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On Neutrosophic Generalized Semi Generalized Closed Sets

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Abstract

The article considers a new generalization of closed sets in neutrosophic topological space. This generalization is called neutrosophic gsg -closed set. Moreover, we discuss its essential features in neutrosophic topological spaces. Furthermore, we extend the research by displaying new related definitions such as neutrosophic gsg -closure and neutrosophic gsg -interior and debating their powerful characterizations and relationships.

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Keywords: $Neu^{gsg}CS$, $Neu^{gsg}OS$, Neu^{gsg} -closure and Neu^{gsg} -interior.

1. Introduction

The neutrosophic set theory was contributed by Smarandache in [1,2]. The neutrosophic topological space (simply Neu^{TS}) was offered by Salama et al. in [3]. The definition of semi- α -open sets in neutrosophic topological spaces was displayed by Imran et al. in [4]. The neutrosophic generalized homeomorphism was submitted by PAGE et al. in [5]. The class of generalized neutrosophic closed sets was given by Dhavaseelan et al. in [6]. The concepts of neutrosophic generalized ag -closed sets and neutrosophic generalized ag -continuous functions were provided by Imran et al. in [7]. The objective of this article is to show the sense of neutrosophic gsg -closed set (briefly $Neu^{gsg}CS$) and investigate their main characteristics in Neu^{TS} . Moreover, we argue neutrosophic gsg -closure (in word Neu^{gsg} -closure) and neutrosophic gsg -interior (fleetingly Neu^{gsg} -interior) with revealing several of their vital spots.

2. Preliminaries

In this work, (\mathcal{U}, ζ) (or simply \mathcal{U}) always mean Neu^{TS} . Let \mathfrak{P} be a neutrosophic set in a $Neu^{TS}(\mathcal{U}, \zeta)$, we denote the neutrosophic closure, the neutrosophic interior, and the neutrosophic complement of \mathfrak{P} by $NeuCl(\mathfrak{P})$, $NeuInt(\mathfrak{P})$ and $\mathfrak{P}^c = 1_{Neu} - \mathfrak{P}$, respectively.

Definition 2.1: [3]

The family ζ of neutrosophic subsets of a non-empty neutrosophic set $\mathcal{U} \neq \emptyset$ is called a neutrosophic topology (in short, Neu^T) on \mathcal{U} if it satisfies the below axioms:

- (i) $0_{Neu}, 1_{Neu} \in \zeta$,
- (ii) $\mathfrak{P}_1 \cap \mathfrak{P}_2 \in \zeta$ being $\mathfrak{P}_1, \mathfrak{P}_2 \in \zeta$,
- (iii) $\bigcup \mathfrak{P}_i \in \zeta$ for arbitrary family $\{\mathfrak{P}_i | i \in \Lambda\} \subseteq \zeta$.

In this case, we signified Neu^{TS} by (\mathcal{U}, ζ) or \mathcal{U} . Moreover, the neutrosophic set in ζ is named neutrosophic open (in short, $NeuOS$). Furthermore, for any $NeuOS$ \mathfrak{P} , then \mathfrak{P}^c is titled neutrosophic closed set (briefly, $NeuCS$) in \mathcal{U} .

Definition 2.2:

Let \mathfrak{P} be a neutrosophic subset of a $Neu^{TS}(\mathcal{U}, \zeta)$, then it is called to be:

- (i) a neutrosophic semi-open set and denoted by Neu^sOS if $\mathfrak{P} \subseteq NeuCl(NeuInt(\mathfrak{P}))$. [8]
- (ii) a neutrosophic semi-closed set and denoted by Neu^sCS if $NeuInt(NeuCl(\mathfrak{P})) \subseteq \mathfrak{P}$. The intersection of entire Neu^sCS s, including \mathfrak{P} is named a neutrosophic semi-closure, and it is symbolized by $Neu^sCl(\mathfrak{P})$. [8]
- (iii) a neutrosophic α -open set and denoted by $Neu^\alpha OS$ if $\mathfrak{P} \subseteq NeuInt(NeuCl(NeuInt(\mathfrak{P})))$. [9]
- (iv) a neutrosophic α -closed set and denoted by $Neu^\alpha CS$ if $NeuCl(NeuInt(NeuCl(\mathfrak{P}))) \subseteq \mathfrak{P}$. The intersection of the whole $Neu^\alpha CS$ s including \mathfrak{P} is named neutrosophic α -closure, and it is symbolized by $Neu^\alpha Cl(\mathfrak{P})$. [9]

Definition 2.3:

Let \mathfrak{P} be a neutrosophic subset of a $Neu^{TS}(\mathcal{U}, \zeta)$, and let \mathfrak{M} be a $NeuOS$ in (\mathcal{U}, ζ) such that $\mathfrak{P} \subseteq \mathfrak{M}$ then \mathfrak{P} is called to be:

- (i) a neutrosophic generalized closed set, and it is denoted by Neu^gCS if $NeuCl(\mathfrak{P}) \subseteq \mathfrak{M}$. The complement of a Neu^gCS is a Neu^gOS in (\mathcal{U}, ζ) . [10]
- (ii) a neutrosophic αg -closed set, and it is denoted by $Neu^{\alpha g}CS$ if $Neu^\alpha Cl(\mathfrak{P}) \subseteq \mathfrak{M}$. The complement of a $Neu^{\alpha g}CS$ is a $Neu^{\alpha g}OS$ in (\mathcal{U}, ζ) . [11]
- (iii) a neutrosophic $g\alpha$ -closed set, and it is denoted by $Neu^{g\alpha}CS$ if $Neu^\alpha Cl(\mathfrak{P}) \subseteq \mathfrak{M}$. The complement of a $Neu^{g\alpha}CS$ is a $Neu^{g\alpha}OS$ in (\mathcal{U}, ζ) . [12]
- (iv) a neutrosophic sg -closed set, and it is denoted by $Neu^{sg}CS$ if $Neu^sCl(\mathfrak{P}) \subseteq \mathfrak{M}$. The complement of a $Neu^{sg}CS$ is a $Neu^{sg}OS$ in (\mathcal{U}, ζ) . [13]
- (v) a neutrosophic gs -closed set, and it is denoted by $Neu^{gs}CS$ if $Neu^gCl(\mathfrak{P}) \subseteq \mathfrak{M}$. The complement of a $Neu^{gs}CS$ is a $Neu^{gs}OS$ in (\mathcal{U}, ζ) . [14]

Proposition 2.4:[9,10]

In a $Neu^{TS}(\mathcal{U}, \zeta)$, then the next arguments stand, and the opposite of every argument is not valid:

- (i) Each $NeuOS$ (resp. $NeuCS$) is a $Neu^\alpha OS$ (resp. $Neu^\alpha CS$).
- (ii) Each $NeuOS$ (resp. $NeuCS$) is a $Neu^g OS$ (resp. $Neu^g CS$).

