A Review Study On Neutrosophic AH-Algebraic Structures

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A Review Study On Neutrosophic AH-Algebraic Structures

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Abstract: Neutrosophic Algebraic structures are rich fields for researchers to get many interesting generalizations of classical and fuzzy structures. This Study is dedicated to give the interested reader some of special neutrosophic algebraic substructures of neutrosophic algebraic structures, especially AH-substructures in neutrosophic rings, spaces, modules, and their generalizations.

Key words: Neutrosophic set, neutrosophic number, neutrosophic integer, Neutrosophic Algebra.

1. Introduction

Neutrosophy is considered as a new generalization of fuzzy and intuitionistic fuzzy ideas proposed by Smarandache [1]. Neutrosophic sets were very applicable in many areas of pure and applied mathematic such as matrix theory, topology, number theory, and algebraic structures [2-12].

Neutrosophic algebra began with Smarandache et. al, where they have defined neutrosophic rings, groups, vector spaces, and modules [20-30, 60-75].

In the literature many researchers have studied and contributed to these structures and their generalizations, where we can find many generalizations such as refined neutrosophic structures [13-20, 36-40], and n-refined neutrosophic structures [46-54].

An interesting direction was opened by Abobala. Et. al, where they defined AH-substructures such as AH-subspaces, AH-homomorphisms, AH-ideals, and AH-submodules [6-10]. Recently, AH-substructures were used to

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build neutrosophic system of Euclidean geometry, especially AH-isometries[66], and Turiyam symbolic rings [78-90]. The Turiyam set given a way to represent the data set beyond the Non-Euclidean and NeutroGeoemtry [91-92].

Through this work, we review the basic concepts of AH-substructures in neutrosophic rings, refined neutrosophic rings, neutrosophic vector spaces, n-refined neutrosophic modules, and many other neutrosophic algebraic structures.

Main Discussion

Definition 1:

Let R(I) be a neutrosophic ring and $P = P_0 + P_1 I = \{a_0 + a_1 I;\ a_0 \in P_0, a_1 \in P_1\}$.

(a) We say that P is an AH-ideal if P_0 and P_1 are ideals in the ring R.

(b) We say that P is an AHS-ideal if $P_0 = P_1$.

Definition 2:

Let (R(I_1, I_2), $+,\times$) be a refined neutrosophic ring, and P_0, P_1, P_2 be three ideals in the ring R then the set

$P = (P_0, P_1 I_1, P_2 I_2) = \{(a, bI_1, cI_2); a \in P_0, b \in P_1, c \in P_2\}$ is called a refined neutrosophic AH-ideal.

If $P_0 = P_1 = P_2$ then P is called a refined neutrosophic AHS-ideal.

Definition 3:

Let V(I) = V+VI be a strong/weak neutrosophic vector space, the set

$S = P + QI = \{x + yI; x \in P, y \in Q\}$, where P and Q are subspaces of V is called an AH-subspace of V(I).

If $P = Q$ then S is called an AHS-subspace of V(I).

Example 4:

We have V = R^2 is a vector space, $P =< (0,1) >, Q =< (1,0) >$, are two subspaces of V. The set

$S = P + QI = \{(0, a) + (b, 0)I; a, b \in R\}$ is an AH-subspace of V(I).

The set $L = P + PI = \{(0, a) + (0, b)I\}; a, b \in R$ is an AHS-subspace of V(I).

Theorem 5:

Let $V(I) = V + VI$ be a neutrosophic weak vector space, and let $S = P + QI$ be an AH-subspace of $V(I)$, i.e Q, P are supspaces of V, then S is a subspace by the classical meaning.

Proof:

Suppose that $x = a + bI, y = c + dI \in S; a, c \in P,\ b, d \in Q$, we have

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$x + y = (a + c) + (b + d)I \in S$. For each scalar $m \in K$ we obtain $m.x = m.a + (m.b)I \in S$, since P and Q are subspaces; thus $S = P + QI$ is a subspace of V(I) over the field K.

Theorem 6:

Let V(I) be a neutrosophic strong vector space over a neutrosophic field K(I), let $S = P + PI$ be an AHS-subspace. S is a subspace of V(I).

Proof:

Suppose that $x = a + bI, y = c + dI \in S; a, c, b,$ and $c \in P$, we have

$x + y = (a + c) + (b + d)I \in S$. Let $m = x + yI \in K(I)$ be a neutrosophic scalar, we find

$m.x = (x.a) + (y.a + y.b + x.b)I \in S,$ since $y.a + y.b + x.b \in P$, thus we get the desired result.

Definition 7:

(a) Let V and W be two vector spaces, $L_V: V \to W$ be a linear transformation. The AHS-linear transformation can be defined as follows:

$L: V(I) \to W(I); L(a + bI) = L_V(a) + L_V(b)I.$

(b) If $S = P + QI$ is an AH-subspace of V(I), $L(S) = L_V(P) + L_V(Q)I$.

(c) If $S = P + QI$ is an AH-subspace of W(I), $L^{-1}(S) = L_W^{-1}(P) + L_W^{-1}(Q)I$.

(d) $AH - Ker\ L = KerL_V + KerL_V\ I = \{x + yI; x, y \in KerL_V\}$.

Theorem 8:

Let W(I) and V(I) be two neutrosophic strong/weak vector spaces, and $L: V(I) \to W(I)$ be an AHS-linear transformation, we have:

(a) $AH - Ker\ L$ is an AHS-subspace of V(I).

(b) If $S = P + QI$ is an AH-subspace of V(I), $L(S)$ is an AH-subspace of W(I).

(c) If $S = P + QI$ is an AH-subspace of W(I), $L^{-1}(S)$ is an AH-subspace of V(I).

Proof:

(a) Since $KerL_V$ is a subspace of V, we find that $AH - Ker\ L = KerL_V + KerL_V\ I$ is an AHS-subspace of V(I).

(b) We have $L(S) = L_V(P) + L_V(Q)I$; thus L(S) is an AH-subspace of W(I), since $L_V(P), L_V(Q)$are subspaces of W.

(c) By regarding $L^{-1}(S) = L_W^{-1}(P) + L_W^{-1}(Q)I$, $L_W^{-1}(P)$ and $L_W^{-1}(Q)$ are subspaces of V, we obtain that $L^{-1}(S)$ is an AH-subspace of V(I).

Theorem 9:

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Let W(I) and V(I) be two neutrosophic strong vector spaces over a neutrosophic field K(I), and $L: V(I) \to W(I)$ be an AHS-linear transformation, we have:

$L(x+y) = L(x) + L(y), L(m.x) = m.L(x)$, for all $x, y \in V(I), m \in K(I)$.

Proof:

Suppose $x = a + bI, y = c + dI; a, b, c, d \in V, and\ m = s + tI \in K(I)$, we have

$$L(x+y) = L([a+c] + [b+d]I) = L_V(a+c) + L_V(b+d)I =$$

$$[L_V(a) + L_V(b)I] + [L_V(c) + L_V(d)I] = L(x) + L(y).$$

$m.x = (s.a) + (s.b + t.a + t.b)I,\ L(m.x) = L_V(s.a) + L_V(s.b + t.a + t.b)I$

$= s.L_V(a) + [s.L_V(b) + t.L_V(a) + t.L_V(b)]I = (s + tI).(L_V(a) + L_V(b)I) = m.L(x)$.

Theorem 10:

Let $S = P + QI$ be an AH-subspace of a neutrosophic weak vector space V(I) over a field K, suppose that

$X = \{x_i; 1 \le i \le n\}$ is a bases of P and $Y = \{y_j; 1 \le j \le m\}$ is a bases of Q then $X \cup YI$ is a bases of S.

Proof:

Let $z = x + yI$ be an arbitrary element in S; $x \in P, y \in Q$. Since P and Q are subspaces of V we can write

$x = a_1x_1 + a_2x_2 + \dots + a_nx_n;\ a_i \in K\ and\ x_i \in X,\ y = b_1y_1 + b_2y_2 + \dots + b_my_m;\ b_i \in K, y_i \in Y.$

Now we obtain $z = (a_1x_1 + \dots + a_nx_n) + (b_1y_1I + \dots + b_my_mI)$; thus $X \cup YI$ generates the subspace S.

