 2.9. [16] Let $\mathcal{A}$ and $\mathcal{B}$ be two HPNSs over the universe $\mathfrak{U}$. We define their *union* as a new HPNS $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ such that for any $x \in \mathfrak{U}$ the following holds: $T_{\mathcal{C}}(x) = \max(T_{\mathcal{A}}(x), T_{\mathcal{B}}(x))$, $M_{\mathcal{C}}(x) = \max(M_{\mathcal{A}}(x), M_{\mathcal{B}}(x)), C_{\mathcal{C}}(x) = \max(C_{\mathcal{A}}(x), C_{\mathcal{B}}(x)), U_{\mathcal{C}}(x) = \min(U_{\mathcal{A}}(x), U_{\mathcal{B}}(x)), I_{\mathcal{C}}(x) = \min(I_{\mathcal{A}}(x), I_{\mathcal{B}}(x))$, $K_{\mathcal{C}}(x) = \min(K_{\mathcal{A}}(x), K_{\mathcal{B}}(x))$ and $F_{\mathcal{C}}(x) = \min(F_{\mathcal{A}}(x), F_{\mathcal{B}}(x))$.

Definition 2.10. [16] Let $\mathcal{A}$ and $\mathcal{B}$ be two HPNSs over the universe $\mathfrak{U}$. We define their *intersection* as a new HPNS $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ such that for any $x \in \mathfrak{U}$ the following holds: $T_{\mathcal{C}}(x) = \min(T_{\mathcal{A}}(x), T_{\mathcal{B}}(x))$, $M_{\mathcal{C}}(x) = \min(M_{\mathcal{A}}(x), M_{\mathcal{B}}(x))$, $C_{\mathcal{C}}(x) = \min(C_{\mathcal{A}}(x), C_{\mathcal{B}}(x))$, $U_{\mathcal{C}}(x) = \max(U_{\mathcal{A}}(x), U_{\mathcal{B}}(x))$, $I_{\mathcal{C}}(x) = \max(I_{\mathcal{A}}(x), I_{\mathcal{B}}(x))$, $K_{\mathcal{C}}(x) = \max(K_{\mathcal{A}}(x), K_{\mathcal{B}}(x))$ and $F_{\mathcal{C}}(x) = \max(F_{\mathcal{A}}(x), F_{\mathcal{B}}(x))$.

There are two special types of HPNSs which should be defined in an explicit manner.

Definition 2.11. [16]

A HPNS $\mathcal{A}$ is called an *absolute* HPNS (and denoted as $1_{\mathfrak{U}}$) if and only if its membership functions are defined as such (for any $x \in \mathfrak{U}$):

$T_{\mathcal{A}}(x) = M_{\mathcal{A}}(x) = C_{\mathcal{A}}(x) = 1$ and $U_{\mathcal{A}}(x) = I_{\mathcal{A}}(x) = K_{\mathcal{A}}(x) = F_{\mathcal{A}}(x) = 0$.

Definition 2.12. [16]

A HPNS $\mathcal{A}$ is called an *empty* HPNS (and denoted as $0_{\mathfrak{U}}$) if and only if its membership functions are defined as such (for any $x \in \mathfrak{U}$):

$T_{\mathcal{A}}(x) = M_{\mathcal{A}}(x) = C_{\mathcal{A}}(x) = 0$ and $U_{\mathcal{A}}(x) = I_{\mathcal{A}}(x) = K_{\mathcal{A}}(x) = F_{\mathcal{A}}(x) = 1$.

Many standard algebraic (in particular, Boolean) formulas are true for HPNSs with $\cup$, $\cap$ and complement. These properties have been proven by Radha and S. Arul Mary in their paper. Let us list them down here.

Theorem 2.13. *(see[16]) Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be HPNSs over the universe $\mathfrak{U}$. Then the following properties are true:*

1. $\mathcal{A} \cup \mathcal{B} = \mathcal{B} \cup \mathcal{A}$, $\mathcal{A} \cap \mathcal{B} = \mathcal{B} \cap \mathcal{A}$ (commutativity).
2. $(\mathcal{A} \cup \mathcal{B}) \cup \mathcal{C} = \mathcal{A} \cup (\mathcal{B} \cup \mathcal{C})$, $(\mathcal{A} \cap \mathcal{B}) \cap \mathcal{C} = \mathcal{A} \cap (\mathcal{B} \cap \mathcal{C})$ (associativity).
3. $\mathcal{A} \cup (\mathcal{B} \cap \mathcal{C}) = (\mathcal{A} \cup \mathcal{B}) \cap (\mathcal{A} \cup \mathcal{C})$, $\mathcal{A} \cap (\mathcal{B} \cup \mathcal{C}) = (\mathcal{A} \cap \mathcal{B}) \cup (\mathcal{A} \cap \mathcal{C})$ (distributivity).
4. $\mathcal{A} \cup (\mathcal{A} \cap \mathcal{C}) = \mathcal{A}$, $\mathcal{A} \cap (\mathcal{A} \cup \mathcal{C})$ (absorption).
5. $(\mathcal{A}^c)^c = \mathcal{A}$ (involution).
6. $\mathcal{A} \cap \mathcal{A}^c = \emptyset$ (contradiction).
7. $(\mathcal{A} \cup \mathcal{B})^c = \mathcal{A}^c \cap \mathcal{B}^c$, $(\mathcal{A} \cap \mathcal{B})^c = \mathcal{A}^c \cup \mathcal{B}^c$ (de Morgan laws).

We may treat $\mathcal{A} \setminus \mathcal{B}$ as a shortcut for $\mathcal{A} \cap \mathcal{B}^c$. Now it is easy to prove that:

Theorem 2.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be HPNSs over the universe $\mathfrak{U}$. Assume that $\mathcal{A} \cap \mathcal{B} = \emptyset$. Then $\mathcal{A} \subseteq 1_{\mathfrak{U}} \setminus \mathcal{B}$ and $1_{\mathfrak{U}} \setminus \mathcal{B} = \mathcal{B}^c$.*

Proof. First, let us observe that $\mathcal{C} = 1_{\mathfrak{U}} \setminus \mathcal{B} = 1_{\mathfrak{U}} \cap \mathcal{B}^c$. Then (for any $x \in \mathfrak{U}$) we have $T_{\mathcal{C}}(x) = \min(1, F_{\mathcal{B}}(x)) = F_{\mathcal{B}}(x)$. Similar reasoning leads us to the conclusion that all the membership functions of $\mathcal{C}$ are just like membership functions of $\mathcal{B}^c$. For example, we have $U_{\mathcal{C}}(x) = \max(0, 1 - U_{\mathcal{B}}(x)) = 1 - U_{\mathcal{B}}(x)) = U_{\mathcal{B}^c}(x)$.

Now, for any $x \in \mathfrak{U}$, $T_{\mathcal{A}\cap\mathcal{B}}(x) = T_{\emptyset} = 0 \leq T_{\mathcal{C}}$. We may prove analogous inequalities for other logical values to obtain our expected conclusion.

□

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One can prove the following theorem too.

Theorem 2.15. *Let $\mathcal{A}_{i i\in I}$ be an indexed family of HPNSs. Then:*

1. $1_{\mathfrak{U}} \setminus \bigcup_{i\in I} \mathcal{A}_i = \bigcap_{i\in I}(1_{\mathfrak{U}} \setminus \mathcal{A}_i)$.
2. $1_{\mathfrak{U}} \setminus \bigcap_{i\in I} \mathcal{A}_i = \bigcup_{i\in I}(1_{\mathfrak{U}} \setminus \mathcal{A}_i)$.

In the next subsection we shall show several additional definitions which are connected with the idea of soft set.

2.4 Soft sets

We should start from the very basic definition of Molodtsov.

Definition 2.16. [12] Let $\mathfrak{U}$ be the initial universe and $\mathcal{E}$ be the set of parameters. Consider a non-empty set $\mathcal{A} \subseteq \mathcal{E}$. The collection $(\mathsf{J}, \mathcal{A})$ is termed to be a *soft set* on $\mathfrak{U}$ where J is a mapping given by $\mathsf{J} : \mathcal{A} \to P(\mathfrak{U})$[3].

Definition 2.17. [9] Let $\mathfrak{U}$ be the initial universe and $\mathcal{E}$ be the set of parameters. Consider a non-empty set $\mathcal{A} \subseteq \mathcal{E}$. Let $P_N(\mathfrak{U})$ denote the set of all neutrosophic sets on $\mathfrak{U}$. The collection $(\mathsf{J}, \mathcal{A})$ is termed to be *neutrosophic soft set* on $\mathfrak{U}$ where J is a mapping given by $\mathsf{J} : \mathcal{A} \to P_N(\mathfrak{U})$.

Note that the definition above says that a particular neutrosophic soft set should be identified rather with J than with $\mathcal{A}$. This is because it is possible to define various mappings using the same set of parameters.

Clearly, the very concept of neutrosophic soft set can be extended. This has been shown below.

Definition 2.18. [19] Let $\mathfrak{U}$ be the initial universe and $\mathcal{E}$ be the set of parameters. Consider a non-empty set $\mathcal{A} \subseteq \mathcal{E}$. Let $P_{QN}(\mathfrak{U})$ denote the set of all quadripartitioned neutrosophic sets on $\mathfrak{U}$. The collection $(\mathsf{J}, \mathcal{A})$ is termed to be *quadripartitioned neutrosophic soft set* on $\mathfrak{U}$, where J is a mapping given by $\mathsf{J} : \mathcal{A} \to P_{QN}(\mathfrak{U})$.

Definition 2.19. [20] Let $\mathfrak{U}$ be the initial universe set and $\mathcal{E}$ be the set of parameters. Consider a non-empty set $\mathcal{A} \subseteq \mathcal{E}$. Let $P_{PN}(\mathfrak{U})$ denote the set of all pentapartitioned neutrosophic sets on $\mathfrak{U}$. The collection $(\mathsf{J}, \mathcal{A})$ is termed to be *pentapartitioned neutrosophic soft set* on $\mathfrak{U}$, where J is a mapping given by $\mathsf{J} : \mathcal{A} \to P_{PN}(\mathfrak{U})$.

3 Heptapartitioned neutrosophic sets and seven-valued logic

One could ask, if there is any special reason to assume that there are exactly seven logical values in our heptapartitioned neutrosophic set (or, equivalently, seven membership functions). Now we shall try to give at least partial answer to this question.

First, it all depends on needs and applications. Any researcher (be it economist, sociologist, astronomer, psychologist or any other decision maker) should evaluate how many values are necessary in his research. The interpretation presented in the preceding chapter (namely, in terms of absolute and relative truth, contradiction, unknownness etc.) is not the only one possible.

In general, every researcher is free to set up his own interpretation. For example, those seven values can be treated as (respectively, starting from $T_{\mathcal{A}}(x)$ and finishing on $F_{\mathcal{A}}(x)$): strong (unconditional) agreement, weak (conditional) agreement, veto, confusion, indifference, weak disagreement and strong disagreement.

