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Interval Complex Neutrosophic Set: Formulation and Applications in Decision-Making

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Abstract Neutrosophic set is a powerful general formal framework which generalizes the concepts of classic set, fuzzy set, interval-valued fuzzy set, intuitionistic fuzzy set, etc. Recent studies have developed systems with complex fuzzy sets, for better designing and modeling real-life applications. The single-valued complex neutrosophic set, which is an extended form of the single-valued complex fuzzy set and of the single-valued complex intuitionistic fuzzy set, presents difficulties to defining a crisp neutrosophic membership degree as in the single-valued neutrosophic set. Therefore, in this paper we propose a new notion, called interval complex neutrosophic set (ICNS), and examine its characteristics. Firstly, we define several set theoretic operations of ICNS, such as union, intersection and complement, and afterward the operational rules. Next, a decision-making procedure in ICNS and its applications to a green supplier selection are investigated. Numerical examples based on real dataset of Thuan Yen JSC, which is a small-size trading service and transportation company, illustrate the efficiency and the applicability of our approach.

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Abbreviations

NS	Neutrosophic set
INS	Interval neutrosophic set
CFS	Complex fuzzy set
CIFS	Complex intuitionistic fuzzy set
IVCFS	Interval-valued complex fuzzy set
CNS	Complex neutrosophic set
ICNS	Interval-valued complex neutrosophic set, or interval complex neutrosophic set
SVCNS	Single-valued complex neutrosophic set
MCDM	Multi-criteria decision-making
MCGDM	Multi-criteria group decision-making
$\vee$	Maximum operator (t-conorm)
$\wedge$	Minimum operator (t-norm)

1 Introduction

Smarandache [12] introduced the Neutrosophic Set (NS) as a generalization of classical set, fuzzy set, and intuitionistic fuzzy set. The neutrosophic set handles indeterminate data, whereas the fuzzy set and the intuitionistic fuzzy set fail to work when the relations are indeterminate. Neutrosophic set has been successfully applied in different fields, including decision-making problems [2, 5–8, 11, 14–16, 19–24, 27, 28]. Since the neutrosophic set is difficult to be directly used in real-life applications, Smarandache [12] and Wang et al. [18] proposed the concept of single-valued neutrosophic set and provided its theoretic operations and properties. Nonetheless, in many real-life problems, the degrees of truth, falsehood, and indeterminacy of a certain statement may be suitably presented by interval forms, instead of real numbers [17]. To deal with this situation, Wang et al. [17] proposed the concept of Interval Neutrosophic Set (INS), which is characterized by the degrees of truth, falsehood and indeterminacy, whose values are intervals rather than real numbers. Ye [19] presented the Hamming and Euclidean distances between INSs and the similarity measures between INSs based on the distances. Tian et al. [16] developed a multi-criteria decision-making (MCDM) method based on a cross-entropy with INSs [3, 10, 19, 25].

Recent studies in NS and INS have concentrated on developing systems using complex fuzzy sets [9, 10, 26] for better designing and modeling real-life applications. The functionality of 'complex' is for handling the information of uncertainty and periodicity simultaneously. By adding complex-valued non-membership grade to the definition of complex fuzzy set, Salleh [13] introduced the concept of complex intuitionistic fuzzy set. Ali and Smarandache [1] proposed a complex neutrosophic set (CNS), which is an extension form of complex fuzzy set and of complex intuitionistic fuzzy set. The complex neutrosophic set can handle the redundant nature of uncertainty, incompleteness, indeterminacy, inconsistency, etc., in periodic data. The advantage of CNS over the NS is the fact that, in addition to the membership degree provided by the NS and represented in the CNS by amplitude, the CNS also provides the phase, which is an attribute degree characterizing the amplitude.

Yet, in many real-life applications, it is not easy to find a crisp (exact) neutrosophic membership degree (as in the single-valued neutrosophic set), since we deal with unclear and vague information. To overcome this, we must create a new notion, which uses an *interval neutrosophic membership degree*. This paper aims to introduce a new concept of Interval-Valued Complex Neutrosophic Set or shortly Interval Complex Neutrosophic Set (ICNS), that is more flexible and adaptable to real-life applications than those of SVCNS and INS, due to the fact that many applications require elements to be represented by a more accurate form, such as in the decision-making problems [4, 7, 16, 17, 20, 25]. For example, in the green supplier selection, the linguistic rating set should be encoded by ICNS rather than by INS or by SVCNS, to reflect the hesitancy and indeterminacy of the decision.

This paper is the *first attempt* to define and use the ICNS in decision-making. The contributions and the tidings of this paper are highlighted as follows: First, we define the Interval Complex Neutrosophic Set (Sect. 3.1). Next, we define some set theoretic operations, such as union, intersection and complement (Sect. 3.2). Further, we establish the operational rules of ICNS (Sect. 3.3). Then, we aggregate ratings of alternatives versus criteria, aggregate the importance weights, aggregate the weighted ratings of alternatives versus criteria, and define a score function to rank the alternatives. Last, a decision-making procedure in ICNS and an application to a green supplier selection are presented (Sects. 4, 5).

Green supplier selection is a well-known application of decision-making. One of the most important issues in supply chain to make the company operation efficient is the selection of appropriate suppliers. Due to the concerns over the changes in world climate, green supplier selection is considered as a key element for companies to contribute toward the world environment protection, as well as to maintain their competitive advantages in the global market. In order to select the appropriate green supplier, many potential economic and environmental criteria should be taken into consideration in the selection procedure. Therefore, green supplier selection can be regarded as a multi-criteria decision-making (MCDM) problem. However, the majority of criteria is generally evaluated by personal judgement and thus might suffer from subjectivity. In this situation, ICNS can better express this kind of information.

The *advantages* of the proposal over other possibilities are highlighted as follows:

(a) The complex neutrosophic set is a generalization of interval complex fuzzy set, interval complex intuitionistic fuzzy sets, single-valued complex neutrosophic set and so on. For more detail, we refer to Fig. 1 in Sect. 3.1.
(b) In many real-life applications, it is not easy to find a crisp (exact) neutrosophic membership degree (as in the single-valued neutrosophic set), since we deal with unclear and vague periodic information. To overcome this, the complex interval neutrosophic set is a better representation.
(c) In order to select the appropriate green supplier, many potential economic and environmental criteria should be taken into consideration in the selection procedure. Therefore, green supplier selection can be regarded as a multi-criteria decision-making (MCDM) problem. However, the majority of criteria are generally evaluated by personal judgment, and thus, it might suffer from subjectivity. In this

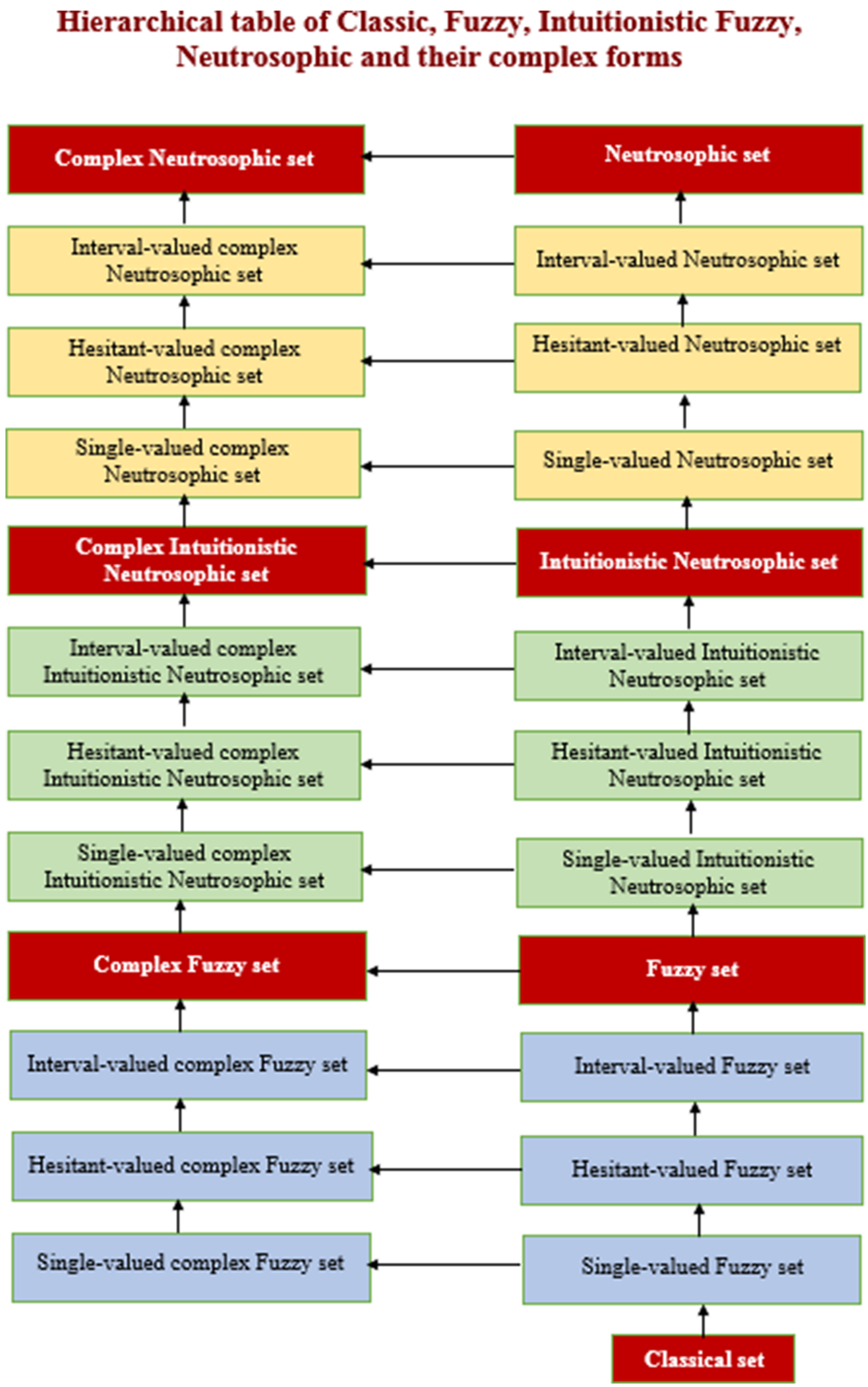

Fig. 1 Relationship of complex neutrosophic set with different types of fuzzy sets

situation, ICNS can better express this kind of information.

(d) The amplitude and phase (attribute) of ICNS have the ability to better catch the unsure values of the membership. Consider an example that we have a car component factory where each worker receives 10 car components per day to polish. The factory needs to have one worker coming in the weekend to work for a day, in order to finish a certain order from a customer. Again, the manager asks for a volunteer worker W. It turns out that the number of car components that will be done over one weekend day is W([0.6, 0.9], [0.1, 0.2], [0.0, 0.2]), which are actually the amplitudes for T, I, F. But what will be their quality? Indeed, their quality will be W([0.6, 0.9] × $e^{[0.6,\ 0.7]}$, [0.1, 0.2] × $e^{[0.4,\ 0.5]}$, [0.0, 0.2] × $e^{[0.0,\ 0.1]}$), by taking the [min, max] for each corresponding phase of T, I, F, respectively, for all workers. The new notion is indeed better in solving the decision-making problem. Unfortunately, other existing approaches cannot handle this type of information.