$X \cup YI$ is linearly independent set. Assume that $(a_1x_1 + \dots + a_nx_n) + (b_1y_1I + \dots + b_my_mI) = 0$, this implies

$a_1x_1 + a_2x_2 + \dots + a_nx_n = 0$ and $(b_1y_1 + b_2y_2 + \dots + b_my_m)I = 0$. Since X and Y are linearly independent sets over K, we get $a_i = b_j = 0\ for\ all\ i, j$ and $X \cup YI$ is linearly independent then it is a bases of S.

Definition 11:

Let V(I) be a neutrosophic strong/weak vector space, $S = P + QI$ be an AH-subspace of V(I), we define

AH-Quotient as:

$V(I)/S = V/P + (V/Q)\,I = (x + P) + (y + Q)I; x, y \in V.$

Theorem 12:

Let V(I) be a neutrosophic weak vector space over a field K, and $S = P + QI$ be an AH-subspace of V(I). The AH-Quotient $V(I)/S$ is a vector space over the field K with respect to the following operations:

Addition: $[(x+P) + (y+Q)I] + [(a+P) + (b+Q)]I = (x + a + P) + (y + b + Q)I; x, y, a, b \in V.$

Multiplication by a scalar: $(m).[(x+P) + (y+Q)I] = (m.x + P) + (m.y + Q)I;$

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$x, y \in V$ and $m \in K$.

Definition 13:

Let (V,+,.) be a vector space over a field K, $V_n(I)$ be the corresponding weak n-refined neutrosophic vector space over K. Consider the set $\{M_i; 0 \leq i \leq n\}$, where M_i is a subspace of V. We say:

$M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n = \{m_0 + m_1 I_1 + \cdots + m_n I_n;\ m_i \in M_i\}$ is a weak n-refined AH-subspace of the weak n-refined vector space $V_n(I)$.

We say that $M_n(I)$ is a weak n-refined AH-subspace if $M_j = M_i\ for\ all\ i, j$.

Definition 14:

Let (V,+,.) be a vector space over a field K, $V_n(I)$ be the corresponding strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$. Consider the set $\{M_i; 0 \leq i \leq n\}$, where M_i is a subspace of V. We say:

$M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n = \{m_0 + m_1 I_1 + \cdots + m_n I_n;\ m_i \in M_i\}$ is a strong n-refined AH-subspace of the strong n-refined vector space $V_n(I)$.

We say that $M_n(I)$ is a strong n-refined AH-subspace if $M_j = M_i\ for\ all\ i, j$.

Theorem 15:

Let (V,+,.) be a vector space over a field K, $V_n(I)$ be the corresponding weak n-refined neutrosophic vector space over K, $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$ be a weak n-refined AH-subspace. Then

(a) $M_n(I)$ is a vector subspace of $V_n(I)$.

(b) If X_i is a bases of M_i, $X = \bigcup_{i=0}^{n} X_i I_i$ is a bases of $M_n(I)$.

(c) $\dim\big(M_n(I)\big) = \sum_{i=0}^{n} \dim(M_i)$.

Proof:

(a) Let $x = \sum_{i=0}^{n} a_i I_i$, $y = \sum_{i=0}^{n} b_i I_i;\ b_i, a_i \in M_i$ be two arbitrary elements in $M_n(I)$, r be an arbitrary element in K, we have:

$x + y = \sum_{i=0}^{n} (a_i + b_i) I_i \in M_n(I)$, since $a_i + b_i \in M_i$ because M_i is a subspace of V.

$r.x = \sum_{i=0}^{n} r a_i I_i \in M_n(I)$, since $r a_i \in M_i$ because M_i is a subspace of V. Thus $M_n(I)$ is a vector subspace of $V_n(I)$.

(b) Suppose that $X_0 = \left\{x_1^{(0)}, \dots, x_{s_0}^{(0)}\right\}, X_1 = \left\{x_1^{(1)}, \dots, x_{s_1}^{(1)}\right\}, \dots \dots, X_n = \{x_1^{(n)}, \dots, x_{s_n}^{(n)}\}$, let $x = \sum_{i=0}^{n} a_i I_i$ be an arbitrary element of $M_n(I)$, since X_i is a basis of M_i for all i. We can write:

$a_i = \sum_{j=0}^{s_i} t_j^{(i)} x_j^{(i)};\ t_j \in K$, so $x = \sum_{j=0}^{s_0} t_j^{(0)} x_j^{(0)} + \sum_{j=0}^{s_1} t_j^{(1)} x_j^{(1)} I_1 + \cdots + \sum_{j=0}^{s_n} t_j^{(n)} x_j^{(n)} I_n$. This implies that X is a generating set of $M_n(I)$.

Now we prove that X is linearly independent. For our purpose we assume

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$\sum_{j=0}^{s_0} t_j^{(0)} x_j^{(0)} + \sum_{j=0}^{s_1} t_j^{(1)} x_j^{(1)} I_1 + \cdots + \sum_{j=0}^{s_n} t_j^{(n)} x_j^{(n)} I_n = 0$, by definition of n-refined vector space we find

$\sum_{j=0}^{s_i} t_j^{(i)} x_j^{(i)}$ for all i, hence $t_j^{(i)} = 0$ for all i,j, since each X_i is linearly independent itself. Thus our proof is complete.

(c) It holds directly from (b).

Theorem 16:

Let (V,+,.) be a vector space over a field K, $V_n(I)$ be the corresponding strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, $M_n(I) = M + MI_1 + \cdots + MI_n$ be a strong n-refined AHS-subspace. Then:

(a) $M_n(I)$ is a submodule of $V_n(I)$.

(b) If Y is a bases of M, $X = \cup_{i=0}^{n} YI_i$ is a bases of $M_n(I)$.

(c) $\dim\left(M_n(I)\right) = \sum_{i=0}^{n} \dim(M) = n.\dim(M)$.

Remark 17:

If $V_n(I)$ is a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, and

$M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$ is a strong n-refined AH-subspace, then it is not supposed to be a submodule.

Example18 :

Let $V = R^2$ be a vector space over R, $V_2(I) = R_2^2(I) = \{(a,b) + (c,d)I_1 + (e,f)I_2; a,b,c,d,e,f \in R\}$ be the corresponding strong 2-refined neutrosophic vector space over the neutrosophic field $R_2(I)$.

$M = < 0,1 >, N = < (1,0) >$ are two subspaces of V, $T = M + NI_1 + NI_2$ is a strong AH-subspace of $V_2(I)$.

$x = (0,1) + (2,0)I_1 + (1,0)I_2 \in T, r = 1 + 1.I_1 + 1.I_2 \in R_2(I)$,

$r.x = 1.(0,1) + 1.(0,1)I_1 + 1.(0,1)I_2 + 1.(2,0)I_1 I_1 + 1.(2,0)I_1 + 1.(1,0)I_1 I_2 + 1.(0,1)I_2 + 1.(2,0)I_1 I_2 + 1.(2,0)I_2 I_2 = (0,1) + [(0,1) + (2,0) + (1,0) + (2,0)]I_1 + [(0,1) + (0,1) + (2,0)]I_2 =$

$(0,1) + (5,1)I_1 + (2,2)I_2$, $r.x$ does not belong to T, thus T is not a submodule.

Definition 19:

Let $V_n(I)$ be a weak/strong n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$,

$W_n(I) = W_0 + W_1 I_1 + \cdots + W_n I_n$ be two weak/strong AH-subspaces of $V_n(I)$, we define:

(a) $M_n(I) \cap W_n(I) = (M_0 \cap W_0) + (M_1 \cap W_1)I_1 + \cdots + (M_n \cap W_n)I_n$.

(b) $M_n(I) + W_n(I) = (M_0 + W_0) + (M_1 + W_1)I_1 + \cdots + (M_n + W_n)I_n$.