However, this researcher should ensure himself that his interpretation is not at odds with the definitions of complement, union, intersection and difference of HPNSs. For example, the absolute truth function of the

[3]Some authors always assume that $\mathcal{A} = \mathcal{E}$ (or rather, that J is a mapping with the domain $\mathcal{E}$). Note that it is possible that $\mathsf{J}(e) = \emptyset$ for at least some of the parameters.

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complement $\mathcal{A}^c$ is just like the absolute falsity function of the original HPNS $\mathcal{A}$. Analogously, one of the conditions required for the inclusion $\mathcal{A} \subseteq \mathcal{B}$ is that $U_{\mathcal{A}}(x) \geq U_{\mathcal{B}}(x)$. Of course there are other conditions too. Hence, the researcher should check if his interpretation makes sense in the light of these operations. Alternatively, he can define his own operations but this is beyond the scope of the present paper.

Second, we would like to sketch briefly certain rather unexpected relationship between the theory of heptapartitioned neutrosophic sets and seven-valued Jaina logic. Jainism is one of the ancient religions of India (together with Buddhism and Hinduism). Jaina philosophers invented so-called *Saptabhangivada*: the theory of seven predicates. An excellent overview of this system has been presented by Ganeri in.[4] In general, Jaina logic has been analyzed by many authors. For example, see[6] and.[7] In the latter paper there are some interesting applications of Jaina logic in computer science.

Basically, Jaina logic describes the situation in which a certain statement or object can be arguably (or conditionally, that is, from some point of view) considered as existing, non-existing or assertible. There is no room here for the detailed study of all the philosophical subtleties of this concept (but we encourage the reader to fully explore the literature). However, we would like to show some analogies between seven Jaina claims (or predicates) and the standard interpretation of membership functions in HNSs.

Typically, Jaina predicates are listed in the following order (see[4]):

1. Arguably, some object φ exists (or some sentence φ is true). This seems to correspond with $T_{\mathcal{A}}(x)$.
2. Arguably, it does not exist. This is like $F_{\mathcal{A}}(x)$.
3. Arguably, it exists and, arguably, it does not exist. This is like $C_{\mathcal{A}}(x)$.
4. Arguably, it is non-assertible. The closest analogy is $U_{\mathcal{A}}(x)$.
5. Arguably, it exists and, arguably, it is non-assertible. This is like $M_{\mathcal{A}}(x)$.
6. Arguably, it does not exist and, arguably, it is non-assertible. This is like $K_{\mathcal{A}}(x)$.
7. Arguably, it exists; arguably, it does not exist and, arguably, it is non-assertible. This is like $I_{\mathcal{A}}(x)$.

Let us repeat once again that we do not insist that those relationships between Jaina predicates and heptapartitioned membership functions are evident and undisputed. However, we think that they should be studied. Be as it may, two things should be noted.

First, Jaina logical values are not fuzzy. It seems that there are just seven possibilities which are distinct and do not appear simultaneously. Contrary to this, in our case we can measure the extent to which we can admit that, say φ exists or φ is non-assertible.

Second, some authors think that those seven Jaina values collapse into three basic ones, that is T (truth / existence), F (falsity / non-existence) and U (being non-assertible or non-describable). Other authors argue that those values are indeed distinct. Clearly, our present study cannot discuss all those matters. Nonetheless, let us finish this section with some example.

Example 3.1. Suppose that we have a non-empty universe which consists of three objects, that is $\mathfrak{U} = \{x_1, x_2, x_3\}$. Suppose that we have two observers, namely $\mathcal{A}$ and $\mathcal{B}$. Each of them has to evaluate the extent (that is, neutrosophic probability, to use the Smarandache's notion) to which a given object may be considered as existing, non-existing, existing and non-assertible etc. In other words, each of them has to formulate his own heptapartitioned neutrosophic set interpreted in terms of Jaina predicates. We said that those predicates refer to the potential existence of some standpoints (e.g. the standpoint from which it is possible to say that x_1 does not exist). Suppose that the sets in question are:

$\mathcal{A} = \{\langle x_1, 0.50, 0.40, 0.03, 0.60, 0.10, 0.17, 0.16\rangle, \langle x_2, 0.10, 0.05, 0.03, 0.04, 0.10, 0.12, 0.40\rangle,$
$\langle x_3, 0.10, 0.09, 0.03, 0.57, 0.05, 0.11, 0.05\rangle\}$

$\mathcal{B} = \{\langle x_1, 0.68, 0.10, 0.02, 0.07, 0.03, 0.04, 0.06\rangle, \langle x_2, 0.25, 0.05, 0.08, 0.04, 0.08, 0.02, 0.48\rangle,$
$\langle x_3, 0.10, 0.07, 0.03, 0.59, 0.05, 0.10, 0.06\rangle.$

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What is specific for observer $\mathcal{B}$, is that in each case he considers his information as complete. This is because in each case (that is, for x_1, x_2 and x_3) all the values sum up to 1. As for the observer $\mathcal{A}$, he allows for some kind of paraconsistency or confusion in his evaluation of x_1 (the values sum up to 1.96). Moreover, he considers his information about x_2 as incomplete (here the sum is 0.84). Only in the last case (the one of x_3) his information is complete (at least in his own eyes). Besides, it is very similar to the evaluation given by $\mathcal{A}$ (albeit not identical).

Note that here we do not know *how* the observers prepared their evaluations. This is because our study is purely mathematical. Clearly, any genuinely practical study should take this question into account. Basically, we may believe that our observers used their intuition. However, they could use more systematic methods. For example, they could conduct surveys (that is, polls) among certain groups they consider to be credible.

4 Heptapartitioned neutrosophic soft set

In this section we would like to introduce the notion of heptapartitioned neutrosophic soft set. It will allow us to analyze those situations where there are several decision makers and each of them evaluates each element of the initial universe in the light of several criteria and exactly seven logical values.

4.1 Basic notions

Let us start from the crucial definition.

Definition 4.1. Let $\mathfrak{U}$ be a non-empty universe and $\mathcal{E}$ be the set of parameters. Consider a non-empty set $\mathcal{A} \subseteq \mathcal{E}$. Let $P_{HN}(\mathfrak{U})$ denote the set of all heptapartitioned neutrosophic sets of $\mathfrak{U}$. The collection $(\mathsf{J}, \mathcal{A})$ is termed to be *heptapartitioned neutrosophic soft set* (that is, HPNSS) over $\mathfrak{U}$, where J is a mapping given by $\mathsf{J} : \mathcal{A} \to P_{HN}(\mathfrak{U})$.

In general, we should write $T_{(\mathsf{J},\mathcal{A})(e)}(x)$ to describe the truth value of x when x is evaluated from the perspective of criterion e. Here we use the notion of "criterion" because our parameters can be considered as some criteria or points of view through the lens of which we may perceive our objects (that is, the elements of universe). However, if we remember that our mapping J is connected with the domain $\mathcal{A}$, then we may simplify our notation and write shortly $T_{\mathsf{J}(e)}(x)$. Of course those remarks refer to other logical values too.

Definition 4.2. We say that a HPNSS $(\mathsf{J}, \mathcal{A})$ is *contained* in another HPNSS $(\mathsf{D}, \mathcal{B})$, i.e. $(\mathsf{J}, \mathcal{A}) \subseteq (\mathsf{D}, \mathcal{B})$ if and only if $\mathcal{A} \subseteq \mathcal{B}$ and for any $e \in \mathcal{A}$ (and for any $x \in \mathfrak{U}$) the following holds:

$T_{\mathsf{J}(e)}(x) \leq T_{\mathsf{D}(e)}(x)$, $M_{\mathsf{J}(e)}(x) \leq M_{\mathsf{D}(e)}(x)$, $C_{\mathsf{J}(e)}(x) \leq C_{\mathsf{D}(e)}(x)$, $U_{\mathsf{J}(e)}(x) \geq U_{\mathsf{D}(e)}(x)$, $I_{\mathsf{J}(e)}(x) \geq I_{\mathsf{D}(e)}(x)$, $K_{\mathsf{J}(e)}(x) \geq K_{\mathsf{D}(e)}(x)$, $F_{\mathsf{J}(e)}(x) \geq F_{\mathsf{D}(e)}(x)$.

Example 4.3. Let us consider the following example from social science research. We have universe $\mathfrak{U} = \{x_1, x_2, x_3\}$ which consists of three candidates (or some political leaders). There are also three political scientists or sociologists, call them $\mathsf{J}, \mathsf{D}, \mathsf{H}$. Their task is to analyze and evaluate the leaders in the light of the following criteria: *rightist*, *leftist*, *sincere in his declarations*, *having a real chance of winning*, *radical against the status quo*, *experienced as a politician* and *resistant to stress*. These are parameters which form the set $\mathcal{E} = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. Each of the scientists has determined which criteria he is able to evaluate (in terms of seven logical values). Hence, each of them formed his HPNSS[4]. Assume that they are as such:

1) Scientist J declares that he is able to evaluate (in heptaneutrosophic terms) the degree of leftism and rightism of each leader. He refuses to judge other criteria. Hence, his subset of parameters is $\mathcal{A} = \{e_1, e_2\}$. His heptapartitioned neutrosophic soft set may be written in the following tabular form:

[4] Again, we do not concentrate here on their methods of evaluation. They can vary from pure intuition to some kind of statistical and mathematical modelling.

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As we can see, the first scientist considers his knowledge as complete (in each case his seven values sum up to 1). One can check his evaluations of x_1: it seems that he considers x_1 as rather strongly rightist and not too much leftist. Moreover, it seems that for him being leftist is close to the concept of not being rightist. Note that $T_{\mathsf{J}(e_1)}(x_1) = 0.65$ and this value is very close to $F_{\mathsf{J}(e_2)}(x_1) = 0.63$.

2) Scientist D considers the notions of leftism and rightism as useless. However, he would like to evaluate sincerity, radicalism and chances of each candidate. Hence, his subset of parameters is $\mathcal{B} = \{e_3, e_4, e_5\}$. His HPNSS is presented below in a tabular form:

As we can see, his evaluations are "more neutrosophic" than the evaluations of the first scientist. We mean that some of his sets contain complete or paraconsistent information. For example, he considers sincerity of the third candidate as very enigmatic: note that logical values in this case sum up to 4.57. Moreover, almost each logical value got high estimation level, e.g. we have 0.78 for absolute truth (of the statement about sincerity of x_3), 0.70 for unknownness and 0.82 for absolute falsity.

3) Scientist H decided to evaluate leftism, sincerity and stress-resistance. Hence, his subset of parameters is $\mathcal{C} = \{e_2, e_3, e_7\}$. Note that this set has non-empty intersection both with $\mathcal{A}$ and $\mathcal{B}$. Now the HPNSS is:

Hence, we get three heptapartitioned neutrosophic soft sets. One can use them in further multi-criteria analysis.