(iii) Each $Neu^\alpha OS$ (resp. $Neu^\alpha CS$) is a $Neu^s OS$ (resp. $Neu^s CS$).

Proposition 2.5:[11,12]

In a $Neu^{TS}(\mathcal{U}, \zeta)$, then the next arguments stand, and the opposite of every argument is not valid:

- (i) Each $Neu^g OS$ (resp. $Neu^g CS$) is a $Neu^{\alpha g} OS$ (resp. $Neu^{\alpha g} CS$).
- (ii) Each $Neu^\alpha OS$ (resp. $Neu^\alpha CS$) is a $Neu^{g\alpha} OS$ (resp. $Neu^{g\alpha} CS$).
- (iii) Each $Neu^{g\alpha} OS$ (resp. $Neu^{g\alpha} CS$) is a $Neu^{\alpha g} OS$ (resp. $Neu^{\alpha g} CS$).

Proposition 2.6:[13-15]

In a $Neu^{TS}(\mathcal{U}, \zeta)$, then the next arguments stand, and the opposite of every argument is not valid:

- (i) Each $Neu^s OS$ (resp. $Neu^s CS$) is a $Neu^{gs} OS$ (resp. $Neu^{gs} CS$).
- (ii) Each $Neu^s OS$ (resp. $Neu^s CS$) is a $Neu^{sg} OS$ (resp. $Neu^{sg} CS$).
- (iii) Each $Neu^{sg} OS$ (resp. $Neu^{sg} CS$) is a $Neu^{gs} OS$ (resp. $Neu^{gs} CS$).
- (iv) Each $Neu^{g\alpha} OS$ (resp. $Neu^{g\alpha} CS$) is a $Neu^{gs} OS$ (resp. $Neu^{gs} CS$).

3. Neutrosophic Generalized sg -Closed Sets

In this sector, we present and analyse the neutrosophic generalized sg -closed sets and some of their features.

Definition 3.1:

Suppose that \mathfrak{P} is a neutrosophic set in a $Neu^{TS}(\mathcal{U}, \zeta)$ and assume that \mathfrak{M} is a $Neu^{sg} OS$ in (\mathcal{U}, ζ) where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. The set \mathfrak{P} is termed as a neutrosophic generalized sg -closed set, and it is signified by $Neu^{gs} CS$ if $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The collection of all $Neu^{gs} CS$ s in a $Neu^{TS}(\mathcal{U}, \zeta)$ is signified by $Neu^{gs} C(\mathcal{U})$.

Theorem 3.2:

In a $Neu^{TS}(\mathcal{U}, \zeta)$, the subsequent arguments are valid:

- (i) Each $NeuCS$ is a $Neu^{gs} CS$.
- (ii) Each $Neu^{gs} CS$ is a $Neu^g CS$.

Proof:

(i) Let $NeuCS \mathfrak{P}$ and $Neu^{sg} OS \mathfrak{M}$ be in a $Neu^{TS}(\mathcal{U}, \zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Then $NeuCl(\mathfrak{P}) = \mathfrak{P} \sqsubseteq \mathfrak{M}$. Therefore \mathfrak{P} is a $Neu^{gs} CS$.

(ii) Let $Neu^{gs} CS \mathfrak{P}$ and $NeuOS \mathfrak{M}$ be in a $Neu^{TS}(\mathcal{U}, \zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $NeuOS$ is a $Neu^{sg} OS$, we get $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, \mathfrak{P} is a $Neu^g CS$. ▀

The reverse of the above theorem is inaccurate, as displayed in the subsequent instances.

Example 3.3:

Suppose that $\mathcal{U} = \{u_1, u_2\}$ is a set and assume that $\zeta = \{0_{Neu}, \mathfrak{P}_1, \mathfrak{P}_2, 1_{Neu}\}$ is a Neu^T defined on \mathcal{U} . Suppose that we have the sets $\mathfrak{P}_1 = \langle u, (0.6, 0.7), (0.1, 0.1), (0.4, 0.2) \rangle$ and $\mathfrak{P}_2 = \langle u, (0.1, 0.2), (0.1, 0.1), (0.8, 0.8) \rangle$ are given. Then the neutrosophic set $\mathfrak{P}_3 = \langle u, (0.2, 0.2), (0.1, 0.1), (0.6, 0.7) \rangle$ is a $Neu^{gs} CS$. However, this latter set is not a $NeuCS$.

Example 3.4:

Suppose that $\mathcal{U} = \{u_1, u_2, u_3\}$ is a set and assume that $\zeta = \{0_{Neu}, \mathfrak{P}_1, \mathfrak{P}_2, 1_{Neu}\}$ is a Neu^T defined on \mathcal{U} . Suppose that we have the following sets $\mathfrak{P}_1 = \langle u, (0.5, 0.5, 0.4), (0.7, 0.5, 0.5), (0.4, 0.5, 0.5) \rangle$ and $\mathfrak{P}_2 =$

$\langle u, (0.3, 0.4, 0.4), (0.4, 0.5, 0.5), (0.3, 0.4, 0.6) \rangle$ are given. Then the neutrosophic set $\mathfrak{P}_3 = \langle u, (0.4, 0.6, 0.5), (0.4, 0.3, 0.5), (0.5, 0.6, 0.4) \rangle$ is a $Neu^g CS$. However, this latter set is not a $Neu^{sg} CS$.

Theorem 3.5:

In a $Neu^{TS}(\mathcal{U}, \zeta)$, the subsequent arguments are valid:

- (i) Each $Neu^{sg} CS$ is a $Neu^g CS$.
- (ii) Each $Neu^{sg} CS$ is a $Neu^{ga} CS$.
- (iii) Each $Neu^{sg} CS$ is a $Neu^s CS$.
- (iv) Each $Neu^{sg} CS$ is a $Neu^{gs} CS$.