Theorem 20:

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Let $V_n(I)$ be a weak n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n$,

$W_n(I) = W_0 + W_1I_1 + \cdots + W_nI_n$ be two weak AH-subspaces of $V_n(I)$. Then:

$M_n(I) \cap W_n(I), M_n(I) + W_n(I)$ are two weak AH-subspaces of $V_n(I)$.

Proof:

Since $M_i \cap W_i, M_i + W_i$ are subspaces of V for all i, we obtain the proof.

Theorem 21:

Let $V_n(I)$ be a strong n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n$,

$W_n(I) = W_0 + W_1I_1 + \cdots + W_nI_n$ be two strong AH-subspaces of $V_n(I)$. Then:

(a) $M_n(I) \cap W_n(I)$ is a strong AH-subspaces of $V_n(I)$.

(b) $M_n(I) + W_n(I)$ is not supposed to be a strong AH-subspace of $V_n(I)$.

Definition 22:

Let V,W be two vector spaces over the field K, $f_i: V \to W; 0 \leq i \leq n+1$ be $n+1$ linear transformations, $V_n(I), W_n(I)$ be the corresponding weak n-refined neutrosophic vector spaces over the field K respectively. We say:

(a) $f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_iI_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ is a weak AH-linear transformation.

(b) If $f_i = f_j \; for \; all \; i, j$, we call f a weak AHS-linear transformation.

Example 23:

(a) Let $V = R^3, W = R^2$ be two vector spaces over the field R, $V_2(I) = R_2^3(I) = \{\, (x_0, y_0z_0) + (x_1, y_1, z_1)I_1 + (x_2, y_2, z_2)I_2;\; x_i, y_i, z_i \in R\}$,

$W_2(I) = \{(x_0, y_0) + (x_1, y_1)I_1 + (x_2, y_2)I_2;\; x_i, y_i \in R\}$ be the corresponding weak 2-refined neutrosophic vector spaces. We have $g: V \to W; g(a, b, c) = (b, c), h: V \to W; h(a, b, c) = (2a, 0)$

$s: V \to W; s(a, b, c) = (2b, 3c)$ are three linear transformations.

(b) $f: V_2(I) \to W_2(I); f(m + nI_1 + qI_2) = g(m) + h(n)I_1 + s(q)I_2; m, n, q \in V$ is a weak AH-linear transformation.

(c) We clarify f as follows:

$x = (1,2,2) + (1,0,1)I_1 + (3, -1,0)I_2 \in V_2(I)$,

$f(x) = g(1,2,2) + [h(1,0,1)]I_1 + [s(3, -1,0)]I_2 = (2,2) + (2,0)I_1 + (-2,0)I_2$.

(d) $k: V_2(I) \to W_2(I); k(m + nI_1 + qI_2) = g(m) + g(n)I_1 + g(q)I_2; m, n, q \in V$ is a weak AHS-linear transformation.

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Definition 24:

Let V,W be two vector spaces over the field K, $f_i: V \to W; 0 \le i \le n+1$ be $n+1$ linear transformations, $V_n(I), W_n(I)$ be the corresponding strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K_n(I)$ respectively. We say:

(a) $f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ is a strong AH-linear transformation.

(b) If $f_i = f_j \; for \; all \; i, j$, we call f a strong AHS-linear transformation.

Example 25:

(a) Let $V = R^3, W = R^2$ be two vector spaces over the field R, $V_2(I) = R_2^3(I) = \{(x_0, y_0 z_0) + (x_1, y_1, z_1)I_1 + (x_2, y_2, z_2)I_2;\; x_i, y_i, z_i \in R\}$,

$W_2(I) = \{(x_0, y_0) + (x_1, y_1)I_1 + (x_2, y_2)I_2;\; x_i, y_i \in R\}$ be the corresponding strong 2-refined neutrosophic vector spaces over the 2-refined neutrosophic field $R_2(I)$. We have $g: V \to W; g(a, b, c) = (b, c), h: V \to W; h(a, b, c) = (2a, 0)$,

$s: V \to W; s(a, b, c) = (2b, 3c)$ are three linear transformations.

(b) $f: V_2(I) \to W_2(I); f(m + nI_1 + qI_2) = g(m) + h(n)I_1 + s(q)I_2; m, n, q \in V$ is a strong AH-linear transformation.

(c) We clarify f as follows:

$x = (1,2,2) + (1,0,1)I_1 + (3, -1,0)I_2 \in V_2(I)$,

$f(x) = g(1,2,2) + [h(1,0,1)]I_1 + [s(3, -1,0)]I_2 = (2,2) + (2,0)I_1 + (-2,0)I_2$.

(d) $k: V_2(I) \to W_2(I); k(m + nI_1 + qI_2) = g(m) + g(n)I_1 + g(q)I_2; m, n, q \in V$ is a strong AHS-linear transformation.

Definition 26:

Let $V_n(I), W_n(I)$ be two weak/strong n-refined neutrosophic vector spaces,

$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ be a weak/strong AH-linear transformation. We define:

(a) $AH - Ker(f) = Ker(f_0) + Ker(f_1)I_1 + \cdots + Ker(f_n)I_n$.

(b) $AH - Im(f) = Im(f_0) + Im(f_1)I_1 + \cdots + Im(f_n)I_n$.

Theorem 27:

Let $V_n(I), W_n(I)$ be two weak n-refined neutrosophic vector spaces,

$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ be a weak AH-linear transformation. Then:

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(a) $AH-Ker(f)$ is a weak AH-subspace of $V_n(I)$.

(b) $AH-Im(f)$ is a weak AH-subspace of $W_n(I)$.

(c) If $M_n(I)=M_0+M_1I_1+\cdots+M_nI_n$ is a weak AH-subspace of $V_n(I)$, $f(M_n(I))$ is a weak AH-subspace of $W_n(I)$.

Proof:

(a) Since $Ker(f_i)$ is a subspace of V, we find that

$AH-Ker(f)=Ker(f_0)+Ker(f_1)I_1+\cdots+Ker(f_n)I_n$ is a weak AH-subspace of $V_n(I)$.

(b) Since $Im(f_i)$ is a subspace of W, we find that $AH-Im(f)=Im(f_0)+Im(f_1)I_1+\cdots+Im(f_n)I_n$ is a weak AH-subspace of $W_n(I)$.

(c) It is known that $f_i(M_i)$ is a subspace of W, hence

$f\big(M_n(I)\big)=f_0(M_0)+f_1(M_1)I_1+\cdots+f_n(M_n)I_n$ is a weak AH-subspace of $W_n(I)$.

Theorem 28:

Let $V_n(I),W_n(I)$ be two strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K_n(I)$,

$f:V_n(I)\to W_n(I);f(\sum_{i=0}^{n}a_iI_i)=f_0(a_0)+f_1(a_1)I_1+\cdots+f_n(a_n)I_n=\sum_{i=0}^{n}f_i(a_i)I_i$ be a strong AH-linear transformation. Then:

(a) $AH-Ker(f)$ is a strong AH-subspace of $V_n(I)$.

(b) $AH-Im(f)$ is a strong AH-subspace of $W_n(I)$.

(c) If $M_n(I)=M_0+M_1I_1+\cdots+M_nI_n$ is a strong AH-subspace of $V_n(I)$, $f(M_n(I))$ is a strong AH-subspace of $W_n(I)$.

Theorem 29 :

Let $V_n(I),W_n(I)$ be two weak n-refined neutrosophic vector spaces over the field K,

$f:V_n(I)\to W_n(I);f(\sum_{i=0}^{n}a_iI_i)=f_0(a_0)+f_1(a_1)I_1+\cdots+f_n(a_n)I_n=\sum_{i=0}^{n}f_i(a_i)I_i$ be a weak AH-linear transformation. Then:

$f(x+y)=f(x)+f(y),f(r.x)=r.f(x)$ for all $x,y\in V_n(I),r\in K$.

