On The Group of Units Classification In 3-Cyclic and 4-cyclic Refined Rings of Integers And The Proof of Von Shtawzens' Conjectures

Article *in* International Journal of Neutrosophic Science · August 2023

DOI: 10.54216/IJNS.210414

Mohammad Abobala
Tishreen University

On The Group of Units Classification In 3-Cyclic and 4-cyclic Refined Rings of Integers And The Proof of Von Shtawzens' Conjectures

Hasan Sankari, Mohammad Abobala

Tishreen University, Department Of Mathematics, Latakia, Syria

Emails: Hasan2sankari@gmail.com; Mohammadabobala777@gmail.com

Abstract

First Von Shtawzen's Diophantine equation is a non-linear Diophantine equation with three variables . This equation has been conjectured that it has a finite number of integer solutions, and this number of solutions is divisible by 6. Second Von Shtawzen's Diophantine equation is a non-linear Diophantine equation with four variables. This equation has been conjectured that it has a finite number of integer solutions, and this number of solutions is divisible by 8. In this paper, we prove that first Von Shtawzen's conjecture is true, where we show that first Von Shtawzen's Diophantine equations has exactly 12 solutions. On the other hand, we find all solutions of this Diophantine equations. In addition, we provide a full proof of second Von Shtawzen's conjecture, where we prove that the previous Diophantine equation has exactly 16 solutions, and we determine all of its possible solutions.

Keywords: n-cyclic refined ring; first Von Shtawzen's conjecture; group of units; second Von Shtawzen's conjecture

1. Introduction

In every ring R, the set of all invertible elements has a group structure under multiplication, which is called the group of units of the ring R. It is denoted as U(R) [1].

The concept of n-cyclic refined neutrosophic ring (or n-cyclic refined ring) was defined [2] as follows:

If R is a ring, the following set $R_n(I) = \{a_0 + a_1I_1 + \cdots + a_nI_n\ ;\ a_i \in R\}$ is called the n-cyclic refined ring. The operations on $R_n(I)$ are defined as follows:

$(a_0 + a_1I_1 + \cdots + a_nI_n)$+$(b_0 + b_1I_1 + \cdots + b_nI_n)$=

$a_0 + b_0 + I_1[a_1 + b_1]+..+I_n[a_n + b_n]$.

$(a_0 + a_1I_1 + \cdots + a_nI_n).(b_0 + b_1I_1 + \cdots + b_nI_n)= a_0b_0 + I_1\left[\sum_{i+j\equiv 1 \bmod n} a_ib_j\right] + I_2\left[\sum_{i+j\equiv 2 \bmod n} a_ib_j\right]+..+I_n\left[\sum_{i+j\equiv n \bmod n} a_ib_j\right]$.

The n-cyclic refined ring is a ring in the algebraic meaning, then the invertible elements (Units) have a group structure under multiplication. It is denoted by U($R_n(I)$) .

The group of units of the 2-cyclic refined rings of integers, rationals, and reals was studied firstly by Sadiq in [2]. Where he classified the group of units of the 2-cyclic refined group of integer units as a direct product of cyclic groups.

In [3,5], Von Shtawzen has studied the problem of 3-cyclic and 4-cyclic refined group of integer units. Also, he has presented the following conjectures:

Doi: https://doi.org/10.54216/IJNS.210414
Received: February 06, 2023 Revised: May 12, 2023 Accepted: July 28, 2023

First Von Shtawzen's Conjecture:

Let $Z_3(I) = \{a_0 + a_1I_1 + a_2I_2 + a_3I_3;\ a_i \in Z\}$ be the 3-cyclic refined neutrosophic ring of integers, then the group of units $U(Z_3(I))$ has a finite order and it is divisible by 6. The equivalent formula of the previous conjecture can be written as follows:

$(a_0 + a_3)^3 + (a_1)^3 + (a_2)^3 - 3a_1a_2(a_0 + a_3) = 1\ or - 1$. With $a_0 \in \{1, -1\}$. Has a finite number of solutions and this number is divisible by 6.