Proof:

- (i) Let $Neu^{sg} CS \mathfrak{P}$ and $NeuOS \mathfrak{M}$ be in a $Neu^{TS}(\mathcal{U}, \zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $NeuOS$ is a $Neu^g CS$, we get $Neu^g Cl(\mathfrak{P}) \sqsubseteq NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The latter implies $Neu^g Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, \mathfrak{P} is a $Neu^g CS$.
- (ii) Let $Neu^{sg} CS \mathfrak{P}$ and $Neu^a OS \mathfrak{M}$ be in a $Neu^{TS}(\mathcal{U}, \zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $Neu^a OS$ is a $Neu^s OS$, which is a $Neu^{sg} OS$, we get $Neu^g Cl(\mathfrak{P}) \sqsubseteq NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The latter implies $Neu^g Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, \mathfrak{P} is a $Neu^{ga} CS$.
- (iii) Let $Neu^{sg} CS \mathfrak{P}$ and $Neu^s OS \mathfrak{M}$ be in a $Neu^{TS}(\mathcal{U}, \zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $Neu^s OS$ is a $Neu^{sg} OS$, we get $Neu^s Cl(\mathfrak{P}) \sqsubseteq NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. The latter implies $Neu^s Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, \mathfrak{P} is a $Neu^s CS$.
- (iv) Let $Neu^{sg} CS \mathfrak{P}$ and $NeuOS \mathfrak{M}$ be in a $Neu^{TS}(\mathcal{U}, \zeta)$ where $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because each $NeuOS$ is a $Neu^{sg} OS$, we get $Neu^s Cl(\mathfrak{P}) \sqsubseteq NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. That implies $Neu^s Cl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Consequently, \mathfrak{P} is a $Neu^{gs} CS$. ■

The reverse of the above theorem is inaccurate, as displayed in the subsequent instances.

Example 3.6:

Let $\mathcal{U} = \{u_1, u_2\}$ be a set and assume that $\zeta = \{0_{Neu}, \mathfrak{P}_1, \mathfrak{P}_2, 1_{Neu}\}$ is a Neu^T defined on \mathcal{U} . Suppose that we have the following sets $\mathfrak{P}_1 = \langle u, (0.5, 0.6), (0.3, 0.2), (0.4, 0.1) \rangle$ and $\mathfrak{P}_2 = \langle u, (0.4, 0.4), (0.4, 0.3), (0.5, 0.4) \rangle$ are given. Then the neutrosophic set $\mathfrak{P}_3 = \langle u, (0.5, 0.4), (0.4, 0.4), (0.4, 0.5) \rangle$ is a $Neu^g CS$ and hence $Neu^{ga} CS$ but not a $Neu^{sg} CS$.

Example 3.7:

Let $\mathcal{U} = \{u_1, u_2\}$ and let $\zeta = \{0_{Neu}, \mathfrak{P}_1, 1_{Neu}\}$ be a Neu^T on \mathcal{U} . Take $\mathfrak{P}_1 = \langle u, (0.3, 0.4, 0.6), (0.6, 0.6, 0.4) \rangle$. Then the neutrosophic set $\mathfrak{P}_2 = \langle u, (0.3, 0.2, 0.5), (0.6, 0.6, 0.8) \rangle$ is a $Neu^s CS$ but not a $Neu^{sg} CS$.

Example 3.8:

Let $\mathcal{U} = \{u_1, u_2\}$ and let $\zeta = \{0_{Neu}, \mathfrak{P}_1, 1_{Neu}\}$ be a Neu^T on \mathcal{U} . Where $\mathfrak{P}_1 = \langle u, (0.3, 0.2, 0.3), (0.8, 0.6, 0.7) \rangle$. Then the neutrosophic set $\mathfrak{P}_2 = \langle u, (0.3, 0.2, 0.6), (0.8, 0.9, 0.8) \rangle$ is a $Neu^{gs} CS$. However, this latter set is not a $Neu^{sg} CS$.

Remark 3.9:

The $Neu^{sg} CS$ are independent of $Neu^g CS$ and $Neu^s CS$.

Definition 3.10:

A neutrosophic subset \mathfrak{P} of a $Neu^{TS}(\mathcal{U}, \zeta)$ is called a neutrosophic generalized sg -open set (in short, $Neu^{sg} OS$) iff $1_{Neu} - \mathfrak{P}$ is a $Neu^{sg} CS$. The collection of entire $Neu^{sg} OS$ s of a $Neu^{TS}(\mathcal{U}, \zeta)$ is signified by $Neu^{sg} O(\mathcal{U})$.

Proposition 3.11:

Let \mathfrak{P} be a $NeuOS$ in $Neu^{TS}(\mathcal{U}, \zeta)$, then this set \mathfrak{P} is $Neu^{sg} OS$ in the space (\mathcal{U}, ζ) .

Proof:

Let \mathfrak{P} be a *NeuOS* in a $Neu^{TS}(\mathcal{U}, \zeta)$, then $1_{Neu} - \mathfrak{P}$ is a *NeuCS* in (\mathcal{U}, ζ) . According to theorem (3.2), point (i), $1_{Neu} - \mathfrak{P}$ is a *Neu^{gs}CS*. Therefore, \mathfrak{P} is a *Neu^{gs}OS* in (\mathcal{U}, ζ) . ■

Proposition 3.12:

Let \mathfrak{P} be a *Neu^{gs}OS* in $Neu^{TS}(\mathcal{U}, \zeta)$, then this set \mathfrak{P} is *Neu^gOS* in the space (\mathcal{U}, ζ) .

Proof:

Let \mathfrak{P} be a *Neu^{gs}OS* in a $Neu^{TS}(\mathcal{U}, \zeta)$, then $1_{Neu} - \mathfrak{P}$ is a *Neu^{gs}CS* in (\mathcal{U}, ζ) . According to theorem (3.2), point (ii), $1_{Neu} - \mathfrak{P}$ is a *Neu^gCS*. Therefore, \mathfrak{P} is a *Neu^gOS* in (\mathcal{U}, ζ) . ■

Theorem 3.13:

In a $Neu^{TS}(\mathcal{U}, \zeta)$, the subsequent arguments are valid:

(i) Each *Neu^{gs}OS* is a *Neu^αOS* and *Neu^αOS*.

(ii) Each *Neu^{gs}OS* is a *Neu^sOS* and *Neu^sOS*.

Proof:

Similar to above proposition. ■

Proposition 3.14:

If \mathfrak{P} and \mathfrak{Q} are *Neu^{gs}CSs* in a $Neu^{TS}(\mathcal{U}, \zeta)$, then $\mathfrak{P} \sqcup \mathfrak{Q}$ is a *Neu^{gs}CS*.