(e) The modified score function, accuracy function and certainty function of ICNS are more general in nature as compared to classical score, accuracy and certainty functions of existing methods. In modified forms of these functions, we have defined them for both amplitude and phase terms while it is not possible in the traditional case.

The rest of this paper is organized as follows. Section 2 recalls some basic concepts of neutrosophic set, interval neutrosophic set, complex neutrosophic set, and their operations. Section 3 presents the formulation of the interval complex neutrosophic set and its operations. Section 4 proposes a multi-criteria group decision-making model in ICNS. Section 5 demonstrates a numerical example of the procedure for green supplier selection on a real dataset. Section 6 delineates conclusions and suggests further studies.

2 Basic Concepts

Definition 1 [12] Neutrosophic set (NS)

Let X be a space of points and let $x \in X$. A neutrosophic set $\overline{S}$ in X is characterized by a truth membership function $T_{\overline{S}}$, an indeterminacy membership function $I_{\overline{S}}$, and a falsehood membership function $F_{\overline{S}}$. $T_{\overline{S}}$, $I_{\overline{S}}$ and $F_{\overline{S}}$ are real standard or non-standard subsets of $]0^-, 1^+[$. To use neutrosophic set in some real-life applications, such as engineering and scientific problems, it is necessary to consider the interval $[0, 1]$ instead of $]0^-, 1^+[$, for technical applications. The neutrosophic set can be represented as:

$$\overline{S} = \left\{ \left(x, T_{\overline{S}}(x), I_{\overline{S}}(x), F_{\overline{S}}(x) \right) : x \in X \right\},$$

where one has that $0 \le \sup T_{\overline{S}}(x) + \sup I_{\overline{S}}(x) + \sup F_{\overline{S}}(x) \le 3$, and $T_{\overline{S}}$, $I_{\overline{S}}$ and $F_{\overline{S}}$ are subsets of the unit interval [0, 1].

Definition 2 [9, 10] Complex fuzzy set (CFS)

A complex fuzzy set $\overline{S}$, defined on a universe of discourse X, is characterized by a membership function $\eta_{\overline{S}}(x)$ that assigns to any element $x \in X$ a complex-valued grade of membership in $\overline{S}$. The values $\eta_{\overline{S}}(x)$ lie within the unit circle in the complex plane, and thus, all forms $p_{\overline{S}}(x) \cdot e^{j \cdot \mu_{\overline{S}}(x)}$ where $p_{\overline{S}}(x)$ and $\mu_{\overline{S}}(x)$ are both real-valued and $p_{\overline{S}}(x) \in [0, 1]$. The term $p_{\overline{S}}(x)$ is termed as amplitude term, and $e^{j \cdot \mu_{\overline{S}}(x)}$ is termed as phase term. The complex fuzzy set can be represented as:

$$\overline{S} = \left\{ \left(x, \eta_{\overline{S}}(x) \right) : x \in X \right\}.$$

Definition 3 [13] Complex intuitionistic fuzzy set (CIFS)

A complex intuitionistic fuzzy set $\overline{S}$, defined on a universe of discourse X, is characterized by a membership function $\eta_{\overline{S}}(x)$ and a non-membership function $\zeta_{\overline{S}}(x)$, respectively, assigning to an element $x \in X$ a complex-valued grade to both membership and non-membership in $\overline{S}$. The values of $\eta_{\overline{S}}(x)$ and $\zeta_{\overline{S}}(x)$ lie within the unit circle in the complex plane and are of the form $\eta_{\overline{S}}(x) = p_{\overline{S}}(x) \cdot e^{j \cdot \mu_{\overline{S}}(x)}$ and $\zeta_{\overline{S}}(x) = r_{\overline{S}}(x) \cdot e^{j \cdot \omega_{\overline{S}}(x)}$ where $p_{\overline{S}}(x), r_{\overline{S}}(x)$, $\mu_{\overline{S}}(x)$ and $\omega_{\overline{S}}(x)$ are all real-valued and $p_{\overline{S}}(x)$, $r_{\overline{S}}(x) \in [0, 1]$ with $j = \sqrt{-1}$. The complex intuitionistic fuzzy set can be represented as:

$$\overline{S} = \left\{ \left(x, \eta_{\overline{S}}(x), \zeta_{\overline{S}}(x) \right) : x \in X \right\}.$$

Definition 4 [4] Interval-valued complex fuzzy set (IVCFS)

An interval-valued complex fuzzy set $\bar{A}$ is defined over a universe of discourse X by a membership function

$$\mu_{\bar{A}} : X \to \Gamma^{[0,1]} \times R,$$
$$\mu_{\bar{A}}(x) = r_{\bar{A}}(x) \cdot e^{j\omega_{\bar{A}}(x)}$$

In the above equation, $\Gamma^{[0,1]}$ is the collection of interval fuzzy sets and R is the set of real numbers. $r_{\overline{S}}(x)$ is the interval-valued membership function while $e^{j\omega_{\bar{A}}(x)}$ is the phase term, with $j = \sqrt{-1}$.

Definition 5 [1] Single-valued complex neutrosophic set (SVCNS)

A single-valued complex neutrosophic set $\overline{S}$, defined on a universe of discourse X, is expressed by a truth membership function $T_{\overline{S}}(x)$, an indeterminacy membership function $I_{\overline{S}}(x)$ and a falsity membership function $F_{\overline{S}}(x)$, assigning a complex-valued grade of $T_{\overline{S}}(x)$, $I_{\overline{S}}(x)$ and $F_{\overline{S}}(x)$ in $\overline{S}$ for any $x \in X$. The values $T_{\overline{S}}(x)$, $I_{\overline{S}}(x)$, $F_{\overline{S}}(x)$ and their sum may all be within the unit circle in the complex plane, and so it is of the following form:

$$T_{\overline{S}}(x) = p_{\overline{S}}(x) \cdot e^{j\mu_{\overline{S}}(x)},\ I_{\overline{S}}(x) = q_{\overline{S}}(x) \cdot e^{j\nu_{\overline{S}}(x)} \text{ and } F_{\overline{S}}(x) = r_{\overline{S}}(x) \cdot e^{j\omega_{\overline{S}}(x)},$$

where $p_{\overline{S}}(x)$, $q_{\overline{S}}(x)$, $r_{\overline{S}}(x)$ and $\mu_{\overline{S}}(x)$, $\nu_{\overline{S}}(x)$, $\omega_{\overline{S}}(x)$ are, respectively, real values and $p_{\overline{S}}(x), q_{\overline{S}}(x), r_{\overline{S}}(x) \in [0,1]$, such that $0 \le p_{\overline{S}}(x) + q_{\overline{S}}(x) + r_{\overline{S}}(x) \le 3$. The single-valued complex neutrosophic set S can be represented in set form as:

$$\overline{S} = \left\{ \left(x, T_{\overline{S}}(x), I_{\overline{S}}(x), F_{\overline{S}}(x)\right) : x \in X \right\}.$$

Definition 6 [1] Complement of single-valued complex neutrosophic set

Let $\overline{S} = \left\{ \left(x, T_{\overline{S}}(x), I_{\overline{S}}(x), F_{\overline{S}}(x)\right) : x \in X \right\}$ be a single-valued complex neutrosophic set in X. Then, the complement of a SVCNS $\overline{S}$ is denoted as $\overline{S}^c$ and is defined by:

$$\overline{S}^c = \left\{ \left(x, T_{\overline{S}^c}(x), I_{\overline{S}^c}(x), F_{\overline{S}^c}(x)\right) : x \in X \right\},$$

where $T_{\overline{S}^c}(x) = p_{\overline{S}^c}(x) \cdot e^{j\cdot\mu_{\overline{S}^c}(x)}$ is such that $p_{\overline{S}^c}(x) = r_{\overline{S}}(x)$ and $\mu_{\overline{S}^c}(x) = \mu_{\overline{S}}(x), 2\pi - \mu_{\overline{S}}(x)$ or $\mu_{\overline{S}}(x) + \pi$. Similarly, $I_{\overline{S}^c}(x) = q_{\overline{S}^c}(x) \cdot e^{j\cdot\nu_{\overline{S}^c}(x)}$, where $q_{\overline{S}^c}(x) = 1 - q_{\overline{S}}(x)$ and $\nu_{\overline{S}^c}(x) = \nu_{\overline{S}}(x), 2\pi - \nu_{\overline{S}^c}(x)$ or $\nu_{\overline{S}^c}(x) + \pi$. Finally, $F_{\overline{S}^c}(x) = r_{\overline{S}^c}(x) \cdot e^{j\cdot\omega_{\overline{S}^c}(x)}$, where $r_{\overline{S}^c}(x) = p_{\overline{S}}(x)$ and $\omega_{\overline{S}^c}(x) = \omega_{\overline{S}}(x), 2\pi - \omega_{\overline{S}}(x)$ or $\omega_{\overline{S}}(x) + \pi$

Definition 7 [1] Union of single-valued complex neutrosophic sets

Let $\bar{A}$ and $\bar{B}$ be two SVCNSs in X. Then:

$$\overline{A} \cup \overline{B} = \left\{ \left(x, T_{\overline{A}\cup\overline{B}}(x), I_{\overline{A}\cup\overline{B}}(x), F_{\overline{A}\cup\overline{B}}(X)\right) : x \in X \right\},$$

where

$$T_{\overline{A}\cup\overline{B}}(x) = \left[\left(p_{\overline{A}}(x) \vee p_{\overline{B}}(x)\right)\right] \cdot e^{j\cdot\mu_{T_{\overline{A}\cup\overline{B}}}(x)},$$
$$I_{\overline{A}\cup\overline{B}}(x) = \left[\left(q_{\overline{A}}(x) \wedge q_{\overline{B}}(x)\right)\right] \cdot e^{j\cdot\nu_{I_{\overline{A}\cup\overline{B}}}(x)},$$
$$F_{\overline{A}\cup\overline{B}}(x) = \left[\left(r_{\overline{A}}(x) \wedge r_{\overline{B}}(x)\right)\right] \cdot e^{j\cdot\omega_{F_{\overline{A}\cup\overline{B}}}(x)}$$

where $\vee$ and $\wedge$ denote the max and min operators, respectively. To calculate the phase terms $e^{j\cdot\mu_{A\cup B}(x)}$, $e^{j\cdot\nu_{A\cup B}(x)}$ and $e^{j\cdot\omega_{A\cup B}(x)}$, we refer to [1].

Definition 8 [1] Intersection of single-valued complex neutrosophic sets

Let $\bar{A}$ and $\overline{B}$ be two SVCNSs in X. Then:

$$\bar{A} \cap \bar{B} = \{(x, T_{\bar{A}\cap\bar{B}}(x), I_{\bar{A}\cap\bar{B}}(x), F_{\bar{A}\cap\bar{B}}(X)) : x \in X\},$$

where

$$T_{\bar{A}\cap\bar{B}}(x) = [(p_{\bar{A}}(x) \wedge p_{\bar{B}}(x))] \cdot e^{j\cdot\mu_{T_{\bar{A}\cap\bar{B}}}(x)},$$
$$I_{\bar{A}\cap\bar{B}}(x) = [(q_{\bar{A}}(x) \vee q_{\bar{B}}(x))] \cdot e^{j\cdot\nu_{I_{\bar{A}\cap\bar{B}}}(x)},$$
$$F_{\bar{A}\cap\bar{B}}(x) = [(r_{\bar{A}}(x) \vee r_{\bar{B}}(x))] \cdot e^{j\cdot\omega_{F_{\bar{A}\cap\bar{B}}}(x)}$$

where $\vee$ and $\wedge$ denote the max and min operators, respectively. To calculate the phase terms $e^{j\cdot\mu_{A\cup B}(x)}$, $e^{j\cdot\nu_{A\cup B}(x)}$ and $e^{j\cdot\omega_{A\cup B}(x)}$, we refer to [1].