Proof:

Let $x=\sum_{i=0}^{n}a_iI_i$, $y=\sum_{i=0}^{n}b_iI_i$ be two arbitrary elements in $V_n(I)$, $r\in K$ be any element in the field K, we have:

$f(x+y)=f(\sum_{i=0}^{n}(a_i+b_i)I_i)=\sum_{i=0}^{n}f_i(a_i+b_i)I_i=\sum_{i=0}^{n}f_i(a_i)I_i+\sum_{i=0}^{n}f_i(b_i)I_i=f(x)+f(y)$.

$f(r.x)=f(\sum_{i=0}^{n}ra_iI_i)=\sum_{i=0}^{n}f_i(ra_i)I_i=r.\sum_{i=0}^{n}f_i(a_i)I_i=r.f(x)$.

Theorem 30:

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Let $V_n(I), W_n(I)$ be two strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K_n(I)$,

$f: V_n(I) \rightarrow W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ be a strong AH-linear transformation. Then:

$f(x+y) = f(x) + f(y), f(r.x) = r.f(x)$ for all $x, y \in V_n(I), r \in K_n(I)$.

Proof:

Let $x = \sum_{i=0}^{n} a_i I_i$, $y = \sum_{i=0}^{n} b_i I_i$ be two arbitrary elements in $V_n(I)$, $r = \sum_{i=0}^{n} r_i I_i \in K_n(I)$ be any element in the n-refined neutrosophic field $K_n(I)$, we have:

$f(x+y) = f(\sum_{i=0}^{n}(a_i + b_i)I_i) = \sum_{i=0}^{n} f_i(a_i + b_i)I_i = \sum_{i=0}^{n} f_i(a_i)I_i + \sum_{i=0}^{n} f_i(b_i)I_i = f(x) + f(y)$.

For the proof of the second proposition we use induction on n. If n=0, the theorem is true clearly.

Suppose that it is true for n-1, we must prove it for n.

$f(r.x) = f\left(\sum_{i,j=0}^{n} r_i a_j I_i\, I_j\right) = f(\sum_{i,j=0}^{n-1} r_i a_j I_i\, I_j + (\sum_{i=0}^{n} r_i I_i)a_n I_n)$, we can write

$\sum_{i,j=0}^{n-1} r_i a_j I_i\, I_j = m_0 + m_1 I_1 + \cdots + m_{n-1} I_{n-1}$,

$(\sum_{i=0}^{n} r_i I_i)a_n I_n = r_1 a_n I_1 + r_2 a_n I_2 + \cdots + (r_0 a_n + r_n a_n)I_n$,

$r.x = \sum_{i,j=0}^{n-1} r_i a_j I_i\, I_j + (\sum_{i=0}^{n} r_i I_i)a_n I_n = \mathrm{m}_0 + (\mathrm{m}_1 + \mathrm{r}_1\mathrm{a}_\mathrm{n})\mathrm{I}_1 + (\mathrm{m}_2 + \mathrm{r}_2\mathrm{a}_\mathrm{n})\mathrm{I}_2 + \cdots + (r_0 a_n + r_n a_n)I_n$,

$f(r.x) = f_0(m_0) + f_1(m_1 + r_1 a_n)I_1 + f_2(m_2 + r_2 a_n)I_2 + \cdots f_n(r_0 a_n + r_n a_n)I_n =$

$f_0(m_0) + [f_1(m_1) + r_1 f_1(a_n)]I_1 + \cdots + [r_0 f_n(a_n) + r_n f_n(a_n)]I_n = r.f(x)$.

Theorem 31:

Let $V_n(I), W_n(I), U_n(I)$ be three weak n-refined neutrosophic vector spaces over the field K,

$f: W_n(I) \rightarrow U_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$,

$g: V_n(I) \rightarrow W_n(I); g(\sum_{i=0}^{n} a_i I_i) = g_0(a_0) + g_1(a_1)I_1 + \cdots + g_n(a_n)I_n = \sum_{i=0}^{n} g_i(a_i)I_i$,

be two weak AH-linear transformations. Then:

(a) $fog = \sum_{i=0}^{n}(f_i o g_i)$.

(b) fog is a weak AH-linear transformation between $V_n(I), U_n(I)$.

Proof:

(a) Let $x = \sum_{i=0}^{n} a_i I_i \in V_n(I)$, $fog(x) = f(\sum_{i=0}^{n} g_i(a_i)I_i) = f(g_0(a_0) + g_1(a_1)I_1 + \cdots + g_n(a_n)I_n) =$

$f_0\big(g_0(a_0)\big) + f_1\big(g_1(a_1)\big)I_1 + \cdots + f_n\big(g_n(a_n)\big)I_n = \sum_{i=0}^{n}(f_i o g_i)\,(a_i)I_i$.

(b) Since $f_i o g_i$ is a linear transformation for all i, then we get the proof.

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Theorem 32:

Let $V_n(I), W_n(I), U_n(I)$ be three strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field K,

$f: W_n(I) \to U_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$,

$g: V_n(I) \to W_n(I); g(\sum_{i=0}^{n} a_i I_i) = g_0(a_0) + g_1(a_1)I_1 + \cdots + g_n(a_n)I_n = \sum_{i=0}^{n} g_i(a_i)I_i$,

be two strong AH-linear transformations. Then:

(a) $fog = \sum_{i=0}^{n}(f_i o g_i)$.

(b) fog is a strong AH-linear transformation between $V_n(I), U_n(I)$.

Definition 33:

Let $M(I_1, I_2)$ be a strong refined neutrosophic module over the refined neutrosophic ring $R(I_1, I_2)$, P,Q,S be three submodules of M. The set N = $(P, QI_1, SI_2) = \{(a, bI_1, cI_2); a \in P, b \in Q, c \in S\}$ is called a strong AH-submodule of the strong refined neutrosophic module $M(I_1, I_2)$.

If P = Q = S, we call N a strong AHS-submodule.

Theorem 34:

Let $M(I_1, I_2)$ be a strong refined neutrosophic module over the refined neutrosophic ring $R(I_1, I_2)$,

N = (P, PI_1, PI_2) be a strong AHS-submodule. Then N is a submodule by classical meaning.

Definition 35:

Let M, W be two modules over the ring R, $M(I_1, I_2)$ and $W(I_1, I_2)$ be the corresponding strong refined neutrosophic modules over the refined neutrosophic ring $R(I_1, I_2)$. Let $f, g, h: M \to W$ be three homomorphisms, then $[f, g, h]: M(I_1, I_2) \to W(I_1, I_2); [f, g, h](a, bI_1, cI_2) = (f(a), g(b)I_1, h(c)I_2)$ is called a strong AH-homomorphism. If $f = g = h$, we get the strong AHS-homomorphism.

Definition 36:

Let $M(I_1, I_2), W(I_1, I_2)$ be two strong refined neutrosophic modules over the refined neutrosophic ring $R(I_1, I_2)$, $[f, g, h]: M(I_1, I_2) \to W(I_1, I_2)$ be a strong AH-homomorphism, we define

(a) $AH - Ker[f, g, h] = (Ker(f), Ker(g)I_1, Ker(h)I_2) = \{(a, bI_1, cI_2); a \in Ker(f), b \in Ker(g), c \in Ker(h)\}$.

(b) $AH - Im[f, g, h] = (Im(f), Im(g)I_1, Im(h)I_2)$.

Theorem 37:

Let $M(I_1, I_2), W(I_1, I_2)$ be two strong refined neutrosophic modules over the refined neutrosophic ring $R(I_1, I_2)$, $[f, g, h]: M(I_1, I_2) \to W(I_1, I_2)$ be a strong AH-homomorphism.

(a) If N = (P, QI_1, SI_2) is a strong AH-submodule of $M(I_1, I_2)$, then $[f, g, h](N)$ is a strong AH-submodule of $W(I_1, I_2)$.

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(b) $[f, g, h]$ is a classical module homomorphism.

(c) $AH - Ker[f, g, h]$ is a strong AH-submodule of $M(I_1, I_2)$.