Proof:

Let \mathfrak{P} and \mathfrak{Q} be two *Neu^{gs}CSs* in a $Neu^{TS}(\mathcal{U}, \zeta)$ and let \mathfrak{M} be any *Neu^sOS* in \mathcal{U} such that $\mathfrak{P} \sqsubseteq \mathfrak{M}$ and $\mathfrak{Q} \sqsubseteq \mathfrak{M}$. Then we have $\mathfrak{P} \sqcup \mathfrak{Q} \sqsubseteq \mathfrak{M}$. Since \mathfrak{P} and \mathfrak{Q} are *Neu^{gs}CSs* in \mathcal{U} , $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$ and $NeuCl(\mathfrak{Q}) \sqsubseteq \mathfrak{M}$. Now, $NeuCl(\mathfrak{P} \sqcup \mathfrak{Q}) = NeuCl(\mathfrak{P}) \sqcup NeuCl(\mathfrak{Q}) \sqsubseteq \mathfrak{M}$ and so $NeuCl(\mathfrak{P} \sqcup \mathfrak{Q}) \sqsubseteq \mathfrak{M}$. Hence $\mathfrak{P} \sqcup \mathfrak{Q}$ is a *Neu^{gs}CS* in \mathcal{U} . ■

Proposition 3.15:

If \mathfrak{P} is a *Neu^{gs}CS* in a $Neu^{TS}(\mathcal{U}, \zeta)$, then $NeuCl(\mathfrak{P}) - \mathfrak{P}$ does not include non-empty *NeuCS* in (\mathcal{U}, ζ) .

Proof:

Let \mathfrak{P} be a *Neu^{gs}CS* in a $Neu^{TS}(\mathcal{U}, \zeta)$ and let \mathfrak{F} be any *NeuCS* in (\mathcal{U}, ζ) such that $\mathfrak{F} \sqsubseteq NeuCl(\mathfrak{P}) - \mathfrak{P}$. Since \mathfrak{P} is a *Neu^{gs}CS*, we have $NeuCl(\mathfrak{P}) \sqsubseteq 1_{Neu} - \mathfrak{F}$. This implies $\mathfrak{F} \sqsubseteq 1_{Neu} - NeuCl(\mathfrak{P})$. Then $\mathfrak{F} \sqsubseteq NeuCl(\mathfrak{P}) \cap (1_{Neu} - NeuCl(\mathfrak{P})) = 0_{Neu}$. Thus, $\mathfrak{F} = 0_{Neu}$. Hence $NeuCl(\mathfrak{P}) - \mathfrak{P}$ does not include non-empty *NeuCS* in (\mathcal{U}, ζ) . ■

Proposition 3.16:

A neutrosophic set \mathfrak{P} is *Neu^{gs}CS* in a $Neu^{TS}(\mathcal{U}, \zeta)$ iff $NeuCl(\mathfrak{P}) - \mathfrak{P}$ does not include non-empty *Neu^sCS* in (\mathcal{U}, ζ) .

Proof:

Let \mathfrak{P} be a *Neu^{gs}CS* in a $Neu^{TS}(\mathcal{U}, \zeta)$ and let \mathfrak{R} be any *Neu^sCS* in (\mathcal{U}, ζ) such that $\mathfrak{R} \sqsubseteq NeuCl(\mathfrak{P}) - \mathfrak{P}$. Since \mathfrak{P} is a *Neu^{gs}CS*, we have $NeuCl(\mathfrak{P}) \sqsubseteq 1_{Neu} - \mathfrak{R}$. This implies $\mathfrak{R} \sqsubseteq 1_{Neu} - NeuCl(\mathfrak{P})$. Then $\mathfrak{R} \sqsubseteq NeuCl(\mathfrak{P}) \cap (1_{Neu} - NeuCl(\mathfrak{P})) = 0_{Neu}$. Thus, \mathfrak{R} is empty.

Conversely, suppose that $NeuCl(\mathfrak{P}) - \mathfrak{P}$ does not include non-empty *Neu^sCS* in (\mathcal{U}, ζ) . Let $\mathfrak{P} \sqsubseteq \mathfrak{M}$ and \mathfrak{M} is *Neu^sOS*. If $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$ then $NeuCl(\mathfrak{P}) \cap (1_{Neu} - \mathfrak{M})$ is non-empty. Since $NeuCl(\mathfrak{P})$ is *NeuCS* and $1_{Neu} - \mathfrak{M}$ is *Neu^sCS*, we have $NeuCl(\mathfrak{P}) \cap (1_{Neu} - \mathfrak{M})$ is not empty *Neu^sCS* of $NeuCl(\mathfrak{P}) - \mathfrak{P}$, which is a contradiction. Therefore $NeuCl(\mathfrak{P}) \not\sqsubseteq \mathfrak{M}$. Hence \mathfrak{P} is a *Neu^{gs}CS*. ■

Proposition 3.17:

If \mathfrak{P} is a $Neu^{gs\mathcal{G}}CS$ in a $Neu^{TS}(\mathfrak{U}, \zeta)$ and $\mathfrak{P} \sqsubseteq \mathfrak{Q} \sqsubseteq NeuCl(\mathfrak{P})$, then \mathfrak{Q} is a $Neu^{gs\mathcal{G}}CS$ in (\mathfrak{U}, ζ) .

Proof:

Assume the set \mathfrak{P} is a $Neu^{gs\mathcal{G}}CS$ in a $Neu^{TS}(\mathfrak{U}, \zeta)$. Suppose the set \mathfrak{M} is a $Neu^{s\mathcal{G}}OS$ in (\mathfrak{U}, ζ) where $\mathfrak{Q} \sqsubseteq \mathfrak{M}$. So, $\mathfrak{P} \sqsubseteq \mathfrak{M}$. Because \mathfrak{P} is a $Neu^{gs\mathcal{G}}CS$, it observes that $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$. Currently, $\mathfrak{Q} \sqsubseteq NeuCl(\mathfrak{P})$ suggests $NeuCl(\mathfrak{Q}) \sqsubseteq NeuCl(NeuCl(\mathfrak{P})) = NeuCl(\mathfrak{P})$. Thus, $NeuCl(\mathfrak{Q}) \sqsubseteq \mathfrak{M}$. Hence \mathfrak{Q} is a $Neu^{gs\mathcal{G}}CS$. ■

Proposition 3.18:

Let $\mathfrak{P} \sqsubseteq \mathfrak{D} \sqsubseteq \mathfrak{U}$ and if \mathfrak{P} is a $Neu^{gs\mathcal{G}}CS$ in \mathfrak{U} then \mathfrak{P} is a $Neu^{gs\mathcal{G}}CS$ relative to \mathfrak{D} .

Proof:

$\mathfrak{P} \sqsubseteq \mathfrak{D} \cap \mathfrak{M}$ where \mathfrak{M} is a $Neu^{s\mathcal{G}}OS$ in \mathfrak{U} . Then $\mathfrak{P} \sqsubseteq \mathfrak{M}$ and hence $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{M}$.