Second Von Sthawzen's Conjecture:

Let $Z_4(I) = \{a_0 + a_1I_1 + a_2I_2 + a_3I_3 + a_4I_4;\ a_i \in Z\}$ be the 4-cyclic refined neutrosophic ring of integers**,** then the group of units $U(Z_4(I))$ has a finite order and it is divisible by 8. The equivalent formula of the previous conjecture can be written as follows:

$(a_0 + a_4)^4 - 4a_1a_3(a_0 + a_4)^2 + 4a_2{a_3}^2(a_0 + a_4) - 4a_1a_3{a_2}^2 - {a_1}^4 - {a_3}^4 + {a_2}^4 + 4a_2{a_1}^2(a_0 + a_4) + 2{a_1}^2{a_3}^2 - 2{a_2}^2(a_0 + a_4)^2 = 1\ or - 1, with\ a_0 \in \{1, -1\}$. With $a_0 \in \{1, -1\}$ has a finite number of solutions and this number is divisible by 8.

In [4], Abd Alrida et.al, have generalized the first Von Shtawzen's conjecture by the following formula:

If $Z_n(I) = \{a_0 + a_1I_1 + a_2I_2 + .. + a_nI_n;\ a_i \in Z\}$, then the order of Von Shtawzens' group is finite and divisible by 2n.

In addition, another review of units in n-cyclic rings was supposed through Diophantine equations in [6-8], with many other open problems about n-cyclic refined rings especially those are related to number theory and linear spaces theory.

In this work, we present a complete proof of first/second Von Shtawzens' conjectures, where we prove that the order of the units group $U(Z_3(I))$ is 12., we determine all 12 units in $U(Z_3(I))$. This determines the all 12 solutions of the first Von Shtawzen's Diophantine equation showed above. Also, we prove that the order of the units group $U(Z_4(I))$ is 16, and we determine all 16 units in $U(Z_4(I))$. This determines al 16 solutions for the second Von Shtawzens's Diophantine equation. .

2. Main Discussion

Theorem:

Let $Z_3(I) = \{a_0 + a_1I_1 + a_2I_2 + a_3I_3;\ a_i \in Z\}$ be the 3-cyclic refined neutrosophic ring of integers, then the group of units $U(Z_3(I))$ is finite with order 12.

Proof:

Let $Z_3(I) = \{a + bI_1 + cI_2 + dI_3; a, b, c, d \in Z\}$ be the 3-cyclic refined ring of integers, we define the mapping: $f: Z_3(I) \to Z \times Z$ such that:

$f(a_0 + a_1I_1 + a_2I_2 + a_3I_3) = (a_0, a_0 + a_1 + a_2 + a_3)$. We have the following properties:

f is well defined, that is because if we supposed that

$a_0 + a_1I_1 + a_2I_2 + a_3I_3 = b_0 + b_1I_1 + b_2I_2 + b_3I_3$, then we get $a_i = b_i$ for all $0 \le i \le 3$.

Thus $(a_0, a_0 + a_1 + a_2 + a_3)$ =($b_0, b_0 + b_1 + b_2 + b_3)$

i.e. $f(a_0 + a_1I_1 + a_2I_2 + a_3I_3)$=$f(b_0 + b_1I_1 + b_2I_2 + b_3I_3)$.

(2) f preserves addition, that is because:

$$f[(a_0 + a_1I_1 + a_2I_2 + a_3I_3) + (b_0 + b_1I_1 + b_2I_2 + b_3I_3)] =$$

$f[(a_0 + b_0) + (a_1 + b_1)I_1 + (a_2 + b_2)I_2 + (a_3 + b_3)I_3]$=

$$(a_0 + b_0, a_0 + b_0 + a_1 + b_1 + a_2 + b_2 + a_3 + b_3) =$$

Doi: https://doi.org/10.54216/IJNS.210414
Received: February 06, 2023 Revised: May 12, 2023 Accepted: July 28, 2023

$$(a_0, a_0 + a_1 + a_2 + a_3) + (b_0, b_0 + b_1 + b_2 + b_3) =$$

$f(a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3) + f(b_0 + b_1 I_1 + b_2 I_2 + b_3 I_3)$.