4.2 Fundamental algebraic operations

In this subsection we shall see how it is possible to perform algebraic operations on the sets in question. Let us start from the complement of HPNSS. To certain extent this definition is arbitrary. However, it seems to fit to the basic interpretation of seven logical values (the one presented in chapter 2). Moreover, it is based on the original definition from[16] which has been established for non-soft HPNSs.

Definition 4.4. The complement of HPNSS $(\mathsf{J}, \mathcal{A})$ is denoted by $(\mathsf{J}, \mathcal{A})^c$ (or just by J^c) and its membership functions are defined in the following way:

$T_{\mathsf{J}^c(e)}(x) = F_{\mathsf{J}(e)}(x)$, $M_{\mathsf{J}^c(e)}(x) = K_{\mathsf{J}(e)}(x)$, $C_{\mathsf{J}^c(e)}(x) = I_{\mathsf{J}(e)}(x)$, $U_{\mathsf{J}^c(e)}(x) = 1 - U_{\mathsf{J}(e)}(x)$, $I_{\mathsf{J}^c(e)}(x) = C_{\mathsf{J}(e)}(x)$, $K_{\mathsf{J}^c(e)}(x) = M_{\mathsf{J}(e)}(x)$, $F_{\mathsf{J}^c(e)}(x) = T_{\mathsf{J}(e)}(x)$.

Clearly, the set of parameters is the same but membership functions are different. Of course, they fully rely on the original functions. One can easily grasp the idea. We can add for the next time that one can define complement in several ways, depending on applications and needs. The definition presented here is an exemplary one. However, there is some consistent logic behind it. For example, it is very natural to assume that the truth membership function of complement takes the same value as the falsity membership function of the original set.

Now we can define intersection and union.

Definition 4.5. Let $\mathfrak{U}$ be a non-empty set. Consider two HPNSSs, namely $(\mathsf{J}, \mathcal{A})$ and $(\mathsf{D}, \mathcal{B})$. Then the *intersection* of these two sets is defined as: $(\mathsf{J}, \mathcal{A}) \cap (\mathsf{D}, \mathcal{B}) = (\mathsf{K}, \mathcal{C})$, where $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$, K is a mapping given by $\mathsf{K} : C \to P_{HN}(\mathfrak{U})$ and membership functions (for any $e \in \mathcal{C}$ and any $x \in \mathfrak{U}$) are as follows:

$T_{\mathsf{K}(e)}(x) = \min(T_{\mathsf{J}(e)}(x), T_{\mathsf{D}(e)}(x))$, $M_{\mathsf{K}(e)}(x) = \min(M_{\mathsf{J}(e)}(x), M_{\mathsf{D}(e)}(x))$, $C_{\mathsf{K}(e)}(x) = \min(C_{\mathsf{J}(e)}(x), C_{\mathsf{D}(e)}(x))$, $U_{\mathsf{K}(e)}(x) = \max(U_{\mathsf{J}(e)}(x), U_{\mathsf{D}(e)}(x))$, $I_{\mathsf{K}(e)}(x) = \max(I_{\mathsf{J}(e)}(x), I_{\mathsf{D}(e)}(x))$, $K_{\mathsf{K}(e)}(x) = \max(K_{\mathsf{J}(e)}(x), K_{\mathsf{D}(e)}(x))$, $F_{\mathsf{K}(e)}(x) = \max(F_{\mathsf{J}(e)}(x), F_{\mathsf{D}(e)}(x))$.

As for the union, it may be defined in somewhat analogous manner.

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Leader / parameter	e_1	e_2
x_1	<0.65, 0.05, 0.02, 0.08, 0.03, 0.07, 0.10>	<0.11, 0.06, 0.05, 0.03, 0.07, 0.05, 0.63>
x_2	<0.35, 0.10, 0.03, 0.05, 0.04, 0.08, 0.35>	<0.34, 0.09, 0.03, 0.06, 0.04, 0.07, 0.37>
x_3	<0.20, 0.10, 0.05, 0.25, 0.20, 0.07, 0.13>	<0.20, 0.10, 0.05, 0.30, 0.15, 0.06, 0.14>

Leader / parameter	e_3	e_4	e_5
x_1	<0.70, 0.08, 0.02, 0.03, 0.02, 0.05, 0.00>	<0.60, 0.10, 0.05, 0.15, 0.03, 0.10, 0.20>	<0.34, 0.10, 0.08, 0.60, 0.09, 0.11, 0.13>
x_2	<0.05, 0.09, 0.30, 0.20, 0.20, 0.34, 0.70>	<0.70, 0.03, 0.02, 0.10, 0.04, 0.03, 0.08>	<0.60, 0.17, 0.03, 0.10, 0.02, 0.08, 0.10>
x_3	<0.78, 0.70, 0.35, 0.70, 0.58, 0.64, 0.82>	<0.45, 0.30, 0.50, 0.60, 0.30, 0.41, 0.34>	<0.55, 0.10, 0.09, 0.10, 0.03, 0.03, 0.10>

Leader / parameter	e_2	e_3	e_7
x_1	(0.10, 0.07, 0.04, 0.04, 0.07, 0.07, 0.61)	(0.65, 0.08, 0.03, 0.02, 0.06, 0.06, 0.10)	(0.40, 0.30, 0.05, 0.05, 0.05, 0.11, 0.04)
x_2	(0.33, 0.07, 0.09, 0.02, 0.10, 0.06, 0.33)	(0.67, 0.16, 0.05, 0.10, 0.09, 0.12, 0.13)	(0.13, 0.07, 0.20, 0.10, 0.10, 0.32, 0.28)
x_3	(0.15, 0.07, 0.08, 0.33, 0.19, 0.08, 0.10)	(0.61, 0.25, 0.10, 0.14, 0.31, 0.45, 0.70)	(0.81, 0.70, 0.30, 0.50, 0.27, 0.49, 0.47)

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Definition 4.6. Let $\mathfrak{U}$ be a non-empty set. Consider two HPNSSs, namely $(\mathsf{J}, \mathcal{A})$ and $(\mathsf{D}, \mathcal{B})$. Then the *union* of these two sets is defined as: $(\mathsf{J}, \mathcal{A}) \cup (\mathsf{D}, \mathcal{B}) = (\mathsf{K}, \mathcal{C})$, where $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$, K is a mapping given by $\mathsf{K} : C \to P_{HN}(\mathfrak{U})$ and membership functions (for any $e \in \mathcal{C}$ and any $x \in \mathfrak{U}$) are as follows:

$$T_{\mathsf{K}(e)}(x) = \begin{cases} T_{\mathsf{J}(e)}(x), & e \in \mathcal{A} \setminus \mathcal{B} \\ T_{\mathsf{D}(e)}(x), & e \in \mathcal{B} \setminus \mathcal{A} \\ \max(T_{\mathsf{J}(e)}(x), T_{\mathsf{D}(e)}(x)), & e \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

$$M_{\mathsf{K}(e)}(x) = \begin{cases} M_{\mathsf{J}(e)}(x), & e \in \mathcal{A} \setminus \mathcal{B} \\ M_{\mathsf{D}(e)}(x), & e \in \mathcal{B} \setminus \mathcal{A} \\ \max(M_{\mathsf{J}(e)}(x), M_{\mathsf{D}(e)}(x)), & e \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

$$C_{\mathsf{K}(e)}(x) = \begin{cases} C_{\mathsf{J}(e)}(x), & e \in \mathcal{A} \setminus \mathcal{B} \\ C_{\mathsf{D}(e)}(x), & e \in \mathcal{B} \setminus \mathcal{A} \\ \max(C_{\mathsf{J}(e)}(x), C_{\mathsf{D}(e)}(x)), & e \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

$$U_{\mathsf{K}(e)}(x) = \begin{cases} U_{\mathsf{J}(e)}(x), & e \in \mathcal{A} \setminus \mathcal{B} \\ U_{\mathsf{D}(e)}(x), & e \in \mathcal{B} \setminus \mathcal{A} \\ \min(U_{\mathsf{J}(e)}(x), U_{\mathsf{D}(e)}(x)), & e \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

$$I_{\mathsf{K}(e)}(x) = \begin{cases} I_{\mathsf{J}(e)}(x), & e \in \mathcal{A} \setminus \mathcal{B} \\ I_{\mathsf{D}(e)}(x), & e \in \mathcal{B} \setminus \mathcal{A} \\ \min(I_{\mathsf{J}(e)}(x), I_{\mathsf{D}(e)}(x)), & e \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

$$K_{\mathsf{K}(e)}(x) = \begin{cases} K_{\mathsf{J}(e)}(x), & e \in \mathcal{A} \setminus \mathcal{B} \\ K_{\mathsf{D}(e)}(x), & e \in \mathcal{B} \setminus \mathcal{A} \\ \min(K_{\mathsf{J}(e)}(x), K_{\mathsf{D}(e)}(x)), & e \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

$$F_{\mathsf{K}(e)}(x) = \begin{cases} F_{\mathsf{J}(e)}(x), & e \in \mathcal{A} \setminus \mathcal{B} \\ F_{\mathsf{D}(e)}(x), & e \in \mathcal{B} \setminus \mathcal{A} \\ \min(F_{\mathsf{J}(e)}(x), F_{\mathsf{D}(e)}(x)), & e \in \mathcal{A} \cap \mathcal{B} \end{cases}$$

Again, we think that these definitions are rather intuitive. However, it would be possible to define similar but not equivalent operations of this kind. For example, one could say that in case of logical value I (and maybe U) it would be reasonable to use arithmetic mean instead of maximum (minimum) function. This approach was presented by Maji in[9] (with respect to standard neutrosophic sets).

Be as it may, we should show some examples.

Example 4.7. Let us consider those HPNSSs which were presented in Example 4.3. Let us calculate $(\mathsf{D}, \mathcal{B}) \cap (\mathsf{H}, \mathcal{C})$. First, we see that $\mathcal{B} \cap \mathcal{C} = \{e_3\}$. After some calculations we get:

Now let us calculate $(\mathsf{D}, \mathcal{B}) \cup (\mathsf{H}, \mathcal{C})$. Note that $\mathcal{B} \cup \mathcal{C} = \{e_2, e_3, e_4, e_5, e_7\}$.

The reader can check if the calculations are accurate and in the accordance with the definition presented earlier. Moreover, he may perform other calculations of this type.

If we have union, intersection and complement, then we are able to define difference of HPNSSs.

Definition 4.8. Assume that $(\mathsf{J}, \mathcal{A})$ and $(\mathsf{D}, \mathcal{B})$ are two HPNSSs. Then we define their difference, that is $(\mathsf{J}, \mathcal{A}) \setminus (\mathsf{D}, \mathcal{B})$ as $(\mathsf{J}, \mathcal{A}) \cap (\mathsf{D}, \mathcal{B})^c$.

Having union and intersection defined, we may introduce two other operations.