(d) $AH - Im[f, g, h]$ is a strong AH-submodule of $W(I_1, I_2)$.

Proof:

(a) Since $f(P), g(Q), h(S)$ are submodules of N, we find that $[f, g, h](N) = (f(P), g(Q)I_1, h(S)I_2)$ is a strong AH-submodule of $W(I_1, I_2)$.

(b) Let $m = (x, yI_1, zI_2), n = (a, bI_1, cI_2)$ be two arbitrary elements in $M(I_1, I_2)$, $r = (t, uI_1, vI_2)$ be any element in $R(I_1, I_2)$,

$m + n = (x + a, [y + b]I_1, [z + c]I_2), r.m = (tx, [xu + yt + yu + yv + zu]I_1, [xv + zt + zv]I_2)$,

$[f, g, h](m + n) = (f(x + a), g([y + b])I_1, h([z + c])I_2)$=$(f(x), g(y)I_1, h(z)I_2) + (f(a), g(b)I_1, h(c)I_2) = [f, g, h](m) + [f, g, h](n)$.

$[f, g, h](r.m) = (f(tx), g([xu + yt + yu + yv + zu])I_1, h([xv + zt + zv])I_2)$=

$(t, uI_1, vI_2).(f(x), g(y)I_1, h(z)I_2) = r.[f, g, h](m)$. Thus $[f, g, h]$ is a classical homomorphism.

(c) Since $Ker(f), Ker(g), Ker(h)$ are submodules of M, we get $AH - Ker[f, g, h] = (Ker(f), Ker(g)I_1, Ker(h)I_2)$ as a strong AH-submodule of $M(I_1, I_2)$.

(d) Since $Im(f), Im(g), Im(h)$ are submodules of W, we get $AH - Im[f, g, h] = (Im(f), Im(g)I_1, Im(h)I_2)$ as a strong AH-submodule of $W(I_1, I_2)$.

Example 38:

(a) Let $M = R^2, W = R$ be two modules over the ring R,

$f: M \to W; f(x, y) = 2x, g: M \to W; g(x, y) = 3y, h: M \to W; h(x, y) = x + y$ are three homomorphisms.

(b) $[f, g, h]: M(I_1, I_2) \to W(I_1, I_2); [f, g, h]\big((x, y), (z, t)I_1, (s, m)I_2\big) = (f(x, y), g(z, t)I_1, h(s, m)I_2)$=

$(2x, 3tI_1, [s + m]I_2)$ is a strong AH-homomorphism, where $x, y, z, t, s, m \in R$.

(c) $P = \{(0, x); x \in R\}, Q = \{(x, 0); x \in R\}$ are two submodules of M,

$N = (P, PI_1, QI_2) = \{((0, x), (0, y)I_1, (z, 0)I_2; x, y, z \in R\}$ is a strong AH-submodule of $M(I_1, I_2)$.

(d) $f(P) = \{0\}, g(P) = \{3y; y \in R\} = R, h(Q) = \{z; z \in R\} = R$,

$[f, g, h](N) = (f(P), g(P)I_1, h(Q)I_2) = (0, RI_1, RI_2) = \{(0, xI_1, yI_2); x, y \in R\}$ is a strong AH-submodule of $W(I_1, I_2)$.

(e) $Ker(f) = \{(0, x); x \in R\}, Ker(g) = \{(x, 0); x \in R\}, Ker(h) = \{(y, -y); y \in R\}$,

$AH - Ker[f, g, h] = (\ker(f), Ker(g)I_1, Ker(h)I_2) = \{(0, x), (y, 0)I_1, (z, -z)I_2); x, y, z \in R\}$.

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Definition 39:

Let $M(I) = M + MI$ be a strong/weak neutrosophic module, the set

$S = P + QI = \{x + yI; x \in P, y \in Q\}$, where P and Q are submodules of V is called an AH-submodule of M(I).

If $P = Q$, S is called an AHS-submodule of M(I).

Example 40:

We have M = $Z^2 = Z \times Z$ is a module over R, $P =< (0,1) >, Q =< (1,0) >$, are two submodules of M. The set

$S = P + QI = \{(0, a) + (b, 0)I; a, b \in Z\}$ is an AH-submodule of M(I).

The set $L = P + PI = \{(0, a) + (0, b)I\}; a, b \in Z$ is an AHS-submodule of M(I).

Theorem 41:

Let $M(I) = M + MI$ be a neutrosophic weak module over the ring R, and let $S = P + QI$ be an AH-submodule of $M(I)$, then S is a submodule.

Proof:

Suppose that $x = a + bI, y = c + dI \in S; a, c \in P,\ b, d \in Q$,

$x + y = (a + c) + (b + d)I \in S$. For each scalar $m \in R$ we obtain $m.x = m.a + (m.b)I \in S$, since P and Q are submodules; thus $S = P + QI$ is a submodule of M(I) over the ring R.

Theorem 42:

Let M(I) be a neutrosophic strong module over a neutrosophic ring R(I), let $S = P + PI$ be an AHS-submodule. Then S is a submodule of M(I).

Definition 43:

(a) Let M and W be two modules, $L_M: M \to W$ be a homomorphism. The AHS-homomorphism can be defined as follows:

$L: M(I) \to W(I); L(a + bI) = L_M(a) + L_M(b)I.$

(b) If $S = P + QI$ is an AH-submodule of M(I), $L(S) = L_M(P) + L_M(Q)I$.

(c) If $S = P + QI$ is an AH-submodule of W(I), $L^{-1}(S) = L_W^{-1}(P) + L_W^{-1}(Q)I$.

(d) $AH - Ker(L) = Ker(L_M) + Ker(L_M)\ I = \{x + yI; x, y \in Ker(L_M)\}$.

Theorem 44:

Let W(I) and M(I) be two neutrosophic strong/weak modules, and $L: M(I) \to W(I)$ be an AHS-homomorphism:

(a) $AH - Ker(L)$ is an AHS-submodule of M(I).

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(b) If $S = P + QI$ is an AH-submodule of M(I), $L(S)$ is an AH-submodule of W(I).

(c) If $S = P + QI$ is an AH-submodule of W(I), $L^{-1}(S)$ is an AH-submodule of M(I).

Proof:

(a) Since $Ker(L_M)$ is a submodule of M, we find that $AH - Ker(L) = Ker(L_M) + Ker(L_M)I$ is an AHS-submodule of M(I).

(b) We have $L(S) = L_M(P) + L_M(Q)I$; thus L(S) is an AH-submodule of W(I), since $L_M(P), L_M(Q)$are submodules of W.

(c) By regarding that $L^{-1}(S) = L_W^{-1}(P) + L_W^{-1}(Q)I$, $L_W^{-1}(P)$ and $L_W^{-1}(Q)$ are submodules of M, we obtain that $L^{-1}(S)$ is an AH-subModule of M(I).

Theorem 45:

Let W(I) and M(I) be two neutrosophic strong modules over a neutrosophic ring R(I), and $L: M(I) \to W(I)$ be an AHS-homomorphism. Then

$L(x + y) = L(x) + L(y), L(m.x) = m.L(x)$, for all $x, y \in M(I), m \in R(I)$.

Proof:

Suppose $x = a + bI, y = c + dI; a, b, c, d \in M, and\ m = s + tI \in K(I)$, we have

$L(x + y) = L([a + c] + [b + d]I) = L_M(a + c) + L_M(b + d)I = [L_M(a) + L_M(b)I] + [L_M(c) + L_M(d)I] = L(x) + L(y)$.

$m.x = (s.a) + (s.b + t.a + t.b)I, L(m.x) = L_M(s.a) + L_M(s.b + t.a + t.b)I$

$= s.L_M(a) + [s.L_M(b) + t.L_M(a) + t.L_M(b)]I = (s + tI).(L_M(a) + L_M(b)I) = m.L(x)$.