3 Interval Complex Neutrosophic Set with Set Theoretic Properties

3.1 Interval Complex Neutrosophic Set

Before we present the definition, let us consider an example below to see the advantages of the new notion ICNS.

Example 1 Suppose we have a car component factory. Each worker from this factory receives 10 car components per day to polish.

- *NS* The best worker, John, successfully polishes 9 car components, 1 car component is not finished, and he wrecks 0 car component. Then, John's neutrosophic work is (0.9, 0.1, 0.0). The worst worker, George, successfully polishes 6, not finishing 2, and wrecking 2. Thus, George's neutrosophic work is (0.6, 0.2, 0.2).
- *INS* The factory needs to have one worker coming in the weekend, to work for a day in order to finish a required order from a customer. Since the factory management cannot impose the weekend overtime to workers, the manager asks for a volunteer. How many car components are to be polished during the weekend? Since the manager does not know which worker (W) will volunteer, he estimates that the work to be done in a weekend day will be: W([0.6, 0.9], [0.1, 0.2], [0.0, 0.2]), i.e., an interval for each T, I, F, respectively, between the minimum and maximum values of all workers.
- *CNS* The factory's quality control unit argues that although many workers correctly/successfully polish their car components, some of the workers do a work of a better quality than the others. Going back to John and George, the factory's quality control unit measures the work quality of each of them and finds out that: John's work is ($0.9 \times e^{0.6}$, $0.1 \times e^{0.4}$, $0.0 \times e^{0.0}$), and George's work is ($0.6 \times e^{0.7}$, $0.2 \times e^{0.5}$, $0.2 \times e^{0.1}$). Thus, although John polishes successfully 9 car components, more than George's 6 successfully polished

car components, the quality of John's work (0.6, 0.4, 0.0) is less than the quality of George's work (0.7, 0.5, 0.1).

It is clear from the above example that the amplitude and phase (attribute) of CNS should be represented by intervals, which better catch the unsure values of the membership. Let us come back to Example 1, where the factory needs to have one worker coming in the weekend to work for a day, in order to finish a certain order from a customer. Again, the manager asks for a volunteer worker W. We find out that the number of car components that will be done over one weekend day is W([0.6, 0.9], [0.1, 0.2], [0.0, 0.2]), which are actually the amplitudes for T, I, F. But what will be their quality? Indeed, their quality will be W([0.6, 0.9] × $e^{[0.6,\ 0.7]}$, [0.1, 0.2] × $e^{[0.4,\ 0.5]}$, [0.0, 0.2] × $e^{[0.0,\ 0.1]}$), by taking the [min, max] for each corresponding phases for T, I, F, respectively, for all workers. Therefore, we should propose a new notion for such the cases of decision-making problems.

Definition 9 Interval complex neutrosophic set.

An interval complex neutrosophic set is defined over a universe of discourse X by a truth membership function $T_{\overline{S}}$, an indeterminate membership function $I_{\overline{S}}$, and a falsehood membership function $F_{\overline{S}}$, as follows:

$$\left.\begin{aligned} T_{\overline{S}} &: X \to \Gamma^{[0,1]} \times R, T_{\overline{S}}(x) = t_{\overline{S}}(x) \cdot \mathrm{e}^{j\alpha\omega_{\overline{S}}(x)} \\ I_{\overline{S}} &: X \to \Gamma^{[0,1]} \times R, I_{\overline{S}}(x) = i_{\overline{S}}(x) \cdot \mathrm{e}^{j\beta\psi_{\overline{S}}(x)} \\ F_{\overline{S}} &: X \to \Gamma^{[0,1]} \times R, F_{\overline{S}}(x) = f_{\overline{S}}(x) \cdot \mathrm{e}^{j\gamma\phi_{\overline{S}}(x)} \end{aligned}\right\} \tag{1}$$

In the above Eq. (1), $\Gamma^{[0,1]}$ is the collection of interval neutrosophic sets and R is the set of real numbers, $t_{\overline{S}}(x)$ is the interval truth membership function, $i_{\overline{S}}(x)$ is the interval indeterminate membership and $f_{\overline{S}}(x)$ is the interval falsehood membership function, while $\mathrm{e}^{j\alpha\omega_{\overline{S}}(x)}$, $\mathrm{e}^{j\beta\psi_{\overline{S}}(x)}$ and $\mathrm{e}^{j\gamma\phi_{\overline{S}}(x)}$ are the corresponding interval-valued phase terms, respectively, with $j = \sqrt{-1}$. The scaling factors α, β and γ lie within the interval $(0, 2\pi]$. This study assumes that the values $\alpha, \beta, \gamma = \pi$. In set theoretic form, an interval complex neutrosophic set can be written as:

$$\overline{S} = \left\{ \left\langle \frac{T_{\overline{S}}(x) = t_{\overline{S}}(x) \cdot \mathrm{e}^{j\alpha\omega_{\overline{S}}(x)}, I_{\overline{S}}(x) = i_{\overline{S}}(x) \cdot \mathrm{e}^{j\beta\psi_{\overline{S}}(x)}, F_{\overline{S}}(x) = f_{\overline{S}}(x) \cdot \mathrm{e}^{j\gamma\phi_{\overline{S}}(x)}}{x} \right\rangle : x \in X \right\} \tag{2}$$

In (2), the amplitude interval-valued terms $t_{\overline{S}}(x), i_{\overline{S}}(x), f_{\overline{S}}(x)$ can be further split as $t_{\overline{S}}(x) = \left[t_{\overline{S_L}}(x), t_{\overline{S_U}}(x)\right]$, $i_{\overline{S}}(x) = \left[i_{\overline{S_L}}(x), i_{\overline{S_U}}(x)\right]$ and $f_{\overline{S}}(x) = \left[f_{\overline{S_L}}(x), f_{\overline{S_U}}(x)\right]$, where $t_{\overline{S_U}}(x), i_{\overline{S_U}}(x), f_{\overline{S_U}}(x)$ represents the upper bound, while $t_{\overline{S_L}}(x), i_{\overline{S_L}}(x), f_{\overline{S_L}}(x)$ represents the lower bound in each interval, respectively. Similarly, for the phases: $\omega_{\overline{S}}(x) = \left[\omega_{\overline{S_L}}(x), \omega_{\overline{S_U}}(x)\right]$, $\psi_{\overline{S}}(x) = \left[\psi_{\overline{S_L}}(x), \psi_{\overline{S_U}}(x)\right]$, and $\varphi_{\overline{S}}(x) = \left[\varphi_{\overline{S_L}}(x), \varphi_{\overline{S_U}}(x)\right]$.

Example 2 Let $X = \{x_1, x_2, x_3, x_4\}$ be a universe of discourse. Then, an interval complex neutrosophic set $\overline{S}$ can be given as follows:

$$\overline{S} = \left\{ \begin{aligned} &\frac{[0.4, 0.6] \cdot \mathrm{e}^{j\pi[0.5,0.6]}, [0.1, 0.7] \cdot \mathrm{e}^{j\pi[0.1,0.3]}, [0.3, 0.5] \cdot \mathrm{e}^{j\pi[0.8,0.9]}}{x_1}, \frac{[0.2, 0.4] \cdot \mathrm{e}^{j\pi[0.3,0.6]}, [0.1, 0.1] \cdot \mathrm{e}^{j\pi[0.7,0.9]}, [0.5, 0.9] \cdot \mathrm{e}^{j\pi[0.2,0.5]}}{x_2}, \\ &\frac{[0.3, 0.4].e^{j\pi[0.7,0.8]}, [0.6, 0.7] \cdot \mathrm{e}^{j\pi[0.6,0.7]}, [0.2, 0.6] \cdot \mathrm{e}^{j\pi[0.6,0.8]}}{x_3}, \frac{[0, 0.9] \cdot \mathrm{e}^{j\pi[0.9,1]}, [0.2, 0.3] \cdot \mathrm{e}^{j\pi[0.7,0.8]}, [0.3, 0.5] \cdot \mathrm{e}^{j\pi[0.4,0.5]}}{x_4} \end{aligned} \right\}$$

Further on, we present the connections among different types of fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, to complex neutrosophic set (in Fig. 1). The arrows (→) refer to the generalization of the preceding term to the next term, e.g., the fuzzy set is the generalization of the classic set, and so on.

3.2 Set Theoretic Operations of Interval Complex Neutrosophic Set

Definition 10 Let $\bar{A}$ and $\bar{B}$ be two interval complex neutrosophic set over X which are defined by $T_{\bar{A}}(x) = t_{\bar{A}}(x) \cdot \mathrm{e}^{j\pi\omega_{\bar{A}}(x)}$, $I_{\bar{A}}(x) = i_{\bar{A}}(x) \cdot \mathrm{e}^{j\pi\psi_{\bar{A}}(x)}$, $F_{\bar{A}}(x) = f_{\bar{A}}(x) \cdot \mathrm{e}^{j\pi\phi_{\bar{A}}(x)}$ and $T_{\bar{B}}(x) = t_{\bar{B}}(x) \cdot \mathrm{e}^{j\pi\omega_{\bar{B}}(x)}$, $I_{\bar{B}}(x) = i_{\bar{B}}(x) \cdot \mathrm{e}^{j\pi\psi_{\bar{B}}(x)}$, $F_{\overline{S}}(x) = f_{\overline{S}}(x) \cdot \mathrm{e}^{j\pi\phi_{\overline{S}}(x)}$, respectively. The *union* of $\bar{A}$ and $\bar{B}$ is denoted as
$\bar{A} \cup \bar{B}$, and it is defined as:

$$\begin{aligned} T_{\bar{A}\cup\bar{B}}(x) &= [\inf t_{\bar{A}\cup\bar{B}}(x), \sup t_{\bar{A}\cup\bar{B}}(x)] \cdot \mathrm{e}^{j\pi\omega_{\bar{A}\cup\bar{B}}(x)}, \\ I_{\bar{A}\cup\bar{B}}(x) &= [\inf i_{\bar{A}\cup\bar{B}}(x), \sup i_{\bar{A}\cup\bar{B}}(x)] \cdot \mathrm{e}^{j\pi\psi_{\bar{A}\cup\bar{B}}(x)}, \\ F_{\bar{A}\cup\bar{B}}(x) &= [\inf f_{\bar{A}\cup\bar{B}}(x), \sup f_{\bar{A}\cup\bar{B}}(x)] \cdot \mathrm{e}^{j\pi\phi_{\bar{A}\cup\bar{B}}(x)}, \end{aligned}$$

where

$$\begin{aligned} &\inf t_{\bar{A}\cup\bar{B}}(x) = \vee(\inf t_{\bar{A}}(x), \inf t_{\bar{B}}(x)), \sup t_{\bar{A}\cup\bar{B}}(x) = \vee(\sup t_{\bar{A}}(x), \sup t_{\bar{B}}(x)); \\ &\inf i_{\bar{A}\cup\bar{B}}(x) = \wedge(\inf i_{\bar{A}}(x), \inf i_{\bar{B}}(x)),\ \sup i_{\bar{A}\cup\bar{B}}(x) = \wedge(\sup i_{\bar{A}}(x), \sup i_{\bar{B}}(x)); \\ &\inf f_{\bar{A}\cup\bar{B}}(x) = \wedge(\inf f_{\bar{A}}(x), \inf f_{\bar{B}}(x)),\ \sup f_{\bar{A}\cup\bar{B}}(x) = \wedge(\sup f_{\bar{A}}(x), \sup f_{\bar{B}}(x)); \end{aligned}$$

for all $x \in X$. The union of the phase terms remains the same as defined for single-valued complex neutrosophic set, with the distinction that instead of subtractions and additions of numbers, we now have subtractions and additions of intervals. The symbols ∨,∧ represent max and min operators.