(3) f preserves multiplication. For this goal, we assume that

$x = a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3, y = b_0 + b_1 I_1 + b_2 I_2 + b_3 I_3$. We have:

$x.y = a_0 b_0 + I_1[a_0 b_1 + a_1 b_0 + a_1 b_3 + a_3 b_1 + a_2 b_2] + I_2[a_0 b_2 + a_2 b_0 + a_1 b_1 + a_2 b_3 + a_3 b_2] + I_3[a_0 b_3 + a_3 b_0 + a_1 b_2 + a_2 b_1 + a_3 b_3]$.

$$f(xy) = (a_0 b_0, a_0 b_0 + a_0 b_1 + a_1 b_0 + a_1 b_3 + a_3 b_1 + a_2 b_2 + a_0 b_2 + a_2 b_0 + a_1 b_1 + a_2 b_3 + a_3 b_2 + a_0 b_3 + a_3 b_0 + a_1 b_2 + a_2 b_1 + a_3 b_3\ ) =$$

$(a_0, a_0 + a_1 + a_2 + a_3)(b_0, b_0 + b_1 + b_2 + b_3) = f(x)f(y)$.

This implies that the mapping f is a ring homomorphism.

Now, let $U(Z_3(I))$ be the group of units of the 3-cyclic refined ring of integers. Suppose that the mapping g is the restriction of the homomorphism f on $(Z_3(I))$. i.e. $g = f|_{U(Z_3(I))}: U(Z_3(I)) \to U(Z) \times U(Z) \cong Z_2 \times Z_2$.

The mapping g is a group homomorphism since it is well defined and preserves multiplication.

Ker(g)=$\{x = a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3 \in U\big(Z_3(I)\big); g(x) = (1,1)\}$

=$\{\ x = 1 + a_1 I_1 + a_2 I_2 + a_3 I_3 \in U\big(Z_3(I)\big);\ a_1 + a_2 + a_3 = 0\}$

=$\{\ x = 1 + a_1 I_1 + a_2 I_2 + (-a_1 - a_2) I_3 \in U\big(Z_3(I)\big)\}$.

According to the isomorphism theorem in groups, we can write:

$$U\big(Z_3(I)\big)\Big/_{Ker(g)} \cong g(U\big(Z_3(I)\big)) \leq Z_2 \times Z_2.$$

Now, we are forced to determine the elements of the group Ker(g) and its classification as a finite abelian group.

Let $x = 1 + a_1 I_1 + a_2 I_2 + (-a_1 - a_2) I_3$ be a unit in Ker(g), this is possible if and only if there exists $x = 1 + b_1 I_1 + b I_2 + (-b_1 - b_2) I_3 \in Ker(g)$ such that $xy = 1$.

On the other hand, we have:

$xy = 1 + I_1[b_1 + a_1 - a_1 b_1 - a_1 b_2 + a_2 b_2 - a_1 b_1 - a_2 b_1] + I_2[b_2 + a_2 + a_1 b_1 - a_2 b_1 - a_2 b_2 - a_1 b_2 - a_2 b_2] + I_3[-b_1 - b_2 + a_1 b_2 + a_2 b_1 + a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2]$.