Theorem 46:

Let $S = P + QI$ be an AH-submodule of a neutrosophic weak module M(I) over a ring R, suppose that

$X = \{x_i; 1 \le i \le n\}$ is a bases of P and $Y = \{y_j; 1 \le j \le m\}$ is a bases of Q then $X \cup YI$ is a bases of S.

Proof:

Let $z = x + yI$ be an arbitrary element in S; $x \in P, y \in Q$. Since P and Q are submodules of M we can write

$x = a_1x_1 + a_2x_2 + \cdots + a_nx_n;\ a_i \in R\ and\ x_i \in X,\ y = b_1y_1 + b_2y_2 + \cdots + b_my_m;\ b_i \in K, y_i \in Y$.

Now we obtain $z = (a_1x_1 + \cdots + a_nx_n) + (b_1y_1I + \cdots + b_my_mI)$; thus $X \cup YI$ generates the subspace S.

$X \cup YI$ is linearly independent set. Assume that $(a_1x_1 + \cdots + a_nx_n) + (b_1y_1I + \cdots + b_my_mI) = 0$, this implies

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $(b_1y_1 + b_2y_2 + \cdots + b_my_m)I = 0$. Since X and Y are linearly independent sets over R, we get $a_i = b_j = 0\ for\ all\ i, j$ and $X \cup YI$ is linearly independent then it is a basis of S.

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Result 47:

Let $S = P + QI$ be an AH-submodule of a neutrosophic weak module M(I) with finite dimension over a ring R, from Theorem 3.8 and the fact that $X \cap YI = \emptyset$, we find $\dim(S) = \dim(P) + \dim(Q)$.

Example 48:

Let M $= Z^3 = Z \times Z \times Z$ is a module over the ring Z, $P =< (0,0,1) >, Q =< (0,1,0) >$ be two submodules of M,

(a) $S = P + QI = \{(0,0,m) + (0,n,0)I;\ m,n \in Z\}$is an AH-submodule of M(I).

(b) The set $\{(0,0,1),(0,1,0)I\}$ is a bases of S, $\dim(S) = \dim(P) + \dim(Q) = 1 + 1 = 2$.

(c) $L_M: M \to M;\ L_M(x,y,z) = (x + y, y, z) for\ all\ x, y, z \in Z$ is a homomorphism, the corresponding AHS-homomorphism is

$L: M(I) \to M(I); L[(x,y,z) + (a,b,c)I] = L_M(x,y,z) + L_M(a,b,c)I = (x + y, y, z) + (a + b, b, c)I.$

(d) $L(S) = L_M(P) + L_M(Q) = L_M\{(0,0,m)\} + L_M\{(0,n,0)\}I = \{(0,0,m) + (n,n,0)I\}; m,n \in Z$, which is an AH-submodule of M(I).

Example 49:

Let M$=Z^2 = Z \times Z$, W$=Z^3 = Z \times Z \times Z$ be two modules over the ring Z, $L_M: M \to W;\ L_M(x,y) = (x + y, x + y, x + y)$ is a homomorphism. The corresponding AHS-homomorphism is

$L: M(I) \to W(I); L[(x,y) + (a,b)I] = (x + y, x + y, x + y) + (a + b, a + b, a + b)I.$

$KerL_M =< (1,-1) >, AH - Ker(L) = Ker(L_M) + Ker(L_M)\ I =< (1,-1) > +< (1,-1) > I =$

$\{(a,-a) + (b,-b)I; a,b \in Z\}$ which is an AHS-submodule of M(I).

We find $\dim(Ker(L)) = 1 + 1 = 2$.

Definition 50:

Let M(I) be a neutrosophic strong/weak module, $S = P + QI$ be an AH-submodule of M(I), we define

AH-Quotient module as:

$M(I)/S = M/P + (M/Q)\, I = (x + P) + (y + Q)I; x, y \in M.$

Theorem 51:

Let M(I) be a neutrosophic weak module over a ring R, and $S = P + QI$ be an AH-submodule of M(I). The AH-Quotient $M(I)/S$ is a module with respect to the following operations:

Addition: $[(x + P) + (y + Q)I] + [(a + P) + (b + Q)]I = (x + a + P) + (y + b + Q)I; x, y, a, b \in M.$

Multiplication by a scalar: $(m).[(x + P) + (y + Q)I] = (m.x + P) + (m.y + Q)I;$

$x, y \in M$ and $m \in R$.

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Proof:

It is easy to check that operations are well defined, and $(M(I)/S, +)$ is abelian group.

Let $z = [(x + P) + (y + Q)I] \in M(I)/S$, we have $1. z = z$.

Assume that $m, n \in R$, we have $m.(n.z) = m.[(n.x + P) + (n.y + Q)I] = (m.n.x + P) + (m.n.y + Q)I = (m.n).z$.

$(m + n).z = [(m + n).x + P] + [(m + n).y + Q]I = m.z + n.z.$

Let $h = [(a + P) + (b + Q)I] \in M(I)/S, z + h = (x + a + P) + (y + b + Q)I,$

$m.(z + h) = (m.x + m.a + P) + (m.y + m.b + Q)m.z + m.h.$

Conclusion

In this review, we have discussed many neutrosophic algebraic AH-structures, such as AH-subspaces, AH-submodules, AH-ideals and n-refined AH-ideals and spaces.

This work maybe very useful in the future studies in Turiyam algebraic AH-structures, especially spaces and modules, where they can be built as new generalizations of corresponding neutrosophic ones.

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References

[1] Smarandache, F., " A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability", American Research Press. Rehoboth, 2003.

[2] Alhamido, R., and Gharibah, T., "Neutrosophic Crisp Tri-Topological Spaces", Journal of New Theory, Vol. 23 , pp.13-21. 2018.

[3] Edalatpanah. S.A., "Systems of Neutrosophic Linear Equations", Neutrosophic Sets and Systems, Vol. 33, pp. 92-104. 2020.

[4] Sankari, H., and Abobala, M., "Neutrosophic Linear Diophantine Equations With two Variables", Neutrosophic Sets and Systems, Vol. 38, pp. 22-30, 2020.

Doi : https://doi.org/10.54216/JNFS.020105
Received July 15, 2021 Accepted: Jan 10, 2022

[5] Sankari, H., and Abobala, M." *n*-Refined Neutrosophic Modules", Neutrosophic Sets and Systems, Vol. 36, pp. 1-11. 2020.

[6] Alhamido, R., and Abobala, M., "AH-Substructures in Neutrosophic Modules", International Journal of Neutrosophic Science, Vol. 7, pp. 79-86 . 2020.

[7] Abobala, M., "AH-Subspaces in Neutrosophic Vector Spaces", International Journal of Neutrosophic Science, Vol. 6 , pp. 80-86. 2020.

[8] Abobala, M.,. "A Study of AH-Substructures in *n*-Refined Neutrosophic Vector Spaces", International Journal of Neutrosophic Science", Vol. 9, pp.74-85. 2020.

[9] Hatip, A., Alhamido, R., and Abobala, M., "A Contribution to Neutrosophic Groups", International Journal of Neutrosophic Science", Vol. 0, pp. 67-76 . 2019.

[10] Abobala, M., " *n*-Refined Neutrosophic Groups I", International Journal of Neutrosophic Science, Vol. 0, pp. 27-34. 2020.

[11] Kandasamy, V.W.B., and Smarandache, F., "Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures", Hexis, Phonex, Arizona, 2006.

[12] Agboola, A.A.A., Akinola, A.D., and Oyebola, O.Y., " Neutrosophic Rings I" , International J.Mathcombin, Vol 4,pp 1-14. 2011

[13] Agboola, A.A.A., "On Refined Neutrosophic Algebraic Structures," Neutrosophic Sets and Systems,Vol.10, pp. 99-101. 2015.

[14] Abobala, M., "Classical Homomorphisms Between Refined Neutrosophic Rings and Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp. 72-75. 2020.

[15] Smarandache, F., and Abobala, M., n-Refined neutrosophic Rings, International Journal of Neutrosophic Science, Vol. 5 , pp. 83-90, 2020.