This implies that $\mathfrak{D} \cap NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{D} \cap \mathfrak{M}$. Thus \mathfrak{P} is a $Neu^{gs\mathcal{G}}CS$ relative to \mathfrak{D} . ■

Proposition 3.19:

If \mathfrak{P} is a $Neu^{s\mathcal{G}}OS$ and a $Neu^{gs\mathcal{G}}CS$ in a $Neu^{TS}(\mathfrak{U}, \zeta)$, then \mathfrak{P} is a $NeuCS$ in (\mathfrak{U}, ζ) .

Proof:

Suppose that \mathfrak{P} is a $Neu^{s\mathcal{G}}OS$ and a $Neu^{gs\mathcal{G}}CS$ in a $Neu^{TS}(\mathfrak{U}, \zeta)$, then $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{P}$ and since $\mathfrak{P} \sqsubseteq NeuCl(\mathfrak{P})$. Thus, $NeuCl(\mathfrak{P}) = \mathfrak{P}$. Hence \mathfrak{P} is a $NeuCS$. ■

Theorem 3.20:

For each $u \in \mathfrak{U}$ either $\langle u, (0.1, 0.1) \rangle$ is a $Neu^{s\mathcal{G}}CS$ or $1_{Neu} - \langle u, (0.1, 0.1) \rangle$ is a $Neu^{gs\mathcal{G}}CS$ in \mathfrak{U} .

Proof:

If $\langle u, (0.1, 0.1) \rangle$ is not a $Neu^{s\mathcal{G}}CS$ in \mathfrak{U} then $1_{Neu} - \langle u, (0.1, 0.1) \rangle$ is not a $Neu^{s\mathcal{G}}OS$ and the only $Neu^{s\mathcal{G}}OS$ containing $1_{Neu} - \langle u, (0.1, 0.1) \rangle$ is the space \mathfrak{U} itself. Therefore $NeuCl(1_{Neu} - \langle u, (0.1, 0.1) \rangle) \sqsubseteq 1_{Neu}$ and so $1_{Neu} - \langle u, (0.1, 0.1) \rangle$ is a $Neu^{gs\mathcal{G}}CS$ in \mathfrak{U} . ■

Proposition 3.21:

If \mathfrak{P} and \mathfrak{Q} are $Neu^{gs\mathcal{G}}OS$ s in a $Neu^{TS}(\mathfrak{U}, \zeta)$, then $\mathfrak{P} \cap \mathfrak{Q}$ is a $Neu^{gs\mathcal{G}}OS$.

Proof:

Let \mathfrak{P} and \mathfrak{Q} be $Neu^{gs\mathcal{G}}OS$ s in a $Neu^{TS}(\mathfrak{U}, \zeta)$. Then $1_{Neu} - \mathfrak{P}$ and $1_{Neu} - \mathfrak{Q}$ are $Neu^{gs\mathcal{G}}CS$ s. By proposition (3.14), $(1_{Neu} - \mathfrak{P}) \sqcup (1_{Neu} - \mathfrak{Q})$ is a $Neu^{gs\mathcal{G}}CS$. Since $(1_{Neu} - \mathfrak{P}) \sqcup (1_{Neu} - \mathfrak{Q}) = 1_{Neu} - (\mathfrak{P} \cap \mathfrak{Q})$. Hence $\mathfrak{P} \cap \mathfrak{Q}$ is a $Neu^{gs\mathcal{G}}OS$. ■

Theorem 3.22:

A neutrosophic set \mathfrak{P} is $Neu^{gs\mathcal{G}}OS$ iff $\mathfrak{S} \sqsubseteq NeuInt(\mathfrak{P})$ where \mathfrak{S} is a $Neu^{gs\mathcal{G}}CS$ and $\mathfrak{S} \sqsubseteq \mathfrak{P}$.

Proof:

Suppose that $\mathfrak{S} \sqsubseteq NeuInt(\mathfrak{P})$ where \mathfrak{S} is a $Neu^{gs\mathcal{G}}CS$ and $\mathfrak{S} \sqsubseteq \mathfrak{P}$. Then $1_{Neu} - \mathfrak{P} \sqsubseteq 1_{Neu} - \mathfrak{S}$ and $1_{Neu} - \mathfrak{S}$ is a $Neu^{s\mathcal{G}}OS$ by theorem (3.13) part (ii). Now, $NeuCl(1_{Neu} - \mathfrak{P}) = 1_{Neu} - NeuInt(\mathfrak{P}) \sqsubseteq 1_{Neu} - \mathfrak{S}$. Then $1_{Neu} - \mathfrak{P}$ is a $Neu^{gs\mathcal{G}}CS$. Hence \mathfrak{P} is a $Neu^{gs\mathcal{G}}OS$.

Conversely, let \mathfrak{P} be a $Neu^{gs\mathcal{G}}OS$ and \mathfrak{S} be a $Neu^{gs\mathcal{G}}CS$ and $\mathfrak{S} \sqsubseteq \mathfrak{P}$. Then $1_{Neu} - \mathfrak{P} \sqsubseteq 1_{Neu} - \mathfrak{S}$. Since $1_{Neu} - \mathfrak{P}$ is a $Neu^{gs\mathcal{G}}CS$ and $1_{Neu} - \mathfrak{S}$ is a $Neu^{s\mathcal{G}}OS$, we have $NeuCl(1_{Neu} - \mathfrak{P}) \sqsubseteq 1_{Neu} - \mathfrak{S}$. Then $\mathfrak{S} \sqsubseteq NeuInt(\mathfrak{P})$. ■

Theorem 3.23:

If $\mathfrak{P} \sqsubseteq \mathfrak{Q} \sqsubseteq \mathfrak{U}$ where \mathfrak{P} is a $Neu^{gs\mathcal{G}}OS$ relative to \mathfrak{Q} and \mathfrak{Q} is a $Neu^{gs\mathcal{G}}OS$ in \mathfrak{U} , then \mathfrak{P} is a $Neu^{gs\mathcal{G}}OS$ in \mathfrak{U} .