The previous formula ensures that the condition $xy = 1$ is equivalent to following system of Diophantine equations:

$$\begin{cases} b_1(1 - 2a_1 - a_2) + b_2(a_2 - a_1) = -a_1 \\ b_1(a_1 - a_2) + b_2(1 - 2a_2 - a_1) = -a_2 \end{cases}.$$

Since x=1 is trivial unit, we can suppose that $a_1\ or\ a_2 \neq 0$. According to Cramer's rule, the previous system has a unique solution if and only if the determinant $T = \begin{vmatrix} 1 - 2a_1 - a_2 & a_2 - a_1 \\ a_1 - a_2 & 1 - 2a_2 - a_1 \end{vmatrix}$ is a unit in the ring of integers Z. i.e. $T \in \{1, -1\}$.

We have

$T = (1 - 2a_1 - a_2)(1 - 2a_2 - a_1) - (a_2 - a_1)(a_1 - a_2) = 3[(a_1 + a_2)(a_1 + a_2 - 1) - a_1 a_2] + 1$.

Firstly, we assume that $T = -1$, then

$3[(a_1 + a_2)(a_1 + a_2 - 1) - a_1 a_2] = -2$,

which is impossible, that is because the left side is equal to (0 mod 3), and the right side is not.

This means that $T = 1$ is the only possible value of T. The equation T=1 implies:

$3[(a_1 + a_2)(a_1 + a_2 - 1) - a_1 a_2] = 0, thus\ [(a_1 + a_2)(a_1 + a_2 - 1)] = a_1 a_2$.

Doi: https://doi.org/10.54216/IJNS.210414
Received: February 06, 2023 Revised: May 12, 2023 Accepted: July 28, 2023

We will discuss the possible cases of the previous non linear Diophantine equation.

Case (1):

If $a_1 = a_2 = 0$, then we get the trivial unit $x = 1$.

Case (2):

If $a_1 = 0, a_2 \neq 0$, then $a_2 = 1$, and we get the unit $x = 1 + I_2 - I_3$.

Case (3):

If $a_2 = 0, a_1 \neq 0$, then $a_1 = 1$, and we get the unit $x = 1 + I_1 - I_3$.

For discussing all remaining possible cases, we write the equation $[(a_1 + a_2)(a_1 + a_2 - 1)] = a_1 a_2$, by the following formula:

$a_1{}^2 + a_2{}^2 + a_1 a_2 = a_1 + a_2$.

Case (4):

If $a_1, a_2 > 0$, then $a_1{}^2 \geq a_1, a_2{}^2 \geq a_2$, thus $a_1{}^2 + a_2{}^2 + a_1 a_2 > a_1 + a_2$, which is a contradiction.

Case (5):

If $a_1, a_2 < 0$, then $a_1{}^2 \geq a_1, a_2{}^2 \geq a_2$, thus $a_1{}^2 + a_2{}^2 + a_1 a_2 > 0, and\ a_1 + a_2 < 0$, which is a contradiction.

The only remaining possible case is that one of the two integers a_1, a_2 is positive and the other is negative.

Without affecting the generality, we can assume that $a_2 > 0, a_1 < 0$.

Case (6):

If $a_2 > |a_1| = -a_1$, we write the equation $a_1{}^2 + a_2{}^2 + a_1 a_2 = a_1 + a_2$ by the formula

$a_1{}^2 + a_2{}^2 = a_2(1 - a_1) + a_1$.

From the assumption, we get $a_1 < a_1{}^2, \begin{cases} a_2 \leq a_2 \\ 1 - a_1 \leq a_2 \end{cases}$, hence $a_2(1 - a_1) + a_1 < a_1{}^2 + a_2{}^2$ which is a contradiction.

Case (7):

If $a_2 < |a_1| = -a_1$, we write the equation $a_1{}^2 + a_2{}^2 + a_1 a_2 = a_1 + a_2$ by the formula

$a_1{}^2 + a_2{}^2 = -a_1(a_2 - 1) + a_2$.

On the other hand, $a_2 \neq 1$, that is because if $a_2 = 1$, then we get $a_1 = 0$, which is exactly equivalent to the Case (2).