Example 3 Let $X = \{x_1, x_2, x_3, x_4\}$ be a universe of discourse. Let $\bar{A}$ and $\overline{B}$ be two interval complex neutrosophic sets defined on X as follows:

$$\bar{A} = \left\{ \begin{aligned} &\frac{[0.4, 0.6] \cdot \mathrm{e}^{j\pi[0.5,0.6]}, [0.1, 0.7] \cdot \mathrm{e}^{j\pi[0.1,0.3]}, [0.3, 0.5] \cdot \mathrm{e}^{j\pi[0.8,0.9]}}{x_1}, \frac{[0.2, 0.4] \cdot \mathrm{e}^{j\pi[0.3,0.6]}, [0.1, 0.1] \cdot \mathrm{e}^{j\pi[0.7,0.9]}, [0.5, 0.9] \cdot \mathrm{e}^{j\pi[0.2,0.5]}}{x_2}, \\ &\frac{[0.3, 0.4].e^{j\pi[0.7,0.8]}, [0.6, 0.7] \cdot \mathrm{e}^{j\pi[0.6,0.7]}, [0.2, 0.6] \cdot \mathrm{e}^{j\pi[0.6,0.8]}}{x_3}, \frac{[0, 0.9] \cdot \mathrm{e}^{j\pi[0.9,1]}, [0.2, 0.3] \cdot \mathrm{e}^{j\pi[0.7,0.8]}, [0.3, 0.5] \cdot \mathrm{e}^{j\pi[0.4,0.5]}}{x_4} \end{aligned} \right\}$$

$$\bar{B} = \left\{ \begin{aligned} &\frac{[0.3, 0.7] \cdot \mathrm{e}^{j\pi[0.7,0.8]}, [0.4, 0.9] \cdot \mathrm{e}^{j\pi[0.3,0.5]}, [0.6, 0.8] \cdot \mathrm{e}^{j\pi[0.5,0.6]}}{x_1}, \frac{[0.4, 0.4] \cdot \mathrm{e}^{j\pi[0.6,0.7]}, [0.1, 0.9] \cdot \mathrm{e}^{j\pi[0.2,0.4]}, [0.3, 0.8] \cdot \mathrm{e}^{j\pi[0.5,0.6]}}{x_2}, \\ &\frac{[0.37, 0.64] \cdot \mathrm{e}^{j\pi[0.47,0.50}, [0.36, 0.57] \cdot \mathrm{e}^{j\pi[0.64,0.7]}, [0.28, 0.66] \cdot \mathrm{e}^{j\pi[0.16,0.2]}}{x_3}, \frac{[0.15, 0.52] \cdot \mathrm{e}^{j\pi[0.1,0.2]}, [0, 0.5] \cdot \mathrm{e}^{j\pi[0.6,0.7]}, [0.3, 0.3] \cdot \mathrm{e}^{j\pi[0.6,0.7]}}{x_4} \end{aligned} \right\}$$

Then, their union $\bar{A} \cup \bar{B}$ is given by:

$$\bar{A} \cup \bar{B} = \left\{ \begin{array}{c} \frac{[0.4,0.7]\cdot e^{j\pi[0.7,0.8]},\,[0.1,0.7]\cdot e^{j\pi[0.1,0.3]},\,[0.3,0.5]\cdot e^{j\pi[0.5,0.6]}}{x_1}, \frac{[0.4,0.4]\cdot e^{j\pi[0.6,0.7]},\,[0.1,0.1]\cdot e^{j\pi[0.7,0.9]},\,[0.3,0.8]\cdot e^{j\pi[0.5,0.6]}}{x_2}, \\ \frac{[0.37,0.64]\cdot e^{j\pi[0.7,0.8]},\,[0.36,0.57]\cdot e^{j\pi[0.6,0.7]},\,[0.2,0.6]\cdot e^{j\pi[0.16,0.21]}}{x_3}, \frac{[0.15,0.9]\cdot e^{j\pi[0.9,1]},\,[0,0.3]\cdot e^{j\pi[0.6,,0.7]},\,[0.3,0.3]\cdot e^{j\pi[0.4,0.5]}}{x_4} \end{array} \right\}$$

Definition 11 Let $\bar{A}$ and $\bar{B}$ be two interval complex neutrosophic set over X which are defined by $T_{\bar{A}}(x) = t_{\bar{A}}(x) \cdot e^{j\pi\omega_{\bar{A}}(x)}$, $I_{\bar{A}}(x) = i_{\bar{A}}(x) \cdot e^{j\pi\psi_{\bar{A}}(x)}$, $F_{\bar{A}}(x) = f_{\bar{A}}(x) \cdot e^{j\pi\phi_{\bar{A}}(x)}$ and $T_{\bar{B}}(x) = t_{\bar{B}}(x) \cdot e^{j\pi\omega_{\bar{B}}(x)}$, $I_{\bar{B}}(x) = i_{\bar{B}}(x) \cdot e^{j\pi\psi_{\bar{B}}(x)}$, $F_{\bar{S}}(x) = f_{\bar{S}}(x) \cdot e^{j\pi\phi_{\bar{S}}(x)}$, respectively. The *intersection* of $\bar{A}$ and $\bar{B}$ is denoted as $\bar{A} \cap \bar{B}$, and it is defined as:

$$T_{\bar{A}\cap\bar{B}}(x) = [\inf t_{\bar{A}\cap\bar{B}}(x), \sup t_{\bar{A}\cap\bar{B}}(x)] \cdot e^{j\pi\omega_{\bar{A}\cap\bar{B}}(x)},$$
$$I_{\bar{A}\cap\bar{B}}(x) = [\inf i_{\bar{A}\cap\bar{B}}(x), \sup i_{\bar{A}\cap\bar{B}}(x)] \cdot e^{j\pi\psi_{\bar{A}\cap\bar{B}}(x)},$$
$$F_{\bar{A}\cap\bar{B}}(x) = [\inf f_{\bar{A}\cap\bar{B}}(x), \sup f_{\bar{A}\cap\bar{B}}(x)] \cdot e^{j\pi\phi_{\bar{A}\cap\bar{B}}(x)},$$

where

$$\inf t_{\bar{A}\cap\bar{B}}(x) = \wedge(\inf t_{\bar{A}}(x), \inf t_{\bar{B}}(x)),\ \sup t_{\bar{A}\cap\bar{B}}(x) = \wedge(\sup t_{\bar{A}}(x), \sup t_{\bar{B}}(x)),$$
$$\inf i_{\bar{A}\cap\bar{B}}(x) = \vee(\inf i_{\bar{A}}(x), \inf i_{\bar{B}}(x)),\ \sup i_{\bar{A}\cap\bar{B}}(x) = \vee(\sup i_{\bar{A}}(x), \sup i_{\bar{B}}(x)),$$
$$\inf f_{\bar{A}\cap\bar{B}}(x) = \vee(\inf f_{\bar{A}}(x), \inf f_{\bar{B}}(x)),\ \sup f_{\bar{A}\cap\bar{B}}(x) = \vee(\sup f_{\bar{A}}(x), \sup f_{\bar{B}}(x)),$$

for all $x \in X$. Similarly, the intersection of the phase terms remains the same as defined for single-valued complex neutrosophic set, with the distinction that instead of subtractions and additions of numbers we now have subtractions and additions of intervals. The symbols $\vee$,$\wedge$ represent max and min operators.

Example 4 Let X, $\bar{A}$ and $\overline{B}$ be as in Example 3. Then, the intersection $\bar{A} \cap \bar{B}$ is given by:

$$\bar{A} \cap \bar{B} = \left\{ \begin{array}{c} \frac{[0.3,0.6]\cdot e^{j\pi[0.5,0.6]},\,[0.4,0.9]\cdot e^{j\pi[0.3,0.5]},\,[0.6,0.8]\cdot e^{j\pi[0.8,0.9]}}{x_1}, \frac{[0.2,0.4]\cdot e^{j\pi[0.3,0.6]},\,[0.1,0.9]\cdot e^{j\pi[0.7.0.9]},\,[0.5,0.9]\cdot e^{j\pi[0.5,0.6]}}{x_2}, \\ \frac{[0.3,0.4]\cdot e^{j\pi[0.47,0.50]},\,[0.6,0.7]\cdot e^{j\pi[0.64,0.70]},\,[0.28,0.6]6\cdot e^{j\pi[0.6,0.8]}}{x_3}, \frac{[0,0.52]\cdot e^{j\pi[0.1,0.2]},\,[0.2,0.5]\cdot e^{j\pi[0.7,0.8]},\,[0.3,0.5]\cdot e^{j\pi[0.6,0.7]}}{x_4} \end{array} \right\}$$

Definition 12 Let $\bar{A}$ be an interval complex neutrosophic set over X which is defined by $T_{\bar{A}}(x) = t_{\bar{A}}(x) \cdot e^{j\pi\omega_{\bar{A}}(x)}$, $I_{\bar{A}}(x) = i_{\bar{A}}(x) \cdot e^{j\pi\psi_{\bar{A}}(x)}$, $F_{\bar{A}}(x) = f_{\bar{A}}(x) \cdot e^{j\pi\phi_{\bar{A}}(x)}$. The *complement* of $\bar{A}$ is denoted as $\bar{A}^c$, and it is defined as:

$$\bar{A}^c = \left\{ \left\langle \frac{T_{\bar{A}^c}(x) = t_{\bar{A}^c}(x) \cdot e^{j\pi\omega_{\bar{A}^c}(x)}, I_{\bar{A}^c}(x) = i_{\bar{A}^c}(x) \cdot e^{j\pi\psi_{\bar{A}^c}(x)}, F_{\bar{A}^c}(x) = f_{\bar{A}^c}(x) \cdot e^{j\pi\phi_{\bar{A}^c}(x)}}{x} \right\rangle : x \in X \right\},$$

where $t_{\bar{A}^c}(x) = f_{\bar{A}}(x)$ and $\omega_{\bar{A}^c}(x) = 2\pi - \omega_{\bar{A}}(x)$ or $\omega_{\bar{A}}(x) + \pi$. Similarly,$i_{\bar{A}^c}(x) = (\inf i_{\bar{A}^c}(x), \sup i_{\bar{A}^c}(x))$, where $\inf i_{\bar{A}^c}(x) = 1 - \sup i_{\bar{A}}(x)$ and $\sup i_{\bar{A}^c}(x) = 1 - \inf i_{\bar{A}}(x)$, with phase term $\psi_{\bar{A}^c}(x) = 2\pi - \psi_{\bar{A}}(x)$ or $\psi_{\bar{A}}(x) + \pi$. Also, $f_{\bar{A}^c}(x) = i_{\bar{A}^c}(x)$, while the phase term $\phi_{\bar{A}^c}(x) = 2\pi - \phi_{\bar{A}}(x)$ or $\phi_{\bar{A}}(x) + \pi$.