Proof:

Let \mathfrak{F} be a $Neu^{sg}CS$ in \mathfrak{U} and suppose that $\mathfrak{F} \sqsubseteq \mathfrak{P}$. Then $\mathfrak{F} = \mathfrak{F} \sqcap \mathfrak{Q}$ is a $Neu^{sg}CS$ in \mathfrak{Q} . But \mathfrak{P} is a $Neu^{gsg}OS$ relative to \mathfrak{Q} . Therefore $\mathfrak{F} \sqsubseteq NeuInt_{\mathfrak{Q}}(\mathfrak{P})$. Since $NeuInt_{\mathfrak{Q}}(\mathfrak{P})$ is a $NeuOS$ relative to \mathfrak{Q} . We have $\mathfrak{F} \sqsubseteq \mathfrak{M} \sqcap \mathfrak{Q} \sqsubseteq \mathfrak{P}$, for some $NeuOS$ \mathfrak{M} in \mathfrak{U} . Since \mathfrak{Q} is a $Neu^{gsg}OS$ in \mathfrak{U} , we have $\mathfrak{F} \sqsubseteq NeuInt(\mathfrak{Q}) \sqsubseteq \mathfrak{Q}$. Therefore $\mathfrak{F} \sqsubseteq NeuInt(\mathfrak{Q}) \sqcap \mathfrak{M} \sqsubseteq \mathfrak{Q} \sqcap \mathfrak{M} \sqsubseteq \mathfrak{P}$. It follows that $\mathfrak{F} \sqsubseteq NeuInt(\mathfrak{P})$. Thus, \mathfrak{P} is a $Neu^{gsg}OS$ in \mathfrak{U} . ■

Theorem 3.24:

If \mathfrak{P} is a $Neu^{gsg}OS$ in a $Neu^{TS}(\mathfrak{U}, \zeta)$ and $NeuInt(\mathfrak{P}) \sqsubseteq \mathfrak{Q} \sqsubseteq \mathfrak{P}$, then \mathfrak{Q} is a $Neu^{gsg}OS$ in (\mathfrak{U}, ζ) .

Proof:

Suppose that \mathfrak{P} is a $Neu^{gsg}OS$ in a $Neu^{TS}(\mathfrak{U}, \zeta)$ and $NeuInt(\mathfrak{P}) \sqsubseteq \mathfrak{Q} \sqsubseteq \mathfrak{P}$. Then $1_{Neu} - \mathfrak{P}$ is a $Neu^{gsg}CS$ and $1_{Neu} - \mathfrak{P} \sqsubseteq 1_{Neu} - \mathfrak{Q} \sqsubseteq NeuCl(1_{Neu} - \mathfrak{P})$. Then $1_{Neu} - \mathfrak{Q}$ is a $Neu^{gsg}CS$ by proposition (3.17). Hence, \mathfrak{Q} is a $Neu^{gsg}OS$. ■

Theorem 3.25:

For a neutrosophic subset \mathfrak{P} of a $Neu^{TS}(\mathfrak{U}, \zeta)$, the following statements are equivalent:

- (i) \mathfrak{P} is a $Neu^{gsg}CS$.
- (ii) $NeuCl(\mathfrak{P}) - \mathfrak{P}$ contains no non-empty $Neu^{sg}CS$.
- (iii) $NeuCl(\mathfrak{P}) - \mathfrak{P}$ is a $Neu^{gsg}OS$.

Proof:

Follows from proposition (3.16) and proposition (3.18). ■

Remark 3.26:

The subsequent illustration reveals the relative among the diverse kinds of $NeuCS$:

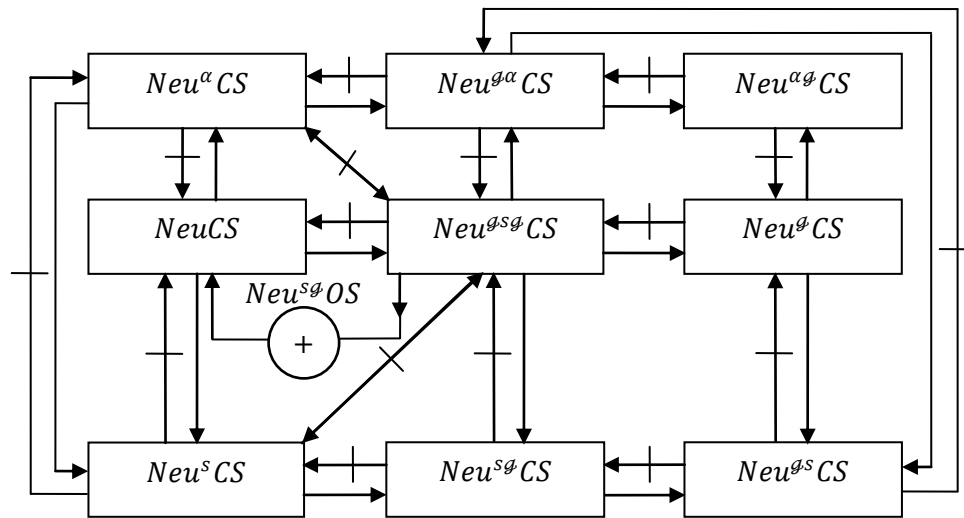


Fig. 3.1

4. Neutrosophic gsg -Closure and Neutrosophic gsg -Interior

We present neutrosophic gsg -closure and neutrosophic gsg -interior and obtain some of its properties in this section.

Definition 4.1:

The intersection of all $Neu^{gs\#}CS$ s in a $Neu^{TS}(\mathcal{U}, \zeta)$ containing \mathfrak{P} is called neutrosophic $gs\#$ -closure of \mathfrak{P} and is denoted by $Neu^{gs\#}Cl(\mathfrak{P})$.

Definition 4.2:

The union of all $Neu^{gs\#}OS$ s in a $Neu^{TS}(\mathcal{U}, \zeta)$ contained in \mathfrak{P} is called neutrosophic $gs\#$ -interior of \mathfrak{P} and is denoted by $Neu^{gs\#}Int(\mathfrak{P})$.

Proposition 4.3:

Let \mathfrak{P} be any neutrosophic set in a $Neu^{TS}(\mathcal{U}, \zeta)$. Then the following properties hold:

- (i) $Neu^{gs\#}Int(\mathfrak{P}) = \mathfrak{P}$ iff \mathfrak{P} is a $Neu^{gs\#}OS$.
- (ii) $Neu^{gs\#}Cl(\mathfrak{P}) = \mathfrak{P}$ iff \mathfrak{P} is a $Neu^{gs\#}CS$.
- (iii) $Neu^{gs\#}Int(\mathfrak{P})$ is the largest $Neu^{gs\#}OS$ contained in \mathfrak{P} .
- (iv) $Neu^{gs\#}Cl(\mathfrak{P})$ is the smallest $Neu^{gs\#}CS$ containing \mathfrak{P} .

Proof:

(i), (ii), (iii) and (iv) are obvious. ■

Proposition 4.4:

Let \mathfrak{P} be any neutrosophic set in a $Neu^{TS}(\mathcal{U}, \zeta)$. Then the following properties hold:

- (i) $Neu^{gs\#}Int(1_{Neu} - \mathfrak{P}) = 1_{Neu} - (Neu^{gs\#}Cl(\mathfrak{P}))$,
- (ii) $Neu^{gs\#}Cl(1_{Neu} - \mathfrak{P}) = 1_{Neu} - (Neu^{gs\#}Int(\mathfrak{P}))$.