From the assumption and since $a_2 \neq 1$, we get $a_2 < a_2{}^2, -a_1 > a_2 - 1$, thus

$(-a_1)(-a_1) > -a_1(a_2 - 1), i.e. a_1{}^2 > -a_1(a_2 - 1)$, this implies that

$a_1{}^2 + a_2{}^2 > -a_1(a_2 - 1) + a_2$, which is a contradiction.

Case (8):

If $a_2 = |a_1| = -a_1$, then we get $a_1 = a_2 = 0$. (It is the trivial case, the unit $x = 1$).

From the discussion above, we get that Ker(g) has exactly 3 elements

$Ker(g) = \{1, 1 + I_1 - I_3, 1 + I_2 - I_3\}$. Thus $Ker(g) \cong Z_3$.

Since $U(Z_3(I)) / Ker(g) \cong g(U(Z_3(I))) \leq Z_2 \times Z_2$, we will have the following:

$O\left(U(Z_3(I))\right) \leq O(Ker(g)) \times O(Z_2) \times O(Z_2) = 3 \times 2 \times 2 = 12$. So that the 3-cyclic refined ring of integers $Z_3(I)$ has 12 units at most. By other words, Von Shtawzen's Diophantine equation has 12 solutions at most.

Doi: https://doi.org/10.54216/IJNS.210414
Received: February 06, 2023 Revised: May 12, 2023 Accepted: July 28, 2023

According to Lagrange's theorem in finite groups, we find that

$O\left(U(Z_3(I))/_{Ker(g)}\right)|O(Z_2 \times Z_2) = 4$, thus $O\left(U(Z_3(I))\right) \in \{3,6,12\}$.

Also, we have shown that $Z_3(I)$ has 3 units $1, 1 + I_1 - I_3, 1 + I_2 - I_3$, with their additive inverses $-1, -1 - I_1 + I_3, -1 - I_2 + I_3$, so that we have at least 6 units. This implies that $O\left(U(Z_3(I))\right) \in \{6,12\}$.

Now, by using the fact that $U(Z_3(I))/_{Ker(g)} \cong g(U(Z_3(I))) \leq Z_2 \times Z_2$, we have at least one unit $x \neq 1\ or\ -1$ with order 2 .i.e $x^2 = 1$.

Let $x = a_0 + a_1I_1 + a_2I_2 + a_3I_3$ be a unit with order 2, then by using the homomorphism g, we can write:

$g(x^2) = g(x)^2 = (a_0{}^2, (a_0 + a_1 + a_2 + a_3)^2) = (1,1)$. Without affecting the generality, we can assume that $a_0 = 1$. So that $a_1 + a_2 + a_3 = 0\ or\ a_1 + a_2 + a_3 = -2$.

On the other side, we compute $x^2 = 1$, as follows:

$1 + I_1[a_2{}^2 + 2a_1 + 2a_1a_3] + I_2[a_1{}^2 + 2a_2 + 2a_3a_2] + I_3[a_3{}^2 + 2a_3 + 2a_1a_2] = 1, this\ implies$ the following system of Diophantine equations:

$$\begin{cases} a_2{}^2 + 2a_1 + 2a_1a_3 = 0\ (1) \\ a_1{}^2 + 2a_2 + 2a_3a_2 = 0\ (2). \\ a_3{}^2 + 2a_3 + 2a_1a_2 = 0\ (3) \end{cases}$$

We start our discussion by considering $a_1 + a_2 + a_3 = 0, thus$

$-a_1 - a_2 = a_3$. The Diophantine equations (1), (2), (3) become

$$\begin{cases} a_2{}^2 - 2a_1{}^2 + 2a_1 - 2a_1a_2 = 0\ (1) \\ a_1{}^2 - 2a_2{}^2 + 2a_2 - 2a_1a_2 = 0\ (2) \\ a_1{}^2 + a_2{}^2 + 4a_1a_2 - 2a_1 - 2a_2 = 0\ (3) \end{cases}$$

. The equation (3) is derived from (1) and (2), thus it has no real value in the proof. We will consider (1) and (2).