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Definition 4.9. If $(\mathsf{J}, \mathcal{A})$ and $(\mathsf{D}, \mathcal{B})$ are two HPNSSs then $(\mathsf{J}, \mathcal{A})$ AND $(\mathsf{D}, \mathcal{B})$ is defined as: $(\mathsf{J}, \mathcal{A}) \wedge (\mathsf{D}, \mathcal{B}) = (\mathsf{H}, \mathcal{A} \times \mathcal{B})$, where the membership functions of H are understood in the following way:

$T_{\mathsf{H}(e,f)}(x) = \min(T_{\mathsf{J}(e)}(x), T_{\mathsf{D}(f)}(x), M_{\mathsf{H}(e,f)}(x) = \min(M_{\mathsf{J}(e)}(x), M_{\mathsf{D}(f)}(x)), C_{\mathsf{H}(e,f)}(x) = \min(C_{\mathsf{J}(e)}(x), C_{\mathsf{D}(f)}(x)),$
$U_{\mathsf{H}(e,f)}(x) = \max(U_{\mathsf{J}(e)}(x), U_{\mathsf{D}(f)}(x)), I_{\mathsf{H}(e,f)}(x) = \max(I_{\mathsf{J}(e)}(x), I_{\mathsf{D}(f)}(x)), K_{\mathsf{H}(e,f)}(x) = \max(K_{\mathsf{J}(e)}(x), K_{\mathsf{D}(f)}(x)),$
$F_{\mathsf{H}(e,f)}(x) = \max(F_{\mathsf{J}(e)}(x), F_{\mathsf{D}(f)}(x)).$

Definition 4.10. If $(\mathsf{J}, \mathcal{A})$ and $(\mathsf{D}, \mathcal{B})$ are two HPNSSs then $(\mathsf{J}, \mathcal{A})$ OR $(\mathsf{D}, \mathcal{B})$ is defined as: $(\mathsf{J}, \mathcal{A}) \vee (\mathsf{D}, \mathcal{B}) = (\mathsf{K}, \mathcal{A} \times \mathcal{B})$, where the membership functions of K are understood in the following way:

$T_{\mathsf{K}(e,f)}(x) = \max(T_{\mathsf{J}(e)}(x), T_{\mathsf{D}(f)}(x), M_{\mathsf{K}(e,f)}(x) = \max(M_{\mathsf{J}(e)}(x), M_{\mathsf{D}(f)}(x)), C_{\mathsf{K}(e,f)}(x) = \max(C_{\mathsf{J}(e)}(x), C_{\mathsf{D}(f)}(x)),$
$U_{\mathsf{K}(e,f)}(x) = \min(U_{\mathsf{J}(e)}(x), U_{\mathsf{D}(f)}(x)), I_{\mathsf{K}(e,f)}(x) = \min(I_{\mathsf{J}(e)}(x), I_{\mathsf{D}(f)}(x)), K_{\mathsf{K}(e,f)}(x) = \min(K_{\mathsf{J}(e)}(x), K_{\mathsf{D}(f)}(x)),$
$F_{\mathsf{K}(e,f)}(x) = \min(F_{\mathsf{J}(e)}(x), F_{\mathsf{D}(f)}(x)).$

Let us show an example of use of these operations.

Example 4.11. Assume that we have a universe $\mathfrak{U} = \{x_1, x_2\}$ and the set of parameters $\mathcal{E} = \{e_1, e_2, e_3\}$. Suppose that there are two HPNSSs defined in this environment, namely $(\mathsf{J}, \mathcal{A})$ and $(\mathsf{D}, \mathcal{B})$. They are presented below in a tabular form:

$(\mathsf{J}, \mathcal{A})$

$(\mathsf{D}, \mathcal{B})$

Then we may calculate $(\mathsf{J}, \mathcal{A}) \wedge (\mathsf{D}, \mathcal{B})$.

Assume that our sets are identified with the opinions of two decision makers, our universe consists of two houses and our parameters (criteria) are (respectively) ”beauty”, ”costliness” and ”equipment”. Then $(\mathsf{J}, \mathcal{A}) \wedge (\mathsf{D}, \mathcal{B})_{\mathsf{H}(e_1,e_2)}(x_1)$ may be interpreted as the joint evaluation of the combined criterion ”(the level of) beauty *and* costliness” given by the decision makers.

4.3 Special types of HPNSSs

Now we can define some special types of HPNSSs.

Definition 4.12. A heptapartitioned neutrosophic soft set $(\mathsf{J}, \mathcal{A})$ over the universe $\mathfrak{U}$ is said to be *empty* HPNSS *with respect to the parameter set* $\mathcal{A}$ if (for any $e \in \mathcal{A}$ and any $x \in \mathfrak{U}$):

$T_{\mathsf{J}(e)}(x) = 0, M_{\mathsf{J}(e)}(x) = 0, C_{\mathsf{J}(e)}(x) = 0, U_{\mathsf{J}(e)}(x) = 1, I_{\mathsf{J}(e)}(x) = 1, K_{\mathsf{J}(e)}(x) = 1, F_{\mathsf{J}(e)}(x) = 1.$

We denote this set by $\emptyset_{\mathcal{A}}$ or by $(\emptyset, \mathcal{A})$[5].

Definition 4.13. A heptapartitioned neutrosophic soft set $(\mathsf{J}, \mathcal{A})$ over the universe $\mathfrak{U}$ is said to be *absolute* HPNSS *with respect to the parameter set* $\mathcal{A}$ if (for any $e \in \mathcal{A}$ and any $x \in \mathfrak{U}$):

$T_{\mathsf{J}(e)}(x) = 1, M_{\mathsf{J}(e)}(x) = 1, C_{\mathsf{J}(e)}(x) = 1, U_{\mathsf{J}(e)}(x) = 0, I_{\mathsf{J}(e)}(x) = 0, K_{\mathsf{J}(e)}(x) = 0, F_{\mathsf{J}(e)}(x) = 0.$

We denote this set by $\mathfrak{U}_{\mathcal{A}}$ or $(\mathfrak{U}, \mathcal{A})$.

Definition 4.14. If $\mathfrak{U}_{\mathcal{A}}$ is an absolute HPNSS with respect to the parameter set $\mathcal{A}$ and $\mathcal{A} = \mathcal{E}$ then we denote it by $\mathfrak{U}_{\mathcal{E}}$ and we call it *absolute* HPNSS over $\mathfrak{U}$.

Definition 4.15. If $\emptyset_{\mathcal{A}}$ is an empty HPNSS with respect to the parameter set $\mathcal{A}$ and $\mathcal{A} = \mathcal{E}$ then we denote it by $\emptyset_{\mathcal{E}}$ and we call it *relative null* HPNSS over $\mathfrak{U}$.

[5]Note that we *did not* assume that all the membership functions take value 0. However, one can discuss the idea of HPNSS in which $T_{\mathsf{J}(e)}(x) = M_{\mathsf{J}(e)}(x) = C_{\mathsf{J}(e)}(x) = U_{\mathsf{J}(e)}(x) = I_{\mathsf{J}(e)}(x) = K_{\mathsf{J}(e)}(x) = F_{\mathsf{J}(e)}(x) = 0$ for any $e \in \mathcal{A}$ and $x \in \mathfrak{U}$. We could call this object *indefinite* HPNSS.

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Leader / parameter	e_3
x_1	(0.65, 0.08, 0.02, 0.03, 0.06, 0.06, 0.10)
x_2	(0.05, 0.09, 0.05, 0.20, 0.20, 0.34, 0.70)
x_3	(0.61, 0.25, 0.10, 0.70, 0.58, 0.64, 0.82)

Leader / parameter	e_2	e_3	e_4	e_5	e_7
x_1	(0.10, 0.07, 0.04, 0.04, 0.07, 0.07, 0.61)	(0.70, 0.08, 0.03, 0.02, 0.02, 0.05, 0.00)	(0.60, 0.10, 0.05, 0.15, 0.03, 0.10, 0.20)	(0.34, 0.10, 0.08, 0.60, 0.09, 0.11, 0.13)	(0.40, 0.30, 0.05, 0.05, 0.05, 0.11, 0.04)
x_2	(0.33, 0.07, 0.09, 0.02, 0.10, 0.06, 0.33)	(0.67, 0.16, 0.30, 0.10, 0.09, 0.12, 0.13)	(0.70, 0.03, 0.02, 0.10, 0.04, 0.03, 0.08)	(0.60, 0.17, 0.03, 0.10, 0.02, 0.08, 0.10)	(0.13, 0.07, 0.20, 0.10, 0.10, 0.32, 0.28)
x_3	(0.15, 0.07, 0.08, 0.33, 0.19, 0.08, 0.10)	(0.78, 0.70, 0.35, 0.14, 0.31, 0.45, 0.70)	(0.45, 0.30, 0.50, 0.60, 0.30, 0.41, 0.34)	(0.55, 0.10, 0.09, 0.10, 0.03, 0.03, 0.10)	(0.81, 0.70, 0.30, 0.50, 0.27, 0.49, 0.47)

Element / parameter	e_1	e_2
x_1	(0.10, 0.15, 0.10, 0.50, 0.05, 0.00, 0.10)	(0.62, 0.34, 0.02, 0.08, 0.10, 0.49, 0.37)
x_2	(0.13, 0.07, 0.17, 0.03, 0.02, 0.09, 0.49)	(0.03, 0.06, 0.13, 0.20, 0.03, 0.10, 0.08)

Element / parameter	e_2	e_3
x_1	(0.47, 0.24, 0.05, 0.10, 0.07, 0.30, 0.21)	(0.10, 0.15, 0.12, 0.30, 0.20, 0.10, 0.03)
x_2	(0.13, 0.05, 0.17, 0.05, 0.04, 0.04, 0.40)	(0.72, 0.08, 0.02, 0.06, 0.02, 0.10, 0.12)

Element / pair of parameters	(e_1, e_2)	(e_1, e_3)	(e_2, e_2)	(e_2, e_3)
x_1	(0.10, 0.15, 0.05, 0.50, 0.07, 0.30, 0.21)	(0.10, 0.15, 0.10, 0.50, 0.20, 0.10, 0.10)	(0.47, 0.24, 0.02, 0.10, 0.10, 0.49, 0.37)	(0.10, 0.15, 0.02, 0.30, 0.20, 0.49, 0.37)
x_2	(0.13, 0.05, 0.17, 0.05, 0.04, 0.09, 0.49)	(0.13, 0.07, 0.02, 0.06, 0.02, 0.10, 0.49)	(0.03, 0.05, 0.13, 0.20, 0.04, 0.10, 0.40)	(0.03, 0.06, 0.02, 0.20, 0.03, 0.10, 0.12)

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4.4 Some algebraic properties

In this subsection we shall list down some of the most important properties of algebraic operations on HPNSSs. In the proofs, we will use extensively theorems 2.13, 2.14 and 2.15 (sometimes openly and sometimes tacitly). In general, these proofs will not be difficult. However, sometimes some subtleties will occur. For example, it is important to distinguish between HPNSs and HPNSSs. Note that each time we talk about *soft* sets, we talk about *families* of our initial sets (in our case, this initial sets are HPNSs). From the formal point of view, there is a difference between, say, $1_{\mathfrak{U}}$ and $\mathfrak{U}_{\mathcal{E}}$. The latter set *is* soft, while the former not[6].