[16] Kandasamy, I., Kandasamy, V., and Smarandache, F., "Algebraic structure of Neutrosophic Duplets in Neutrosophic Rings", Neutrosophic Sets and Systems, Vol. 18, pp. 85-95. 2018.

[17] Yingcang, Ma., Xiaohong Zhang ., Smarandache, F., and Juanjuan, Z., "The Structure of Idempotents in Neutrosophic Rings and Neutrosophic Quadruple Rings", Symmetry Journal (MDPI), Vol. 11. 2019.

[18] Kandasamy, V. W. B,. Ilanthenral, K., and Smarandache, F., "Semi-Idempotents in Neutrosophic Rings", Mathematics Journal (MDPI), Vol. 7. 2019.

[19] Abobala, M., On Some Special Substructures of Neutrosophic Rings and Their Properties, International Journal of Neutrosophic Science", Vol. 4 , pp. 72-81, 2020.

[20] Smarandache, F., " An Introduction To neutrosophic Genetics", International Journal of neutrosophic Science, Vol.13, 2021.

[21] Martin, N, Smarandache, F, and Broumi, S., " Covid 19 Decision Making using Extended Plithogenic hypersoft Sets With Dual Dominent Attributes", International Journal of neutrosophic Science, Vol. 13, 2021.

Doi : https://doi.org/10.54216/JNFS.020105
Received July 15, 2021 Accepted: Jan 10, 2022

[22]Agboola, A.A., "Introduction To Neutro groups", International Journal of neutrosophic Science, Vol. 6, 2020.

[23] Abobala, M., "On Some Special Substructures of Refined Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp. 59-66. 2020.

[24] Smarandache, F., and Ali, M., "Neutrosophic Triplet Group", Neural. Compute. Appl. 2019.

[25] Sankari, H., and Abobala, M.," AH-Homomorphisms In neutrosophic Rings and Refined Neutrosophic Rings", Neutrosophic Sets and Systems, Vol. 38, pp. 101-112, 2020.

[26] Smarandache, F., and Kandasamy, V.W.B., " Finite Neutrosophic Complex Numbers",·Source: arXiv. 2011.

[27]. Abobala, M., " *n*-Refined Neutrosophic Groups II", International Journal of Neutrosophic Science, Vol. 0, 2020.

[28] Ali, Rozina., " On the Concept of Algebraic Actions In Neutrosophic Groups", Resaechgate.net, 2021.

[29] Ali, Rozina., " Neutrosophic Matrices and Their Properties", researchgat.net, 2021.

[30] Ali, Rozina., " Recent Advantages In Neutrosophic Module Theory", researchgate.net, 2021.

[31] Abobala, M, "*n*-Cyclic Refined Neutrosophic Algebraic Systems Of Sub-Indeterminacies, An Application To Rings and Modules", International Journal of Neutrosophic Science, Vol. 12, pp. 81-95 . 2020.

[32] Smarandache, F., "Neutrosophic Set a Generalization of the Intuitionistic Fuzzy Sets", Inter. J. Pure Appl. Math., pp. 287-297. 2005.

[33] M. Ali, F. Smarandache, M. Shabir and L. Vladareanu., "Generalization of Neutrosophic Rings and Neutrosophic Fields", Neutrosophic Sets and Systems, vol. 5, pp. 9-14, 2014.

[34] Anuradha V. S., "Neutrosophic Fuzzy Hierarchical Clustering for Dengue Analysis in Sri Lanka", Neutrosophic Sets and Systems, vol. 31, pp. 179-199. 2020.

[35] Olgun, N., and Hatip, A., "The Effect Of The Neutrosophic Logic On The Decision Making, in Quadruple Neutrosophic Theory And Applications", Belgium, EU, Pons Editions Brussels,pp. 238-253. 2020.

[36] Abobala, M., Bal, M., and Hatip, A.," A Review On Recent Advantages In Algebraic Theory Of Neutrosophic Matrices**",** International Journal of Neutrosophic Science, Vol. 17, 2021.

[37] Turksen, I., "Interval valued fuzzy sets based on normal forms", Fuzzy Sets and Systems, 20, pp.191-210, 1986. 1986.

[38] Chalapathi, T., and Madhavi, L., "Neutrosophic Boolean Rings", Neutrosophic Sets and Systems, Vol. 33, pp. 57-66. 2020.

[39] Abobala, M., "Classical Homomorphisms Between n-refined Neutrosophic Rings", International Journal of Neutrosophic Science", Vol. 7, pp. 74-78. 2020.

[40] Agboola, A.A.A,. Akwu, A.D,. and Oyebo, Y.T., " Neutrosophic Groups and Subgroups", International .J .Math. Combin, Vol. 3, pp. 1-9. 2012.

Doi : https://doi.org/10.54216/JNFS.020105
Received July 15, 2021 Accepted: Jan 10, 2022

[41] Smarandache, F., " *n*-Valued Refined Neutrosophic Logic and Its Applications in Physics", Progress in Physics, 143-146, Vol. 4, 2013.

[42] Adeleke, E.O., Agboola, A.A.A.,and Smarandache, F., "Refined Neutrosophic Rings I", International Journal of Neutrosophic Science, Vol. 2(2), pp. 77-81. 2020.

[43] Hatip, A., and Abobala, M., "AH-Substructures In Strong Refined Neutrosophic Modules", International Journal of Neutrosophic Science, Vol. 9, pp. 110-116 . 2020.

[44] Hatip, A., and Olgun, N., "On Refined Neutrosophic R-Module", International Journal of Neutrosophic Science, Vol. 7, pp.87-96. 2020.

[45] Bal, M., Abobala, M., "On The Representation Of Winning Strategies Of Finite Games By Groups and Neutrosophic Groups", Journal Of Neutrosophic and Fuzzy Systems, 2022.

[46] Smarandache F., and Abobala, M., "*n*-Refined Neutrosophic Vector Spaces", International Journal of Neutrosophic Science, Vol. 7, pp. 47-54. 2020.

[47] Sankari, H., and Abobala, M., "Solving Three Conjectures About Neutrosophic Quadruple Vector Spaces", Neutrosophic Sets and Systems, Vol. 38, pp. 70-77. 2020.

[48] Adeleke, E.O., Agboola, A.A.A., and Smarandache, F., "Refined Neutrosophic Rings II", International Journal of Neutrosophic Science, Vol. 2(2), pp. 89-94. 2020.

[49] Abobala, M., On Refined Neutrosophic Matrices and Their Applications In Refined Neutrosophic Algebraic Equations, Journal Of Mathematics, Hindawi, 2021

[50] Abobala, M., A Study of Maximal and Minimal Ideals of n-Refined Neutrosophic Rings, Journal of Fuzzy Extension and Applications, Vol. 2, pp. 16-22, 2021.

[51] Abobala, M., " Semi Homomorphisms and Algebraic Relations Between Strong Refined Neutrosophic Modules and Strong Neutrosophic Modules", Neutrosophic Sets and Systems, Vol. 39, 2021.

[52] Abobala, M., "On Some Neutrosophic Algebraic Equations", Journal of New Theory, Vol. 33, 2020.

[53] Abobala, M., On The Representation of Neutrosophic Matrices by Neutrosophic Linear Transformations, Journal of Mathematics, Hindawi, 2021.

[54] Abobala, M., "On Some Algebraic Properties of n-Refined Neutrosophic Elements and n-Refined Neutrosophic Linear Equations", Mathematical Problems in Engineering, Hindawi, 2021

[55] Kandasamy V, Smarandache F., and Kandasamy I., Special Fuzzy Matrices for Social Scientists . Printed in the United States of America,2007, book, 99 pages.

[56] Khaled, H., and Younus, A., and Mohammad, A., " The Rectangle Neutrosophic Fuzzy Matrices", Faculty of Education Journal Vol. 15, 2019. (Arabic version).

[57] Abobala, M., Partial Foundation of Neutrosophic Number Theory, Neutrosophic Sets and Systems, Vol. 39 , 2021.