Equation (1) can be written as $a_2{}^2 - 2a_1{}^2 + 2a_1 = 2a_1a_2$. Equation (2) can be written as

$a_1{}^2 - 2a_2{}^2 + 2a_2 = 2a_1a_2$, this implies that $a_2{}^2 - 2a_1{}^2 + 2a_1 = a_1{}^2 - 2a_2{}^2 + 2a_2$, hence we get: $a_2(3a_2 - 2) = a_1(3a_1 - 2)$.

Assume that $a_1 \neq a_2$, then if $a_1, a_2 > 0\ and\ a_1 > a_2$, we get that $a_2(3a_2 - 2) < a_1(3a_1 - 2)$, which is a contradiction.

On the other hand, if $a_1, a_2 < 0\ and\ a_1 > a_2$, we get that $a_2(3a_2 - 2) > a_1(3a_1 - 2)$, which is another contradiction. (We get an easy similar contradiction if $a_1 < a_2$).

If $a_1 > 0\ and\ a_2 < 0\ or\ a_2 > 0\ and\ a_1 < 0$, we get an obvious contradiction.

The only possible case is $a_1 = a_2$, which implies that $a_1 = a_2 = 0\ (the\ trivial\ unit\ case)\ or\ a_1 = a_2 = \frac{2}{3}$ which is not an integer.

So that, the first case $a_1 + a_2 + a_3 = 0$ is impossible.

Now, we consider the second case $a_1 + a_2 + a_3 = -2$, hence

$-a_1 - a_2 - 2 = a_3$. Now, the equations (1), (2), (3) will become

$$\begin{cases} a_2{}^2 - 2a_1{}^2 - 2a_1 - 2a_1a_2 = 0\ (1) \\ a_1{}^2 - 2a_2{}^2 - 2a_2 - 2a_1a_2 = 0\ (2) \\ a_1{}^2 + a_2{}^2 + 4a_1a_2 + 2a_1 + 2a_2 = 0\ (3) \end{cases}.$$

The equation (3) is derived from (1) and (2), thus it has no real value in the proof. We will consider (1) and (2).

By using the same method in discussion, we can get from (1), (2) the following:

Doi: https://doi.org/10.54216/IJNS.210414
Received: February 06, 2023 Revised: May 12, 2023 Accepted: July 28, 2023

$a_2(3a_2+2)=a_1(3a_1+2)$.

We assume that $a_1 \neq a_2$, then if $a_1, a_2 > 0$ and $a_1 > a_2$, we get that $a_2(3a_2+2) < a_1(3a_1+2)$, which is a contradiction.

On the other hand, if $a_1, a_2 < 0$ and $a_1 > a_2$, we get that $a_2(3a_2+2) > a_1(3a_1+2)$, which is another contradiction. (We get an easy similar contradiction if $a_1 < a_2$).

If $a_1 > 0$ and $a_2 < 0$ or $a_2 > 0$ and $a_1 < 0$, we get an obvious contradiction.

The only possible case is $a_1 = a_2$, which implies that

$a_1 = a_2 = -\frac{2}{3}$ (*which is not an integer*) or $a_1 = a_2 = 0$, *thus* $a_3 = -2$. This means that

$x = 1 - 2I_3$ is a unit with order 2. Thus its additive inverse $-x = -1 + 2I_3$ is another unit.

From the previous discussion, we get that has exactly 12 units, thus first Von Shtawzen's conjecture is true.

Until now, we have found 8 units

$\{1, -1, 1-2I_3, -1+2I_3, 1+I_1-I_3, -1-I_1+I_3, 1+I_2-I_3, -1-I_2+I_3\}$.

To find the other 4 units, we can use the group structure of the $U(Z_3(I))$under multiplication.

$(1-2I_3)(1+I_1-I_3) = 1-I_1-I_3$ which is another unit. So that its additive inverse $-1+I_1+I_3$ is a unit.