Attention: in general, in this subsection we shall use round brackets (instead of angle brackets) to denote HPNSs.

Theorem 4.16. *Let* $(\mathsf{J}, \mathcal{A})$ *and* $(\mathsf{D}, \mathcal{A})$ *be two HPNSSs over the universe* $\mathfrak{U}$. *Then the following properties are true.*

1. $(\mathsf{J}, \mathcal{A}) \subseteq (\mathsf{D}, \mathcal{A})$ *iff* $(\mathsf{J}, \mathcal{A}) \cap (\mathsf{D}, \mathcal{A}) = (\mathsf{J}, \mathcal{A})$.
2. $(\mathsf{J}, \mathcal{A}) \subseteq (\mathsf{D}, \mathcal{A})$ *iff* $(\mathsf{J}, \mathcal{A}) \cup (\mathsf{D}, \mathcal{A}) = (\mathsf{D}, \mathcal{A})$.

Proof.

1. Suppose that $(\mathsf{J}, \mathcal{A}) \subseteq (\mathsf{D}, \mathcal{A})$. In particular, it means that for any $e \in \mathcal{A}$, $\mathsf{J}(e) \subseteq \mathsf{D}(e)$ (see Def. 2.7). Now assume that $(\mathsf{J}, \mathcal{A}) \cap (\mathsf{D}, \mathcal{A}) = (\mathsf{H}, \mathcal{A})$. One can easily notice that $\mathsf{H}(e) = \mathsf{J}(e) \cap \mathsf{D}(e)$ for all $e \in \mathcal{A}$ (see Def. 2.10). But then, by the assumption, $(\mathsf{H}, \mathcal{A}) = (\mathsf{J}, \mathcal{A})$.

   Now suppose that $(\mathsf{J}, \mathcal{A}) \cap (\mathsf{D}, \mathcal{A}) = (\mathsf{J}, \mathcal{A})$. Let $(\mathsf{J}, \mathcal{A}) \cap (\mathsf{D}, \mathcal{A}) = (\mathsf{H}, \mathcal{A})$. Since $\mathsf{H}(e) = \mathsf{J}(e) \cap \mathsf{D}(e) = \mathsf{J}(e)$ for all $e \in \mathcal{A}$, we know that $\mathsf{J}(e) \subseteq \mathsf{D}(e)$ for all $e \in \mathcal{A}$. Hence, $(\mathsf{J}, \mathcal{A}) \subseteq (\mathsf{D}, \mathcal{A})$.
2. This is similar to (1).

□

Theorem 4.17. *Let* $(\mathsf{J}, \mathcal{A})$, $(\mathsf{D}, \mathcal{A})$, $(\mathsf{H}, \mathcal{A})$ *and* $(\mathsf{S}, \mathcal{A})$ *be HPNSSs over the universe* $\mathfrak{U}$. *Then the following properties are true.*

1. *If* $(\mathsf{J}, \mathcal{A}) \cap (\mathsf{D}, \mathcal{A}) = \emptyset_{\mathcal{A}}$, *then* $(\mathsf{J}, \mathcal{A}) \subseteq (\mathsf{D}, \mathcal{A})^c$.
2. *If* $(\mathsf{J}, \mathcal{A}) \subseteq (\mathsf{D}, \mathcal{A})$ *and* $(\mathsf{D}, \mathcal{A}) \subseteq (\mathsf{J}, \mathcal{A})$ *then* $(\mathsf{J}, \mathcal{A}) = (\mathsf{D}, \mathcal{A})$.
3. *If* $(\mathsf{J}, \mathcal{A}) \subseteq (\mathsf{D}, \mathcal{A})$ *and* $(\mathsf{H}, \mathcal{A}) \subseteq (\mathsf{S}, \mathcal{A})$ *then* $(\mathsf{J}, \mathcal{A}) \cap (\mathsf{H}, \mathcal{A}) \subseteq (\mathsf{D}, \mathcal{A}) \cap (\mathsf{S}, \mathcal{A})$.
4. $(\mathsf{J}, \mathcal{A}) \subseteq (\mathsf{D}, \mathcal{A})$ *iff* $(\mathsf{D}, \mathcal{A})^c \subseteq (\mathsf{J}, \mathcal{A})^c$.

Proof.

1. Suppose that $(\mathsf{J}, \mathcal{A}) \cap (\mathsf{D}, \mathcal{A}) = \emptyset_{\mathcal{A}}$. Then, in particular, $\mathsf{J}(e) \cap \mathsf{D}(e) = \emptyset$ (for any $e \in \mathcal{A}$). Hence, $\mathsf{J}(e) \subseteq 1_{\mathfrak{U}} \setminus \mathsf{D}(e) = \mathsf{D}^c(e)$. Therefore, we have $(\mathsf{J}, \mathcal{A}) \subseteq (\mathsf{D}, \mathcal{A}^c)$.
2. This is obvious.
3. Again, this is rather trivial.
4. We have the following sequence of equivalences (for any $e \in \mathcal{A}$): $(\mathsf{J}, \mathcal{A}) \subseteq (\mathsf{D}, \mathcal{A}) \Leftrightarrow \mathsf{J}(e) \subseteq \mathsf{D}(e) \Leftrightarrow (\mathsf{D}(e))^c \subseteq (\mathsf{J}(e))^c) \Leftrightarrow \mathsf{D}^c(e) \subseteq \mathsf{J}^c(e) \Leftrightarrow (\mathsf{D}, \mathcal{A})^c \subseteq (\mathsf{J}, \mathcal{A})^c$.

□

Theorem 4.18. *Let* $(\mathsf{J}, \mathcal{A})$ *be a HPNSS over the universe* $\mathfrak{U}$. *Then the following properties are true.*

1. $(\emptyset, \mathcal{A})^c = (\mathfrak{U}, \mathcal{A})$.
2. $(\mathfrak{U}, \mathcal{A})^c = (\emptyset, \mathcal{A})$.

[6] Someone could say that each set can be considered as soft but we do not adhere here to this trivial case.

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Proof. 1. Let $(\emptyset, \mathcal{A})$ be identified with $(\mathsf{J}, \mathcal{A})$ for some function J. Then, by the very definition of $(\emptyset, \mathcal{A})$, we may write that for any $e \in \mathcal{A}$ and for any $x \in \mathfrak{U}$, $T_{\mathsf{J}(e)}(x) = 0$, $M_{\mathsf{J}(e)}(x) = 0$, $C_{\mathsf{J}(e)}(x) = 0$, $U_{\mathsf{J}(e)}(x) = 1$, $I_{\mathsf{J}(e)}(x) = 1$, $K_{\mathsf{J}(e)}(x) = 1$ and $F_{\mathsf{J}(e)}(x) = 1$.

Now, we have that $(\emptyset, \mathcal{A})^c = (\mathsf{J}, \mathcal{A})^c$. According to the definition of (soft) complement, we may write that for any $e \in \mathcal{A}$ and for any $x \in \mathfrak{U}$, $T_{\mathsf{J}^c(e)}(x) = F_{\mathsf{J}(e)}(x) = 1$, $M_{\mathsf{J}^c(e)}(x) = K_{\mathsf{J}(e)}(x) = 1$, $C_{\mathsf{J}^c(e)}(x) = I_{\mathsf{J}(e)}(x) = 1$, $U_{\mathsf{J}^c(e)}(x) = 1 - U_{\mathsf{J}(e)}(x) = 1 - 1 = 0$, $I_{\mathsf{J}^c(e)}(x) = C_{\mathsf{J}(e)}(x) = 0$, $K_{\mathsf{J}^c(e)}(x) = M_{\mathsf{J}(e)}(x) = 0$ and $F_{\mathsf{J}^c(e)}(x) = T_{\mathsf{J}(e)}(x) = 0$.

But this means that $(\emptyset, \mathcal{A})^c = (\mathfrak{U}, \mathcal{A})$.

2. This is similar.

□

Theorem 4.19. *Let $(\mathsf{J}, \mathcal{A})$ be a HPNSS over the universe $\mathfrak{U}$. Then the following properties are true:*

1. $(\mathsf{J}, \mathcal{A}) \cup (\emptyset, \mathcal{A}) = (\mathsf{J}, \mathcal{A})$.
2. $(\mathsf{J}, \mathcal{A}) \cup (\mathfrak{U}, \mathcal{A}) = (\mathfrak{U}, \mathcal{A})$.

Proof. 1. Consider $(\mathsf{J}, \mathcal{A})$. Clearly, we may write this HPNSS in the following form:

$\{\{e, (x, T_{\mathsf{J}(e)}(x), M_{\mathsf{J}(e)}(x), C_{\mathsf{J}(e)}(x), U_{\mathsf{J}(e)}(x), I_{\mathsf{J}(e)}(x), K_{\mathsf{J}(e)}(x), F_{\mathsf{J}(e)}(x)) : x \in \mathfrak{U}\} : e \in \mathcal{A}\}$.

Analogously, we may write that $(\emptyset, \mathcal{A}) = \{\{e, (x, 0, 0, 0, 1, 1, 1, 1) : x \in \mathfrak{U}\} : e \in \mathcal{A}\}$.

Then:

$(\mathsf{J}, \mathcal{A}) \cup (\emptyset, \mathcal{A}) = \{\{e, (x, \max(T_{\mathsf{J}(e)}(x), 0), \max(M_{\mathsf{J}(e)}(x), 0), \max(C_{\mathsf{J}(e)}(x), 1), \min(U_{\mathsf{J}(e)}(x), 1), \min(I_{\mathsf{J}(e)}(x), 1), \min(K_{\mathsf{J}(e)}(x), 1), \min(F_{\mathsf{J}(e)}(x), 1)) : x \in \mathfrak{U}\} : e \in \mathcal{A}\} = \{\{e, (x, T_{\mathsf{J}(e)}(x), M_{\mathsf{J}(e)}(x), C_{\mathsf{J}(e)}(x), U_{\mathsf{J}(e)}(x), I_{\mathsf{J}(e)}(x), K_{\mathsf{J}(e)}(x), F_{\mathsf{J}(e)}(x)) : x \in \mathfrak{U}\} : e \in \mathcal{A}\} = (\mathsf{J}, \mathcal{A})$.

2. This is similar.

□

Theorem 4.20. *Let $(\mathsf{J}, \mathcal{A})$ be a HPNSS over the universe $\mathfrak{U}$. Then the following properties are true.*

1. $(\mathsf{J}, \mathcal{A}) \cap (\emptyset, \mathcal{A}) = (\emptyset, \mathcal{A})$.
2. $(\mathsf{J}, \mathcal{A}) \cap (\mathfrak{U}, \mathcal{A}) = (\mathsf{J}, \mathcal{A})$.