Doi : https://doi.org/10.54216/JNFS.020105
Received July 15, 2021 Accepted: Jan 10, 2022

[58] F. Smarandache, *Neutrosophic Theory and Applications*, Le Quy Don Technical University, Faculty of Information technology, Hanoi, Vietnam, 17th May 2016.

[59] Sankari, H, and Abobala, M., " On A New Criterion For The Solvability of non Simple Finite Groups and m-Abelian Solvability, Journal of Mathematics, Hindawi, 2021.

[60] Giorgio, N, Mehmood, A., and Broumi, S.," Single Valued neutrosophic Filter", International Journal of Neutrosophic Science, Vol. 6, 2020.

[61] Abobala, M., "A Study Of Nil Ideals and Kothe's Conjecture In Neutrosophic Rings", International Journal of Mathematics and Mathematical Sciences, hindawi, 2021

[62] Abobala, M., Hatip, A., Olgun, N., Broumi, S., Salama, A,A., and Khaled, E, H., The algebraic creativity In The Neutrosophic Square Matrices, Neutrosophic Sets and Systems, Vol. 40, pp. 1-11, 2021.

[63]Alhamido, K., R., "A New Approach of neutrosophic Topological Spaces", International Journal of neutrosophic Science, Vol.7, 2020.

[64] Abobala, M., "Neutrosophic Real Inner Product Spaces", Neutrosophic Sets and Systems, Vol. 43, 2021.

[65] Abobala, M., "On Some Special Elements In Neutrosophic Rings and Refined Neutrosophic Rings", Journal of New Theory, vol. 33, 2020.

[66] Abobala, M., and Hatip, A., "An Algebraic Approch To Neutrosophic Euclidean Geometry", Neutrosophic Sets and Systems, Vol. 43, 2021.

[67] Sundar, J., Vadivel, A., " New operators Using Neutrosophic $\vartheta -$Open Sets", Journal Of Neutrosophic and Fuzzy Systems, 2022.

[68] Sankari, H, and Abobala, M, " A Contribution to m-Power Closed Groups", UMM-Alqura University Journal for Applied Sciences, KSA, 2020.

[69] Abobala, M., "On The Characterization of Maximal and Minimal Ideals In Several Neutrosophic Rings", Neutrosophic Sets and Systems, Vol. 45, 2021.

[70] Chellamani, P., and Ajay, D., "Pythagorean neutrosophic Fuzzy Graphs", International Journal of Neutrosophic Science, Vol. 11, 2021.

[71] Milles, S, Barakat, M, and Latrech, A., " Completeness and Compactness In Standard Single Valued neutrosophic Metric Spaces", International Journal of Neutrosophic Science, Vol.12 , 2021.

[72] Es, Haydar, A., "A Note On neutrosophic Soft Menger Topological Spaces", International Journal of Neutrosophic Science, Vol.7, 2020.

[73] Ceven, Y., and Tekin, S., " Some Properties of Neutrosophic Integers", Kırklareli University Journal of Engineering and Science, Vol. 6, pp.50-59, 2020.

[74] Abobala, M., Hatip, A., Bal,M.," A Study Of Some Neutrosophic Clean Rings", International journal of neutrosophic science, 2022.

Doi : https://doi.org/10.54216/JNFS.020105
Received July 15, 2021 Accepted: Jan 10, 2022

[75] Ahmad, K., Bal, M., Hajjari, A., Ali, R.," On Imperfect Duplets In Some refined Neutrosophic Rings", Journal of Neutrosophic and Fuzzy Systems, 2022.

[76] Singh, P,K., " Anti-geometry and NeutroGeometry Characterization of Non-Euclidean Data", Journal of Neutrosophic and Fuzzy Systems, Vol 1, Issue 1, pp. 24-33, 2021, DOI: https://doi.org/10.54216/JNFS.0101012

[77] Singh, P,K., " Data With Turiyam Set for Fourth Dimension Quantum Information Processing", Journal of Neutrosophic and Fuzzy Systems, Vol 1, Issue 1, pp. 9-23, DOI: https://doi.org/10.54216/JNFS.010101

[78] Singh, P, K., Ahmad, K., Bal, M., Aswad, M.," On The Symbolic Turiyam Rings", Journal of Neutrosophic and Fuzzy Systems, Vol. 1 , No. 2 , pp. 80-88 , 2021, Doi : https://doi.org/10.54216/JNFS.010204

[79] Ahmad, K., Bal, M., and Aswad, M.," A Short Note on The Solution Of Fermat's Diophantine Equation In Some Neutrosophic Rings", Journal of Neutrosophic and Fuzzy Systems, Vol. 1, 2022.

[80] Ibrahim, M., and Abobala, M., "An Introduction To Refined Neutrosophic Number Theory", Neutrosophic Sets and Systems, Vol. 45, 2021.

[81] Abobala, M., Bal, M., Aswad, M., "A Short Note On Some Novel Applications of Semi Module Homomorphisms", International journal of neutrosophic science, 2022.

[82] Ahmad, K., Bal, M., and Aswad, M.," The kernel of Fuzzy and Anti Fuzzy Groups",Journal of Neutrosophic and Fuzzy Systems, Vol.1, 2022.

[83] Smarandache, F., and Broumi,M., "Neutro-Intelligent Set is a particular case of the refined neutrosophic set", Journal of Neutrosophic and Fuzzy Systems, Vol. 1, 2022.

[84] Olgun, N., Hatip, A., Bal, M., and Abobala, M., " A Novel Approach To Necessary and Sufficient Conditions For The Diagonalization of Refined Neutrosophic Matrices", International Journal of Neutrosophic Science, Vol. 16, pp. 72-79, 2021.

[85] Singh, P.,K., "NeutroAlgebra and NeutroGeometry for Dealing Heteroclinic Patterns", In: Theory and Applications of NeutroAlgebras as Generalizations of Classical Algebras, IGI Global Publishers, April 2022, Chapter 6, DOI: 10.4018/978-1-6684-3495-6, ISBN13: 9781668434956

[86] Singh, P.,K., "Data with Non-Euclidean Geometry and its Characterization", Journal of Artificial Intelligence and Technology, Vol 2, Issue 1, pp. 3-8, 2022, DOI: 10.37965/jait.2021.12001

[87] Singh, P,K, "Three-way n-valued neutrosophic concept lattice at different granulation", International Journal of Machine Learning and Cybernetics, Vol 9, Issue 11, pp. 1839-1855, 2019.

[88] Bal, M., Ahmad, K., Hajjari, A., Ali, R.," A Short Note On The Kernel Subgroup Of Intuitionistic Fuzzy groups" Journal of Neutrosophic and Fuzzy Systems, 2022.

Doi : https://doi.org/10.54216/JNFS.020105
Received July 15, 2021 Accepted: Jan 10, 2022

[89] Bal, M., Ahmad, K., Hajjari, A., Ali, R.," The Structure Of Imperfect Triplets In Several Refined Neutrosophic Rings" Journal of Neutrosophic and Fuzzy Systems, 2022.

[90] F. Smarandache, S. Broumi, Singh, P,K, CF Liu, V Venkateswara Rao, H.L. Yang, I. Patrascu, A. Elhasounny, Introduction to Neutrosophy and Neutrosophic Environment, In: Y Guo and Amira S ed., Neutrosophic Set in Medical Image Analysis, ISBN: 9780128181485, Nov 2019, pp. 3-29, Elsevier, https://doi.org/10.1016/B978-0-12-818148-5.00001-1

[91] Singh, P,K, "Turiyam set a fourth dimension data representation", Journal of Applied Mathematics and Physics,Vol.9,Issue7,pp.1821-1828, 2021, DOI: 10.4236/jamp.2021.97116,

[92] Singh, P,K, "Fourth dimension data representation and its analysis using Turiyam Context", Journal of Computer and Communications, Vol. 9, no. 6, pp. 222-229, 2021, DOI: 10.4236/jcc.2021.96014

Doi : https://doi.org/10.54216/JNFS.020105
Received July 15, 2021 Accepted: Jan 10, 2022