Also, $(1-2I_3)(1+I_2-I_3) = 1-I_2-I_3$ which is another unit. So that its additive inverse $-1+I_2+I_3$ is a unit.

So that, the elements of the group of units of the 3-cyclic refined neutrosophic ring of integers $Z_3(I)$ are $\{1, -1, 1-2I_3, -1+2I_3, 1+I_1-I_3, -1-I_1+I_3, 1+I_2-I_3, -1-I_2+I_3, 1-I_1-I_3, 1-I_2-I_3, -1+I_2+I_3, -1+I_1+I_3\}$ which is isomorphic to the group $Z_2 \times Z_2 \times Z_3$.

3. Results

The first Von Shtawzen's Diophantine equation

$(a_0+a_3)^3 + (a_1)^3 + (a_2)^3 - 3a_1a_2(a_0+a_3) = 1$ *or* -1. With $a_0 \in \{1, -1\}$.

Has exactly the following 12 solutions:

$(a_0, a_1, a_2, a_3) \in \{(1,0,0,0), (-1,0,0,0), (1,0,1,-1), (-1,0,-1,1), (1,1,0,-1),$

$(-1,-1,0,1), (1,0,0,-2), (-1,0,0,2), (1,-1,0,-1), (-1,1,0,1), (1,0,-1,-1), (-1,0,1,1)\}$.

Theorem:

Let $Z_4(I) = \{a_0 + a_1I_1 + a_2I_2 + a_3I_3 + a_4I_4;\ a_i \in Z\}$ be the 4-cyclic refined neutrosophic ring of integers, then the group of units $U(Z_4(I))$ is finite with order 16.

Proof:

Let $Z_4(I) = \{a + bI_1 + cI_2 + dI_3 + eI_4; a, b, c, d, e \in Z\}$ be the 4-cyclic refined ring of integers, we define the mapping: $f: Z_4(I) \to Z \times Z \times Z$ such that:

$f(a_0 + a_1I_1 + a_2I_2 + a_3I_3 + a_4I_4) = (a_0, a_0 + a_1 + a_2 + a_3 + a_4, a_0 - a_1 + a_2 - a_3 + a_4)$. We can see by a similar discussion of the first theorem that f is a ring homomorphism.

This means that its restriction $g = f|_{U(Z_4(I))}$ on the group of units $U(Z_4(I))$ will be a group homomorphism between the two groups of units, i.e. $g = f|_{U(Z_4(I))}: U(Z_4(I)) \to U(Z) \times U(Z) \times U(Z) \cong Z_2 \times Z_2 \times Z_2$.

Ker(g)=$\{x = a_0 + a_1I_1 + a_2I_2 + a_3I_3 + a_4I_4 \in U(Z_4(I));$

$g(x) = (1,1,1)\}$=$\{x = 1 + a_1I_1 + a_2I_2 + a_3I_3 + a_4I_4 \in U(Z_4(I));$

$a_1 + a_2 + a_3 + a_4 = -a_1 + a_2 - a_3 + a_4 = 0\}$=

Doi: https://doi.org/10.54216/IJNS.210414
Received: February 06, 2023 Revised: May 12, 2023 Accepted: July 28, 2023

$\{ x = 1 + a_1 I_1 + a_2 I_2 + (-a_1) I_3 + (-a_2) I_4 \in U\big(Z_4(I)\big) \}$.

According to the isomorphism theorem in groups, we can write:

$$U\big(Z_3(I)\big) \big/ Ker(g) \cong g(U\big(Z_4(I)\big)) \leq Z_2 \times Z_2 \times Z_2.$$

Now, we are forced to determine the elements of the group Ker(g) and its classification as a finite abelian group.

Let $x = 1 + a_1 I_1 + a_2 I_2 + (-a_1) I_3 + (-a_2) I_4$

be a unit in Ker(g), this is possible if and only if there exists

$y = 1 + b_1 I_1 + b_2 I_2 + (-b_1) I_3 + (-b_2) I_4 \in Ker(g)$

such that $xy = 1$.