Proof. 1. Similarly as in the previous theorem, we may write:

$(\mathsf{J}, \mathcal{A}) \cup (\emptyset, \mathcal{A}) = \{\{e, (x, \min(T_{\mathsf{J}(e)}(x), 0), \min(M_{\mathsf{J}(e)}(x), 0), \min(C_{\mathsf{J}(e)}(x), 1), \max(U_{\mathsf{J}(e)}(x), 1), \max(I_{\mathsf{J}(e)}(x), 1), \max(K_{\mathsf{J}(e)}(x), 1), \max(F_{\mathsf{J}(e)}(x), 1)) : x \in \mathfrak{U}\} : e \in \mathcal{A}\} = \{\{e, (x, 0, 0, 0, 1, 1, 1, 1) : x \in \mathfrak{U}\} : e \in \mathcal{A}\} = (\emptyset, \mathcal{A})$.

2. This is similar.

□

Theorem 4.21. *Let $(\mathsf{J}, \mathcal{A})$ and $(\mathsf{D}, \mathcal{B})$ be HPNSSs over the universe $\mathfrak{U}$. Then the following properties are true.*

1. $(\mathsf{J}, \mathcal{A}) \cup (\emptyset, \mathcal{B}) = (\mathsf{J}, \mathcal{A})$ *iff* $\mathcal{B} \subseteq \mathcal{A}$.
2. $(\mathsf{J}, \mathcal{A}) \cup (\mathfrak{U}, \mathcal{A}) = (\mathfrak{U}, \mathcal{A})$ *iff* $\mathcal{A} \subseteq \mathcal{B}$.

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Proof. 1. ($\supseteq$) Let $(\mathsf{J},\mathcal{A})\cup(\emptyset,\mathcal{B})=(\mathsf{H},\mathcal{C})$ where $\mathcal{C}=\mathcal{A}\cup\mathcal{B}$. Then for any $e\in\mathcal{C}$ we may describe $\mathsf{H}(e)$ in the following way[7]:

$$\mathsf{H}(e)=\begin{cases}\{(x,T_{\mathsf{J}(e)},M_{\mathsf{J}(e)},C_{\mathsf{J}(e)},U_{\mathsf{J}(e)},I_{\mathsf{J}(e)},K_{\mathsf{J}(e)},F_{\mathsf{J}(e)}):x\in\mathfrak{U}\}, & e\in\mathcal{A}\setminus\mathcal{B}\\ \{(x,0,0,0,1,1,1,1):x\in\mathfrak{U}\}, & e\in\mathcal{B}\setminus\mathcal{A}\\ \{(x,\max(T_{\mathsf{J}(e)},0),\max(M_{\mathsf{J}(e)},0),\max(C_{\mathsf{J}(e)},0),\min(U_{\mathsf{J}(e)},1),\\ \min(I_{\mathsf{J}(e)},1),\min(K_{\mathsf{J}(e)},1),\min(F_{\mathsf{J}(e)},1)):x\in\mathfrak{U}\}, & e\in\mathcal{A}\cap\mathcal{B}\end{cases}=$$

$$\begin{cases}\{(x,T_{\mathsf{J}(e)},M_{\mathsf{J}(e)},C_{\mathsf{J}(e)},U_{\mathsf{J}(e)},I_{\mathsf{J}(e)},K_{\mathsf{J}(e)},F_{\mathsf{J}}(e)):x\in\mathfrak{U}\}, & e\in\mathcal{A}\setminus\mathcal{B}\\ \{(x,0,0,0,1,1,1,1):x\in\mathfrak{U}\}, & e\in\mathcal{B}\setminus\mathcal{A}\\ \{(x,T_{\mathsf{J}(e)},M_{\mathsf{J}(e)},C_{\mathsf{J}(e)},U_{\mathsf{J}(e)},I_{\mathsf{J}(e)},K_{\mathsf{J}(e)},F_{\mathsf{J}}(e)):x\in\mathfrak{U}\}, & e\in\mathcal{A}\cap\mathcal{B}\end{cases}$$

However, if $\mathcal{B}\subseteq\mathcal{A}$, then the middle case (that is, $\mathcal{B}\setminus\mathcal{A}$) is irrelevant. Now:

$$\mathsf{H}(e)=\begin{cases}\{(x,T_{\mathsf{J}(e)},M_{\mathsf{J}(e)},C_{\mathsf{J}(e)},U_{\mathsf{J}(e)},I_{\mathsf{J}(e)},K_{\mathsf{J}(e)},F_{\mathsf{J}}(e)):x\in\mathfrak{U}\}, & e\in\mathcal{A}\setminus\mathcal{B}\\ \{(x,T_{\mathsf{J}(e)},M_{\mathsf{J}(e)},C_{\mathsf{J}(e)},U_{\mathsf{J}(e)},I_{\mathsf{J}(e)},K_{\mathsf{J}(e)},F_{\mathsf{J}}(e)):x\in\mathfrak{U}\}, & e\in\mathcal{A}\cap\mathcal{B}\end{cases}.$$

But then it is clear that $\mathsf{H}(e)=\mathsf{J}(e)$ (for any $e\in\mathcal{A}$).

($\subseteq$) Conversely, let $(\mathsf{J},\mathcal{A})\cup(\emptyset,\mathcal{B})=(\mathsf{J},\mathcal{A})$. Then $\mathcal{A}=\mathcal{A}\cup\mathcal{B}$. Hence, $\mathcal{B}\subseteq\mathcal{A}$.

2. This is similar.

□

Theorem 4.22. *Let $(\mathsf{J},\mathcal{A})$ and $(\mathsf{D},\mathcal{B})$ be two HPNSSs over the universe $\mathfrak{U}$. Then the following properties are true.*

1. $(\mathsf{J},\mathcal{A})\cap(\emptyset,\mathcal{B})=(\emptyset,\mathcal{A}\cap\mathcal{B})$.
2. $(\mathsf{J},\mathcal{A})\cap(\mathfrak{U},\mathcal{B})=(\mathsf{J},\mathcal{A}\cap\mathcal{B})$.

Proof. 1. Let $(\mathsf{J},\mathcal{A})\cap(\emptyset,\mathcal{B})=(\mathsf{H},\mathcal{C})$ where $\mathcal{C}=\mathcal{A}\cap\mathcal{B}$. Then for any $e\in\mathcal{C}$ we may write that:

$\mathsf{H}(e)=\{(x,\min(T_{\mathsf{J}(e)},0),\min(M_{\mathsf{J}(e)},0),\min(C_{\mathsf{J}(e)},0),\max(U_{\mathsf{J}(e)},1),$
$\max(I_{\mathsf{J}(e)},1),\max(K_{\mathsf{J}(e)},1),\max(F_{\mathsf{J}(e)},1)):x\in\mathfrak{U}\}=\{(x,0,0,0,1,1,1,1):x\in\mathfrak{U}\}=(\emptyset,\mathcal{C})$.

Thus $(\mathsf{J},\mathcal{A})\cap(\emptyset,\mathcal{B})=(\emptyset,\mathcal{C})=(\emptyset,\mathcal{A}\cap\mathcal{B})$.

2. This is similar.

□

Theorem 4.23. *Let $(\mathsf{J},\mathcal{A})$ and $(\mathsf{D},\mathcal{B})$ be two HPNSSs over the universe $\mathfrak{U}$. Then the following properties are true.*

1. $((\mathsf{J},\mathcal{A})\cup(\mathsf{D},\mathcal{B}))^c\subseteq(\mathsf{J},\mathcal{A})^c\cup(\mathsf{D},\mathcal{B})^c$.
2. $(\mathsf{J},\mathcal{A})^c\cap(\mathsf{D},\mathcal{B})^c\subseteq((\mathsf{J},\mathcal{A})\cap(\mathsf{D},\mathcal{B}))^c$.

Proof. 1. Let $(\mathsf{J},\mathcal{A})\cup(\mathsf{D},\mathcal{B})=(\mathsf{H},\mathcal{C})$ where $\mathcal{C}=\mathcal{A}\cup\mathcal{B}$. Then for any $e\in\mathcal{C}$ we may write:

$$\mathsf{H}(e)=\begin{cases}\{(x,T_{\mathsf{J}(e)},M_{\mathsf{J}(e)},C_{\mathsf{J}(e)},U_{\mathsf{J}(e)},I_{\mathsf{J}(e)},K_{\mathsf{J}(e)},F_{\mathsf{J}(e)}):x\in\mathfrak{U}\},e\in\mathcal{A}\setminus\mathcal{B}\\ \{(x,T_{\mathsf{D}(e)},M_{\mathsf{D}(e)},C_{\mathsf{D}(e)},U_{\mathsf{D}(e)},I_{\mathsf{D}(e)},K_{\mathsf{D}(e)},F_{\mathsf{D}(e)}):x\in\mathfrak{U}\},e\in\mathcal{B}\setminus\mathcal{A}\\ \{(x,\max(T_{\mathsf{J}(e)},T_{\mathsf{D}(e)}),\max(M_{\mathsf{J}(e)},M_{\mathsf{D}(e)}),\max(C_{\mathsf{J}(e)},C_{\mathsf{D}(e)}),\min(U_{\mathsf{J}(e)},U_{\mathsf{D}(e)}),\\ \min(I_{\mathsf{J}(e)},I_{\mathsf{D}(e)}),\min(K_{\mathsf{J}(e)},K_{\mathsf{D}(e)}),\min(F_{\mathsf{J}(e)},F_{\mathsf{D}(e)})):x\in\mathfrak{U}\}, \quad e\in\mathcal{A}\cap\mathcal{B}\end{cases}.$$

But then $((\mathsf{J},\mathcal{A})\cup(\mathsf{D},\mathcal{B}))^c=(\mathsf{H},\mathcal{C})^c$. Now, for any $e\in\mathcal{C}$:

[7] We shall write $T_{\mathsf{J}(e)}$ instead of $T_{\mathsf{J}(e)}(x)$ and the same for other logical values. This is just an obvious shortcut.