Proof:

- (i) By definition, $Neu^{gs\#}Cl(\mathfrak{P}) = \bigcap \{\mathfrak{Q} : \mathfrak{P} \subseteq \mathfrak{Q}, \mathfrak{Q} \text{ is a } Neu^{gs\#}CS\}$
 $1_{Neu} - (Neu^{gs\#}Cl(\mathfrak{P})) = 1_{Neu} - \bigcap \{\mathfrak{Q} : \mathfrak{P} \subseteq \mathfrak{Q}, \mathfrak{Q} \text{ is a } Neu^{gs\#}CS\}$
 $= \bigcup \{1_{Neu} - \mathfrak{Q} : \mathfrak{P} \subseteq \mathfrak{Q}, \mathfrak{Q} \text{ is a } Neu^{gs\#}CS\}$
 $= \bigcup \{\mathfrak{M} : \mathfrak{M} \subseteq 1_{Neu} - \mathfrak{P}, \mathfrak{M} \text{ is a } Neu^{gs\#}OS\}$
 $= Neu^{gs\#}Int(1_{Neu} - \mathfrak{P}).$

(ii) The evidence is analogous to (i). ■

Theorem 4.5:

Let \mathfrak{P} and \mathfrak{Q} be two neutrosophic sets in a $Neu^{TS}(\mathcal{U}, \zeta)$. Then the following properties hold:

- (i) $Neu^{gs\#}Cl(0_{Neu}) = 0_{Neu}$, $Neu^{gs\#}Cl(1_{Neu}) = 1_{Neu}$.
- (ii) $\mathfrak{P} \subseteq Neu^{gs\#}Cl(\mathfrak{P})$.
- (iii) $\mathfrak{P} \subseteq \mathfrak{Q} \Rightarrow Neu^{gs\#}Cl(\mathfrak{P}) \subseteq Neu^{gs\#}Cl(\mathfrak{Q})$.
- (iv) $Neu^{gs\#}Cl(\mathfrak{P} \cap \mathfrak{Q}) \subseteq Neu^{gs\#}Cl(\mathfrak{P}) \cap Neu^{gs\#}Cl(\mathfrak{Q})$.
- (v) $Neu^{gs\#}Cl(\mathfrak{P} \cup \mathfrak{Q}) = Neu^{gs\#}Cl(\mathfrak{P}) \cup Neu^{gs\#}Cl(\mathfrak{Q})$.
- (vi) $Neu^{gs\#}Cl(Neu^{gs\#}Cl(\mathfrak{P})) = Neu^{gs\#}Cl(\mathfrak{P})$.

Proof:

(i) and (ii) are obvious.

(iii) By part (ii), $\mathfrak{Q} \subseteq Neu^{gs\#}Cl(\mathfrak{Q})$. Since $\mathfrak{P} \subseteq \mathfrak{Q}$, we have $\mathfrak{P} \subseteq Neu^{gs\#}Cl(\mathfrak{Q})$. But $Neu^{gs\#}Cl(\mathfrak{Q})$ is a $Neu^{gs\#}CS$. Thus $Neu^{gs\#}Cl(\mathfrak{Q})$ is a $Neu^{gs\#}CS$ containing \mathfrak{P} . Since $Neu^{gs\#}Cl(\mathfrak{P})$ is the smallest $Neu^{gs\#}CS$ containing \mathfrak{P} , we have $Neu^{gs\#}Cl(\mathfrak{P}) \subseteq Neu^{gs\#}Cl(\mathfrak{Q})$.

(iv) We know that $\mathfrak{P} \cap \mathfrak{Q} \subseteq \mathfrak{P}$ and $\mathfrak{P} \cap \mathfrak{Q} \subseteq \mathfrak{Q}$. Therefore, by part (iii), $Neu^{gs\#}Cl(\mathfrak{P} \cap \mathfrak{Q}) \subseteq Neu^{gs\#}Cl(\mathfrak{P})$ and $Neu^{gs\#}Cl(\mathfrak{P} \cap \mathfrak{Q}) \subseteq Neu^{gs\#}Cl(\mathfrak{Q})$. Hence $Neu^{gs\#}Cl(\mathfrak{P} \cap \mathfrak{Q}) \subseteq Neu^{gs\#}Cl(\mathfrak{P}) \cap Neu^{gs\#}Cl(\mathfrak{Q})$.

(v) Since $\mathfrak{P} \subseteq \mathfrak{P} \cup \mathfrak{Q}$ and $\mathfrak{Q} \subseteq \mathfrak{P} \cup \mathfrak{Q}$, it follows from part (iii) that $Neu^{gs\#}Cl(\mathfrak{P}) \subseteq Neu^{gs\#}Cl(\mathfrak{P} \cup \mathfrak{Q})$ and $Neu^{gs\#}Cl(\mathfrak{Q}) \subseteq Neu^{gs\#}Cl(\mathfrak{P} \cup \mathfrak{Q})$. Hence $Neu^{gs\#}Cl(\mathfrak{P}) \cup Neu^{gs\#}Cl(\mathfrak{Q}) \subseteq Neu^{gs\#}Cl(\mathfrak{P} \cup \mathfrak{Q})$ (1)

Since $Neu^{gs\#}Cl(\mathfrak{P})$ and $Neu^{gs\#}Cl(\mathfrak{Q})$ are $Neu^{gs\#}CSs$, $Neu^{gs\#}Cl(\mathfrak{P}) \sqcup Neu^{gs\#}Cl(\mathfrak{Q})$ is also $Neu^{gs\#}CS$ by proposition (3.14). Also $\mathfrak{P} \sqsubseteq Neu^{gs\#}Cl(\mathfrak{P})$ and $\mathfrak{Q} \sqsubseteq Neu^{gs\#}Cl(\mathfrak{Q})$ implies that $\mathfrak{P} \sqcup \mathfrak{Q} \sqsubseteq Neu^{gs\#}Cl(\mathfrak{P}) \sqcup Neu^{gs\#}Cl(\mathfrak{Q})$. Thus $Neu^{gs\#}Cl(\mathfrak{P}) \sqcup Neu^{gs\#}Cl(\mathfrak{Q})$ is a $Neu^{gs\#}CS$ containing $\mathfrak{P} \sqcup \mathfrak{Q}$. Since $Neu^{gs\#}Cl(\mathfrak{P} \sqcup \mathfrak{Q})$ is the smallest $Neu^{gs\#}CS$ containing $\mathfrak{P} \sqcup \mathfrak{Q}$, we have $Neu^{gs\#}Cl(\mathfrak{P} \sqcup \mathfrak{Q}) \sqsubseteq Neu^{gs\#}Cl(\mathfrak{P}) \sqcup Neu^{gs\#}Cl(\mathfrak{Q}) \dots \dots \dots (2)$

From (1) and (2), we have $Neu^{gs\#}Cl(\mathfrak{P} \sqcup \mathfrak{Q}) = Neu^{gs\#}Cl(\mathfrak{P}) \sqcup Neu^{gs\#}Cl(\mathfrak{Q})$.