Proposition 1 *Let $\bar{A}$, $\bar{B}$ and $\overline{C}$ be three interval complex neutrosophic sets over X. Then:*

1. $\bar{A} \cup \bar{B} = \bar{B} \cup \bar{A}$,
2. $\bar{A} \cap \bar{B} = \bar{B} \cap \bar{A}$,
3. $\bar{A} \cup \bar{A} = \bar{A}$,
4. $\bar{A} \cap \bar{A} = \bar{A}$,
5. $\bar{A} \cup (\bar{B} \cup \overline{C}) = (\bar{A} \cup \bar{B}) \cup \overline{C}$,
6. $\bar{A} \cap (\bar{B} \cap \overline{C}) = (\bar{A} \cap \bar{B}) \cap \overline{C}$,
7. $\bar{A} \cup (\bar{B} \cap \overline{C}) = (\bar{A} \cup \bar{B}) \cap (\bar{A} \cup \overline{C})$,
8. $\bar{A} \cap (\bar{B} \cup \overline{C}) = (\bar{A} \cap \bar{B}) \cup (\bar{A} \cap \overline{C})$,
9. $\bar{A} \cup (\bar{A} \cap \bar{B}) = \bar{A}$,
10. $\bar{A} \cap (\bar{A} \cup \bar{B}) = \bar{A}$,
11. $(\bar{A} \cup \bar{B})^c = \bar{A}^c \cap \bar{B}^c$,
12. $(\bar{A} \cap \bar{B})^c = \bar{A}^c \cup \bar{B}^c$,
13. $(\bar{A}^c)^c = \bar{A}$.

Proof All these assertions can be straightforwardly proven.

Theorem 1 *The interval complex neutrosophic set $\bar{A} \cup \bar{B}$ is the smallest one containing both $\bar{A}$ and $\bar{B}$.*

Proof Straightforwardly.

Theorem 2 *The interval complex neutrosophic set $\bar{A} \cap \bar{B}$ is the largest one contained in both $\bar{A}$ and $\bar{B}$.*

Proof Straightforwardly.

Theorem 3 *Let $\overline{P}$ be the power set of all interval complex neutrosophic set. Then, $(\overline{P}, \cup, \cap)$ forms a distributive lattice.*

Proof Straightforwardly.

Theorem 4 *Let $\bar{A}$ and $\bar{B}$ be two interval complex neutrosophic sets defined on X. Then, $\bar{A} \subseteq \bar{B}$ if and only if $\bar{B}^c \subseteq \bar{A}^c$.*

Proof Straightforwardly.

3.3 Operational Rules of Interval Complex Neutrosophic Sets

Let $\bar{A} = ([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U])$ and $\overline{B} = ([T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U])$ be two interval complex neutrosophic sets over X which are defined by $[T_A^L, T_A^U] = [t_A^L(x), t_A^U(x)] \cdot e^{j\pi[\omega_A^L(x), \omega_A^U(x)]}$,$[I_A^L, I_A^U] = [i_A^L(x), i_A^U(x)] \cdot e^{j\pi[\psi_A^L(x), \psi_A^U(x)]}$, $[F_A^L, F_A^U] = [f_A^L(x), f_A^U(x)] \cdot e^{j\pi[\phi_A^L(x), \phi_A^U(x)]}$ and $[T_B^L, T_B^U] = [t_B^L(x), t_B^U(x)] \cdot e^{j\pi[\omega_B^L(x), \omega_B^U(x)]}$, $[I_B^L, I_B^U] = [i_B^L(x), i_B^U(x)] \cdot e^{j\pi[\psi_B^L(x), \psi_B^U(x)]}$, $[F_B^L, F_B^U] = [f_B^L(x), f_B^U(x)] \cdot e^{j\pi[\phi_B^L(x), \phi_B^U(x)]}$, respectively. Then, the operational rules of ICNS are defined as follows:

(a) The *product* of $\bar{A}$ and $\bar{B}$, denoted as $\bar{A} \times \bar{B}$, is:

$$T_{\bar{A}*\bar{B}}(x) = \left[t^L_{\bar{A}}(x)t^L_{\bar{B}}(x), t^U_{\bar{A}}(x)t^U_{\bar{B}}(x)\right] \cdot e^{j\pi[\omega^L_{\bar{A}*\bar{B}}(x), \omega^R_{\bar{A}*\bar{B}}(x)]},$$
$$I_{\bar{A}*\bar{B}}(x) = \left[i^L_{\bar{A}}(x) + i^L_{\bar{B}}(x) - i^L_{\bar{A}}(x)i^L_{\bar{B}}(x), i^R_{\bar{A}}(x) + i^R_{\bar{B}}(x) - i^R_{\bar{A}}(x)i^R_{\bar{B}}(x)\right] \cdot e^{j\pi[\psi^L_{\bar{A}*\bar{B}}(x), \psi^R_{\bar{A}*\bar{B}}(x)]},$$

$F_{\bar{A}\times\bar{B}}(x) = \left[f^L_{\bar{A}}(x) + f^L_{\bar{B}}(x) - f^L_{\bar{A}}(x)f^L_{\bar{B}}(x), \quad f^R_{\bar{A}}(x) + f^R_{\bar{B}}(x) - f^R_{\bar{A}}(x)f^R_{\bar{B}}(x)\right] \cdot e^{j\pi[\phi^L_{\bar{A}\times\bar{B}}(x), \phi^R_{\bar{A}\times\bar{B}}(x)]}$ The product of phase terms is defined below:

$$\omega^L_{\bar{A}\times\bar{B}}(x) = \omega^L_{\bar{A}}(x)\omega^L_{\bar{B}}(x),\ \omega^U_{\bar{A}\times\bar{B}}(x) = \omega^U_{\bar{A}}(x)\omega^U_{\bar{B}}(x)$$
$$\psi^L_{\bar{A}\times\bar{B}}(x) = \psi^L_{\bar{A}}(x)\psi^L_{\bar{B}}(x),\ \psi^U_{\bar{A}\times\bar{B}}(x) = \psi^U_{\bar{A}}(x)\psi^U_{\bar{B}}(x)$$
$$\phi^L_{\bar{A}\times\bar{B}}(x) = \phi^L_{\bar{A}}(x)\phi^L_{\bar{B}}(x),\ \phi^U_{\bar{A}\times\bar{B}}(x) = \phi^U_{\bar{A}}(x)\phi^U_{\bar{B}}(x).$$

(b) The *addition* of $\bar{A}$ and $\bar{B}$, denoted as $\bar{A}+\bar{B}$, is defined as:

$$T_{\bar{A}+\bar{B}}(x) = \left[t^L_{\bar{A}}(x) + t^L_{\bar{B}}(x) - t^L_{\bar{A}}(x)t^L_{\bar{B}}(x), t^U_{\bar{A}}(x) + t^U_{\bar{B}}(x) - t^U_{\bar{A}}(x)t^U_{\bar{B}}(x)\right] \cdot e^{j\pi[\omega^L_{\bar{A}+\bar{B}}(x), \omega^L_{\bar{A}+\bar{B}}(x)]},$$
$$I_{\bar{A}+\bar{B}}(x) = \left[i^L_{\bar{A}}(x)i^L_{\bar{B}}(x), i^U_{\bar{A}}(x)i^U_{\bar{B}}(x)\right] \cdot e^{j\pi[\psi^L_{\bar{A}+\bar{B}}(x), \psi^R_{\bar{A}+\bar{B}}(x)]},$$
$$F_{\bar{A}+\bar{B}}(x) = \left[f^L_{\bar{A}}(x)f^L_{\bar{B}}(x), f^R_{\bar{A}}(x)f^R_{\bar{B}}(x)\right] \cdot e^{j\pi[\phi^L_{\bar{A}+\bar{B}}(x), \phi^R_{\bar{A}+\bar{B}}(x)]}$$

The addition of phase terms is defined below:

$$\omega^L_{\bar{A}+\bar{B}}(x) = \omega^L_{\bar{A}}(x) + \omega^L_{\bar{B}}(x),\ \omega^U_{\bar{A}+\bar{B}}(x) = \omega^U_{\bar{A}}(x) + \omega^U_{\bar{B}}(x)$$
$$\psi^L_{\bar{A}+\bar{B}}(x) = \psi^L_{\bar{A}}(x) + \psi^L_{\bar{B}}(x),\ \psi^U_{\bar{A}+\bar{B}}(x) = \psi^U_{\bar{A}}(x) + \psi^U_{\bar{B}}(x)$$
$$\phi^L_{\bar{A}+\bar{B}}(x) = \phi^L_{\bar{A}}(x) + \phi^L_{\bar{B}}(x),\ \phi^U_{\bar{A}+\bar{B}}(x) = \phi^U_{\bar{A}}(x) + \phi^U_{\bar{B}}(x)$$

(c) The *scalar multiplication* of $\bar{A}$ is an interval complex neutrosophic set denoted as $\overline{C} = k\bar{A}$ and defined as:

$$T_{\overline{C}}(x) = \left[1-(1-t^L_{\bar{A}}(x))^k, 1-(1-t^R_{\bar{A}}(x))^k\right] \cdot e^{j\pi[\omega^L_{\overline{C}}(x), \omega^R_{\overline{C}}(x)]},$$
$$I_{\overline{C}}(x) = [(i^L_{\bar{A}}(x))^k, (i^R_{\bar{A}}(x))^k] \cdot e^{j\pi[\psi^L_{\overline{C}}(x), \psi^R_{\overline{C}}(x)]},$$
$$F_{\overline{C}}(x) = [(f^L_{\bar{A}}(x))^k, (i^R_{\bar{A}}(x))^k] \cdot e^{j\pi[\phi^L_{\overline{C}}(x), \phi^R_{\overline{C}}(x)]}$$

The scalar of phase terms is defined below:

$$\omega^L_{\overline{C}}(x) = \omega^L_{\bar{A}}(x)\cdot k;\quad \omega^R_{\overline{C}}(x) = \omega^R_{\bar{A}}(x)\cdot k,$$
$$\psi^L_{\overline{C}}(x) = \psi^L_{\bar{A}}(x)\cdot k;\quad \psi^R_{\overline{C}}(x) = \psi^R_{\bar{A}}(x)\cdot k,$$
$$\phi^L_{\overline{C}}(x) = \phi^L_{\bar{A}}(x)\cdot k;\quad \phi^R_{\overline{C}}(x) = \phi^R_{\bar{A}}(x)\cdot k$$

4 A Multi-criteria Group Decision-Making Model in ICNS

Definition 13 Let us assume that a committee of h decision-makers $(D_q, q = 1,\ldots,h)$ is responsible for evaluating o alternatives $(A_o, o = 1,\ldots,m)$ under p selection criteria $(C_p, p = 1,\ldots,n)$, where the suitability ratings of alternatives under each criterion, as well as the weights of all criteria, are assessed in IVCNS. The steps of the proposed MCGDM method are as follows:

4.1 Aggregate Ratings of Alternatives Versus Criteria

Let $x_{opq} = ([T^L_{opq}, T^U_{opq}], [I^L_{opq}, I^U_{opq}], [F^L_{opq}, F^U_{opq}])$ be the suitability rating assigned to alternative A_o by decision-maker D_q for criterion C_p, where $[T^L_{opq}, T^U_{opq}] = [t^L_{opq}, t^U_{opq}] \cdot e^{j\pi[\omega^L(x), \omega^U(x)]}$, $[I^L_{opq}, I^U_{opq}] = [i^L_{opq}, i^U_{opq}] \cdot e^{j\pi[\psi^L(x), \psi^U(x)]}$, $[F^L_{opq}, F^U_{opq}] = [f^L_{opq}, f^U_{opq}] \cdot e^{j\pi[\phi^L(x), \phi^U(x)]}$, $o = 1,\ldots,m$; $p = 1,\ldots,n$; $q = 1,\ldots,h$. Using the operational rules of the IVCNS, the averaged suitability rating $x_{op} = ([T^L_{op}, T^U_{op}], [I^L_{op}, I^U_{op}], [F^L_{op}, F^U_{op}])$ can be evaluated as:

$$x_{op} = \frac{1}{h} \otimes (x_{op1} \oplus x_{op2} \oplus \cdots \oplus x_{opq} \oplus \cdots \oplus x_{oph}), \tag{3}$$

where $T_{op} = \left[\wedge\left(\frac{1}{h}\sum_{q=1}^{h} t^L_{opq}, 1\right), \wedge\left(\frac{1}{h}\sum_{q=1}^{h} t^R_{opq}, 1\right),\right] e^{j\pi\left[\frac{1}{h}\sum_{q=1}^{h} w^L_q(x), \frac{1}{h}\sum_{q=1}^{h} w^U_q(x)\right]}$

$$I_{op} = \left[\wedge\left(\frac{1}{h}\sum_{q=1}^{h} i^L_{opq}, 1\right), \wedge\left(\frac{1}{h}\sum_{q=1}^{h} i^R_{opq}, 1\right),\right] e^{j\pi\left[\frac{1}{h}\sum_{q=1}^{h} \psi^L_q(x), \frac{1}{h}\sum_{q=1}^{h} \psi^U_q(x)\right]}$$

$$F_{op} = \left[\wedge\left(\frac{1}{h}\sum_{q=1}^{h} f^L_{opq}, 1\right), \wedge\left(\frac{1}{h}\sum_{q=1}^{h} f^R_{opq}, 1\right),\right] e^{j\pi\left[\frac{1}{h}\sum_{q=1}^{h} \phi^L_q(x), \frac{1}{h}\sum_{q=1}^{h} \phi^U_q(x)\right]}$$

4.2 Aggregate the Importance Weights

Let $w_{pq} = ([T^L_{pq}, T^U_{pq}], [I^L_{pq}, I^U_{pq}], [F^L_{pq}, F^U_{pq}])$ be the weight assigned by decision-maker D_q to criterion C_p, where $[T^L_{pq}, T^U_{pq}] = [t^L_{pq}, t^U_{pq}] \cdot e^{j\pi[\omega^L(x), \omega^U(x)]}$, $[I^L_{pq}, I^U_{pq}] = [i^L_{pq}, i^U_{pq}] \cdot e^{j\pi[\psi^L(x), \psi^U(x)]}$, $[F^L_{pq}, F^U_{pq}] = [f^L_{pq}, f^U_{pq}] \cdot e^{j\pi[\phi^L(x), \phi^U(x)]}$, $F^U_{pq} = f^U_{pq} \cdot e^{j\pi\phi(x)}$, $p = 1,\ldots,n$; $q = 1,\ldots,h$. Using the operational rules of the IVCNS, the average weight $w_p = ([T^L_p, T^U_p], [I^L_p, I^U_p], [F^L_p, F^U_p])$ can be evaluated as:

$$w_p = (\frac{1}{h}) \otimes (w_{p1} \oplus w_{p2} \oplus \cdots \oplus w_{ph}), \tag{4}$$

where $T_p = \left[\wedge\left(\frac{1}{h}\sum_{q=1}^{h} t^L_{pq}, 1\right), \wedge\left(\frac{1}{h}\sum_{q=1}^{h} t^R_{pq}, 1\right),\right] e^{j\pi\left[\frac{1}{h}\sum_{q=1}^{h} w^L_q(x), \frac{1}{h}\sum_{q=1}^{h} w^U_q(x)\right]}$

$$I_p = \left[\wedge\left(\frac{1}{h}\sum_{q=1}^{h} i^L_{pq}, 1\right), \wedge\left(\frac{1}{h}\sum_{q=1}^{h} i^R_{pq}, 1\right)\right] e^{j\pi\left[\frac{1}{h}\sum_{q=1}^{h}\psi^L_q(x), \frac{1}{h}\sum_{q=1}^{h}\psi^U_q(x)\right]}$$

$$F_p = \left[\wedge\left(\frac{1}{h}\sum_{q=1}^{h} f^L_{pq}, 1\right), \wedge\left(\frac{1}{h}\sum_{q=1}^{h} f^R_{pq}, 1\right)\right] e^{j\pi\left[\frac{1}{h}\sum_{q=1}^{h}\phi^L_q(x), \frac{1}{h}\sum_{q=1}^{h}\phi^U_q(x)\right]}$$

4.3 Aggregate the Weighted Ratings of Alternatives Versus Criteria

The weighted ratings of alternatives can be developed via the operations of interval complex neutrosophic set as follows:

$$V_o = \frac{1}{p}\sum_{p=1}^{h} x_{op} \times w_p, \quad o = 1, \ldots, m; \quad p = 1, \ldots, h. \quad (5)$$

4.4 Ranking the Alternatives

In this section, the modified score function, the accuracy function and the certainty function of an ICNS, i.e., $V_o = ([T^L_o, T^U_o], [I^L_o, I^U_o], [F^L_o, F^U_o])$, $o = 1, \ldots, m$, adopted from Ye [20], are developed for ranking alternatives in decision-making problems, where

$[T^L_o, T^U_o] = [t^L_o, t^U_o]e^{j\pi[\omega^L(x), \omega^U(x)]}$, $[I^L_o, I^U_o] = [i^L_o, i^U_o]e^{j\pi[\psi^L(x), \psi^U(x)]}$, $[F^L_o, F^U_o] = [f^L_o, f^U_o]e^{j\pi[\phi^L(x), \phi^U(x)]\phi^U(x)]}$

The values of these functions for amplitude terms are defined as follows:

$$e^a_{V_o} = \frac{1}{6}(4 + t^L_o - i^L_o - f^L_o + t^U_o - i^U_o - f^U_o),\ h^a_{V_o} = \frac{1}{2}(t^L_o - f^L_o + t^U_o - f^U_o), \text{ and } c^a_{V_o} = \frac{1}{2}(t^L_o + t^U_o)$$

The values of these functions for phase terms are defined below:

$$e^p_{V_o} = \pi[\omega^L(x) - \psi^L(x) - \phi^L(x) + \omega^R(x) - \psi^R(x) - \phi^R(x)],$$
$$h^p_{V_o} = \pi[\omega^L(x) - \phi^L(x) + \omega^R(x) - \phi^R(x)], \text{ and } c^p_{V_o} = \pi[\omega^L(x) + \omega^R(x)]$$

Let V_1 and V_2 be any two ICNSs. Then, the ranking method can be defined as follows:

- If $e^a_{V_1} > e^a_{V_2}$, then $V_1 > V_2$
- If $e^a_{V_1} = e^a_{V_2}$ and $e^p_{V_1} > e^p_{V_2}$, then $V_1 > V_2$
- If $e^a_{V_1} = e^a_{V_2}, e^p_{V_1} = e^p_{V_2}$ and $h^a_{V_1} > h^a_{V_2}$, then $V_1 > V_2$
- If $e^a_{V_1} = e^a_{V_2}, e^p_{V_1} = e^p_{V_2}, h^a_{V_1} = h^a_{V_2}$ and $h^p_{V_1} > h^p_{V_2}$, then $V_1 > V_2$
- If $e^a_{V_1} = e^a_{V_2}, e^p_{V_1} = e^p_{V_2}, h^a_{V_1} = h^a_{V_2}, h^p_{V_1} = h^p_{V_2}$ and $c^a_{V_1} > c^a_{V_2}$, then $V_1 > V_2$
- If $e^a_{V_1} = e^a_{V_2}, e^p_{V_1} = e^p_{V_2}, h^a_{V_1} = h^a_{V_2}, h^p_{V_1} = h^p_{V_2}, c^a_{V_1} = c^a_{V_2}$ and $c^p_{V_1} > c^p_{V_2}$, then $V_1 > V_2$
- If $e^a_{V_1} = e^a_{V_2}, e^p_{V_1} = e^p_{V_2}, h^a_{V_1} = h^a_{V_2}, h^p_{V_1} = h^p_{V_2}, c^a_{V_1} = c^a_{V_2}$ and $c^p_{V_1} = c^p_{V_2}$, then $V_1 = V_2$

5 Application of the Proposed MCGDM Approach

This section applies the proposed MCGDM for green supplier selection in the case study of Thuan Yen JSC, which is a small-size trading service and transportation company. The managers of this company would like to effectively manage the suppliers, due to an increasing number of them. Data were collected by conducting semi-structured interviews with managers and department heads. Three managers (decision-makers), i.e., *D*1–*D*3, were requested to separately proceed to their own evaluation for the importance weights of selection criteria and the ratings of suppliers. According to the survey and the discussions with the managers and department heads, five criteria, namely Price/cost (*C*1), Quality (*C*2), Delivery (*C*3), Relationship Closeness (*C*4) and Environmental Management Systems (*C*5), were selected to evaluate the green suppliers. The entire green supplier selection procedure was characterized by the following steps:

5.1 Aggregation of the Ratings of Suppliers Versus the Criteria

Three managers determined the suitability ratings of three potential suppliers versus the criteria using the linguistic rating set $S = \{\text{VL, L, F, G, VG}\}$ where VL = Very Low = ([0.1, 0.2]$e^{j\pi[0.7,0.8]}$, [0.7, 0.8]$e^{j\pi[0.9,1.0]}$, [0.6, 0.7]$e^{j\pi[1.0,1.1]}$), L = Low = ([0.3, 0.4]$e^{j\pi[0.8,0.9]}$, [0.6, 0.7]$e^{j\pi[1.0,1.1]}$, [0.5, 0.6]$e^{j\pi[0.9,1.0]}$), F = Fair = ([0.4, 0.5]$e^{j\pi[0.8,0.9]}$, [0.5, 0.6]$e^{j\pi[0.9,1.0]}$,[0.4, 0.5]$e^{j\pi[0.8,0.9]}$), G = Good = ([0.6, 0.7]$e^{j\pi[0.9,1.0]}$, [0.4, 0.5]$e^{j\pi[0.9,1.0]}$, [0.3, 0.4]$e^{j\pi[0.7,0.8]}$), and VG = Very Good = ([0.7, 0.8] $e^{j\pi[1.1,1.2]}$, [0.2, 0.3]$e^{j\pi[0.8,0.9]}$, [0.1, 0.2]$e^{j\pi[0.6,0.7]}$), to evaluate the suitability of the suppliers under each criteria. Table 1 gives the aggregated ratings of three suppliers (A_1, A_2, A_3) versus five criteria (C_1,…, C_5) from three decision-makers (D_1, D_2, D_3) using Eq. (3).