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$$(\mathsf{H}(e))^c = \begin{cases} (\mathsf{J}(e))^c, & e \in \mathcal{A} \setminus \mathcal{B} \\ (\mathsf{D}(e))^c, & e \in \mathcal{B} \setminus \mathcal{A} \\ (\mathsf{J}(e) \cup \mathsf{D}(e))^c, & e \in \mathcal{A} \cap \mathcal{B} \end{cases}.$$

But then:

$$(\mathsf{H}(e))^c = \begin{cases} \{(x, F_{\mathsf{J}(e)}, K_{\mathsf{J}(e)}, 1 - U_{\mathsf{J}(e)}, C_{\mathsf{J}(e)}, M_{\mathsf{J}(e)}, T_{\mathsf{J}(e)}) : x \in \mathfrak{U}\}, & e \in \mathcal{A} \setminus \mathcal{B} \\ \{(x, F_{\mathsf{D}(e)}, K_{\mathsf{D}(e)}, 1 - U_{\mathsf{D}(e)}, C_{\mathsf{D}(e)}, M_{\mathsf{D}(e)}, T_{\mathsf{D}(e)}) : x \in \mathfrak{U}\}, & e \in \mathcal{B} \setminus \mathcal{A} \\ \{x, \min(F_{\mathsf{J}(e)}, F_{\mathsf{D}(e)}), \min(K_{\mathsf{J}(e)}, K_{\mathsf{D}(e)}), \min(I_{\mathsf{J}(e)}, I_{\mathsf{D}(e)}), 1 - \min(U_{\mathsf{J}(e)}, U_{\mathsf{D}(e)}), & \\ \max(C_{\mathsf{J}(e)}, C_{\mathsf{D}(e)}), \max(M_{\mathsf{J}(e)}, M_{\mathsf{D}(e)}), \max(T_{\mathsf{J}(e)}, T_{\mathsf{D}(e)})) : x \in \mathfrak{U}\}, & e \in \mathcal{A} \cap \mathcal{B} \end{cases}.$$

On the other hand, we may write that $(\mathsf{J}, \mathcal{A})^c) \cup (\mathsf{D}, \mathcal{B})^c = (\mathsf{K}, \mathcal{C})$, where $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ (again) and for any $e \in \mathcal{C}$:

$$\mathsf{K}(e) = \begin{cases} (\mathsf{J}(e))^c, & e \in \mathcal{A} \setminus \mathcal{B} \\ (\mathsf{D}(e))^c, & e \in \mathcal{B} \setminus \mathcal{A} \\ (\mathsf{J}(e))^c \cup (\mathsf{D}(e))^c, & e \in \mathcal{A} \cap \mathcal{B} \end{cases}.$$

Thus:

$$\mathsf{K}(e) = \begin{cases} \{(x, F_{\mathsf{J}(e)}, K_{\mathsf{J}(e)}, 1 - U_{\mathsf{J}(e)}, C_{\mathsf{J}(e)}, M_{\mathsf{J}(e)}, T_{\mathsf{J}(e)}) : x \in \mathfrak{U}\}, & e \in \mathcal{A} \setminus \mathcal{B} \\ \{(x, F_{\mathsf{D}(e)}, K_{\mathsf{D}(e)}, 1 - U_{\mathsf{D}(e)}, C_{\mathsf{D}(e)}, M_{\mathsf{D}(e)}, T_{\mathsf{D}(e)}) : x \in \mathfrak{U}\}, & e \in \mathcal{B} \setminus \mathcal{A} \\ \{x, \max(F_{\mathsf{J}(e)}, F_{\mathsf{D}(e)}), \max(K_{\mathsf{J}(e)}, K_{\mathsf{D}(e)}), \max(I_{\mathsf{J}(e)}, I_{\mathsf{D}(e)}), & \\ \min(1 - U_{\mathsf{J}(e)}, 1 - U_{\mathsf{D}(e)}), \min(C_{\mathsf{J}(e)}, C_{\mathsf{D}(e)}), \min(M_{\mathsf{J}(e)}, M_{\mathsf{D}(e)}), & \\ \min(T_{\mathsf{J}(e)}, T_{\mathsf{D}(e)})) : x \in \mathfrak{U}\}, & e \in \mathcal{A} \cap \mathcal{B} \end{cases}.$$

Now we see that (for example) $\min(F_{\mathsf{J}(e)}, F_{\mathsf{D}(e)}) \leq \max(F_{\mathsf{J}(e)}, F_{\mathsf{D}(e)})$. The same for the logical values K and I. Analogously, $\max(C_{\mathsf{J}(e)}, C_{\mathsf{D}(e)}) \geq \min(C_{\mathsf{J}(e)}, C_{\mathsf{J}(e)})$. The same for M and T. Moreover, $1 - \min(U_{\mathsf{J}(e)}, U_{\mathsf{D}(e)}) \geq \min(1 - U_{\mathsf{J}(e)}, 1 - U_{\mathsf{D}(e)})$.

The last inequality may be proved (in an even more general form) as follows: assume that it is not true that $1 - \min(a, b) \geq \min(1 - a, 1 - b)$, where $a, b \in \mathbb{R}$. Hence, there are some particular x, y such that $1 - \min(x, y) < \min(1 - x, 1 - y)$. Assume without loss of generality that $\min(x, y) = x$. Now, $-\min(x, y) < \min(1 - x, 1 - y) - 1$. Then $\min(x, y) > 1 - \min(1 - x, 1 - y)$. Hence $x > 1 - \min(1 - x, 1 - y)$. On the other hand, our assumption gives us that $x \leq y$, so $1 - x \geq 1 - y$. Thus, $\min(1 - x, 1 - y) = 1 - y$. Now $x > 1 - (1 - y) = y$. This is contradiction.

Finally, we obtain that all the conditions are satisfied which are required to say that $((\mathsf{J}, \mathcal{A}) \cup (\mathsf{D}, \mathcal{B}))^c \subseteq (\mathsf{J}, \mathcal{A})^c \cup (\mathsf{D}, \mathcal{B})^c$.

2. This is similar. Let $(\mathsf{J}, \mathcal{A}) \cap (\mathsf{D}, \mathcal{B}) = (\mathsf{H}, \mathcal{C})$ where $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$. Then for any $e \in \mathcal{C}$ we may write:

   $\mathsf{H}(e) = \mathsf{J}(e) \cap \mathsf{D}(e) = \{(x, \min(T_{\mathsf{J}(e)}, T_{\mathsf{D}(e)}), \min(M_{\mathsf{J}(e)}, M_{\mathsf{D}(e)}), \min(C_{\mathsf{J}(e)}, C_{\mathsf{D}(e)}), \max(U_{\mathsf{J}(e)}, U_{\mathsf{D}(e)}), \max(I_{\mathsf{J}(e)}, I_{\mathsf{D}(e)}), \max(K_{\mathsf{J}(e)}, K_{\mathsf{D}(e)}), \max(F_{\mathsf{J}(e)}, F_{\mathsf{D}(e)})) : x \in \mathfrak{U}\}$.

   Now $((\mathsf{J}, \mathcal{A}) \cap (\mathsf{D}, \mathcal{B}))^c = (\mathsf{H}, \mathcal{C})^c$ and for any $e \in \mathcal{C}$:

   $((\mathsf{H}(e))^c = \{(x, \min(T_{\mathsf{J}(e)}, T_{\mathsf{D}(e)}), \min(M_{\mathsf{J}(e)}, M_{\mathsf{D}(e)}), \min(C_{\mathsf{J}(e)}, C_{\mathsf{D}(e)}), \max(U_{\mathsf{J}(e)}, U_{\mathsf{D}(e)}), \max(I_{\mathsf{J}(e)}, I_{\mathsf{D}(e)}), \max(K_{\mathsf{J}(e)}, K_{\mathsf{D}(e)}), \max(F_{\mathsf{J}(e)}, F_{\mathsf{D}(e)})) : x \in \mathfrak{U}\}^c$.

   Hence:

   $((\mathsf{H}(e))^c = \{(x, \max(F_{\mathsf{J}(e)}, F_{\mathsf{D}(e)}), \max(K_{\mathsf{J}(e)}, K_{\mathsf{D}(e)}), \max(I_{\mathsf{J}(e)}, I_{\mathsf{D}(e)}), 1 - \max(U_{\mathsf{J}(e)}, U_{\mathsf{D}(e)}), \min(C_{\mathsf{J}(e)}, C_{\mathsf{D}(e)}), \min(M_{\mathsf{J}(e)}, M_{\mathsf{D}(e)}), \min(T_{\mathsf{J}(e)}, T_{\mathsf{D}(e)})) : x \in \mathfrak{U}\}$.

   On the other hand, let $(\mathsf{J}, \mathcal{A})^c \cap (\mathsf{D}, \mathcal{A})^c = (\mathsf{K}, \mathcal{C})$. Then for any $e \in \mathcal{C}$:

   $\mathsf{K}(e) = (\mathsf{J}(e))^c \cap (\mathsf{D}(e))^c = \{(x, \min(F_{\mathsf{J}(e)}, F_{\mathsf{D}(e)}), \min(K_{\mathsf{J}(e)}, K_{\mathsf{D}(e)}), \min(I_{\mathsf{J}(e)}, I_{\mathsf{D}(e)}), \max(1 - U_{\mathsf{J}(e)}, 1 - U_{\mathsf{D}(e)}), \max(C_{\mathsf{J}(e)}, C_{\mathsf{D}(e)}), \max(M_{\mathsf{J}(e)}, M_{\mathsf{D}(e)}), \max(T_{\mathsf{J}(e)}, T_{\mathsf{D}(e)})) : x \in \mathfrak{U}\}$.

   One can check (as in the previous point) that all the requirements of our expected inclusion are satisfied. The only less trivial case is that $\max(1 - U_{\mathsf{J}(e)}, 1 - U_{\mathsf{D}(e)}) \geq 1 - \max(U_{\mathsf{J}(e)}, U_{\mathsf{D}(e)})$. However, the reader is already aware how to prove this inequality.

□

Theorem 4.24. *Let* $(\mathsf{J}, \mathcal{A})$ *and* $(\mathsf{D}, \mathcal{A})$ *be two HPNSSs over the universe* $\mathfrak{U}$. *Then the following properties are true*[8].

[8] Note that we assumed that the set of parameters is the same for both sets. The reader is encouraged to ponder on this assumption.

1. $((\mathsf{J},\mathcal{A})\cup(\mathsf{D},\mathcal{A}))^c=(\mathsf{J},\mathcal{A})^c\cap(\mathsf{D},\mathcal{A})^c$.
2. $((\mathsf{J},\mathcal{A})\cap(\mathsf{D},\mathcal{A}))^c=(\mathsf{J},\mathcal{A})^c\cup(\mathsf{D},\mathcal{A})^c$.

Proof. 1. Let $(\mathsf{J},\mathcal{A})\cup(\mathsf{D},\mathcal{A})=(\mathsf{H},\mathcal{A})$. Then for any $e\in\mathcal{A}$:

$\mathsf{H}(e)=\mathsf{J}(e)\cup\mathsf{D}(e)=\{(x,\max(T_{\mathsf{J}(e)},T_{\mathsf{D}(e)}),\max(M_{\mathsf{J}(e)},M_{\mathsf{D}(e)}),\max(C_{\mathsf{J}(e)},C_{\mathsf{D}(e)}),$ $\min(U_{\mathsf{J}(e)},U_{\mathsf{D}(e)}),\min(I_{\mathsf{J}(e)},I_{\mathsf{D}(e)}),\min(K_{\mathsf{J}(e)},K_{\mathsf{D}(e)}),\min(F_{\mathsf{J}(e)},F_{\mathsf{D}(e)})):x\in\mathfrak{U}\}$.