(vi) Since $Neu^{gs\#}Cl(\mathfrak{P})$ is a $Neu^{gs\#}CS$, we have by proposition (4.3) part (ii), $Neu^{gs\#}Cl(Neu^{gs\#}Cl(\mathfrak{P})) = Neu^{gs\#}Cl(\mathfrak{P})$. ■

Theorem 4.6:

Let \mathfrak{P} and \mathfrak{Q} be two neutrosophic sets in a $Neu^{TS}(\mathfrak{U}, \zeta)$. Then the following properties hold:

- (i) $Neu^{gs\#}Int(0_{Neu}) = 0_{Neu}$, $Neu^{gs\#}Int(1_{Neu}) = 1_{Neu}$.
- (ii) $Neu^{gs\#}Int(\mathfrak{P}) \sqsubseteq \mathfrak{P}$.
- (iii) $\mathfrak{P} \sqsubseteq \mathfrak{Q} \Rightarrow Neu^{gs\#}Int(\mathfrak{P}) \sqsubseteq Neu^{gs\#}Int(\mathfrak{Q})$.
- (iv) $Neu^{gs\#}Int(\mathfrak{P} \sqcap \mathfrak{Q}) = Neu^{gs\#}Int(\mathfrak{P}) \sqcap Neu^{gs\#}Int(\mathfrak{Q})$.
- (v) $Neu^{gs\#}Int(\mathfrak{P} \sqcup \mathfrak{Q}) \supseteq Neu^{gs\#}Int(\mathfrak{P}) \sqcup Neu^{gs\#}Int(\mathfrak{Q})$.
- (vi) $Neu^{gs\#}Int(Neu^{gs\#}Int(\mathfrak{P})) = Neu^{gs\#}Int(\mathfrak{P})$.

Proof:

(i), (ii), (iii), (iv), (v) and (vi) are obvious. ■

Definition 4.7:

A $Neu^{TS}(\mathfrak{U}, \zeta)$ is called a neutrosophic $T_{\frac{1}{2}}$ -space (in short, $NeuT_{\frac{1}{2}}$ -space) if each $Neu^{gs\#}CS$ in this space is a $NeuCS$.

Definition 4.8:

A $Neu^{TS}(\mathfrak{U}, \zeta)$ is called a neutrosophic $T_{gs\#}$ -space (in short, $NeuT_{gs\#}$ -space) if each $Neu^{gs\#}CS$ in this space is a $NeuCS$.

Proposition 4.9:

Every $NeuT_{\frac{1}{2}}$ -space is a $NeuT_{gs\#}$ -space.

Proof:

Let (\mathfrak{U}, ζ) be a $NeuT_{\frac{1}{2}}$ -space and let \mathfrak{P} be a $Neu^{gs\#}CS$ in \mathfrak{U} . Then \mathfrak{P} is a $Neu^{gs\#}CS$, by theorem (3.2) part (ii). Since (\mathfrak{U}, ζ) is a $NeuT_{\frac{1}{2}}$ -space, then \mathfrak{P} is a $NeuCS$ in \mathfrak{U} . Hence (\mathfrak{U}, ζ) is a $NeuT_{gs\#}$ -space. ■

Theorem 4.10:

For a $Neu^{TS}(\mathfrak{U}, \zeta)$, the following statements are equivalent:

- (i) (\mathfrak{U}, ζ) is a $NeuT_{gs\#}$ -space.
- (ii) Every singleton of a $Neu^{TS}(\mathfrak{U}, \zeta)$ is either $Neu^{gs\#}CS$ or $NeuOS$.

Proof:

(i) \Rightarrow (ii) Assume that for some $u \in \mathfrak{U}$ the neutrosophic set $\langle u, (0.1, 0.1) \rangle$ is not a $Neu^{gs\#}CS$ in a $Neu^{TS}(\mathfrak{U}, \zeta)$. Then the only $Neu^{gs\#}OS$ containing $1_{Neu} - \langle u, (0.1, 0.1) \rangle$ is the space \mathfrak{U} itself and $1_{Neu} - \langle u, (0.1, 0.1) \rangle$ is a $Neu^{gs\#}CS$ in (\mathfrak{U}, ζ) . By assumption $1_{Neu} - \langle u, (0.1, 0.1) \rangle$ is a $NeuCS$ in (\mathfrak{U}, ζ) or equivalently $\langle u, (0.1, 0.1) \rangle$ is a $NeuOS$.

(ii) \Rightarrow (i) Let \mathfrak{P} be a $Neu^{gs\#}CS$ in (\mathfrak{U}, ζ) and let $u \in NeuCl(\mathfrak{P})$. By assumption $\langle u, (0.1, 0.1) \rangle$ is either $Neu^{s\#}CS$ or $NeuOS$.

Case(1). Suppose $\langle u, (0.1, 0.1) \rangle$ is a $Neu^{s\#}CS$. If $u \notin \mathfrak{P}$ then $NeuCl(\mathfrak{P}) - \mathfrak{P}$ contains a non-empty $Neu^{s\#}CS$ $\langle u, (0.1, 0.1) \rangle$ which is a contradiction to proposition (3.18). Therefore $u \in \mathfrak{P}$.

Case(2). Suppose $\langle u, (0.1, 0.1) \rangle$ is a $NeuOS$. Since $u \in NeuCl(\mathfrak{P})$, $\langle u, (0.1, 0.1) \rangle \cap \mathfrak{P} \neq 0_{Neu}$ and therefore $NeuCl(\mathfrak{P}) \sqsubseteq \mathfrak{P}$ or equivalently \mathfrak{P} is a $NeuCS$ in a $Neu^{TS}(\mathfrak{U}, \zeta)$. ■

5. Conclusion

The concept of $Neu^{gs\#}CS$ identified utilizing $Neu^{s\#}CS$ constructs a neutrosophic topology and sits between the concept of $NeuCS$ and the concept of $Neu^{s\#}CS$. The $Neu^{gs\#}CS$ can be used to derive a new decomposition of $Neu^{gs\#}$ -continuity and new $Neu^{gs\#}$ -separation axioms.

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