5.2 Aggregation of the Importance Weights

After determining the green suppliers criteria, the three company managers are asked to determine the level of importance of each criterion using a linguistic weighting set $Q = \{\text{UI, OI, I, VI, AI}\}$ where UI = Unimportant = ([0.2, 0.3]$e^{j\pi[0.7,0.8]}$, [0.5, 0.6]$e^{j\pi[0.9,1.0]}$, [0.5, 0.6]$e^{j\pi[1.1,1.2]}$), OI = Ordinary Important = ([0.3, 0.4]$e^{j\pi[0.8,0.9]}$, [0.5, 0.6]$e^{j\pi[1.0,1.1]}$, [0.4, 0.5]$e^{j\pi[0.9,1.0]}$), I = Important = ([0.5, 0.6]$e^{j\pi[0.9,1.0]}$, [0.4, 0.5]$e^{j\pi[0.9,1.0]}$, [0.3,

Table 1 Aggregated ratings of suppliers versus the criteria

Criteria	Suppliers	Decision-makers			Aggregated ratings
		D_1	D_2	D_3	
C_1	A_1	G	F	G	([0.542, 0.644]$e^{j\pi[0.867,0.967]}$, [0.431, 0.531]$e^{j\pi[0.9,1.0]}$], [0.33, 0.431]$e^{j\pi[0.733,0.833]}$)
	A_2	F	F	G	([0.476, 0.578]$e^{j\pi[0.833,0.933]}$, [0.464, 0.565]$e^{j\pi[0.9,1.0]}$, [0.363, 0.464]$e^{j\pi[0.767,0.867]}$)
	A_3	VG	G	VG	([0.67, 0.771]$e^{j\pi[1.033,1.133]}$, [0.252, 0.356]$e^{j\pi[0.833,0.933]}$, [0.144, 0.252]$e^{j\pi[0.633,0.733]}$)
C_2	A_1	F	F	F	([0.4, 0.5]$e^{j\pi[0.8,0.9]}$, [0.5, 0.6]$e^{j\pi[0.9,1.0]}$, [0.4, 0.5]$e^{j\pi[0.8,0.9]}$)
	A_2	VG	G	G	([0.637, 0.738]$e^{j\pi[0.967,1.067]}$, [0.317, 0.422]$e^{j\pi[0.867,0.967]}$, [0.208, 0.317]$e^{j\pi[0.667,0.767]}$)
	A_3	F	G	G	([0.542, 0.644]$e^{j\pi[0.867,0.967]}$, [0.431, 0.531]$e^{j\pi[0.9,1.0]}$, [0.33, 0.431]$e^{j\pi[0.733,0.833]}$)
C_3	A_1	L	F	L	([0.335, 0.435]$e^{j\pi[0.8,0.9]}$, [0.565, 0.665]$e^{j\pi[0.967,1.067]}$, [0.464, 0.565]$e^{j\pi[0.867,0.967]}$)
	A_2	G	G	G	([0.6, 0.7]$e^{j\pi[0.9,1.0]}$, [0.4, 0.5]$e^{j\pi[0.9,1.0]}$, [0.3, 0.4]$e^{j\pi[0.7,0.8]}$)
	A_3	F	G	F	([0.476, 0.578]$e^{j\pi[0.833,0.933]}$, [0.464, 0.565]$e^{j\pi[0.9,1.0]}$, [0.363, 0.464]$e^{j\pi[0.767,0.867]}$)
C_4	A_1	G	F	G	([0.542, 0.644]$e^{j\pi[0.867,0.967]}$, [0.431, 0.531]$e^{j\pi[0.9,1.0]}$, [0.33, 0.431]$e^{j\pi[0.733,0.833]}$)
	A_2	F	F	L	([0.368, 0.469]$e^{j\pi[0.8,0.9]}$, [0.531, 0.632]$e^{j\pi[0.933,1.033]}$, [0.431, 0.531]$e^{j\pi[0.833,0.933]}$)
	A_3	G	VG	G	([0.637, 0.738]$e^{j\pi[0.967,1.067]}$, [0.317, 0.422]$e^{j\pi[0.867,0.967]}$, [0.208, 0.317]$e^{j\pi[0.667,0.767]}$)
C_5	A_1	L	F	L	([0.335, 0.435]$e^{j\pi[0.8,0.9]}$, [0.565, 0.665]$e^{j\pi[0.967,1.067]}$, [0.464, 0.565]$e^{j\pi[0.867,0.967]}$)
	A_2	G	G	VG	([0.637, 0.738]$e^{j\pi[0.967,1.067]}$, [0.317, 0.422]$e^{j\pi[0.867,0.967]}$, [0.208, 0.317]$e^{j\pi[0.667,0.767]}$)
	A_3	G	F	F	([0.476, 0.578]$e^{j\pi[0.833,0.933]}$, [0.464, 0.565]$e^{j\pi[0.9,1.0]}$, [0.363, 0.464]$e^{j\pi[0.767,0.867]}$)

Table 2 The importance and aggregated weights of the criteria

Criteria	Decision-makers			Aggregated weights
	D_1	D_2	D_3	
C_1	VI	I	I	([0.578, 0.683]$e^{j\pi[0.9,1.0]}$, [0.363, 0.464]$e^{j\pi[0.9,1.0]}$, [0.262, 0.363]$e^{j\pi[0.767,0.867]}$)
C_2	AI	VI	VI	([0.738, 0.841]$e^{j\pi[0.933,1.033]}$, [0.262, 0.363]$e^{j\pi[0.867,0.967]}$, [0.159, 0.262]$e^{j\pi[0.667,0.767}$)
C_3	VI	VI	I	([0.644, 0.748]$e^{j\pi[0.9,1.0]}$, [0.33, 0.431]$e^{j\pi[0.9,1.0]}$, [0.229, 0.33]$e^{j\pi[0.733,0.833]}$)
C_4	I	I	I	([0.5, 0.6]$e^{j\pi[0.9,1.0]}$], [0.4, 0.5]$e^{j\pi[0.9,1.0]}$, [0.3, 0.4]$e^{j\pi[0.8,0.9]}$)
C_5	I	OI	OI	([0.374, 0.476]$e^{j\pi[0.833,0.933]}$, [0.391, 0.565]$e^{j\pi[0.967,1.067]}$, [0.363, 0.464]$e^{j\pi[0.867,0.967]}$)

Table 3 The final fuzzy evaluation values of each supplier

Suppliers	Aggregated weights
A_1	([0.247, 0.361]$e^{j\pi[0.739,0.921]}$, [0.673, 0.784]$e^{j\pi[0.841,1.034]}$, [0.552, 0.679]$e^{j\pi[0.614,0.78]}$)
A_2	([0.319, 0.449]$e^{j\pi[0.798,0.986]}$, [0.607, 0.733]$e^{j\pi[0.81,1.0]}$, [0.475, 0.617]$e^{j\pi[0.558,0.717]}$)
A_3	([0.322, 0.451]$e^{j\pi[0.811,1.001]}$, [0.6, 0.724]$e^{j\pi[0.798,0.987]}$, [0.465, 0.606]$e^{j\pi[0.547,0.705]}$)

0.4]$e^{j\pi[0.8,0.9]}$), VI = Very Important = ([0.7, 0.8]$e^{j\pi[0.9,1.0]}$, [0.3, 0.4]$e^{j\pi[0.9,1.0]}$, [0.2, 0.3]$e^{j\pi[0.7,0.8]}$), and AI = Absolutely Important = ([0.8, 0.9]$e^{j\pi[1.0,1.1]}$, [0.2, 0.3]$e^{j\pi[0.8,0.9]}$, [0.1, 0.2]$e^{j\pi[0.6,0.7]}$).

Table 2 displays the importance weights of the five criteria from the three decision-makers. The aggregated weights of criteria obtained by Eq. (4) are shown in the last column of Table 2.

5.3 Compute the Total Value of Each Alternative

Table 3 presents the final fuzzy evaluation values of each supplier using Eq. (5).

Table 4 Modified score function of each alternative

Suppliers	Modified score function		Accuracy function		Certainty function		Ranking
	Amplitude term	Phase term	Amplitude term	Phase term	Amplitude term	Phase term	
A_1	0.320	-1.61π	-0.311	0.265π	0.304	1.659π	3
A_2	0.389	-1.301π	-0.162	0.508π	0.384	1.784π	2
A_3	0.396	-1.225π	-0.149	0.56π	0.387	1.811π	1

Table 5 The importance and aggregated weights of the criteria

Criteria	Decision-makers				Aggregated weights
	D_1	D_2	D_3	D_4	
C_1	AI	AI	AI	VI	([0.269, 0.361]$e^{j\pi[0.194,0.214]}$, [0.115, 0.161]$e^{j\pi[0.156,0.175]}$, [0.066, 0.115]$e^{j\pi[0.117,0.136]}$)
C_2	VI	I	I	VI	([0.157, 0.204]$e^{j\pi[0.175,0.194]}$, [0.191, 0.239]$e^{j\pi[0.175,0.194]}$, [0.144, 0.191]$e^{j\pi[0.148,0.168]}$)
C_3	AI	AI	VI	AI	([0.252, 0.336]$e^{j\pi[0.189,0.208]}$, [0.129, 0.176]$e^{j\pi[0.161,0.18]}$, [0.08, 0.129]$e^{j\pi[0.122,0.141]}$)
C_4	VI	VI	I	OI	([0.186, 0.241]$e^{j\pi[0.175,0.194]}$], [0.176, 0.223]$e^{j\pi[0.175,0.194]}$, [0.129, 0.176]$e^{j\pi[0.141,0.161]}$)
C_5	I	I	AI	AI	([0.168, 0.224]$e^{j\pi[0.18,0.2]}$, [0.17, 0.219]$e^{j\pi[0.175,0.194]}$, [0.12, 0.17]$e^{j\pi[0.145,0.164]}$)