Hence, if $((\mathsf{J},\mathcal{A})\cup(\mathsf{D},\mathcal{A}))^c=(\mathsf{H},\mathcal{A})^c$, then:

$(\mathsf{H}(e))^c=\{(x,\min(F_{\mathsf{J}(e)},F_{\mathsf{D}(e)}),\min(K_{\mathsf{J}(e)},K_{\mathsf{D}(e)}),\min(I_{\mathsf{J}(e)},I_{\mathsf{D}(e)}),1-\min(U_{\mathsf{J}(e)},U_{\mathsf{D}(e)}),\max(C_{\mathsf{J}(e)},C_{\mathsf{D}(e)}),$ $\max(M_{\mathsf{J}(e)},M_{\mathsf{D}(e)}),\max(T_{\mathsf{J}(e)},T_{\mathsf{D}(e)})):x\in\mathfrak{U}\}$.

But then, if $(\mathsf{J},\mathcal{A})^c\cap(\mathsf{D},\mathcal{A})=(\mathsf{K},\mathcal{A})$, then (for any $e\in\mathcal{A}$):

$\mathsf{K}(e)=(\mathsf{J}(e))^c\cap(\mathsf{D}(e))^c=\{(x,\min(F_{\mathsf{J}(e)},F_{\mathsf{D}(e)}),\min(K_{\mathsf{J}(e)},K_{\mathsf{D}(e)}),\min(I_{\mathsf{J}(e)},I_{\mathsf{D}(e)}),$ $\max(1-U_{\mathsf{J}(e)},1-U_{\mathsf{D}(e)}),\max(C_{\mathsf{J}(e)},C_{\mathsf{D}(e)}),\max(M_{\mathsf{J}(e)},M_{\mathsf{D}(e)}),\max(T_{\mathsf{J}(e)},T_{\mathsf{D}(e)})):x\in\mathfrak{U}\}$.

The only thing to prove is that $1-\min(U_{\mathsf{J}(e)},U_{\mathsf{D}(e)})=\max(1-U_{\mathsf{J}(e)},1-U_{\mathsf{D}(e)})$. Again, let us generalize it. Assume that there are some x,y such that $\min(x,y)=x$, $\max(x,y)=y$ and $1-\min(x,y)\neq\max(1-x,1-y)$. Now $-\min(x,y)\neq\max(1-x,1-y)-1$. Thus $\min(x,y)\neq 1-\max(1-x,1-y)$. But $\max(1-x,1-y)=1-x$. Hence, $x\neq 1-(1-x)=x$. Contradiction.

2. This is similar.

□

Now let us discuss the idea of indexed families.

Definition 4.25. Let I be an arbitrary index set and $\{(\mathsf{J}_i,\mathcal{A})\}_{i\in I}$ be an indexed subfamily of HPNSSs.

1. We define the union of this family as a new HPNSS $(\mathsf{H},\mathcal{A})$ where $\mathsf{H}(e)=\bigcup_{i\in I}\mathsf{J}_i(e)$ for each $e\in\mathcal{A}$.
   We write $\bigcup_{i\in I}(\mathsf{J}_i,\mathcal{A})=(\mathsf{H},\mathcal{A})$.
2. We define the intersection of this family as a new HPNSS $(\mathsf{M},\mathcal{A})$ where $\mathsf{M}(e)=\bigcap_{i\in I}\mathsf{J}_i(e)$ for each $e\in\mathcal{A}$.
   We write $\bigcap_{i\in I}(\mathsf{J}_i,\mathcal{A})=(\mathsf{M},\mathcal{A})$.

Remark 4.26. Note that in both the definitions above we assumed that our functions (that is, J_i for $i\in I$) may be different but the subset of parameters is the same (namely, $\mathcal{A}$) in each case.

Theorem 4.27. *Let I be an arbitrary index set and $\{(\mathsf{J}_i,\mathcal{A})\}_{i\in I}$ be a subfamily of HPNSSs over the universe $\mathfrak{U}$. Then:*

1. $(\bigcup_{i\in I}(\mathsf{J}_i,\mathcal{A}))^c=\bigcap_{i\in I}(\mathsf{J}_i,\mathcal{A})^c$.
2. $(\bigcap_{i\in I}(\mathsf{J}_i,\mathcal{A}))^c=\bigcup_{i\in I}(\mathsf{J}_i,\mathcal{A})^c$.

Proof. 1. Binary case was proved in an explicit manner in the preceding theorem. Here we shall use some shortcuts (or rather some properties of HPNSs).

We may write that $(\bigcup_{i\in I}(\mathsf{J}_i,\mathcal{A}))^c=(\mathsf{H},\mathcal{A})^c$ (for all $e\in\mathcal{A}$). Now, $\mathsf{H}^c(e)=1_{\mathfrak{U}}\setminus\mathsf{H}(e)=1_{\mathfrak{U}}\setminus\bigcup_{i\in I}\mathsf{J}_i(e)=\bigcap_{i\in I}(1_{\mathfrak{U}}\setminus\mathsf{J}_i(e))$.

On the other hand, for any $i\in I$, $(\mathsf{J}_i,\mathcal{A})^c=(\mathsf{K}_i,\mathcal{A})$ such that (for any $e\in\mathcal{A}$) we have $\mathsf{K}_i(e)=1_{\mathfrak{U}}\setminus\mathsf{J}_i(e)$. Now $\bigcap_{i\in I}(\mathsf{J}_i,\mathcal{A})^c$ can be described as $\bigcap_{i\in I}(\mathsf{K}_i,\mathcal{A})$ and then as $(\mathsf{M},\mathcal{A})$ such that (for any $e\in\mathcal{A}$) we have $\mathsf{M}(e)=\bigcap_{i\in I}\mathsf{K}_i(e)=\bigcap_{i\in I}(1_{\mathfrak{U}}\setminus\mathsf{J}_i(e))$. Hence, we may identify $\mathsf{H}^c(e)$ with $\mathsf{M}(e)$.

2. This is similar to the previous point.

□

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The last theorem in this subsection deals with operations $\wedge$ and $\vee$.

Theorem 4.28. *Let* $(\mathsf{J}, \mathcal{A})$ *and* $(\mathsf{D}, \mathcal{B})$ *are two HPNSSs over the same universe* $\mathfrak{U}$*. The following properties are true:*

1. $((\mathsf{J}, \mathcal{A}) \wedge (\mathsf{D}, \mathcal{B}))^c \subseteq (\mathsf{J}, \mathcal{A})^c \vee (\mathsf{D}, \mathcal{B})^c$.
2. $((\mathsf{J}, \mathcal{A}) \vee (\mathsf{D}, \mathcal{B}))^c \supseteq (\mathsf{J}, \mathcal{A})^c \wedge (\mathsf{D}, \mathcal{B})^c$.

Proof. 1. Let $(\mathsf{J}, \mathcal{A}) \wedge (\mathsf{D}, \mathcal{B}) = (\mathsf{H}, \mathcal{A} \times \mathcal{B})$, where $\mathsf{H}(e, f) = \mathsf{J}(e) \cap \mathsf{D}(f)$ for any $(e, f) \in \mathcal{A} \times \mathcal{B}$.

Thus we may write:

$\mathsf{H}(e, f) = \{(x, \min(T_{\mathsf{J}(e)}, T_{\mathsf{D}(f)}), \min(M_{\mathsf{J}(e)}, M_{\mathsf{D}(f)}), \min(C_{\mathsf{J}(e)}, C_{\mathsf{D}(f)}), \max(U_{\mathsf{J}(e)}, U_{\mathsf{D}(f)}), \max(I_{\mathsf{J}(e)}, I_{\mathsf{D}(f)}), \max(K_{\mathsf{J}(e)}, K_{\mathsf{D}(f)}), \max(F_{\mathsf{J}(e)}, F_{\mathsf{D}(f)})) : x \in \mathfrak{U}\}$.

Then $((\mathsf{J}, \mathcal{A}) \wedge (\mathsf{D}, \mathcal{B}))^c = (\mathsf{H}, \mathcal{A} \times \mathcal{B})^c$. For any $(e, f) \in \mathcal{A} \times \mathcal{B}$ we may write:

$(\mathsf{H}(e, f))^c = \{(x, \min(T_{\mathsf{J}(e)}, T_{\mathsf{D}(f)}), \min(M_{\mathsf{J}(e)}, M_{\mathsf{D}(f)}), \min(C_{\mathsf{J}(e)}, C_{\mathsf{D}(f)}), \max(U_{\mathsf{J}(e)}, U_{\mathsf{D}(f)}), \max(I_{\mathsf{J}(e)}, I_{\mathsf{D}(f)}), \max(K_{\mathsf{J}(e)}, K_{\mathsf{D}(f)}), \max(F_{\mathsf{J}(e)}, F_{\mathsf{D}(f)})) : x \in \mathfrak{U}\}^c =$
$\{(x, \max(F_{\mathsf{J}(e)}, F_{\mathsf{D}(f)}), \max(K_{\mathsf{J}(e)}, K_{\mathsf{D}(f)}), \max(I_{\mathsf{J}(e)}, I_{\mathsf{D}(f)}), 1 - \max(U_{\mathsf{J}(e)}, U_{\mathsf{D}(f)}), \min(C_{\mathsf{J}(e)}, C_{\mathsf{D}(f)}), \min(M_{\mathsf{J}(e)}, M_{\mathsf{D}(f)}), \min(T_{\mathsf{J}(e)}, T_{\mathsf{D}(f)})) : x \in \mathfrak{U}\}$.

Now let $(\mathsf{J}, \mathcal{A})^c \vee (\mathsf{D}, \mathcal{B})^c = (\mathsf{K}, \mathcal{A} \times \mathcal{B})$ where $\mathsf{K}(e, f) = ((\mathsf{J}(e))^c \cup (\mathsf{D}(f))^c$ for any $e \in \mathcal{A}$ and $f \in \mathcal{B}$. Then:

$\mathsf{K}(e, f) = \{(x, \max(F_{\mathsf{J}(e)}, F_{\mathsf{D}(f)}), \max(K_{\mathsf{J}(e)}, K_{\mathsf{D}(f)}), \max(I_{\mathsf{J}(e)}, I_{\mathsf{D}(f)}), \min(1 - U_{\mathsf{J}(e)}, 1 - U_{\mathsf{D}(f)}), \min(C_{\mathsf{J}(e)}, C_{\mathsf{D}(f)}), \min(M_{\mathsf{J}(e)}, M_{\mathsf{D}(f)}), \min(T_{\mathsf{J}(e)}, T_{\mathsf{D}(f)})$.

Now we are able to formulate our expected equality.

2. This is similar.

□

5 Conclusion

In this paper we have introduced and analyzed the notion of heptapartitioned neutrosophic soft set (HPNSS). Each set of this kind may be considered as a family of heptapartitioned neutrosophic sets (that is, HPNSs) which have been investigated earlier by other authors in.[16] Of course, the whole meaning of HPNSSs relies on the fact that they are equipped with some algebraic operations, namely $\cup$, $\cap$, $\vee$, $\wedge$, complement and $\setminus$. We have checked some algebraic properties of these operations and we have pointed out some important and not so obvious subtleties. We gave several examples of potential practical use of our sets. Moreover, we have shown that the particular choose of exactly seven (neutrosophic) logical values has certain philosophical justification in terms of Jaina logic.

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