Table 6 Aggregated ratings of suppliers versus the criteria

Criteria	Suppliers	Decision-makers				Aggregated ratings
		D_1	D_2	D_3	D_4	
C_1	A_1	G	F	G	G	([0.557, 0.659]$e^{j\pi[0.875,0.975]}$, [0.008, 0.019]$e^{j\pi[0.9,1.0]}$], [0.436, 0.532]$e^{j\pi[0.725,0.825]}$)
	A_2	G	G	F	F	([0.510, 0.613]$e^{j\pi[0.85,0.95]}$, [0.01, 0.023]$e^{j\pi[0.9,1.0]}$, [0.436, 0.532]$e^{j\pi[0.75,0.85]}$)
	A_3	L	G	F	L	([0.414, 0.518]$e^{j\pi[0.825,0.925]}$, [0.019, 0.039]$e^{j\pi[0.95,1.05]}$, [0.495, 0.589]$e^{j\pi[0.825,0.925]}$)
	A_4	G	F	G	F	([0.510, 0.613]$e^{j\pi[0.85,0.95]}$, [0.01, 0.023]$e^{j\pi[0.9,1.0]}$, [0.436, 0.532]$e^{j\pi[0.75,0.85]}$)
	A_5	F	G	G	G	([0.557, 0.659]$e^{j\pi[0.875,0.975]}$, [0.008, 0.019]$e^{j\pi[0.9,1.0]}$], [0.436, 0.532]$e^{j\pi[0.725,0.825]}$)
C_2	A_1	G	G	F	G	([0.557, 0.659]$e^{j\pi[0.875,0.975]}$, [0.008, 0.019]$e^{j\pi[0.9,1.025]}$], [0.495, 0.589]$e^{j\pi[0.725,0.825]}$)
	A_2	G	F	L	F	([0.437, 0.539]$e^{j\pi[0.825,0.925]}$, [0.015, 0.033]$e^{j\pi[0.925,1.025]}$], [0.495, 0.589]$e^{j\pi[0.8,0.9]}$)
	A_3	L	G	G	G	([0.54, 0.643]$e^{j\pi[0.875,0.975]}$, [0.01, 0.023]$e^{j\pi[0.925,1.025]}$], [0.461, 0.557]$e^{j\pi[0.75,0.85]}$)
	A_4	F	L	G	L	([0.414, 0.518]$e^{j\pi[0.825,0.925]}$, [0.019, 0.039]$e^{j\pi[0.95,1.05]}$], [0.495, 0.589]$e^{j\pi[0.825,0.925]}$)
	A_5	G	G	F	G	([0.557, 0.659]$e^{j\pi[0.875,0.975]}$, [0.008, 0.019]$e^{j\pi[0.9,1.0]}$], [0.436, 0.532]$e^{j\pi[0.725,0.825]}$)
C_3	A_1	F	F	L	L	([0.352, 0.452]$e^{j\pi[0.8,0.9]}$, [0.023, 0.047]$e^{j\pi[0.95,1.05]}$], [0.532, 0.622]$e^{j\pi[0.85,0.95]}$)
	A_2	G	G	G	G	([0.6, 0.7]$e^{j\pi[0.9,1.0]}$, [0.006, 0.016]$e^{j\pi[0.9,1.0]}$, [0.405, 0.503]$e^{j\pi[0.7,0.8]}$)
	A_3	L	G	F	F	([0.437, 0.539]$e^{j\pi[0.825,0.925]}$, [0.015, 0.033]$e^{j\pi[0.925,1.025]}$], [0.495, 0.589]$e^{j\pi[0.8,0.9]}$)
	A_4	G	F	G	F	([0.51, 0.613]$e^{j\pi[0.85,0.95]}$, [0.01, 0.023]$e^{j\pi[0.9,1.0]}$], [0.436, 0.532]$e^{j\pi[0.75,0.85]}$)
	A_5	F	G	G	G	([0.557, 0.659]$e^{j\pi[0.875,0.975]}$, [0.008, 0.019]$e^{j\pi[0.9,1.0]}$], [0.436, 0.532]$e^{j\pi[0.725,0.825]}$)
C_4	A_1	G	L	F	L	([0.414, 0.518]$e^{j\pi[0.825,0.925]}$, [0.019, 0.039]$e^{j\pi[0.95,1.05]}$, [0.495, 0.589]$e^{j\pi[0.825,0.925]}$)
	A_2	G	G	L	G	([0.54, 0.643]$e^{j\pi[0.875,0.975]}$, [0.01, 0.023]$e^{j\pi[0.925,1.025]}$, [0.461, 0.557]$e^{j\pi[0.75,0.85]}$)
	A_3	F	F	F	F	([0.4, 0.5]$e^{j\pi[0.8,0.9]}$, [0.016, 0.034]$e^{j\pi[0.9,1.0]}$, [0.503, 0.595]$e^{j\pi[0.8,0.9]}$)
	A_4	L	L	F	G	([0.414, 0.518]$e^{j\pi[0.825,0.925]}$, [0.019, 0.039]$e^{j\pi[0.95,1.05]}$, [0.495, 0.589]$e^{j\pi[0.825,0.925]}$)
	A_5	F	G	G	G	([0.557, 0.659]$e^{j\pi[0.875,0.975]}$, [0.008, 0.019]$e^{j\pi[0.9,1.0]}$], [0.436, 0.532]$e^{j\pi[0.725,0.825]}$)
C_5	A_1	L	F	G	L	([0.414, 0.518]$e^{j\pi[0.825,0.925]}$, [0.019, 0.039]$e^{j\pi[0.95,1.05]}$, [0.495, 0.589]$e^{j\pi[0.825,0.925]}$)
	A_2	G	L	G	G	([0.54, 0.643]$e^{j\pi[0.875,0.975]}$, [0.01, 0.023]$e^{j\pi[0.925,1.025]}$], [0.461, 0.557]$e^{j\pi[0.75,0.85]}$)
	A_3	G	G	L	F	([0.491, 0.595]$e^{j\pi[0.85,0.95]}$, [0.012, 0.027]$e^{j\pi[0.925,1.025]}$, [0.461, 0.557]$e^{j\pi[0.775,0.875]}$)
	A_4	L	L	F	G	([0.414, 0.518]$e^{j\pi[0.825,0.925]}$, [0.019, 0.039]$e^{j\pi[0.95,1.05]}$, [0.495, 0.589]$e^{j\pi[0.825,0.925]}$)
	A_5	G	G	G	G	([0.6, 0.7]$e^{j\pi[0.9,1.0]}$, [0.006, 0.016]$e^{j\pi[0.9,1.0]}$, [0.405, 0.503]$e^{j\pi[0.7,0.8]}$)

Table 7 The final fuzzy evaluation values of each supplier

Suppliers	Aggregated weights
A_1	$([0.095, 0.154]e^{j\pi[0.153,0.19]}, [0.166, 0.228]e^{j\pi[0.156,0.192]}, [0.534, 0.639]e^{j\pi[0.106,0.137]})$
A_2	$([0.11, 0.174]e^{j\pi[0.158,0.195]}, [0.162, 0.22]e^{j\pi[0.153,0.189]}, [0.508, 0.616]e^{j\pi[0.101,0.131]})$
A_3	$([0.093, 0.151]e^{j\pi[0.153,0.189]}, [0.166, 0.227]e^{j\pi[0.155,0.191]}, [0.539, 0.643]e^{j\pi[0.106,0.137]})$
A_4	$([0.096, 0.156]e^{j\pi[0.153,0.189]}, [0.165, 0.227]e^{j\pi[0.156,0.192]}, [0.547, 0.651]e^{j\pi[0.107,0.138]})$
A_5	$([0.117, 0.183]e^{j\pi[0.161,0.198]}, [0.16, 0.217]e^{j\pi[0.15,0.187]}, [0.491, 0.6]e^{j\pi[0.097,0.126]})$

Table 8 Modified score function of each alternative

Suppliers	Modified score function		Accuracy function		Certainty function		Ranking
	Amplitude term	Phase term	Amplitude term	Phase term	Amplitude term	Phase term	
A_1	0.447	-0.248π	-0.461	0.100π	0.125	0.344π	3
A_2	0.463	-0.222π	-0.420	0.121π	0.142	0.353π	2
A_3	0.445	-0.247π	-0.469	0.099π	0.122	0.341π	4
A_4	0.444	-0.252π	-0.473	0.096π	0.126	0.342π	5
A_5	0.472	-0.201π	-0.395	0.136π	0.150	0.359π	1

5.4 Ranking the Alternatives

Using the modified ranking method, the final ranking value of each alternative is defined as in Table 4. According to this table, the ranking order of the three suppliers is $A_3 \succ A_2 \succ A_1$.

6 Comparison of the Proposed Method with Another MCGDM Method

6.1 Example 1

This section compares the proposed approach with another MCGDM approach to demonstrate its advantages and applicability by reconsidering the example investigated by Sahin and Yigider [14]. In this example, four decision-makers ($D_1,\ldots,D_4$) have been appointed to evaluate five suppliers ($S_1,\ldots, S_5$) based on five performance criteria including delivery (C_1), quality (C_2), flexibility (C_3), service (C_4) and price (C_5).

The information of weights provided to the five criteria by the four decision-makers are presented in Table 5. The aggregated weights of criteria obtained by Eq. (4) are shown in the last column of Table 5.

Table 6 demonstrates the averaged ratings of suppliers versus the criteria based on the data presented in Tables 4, 5, 6, 7 and 8 in the work of Sahin and Yigider [14] and the proposed method.

Table 7 presents the final fuzzy evaluation values of each supplier using Eq. (5).

Using the proposed modified ranking method, the final ranking value of each alternative is defined as in Table 8. According to this table, the ranking order of the five suppliers is $A_5 \succ A_2 \succ A_1 \succ A_3 \succ A_4$. Obviously, the results in Sahin and Yigider [14] conflict with ours in this paper. The reason for the difference is in the proposed method: IVCNS was used to measure the ratings of the suppliers and the importance weights of criteria.

6.2 Example 2

This section uses a numerical example to compare the proposed approach with Ye's method [21] as follows. Consider two ICNS, i.e., $A_1 = ([0.5, 0.6]e^{j\pi[0.9,1.0]}, [0.4, 0.5]e^{j\pi[0.7,0.8]}], [0.3, 0.4]e^{j\pi[0.5,0.6]}$ and $A_2 = ([0.5, 0.6]e^{j\pi[0.8,0.9]}, [0.4, 0.5]e^{j\pi[0.5,0.6]}], [0.3, 0.4]e^{j\pi[0.7,0.8]}$. It is clear that the truth membership, indeterminacy membership and false-membership of A_1 and A_2 have the same amplitude values. Using the Ye's method [21], the similarity measures between ICNS A_1 and A_2 are: $S_1(A_1, A_2) = 1$ and $S_2(A_1, A_2) = 1$. Therefore, the ranking order of A_1 and A_2 is $A_1 = A_2$. This is not reasonable.

However using the proposed ranking method, the modified score, the accuracy and certainty function of A_1 and A_2 are: $e^a_{V_o}(A_1) = e^a_{V_o}(A_2) = 0.583, h^a_{V_o}(A_1) = h^a_{V_o}(A_2) = 0.2$, $c^a_{V_o}(A_1) = c^a_{V_o}(A_2) = 0.55$ and $e^p_{V_o}(A_1) = -0.7\pi, e^p_{V_o}(A_2) = -0.9\pi$; $h^p_{V_o}(A_1) = 0.8\pi, h^p_{V_o}(A_2) = 0.2\pi$ and $c^p_{V_o}(A_1) = 1.9\pi, c^p_{V_o}(A_2) = 1.7\pi$. Accordingly, the ranking order of ICNS A_1 and A_2 is $A_1 > A_2$. Obviously, the proposed ranking method can also rank ICNS other than INS.

7 Conclusion

It is believed that uncertain, ambiguous, indeterminate, inconsistent and incomplete periodic/redundant information can be dealt better with intervals instead of single values. This paper aimed to propose the interval complex neutrosophic set, which is more adaptable and flexible to real-life problems than other types of fuzzy sets. The definitions of interval complex neutrosophic set, accompanied by the set operations, were defined. The relationship of interval complex neutrosophic set with other existing approaches was presented.

A new decision-making procedure in the interval complex neutrosophic set has been presented and applied to a decision-making problem for the green supplier selection. Comparison between the proposed method and the related methods has been made to demonstrate the advantages and applicability. The results are significant to enrich the knowledge of neutrosophic set in the decision-making applications.

Future work plans to use the decision-making procedure to more complex applications, and to advance the interval complex neutrosophic logic system for forecasting problems.

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