On the other hand, we have:

$$xy = 1 + I_1[a_1 + b_1 - a_1 b_2 - a_2 b_1 - a_2 b_1 - a_1 b_2] + I_2[a_2 + b_2 - a_2 b_2 - a_2 b_2 - a_1 b_1 - a_1 b_1] + I_3[-(a_1 + b_1 - a_1 b_2 - a_2 b_1 - a_2 b_1 - a_1 b_2)] + I_4[-(a_2 + b_2 - a_2 b_2 - a_2 b_2 - a_1 b_1 - a_1 b_1)].$$

The previous formula ensures that the condition $xy = 1$ is equivalent to following system of Diophantine equations:

$$\begin{cases} b_1(1 - 2a_2) + b_2(-2a_1) = -a_1 \\ b_1(-2a_1) + b_2(1 - 2a_2) = -a_2 \end{cases}.$$

Since $a_1 = a_2 = 0$ is equivalent to the trivial unit 1, we can assume that $a_1 \text{ or } a_2 \neq 0$.

The previous system of Diophantine equations is uniquely solvable if and only if

The determinant $\begin{vmatrix} 1 - 2a_2 & -2a_1 \\ -2a_1 & 1 - 2a_2 \end{vmatrix} = (1 - 2a_2)^2 - 4{a_1}^2 \in \{1, -1\}$.

If $(1 - 2a_2)^2 - 4{a_1}^2 = -1$

we get a contradiction, that is because the left side is equal to 1(mod 4); but the right side is not.

So that $(1 - 2a_2)^2 - 4{a_1}^2 = 1$,

this is equivalent to

$(1 - 2a_2 - 2a_1)(1 - 2a_2 + 2a_1) = 1$

This implies that :

$(1 - 2a_2 - 2a_1) = (1 - 2a_2 + 2a_1) = 1$ (equation I),

or $(1 - 2a_2 - 2a_1) = (1 - 2a_2 + 2a_1) = -1$ (equation II).

The only integer solution for the equation I is

$a_1 = a_2 = 0$ which is equivalent to the trivial unit 1.

The integer solutions of equation II are

$\{a_1 = a_2 = 0\}$ or $\{a_1 = 0, a_2 = 1\}$.

This implies that the only non trivial unit in Ker(g) is

$x = 1 + I_2 - I_4$, thus $Ker(g) \cong Z_2$.

According to the isomorphism theorem in groups, we can write

$$U(Z_4(I)) \big/ Z_2 \cong g(U(Z_4(I)) \leq Z_2 \times Z_2 \times Z_2,$$

hence $O(U(Z_4(I))$ is a divisor of 16.

Doi: https://doi.org/10.54216/IJNS.210414
Received: February 06, 2023 Revised: May 12, 2023 Accepted: July 28, 2023

On the other hand, consider the following 4-cyclic refined integer

$x = 1 + I_1 - I_4$,

it is easy to check that

$x^2 = 1 + I_2 - I_4, x^3 = 1 + I_3 - I_4, x^4 = 1$.

Also, the element

$y = 1 - I_2 - I_4$ has the following property:

$y^2 = 1$.

Also, the element $z = 1 - 2I_4$, has the property $z^2 = 1$.

Now, we have at least the following units

$\{1, -1, x, -x, x^2, -x^2, x^3, -x^3, y, -y, z, -z\}$,

which means that the order of the group of units is greater than 8 and divides 16, thus

$O(U(Z_4(I))) = 16$.

We can find the other 4 units by using the multiplication operation between the units, which implies that the other 4 units are

$\{1 - I_1 - I_4, 1 - I_3 - I_4, -(1 - I_1 - I_4), -(1 - I_3 - I_4)\}$.

4. Remarks

The second Von Shtawzen's Diophantine equation has exactly the following 16 solutions:

$$x_1 = (1,0,0,0,0),$$

$$x_2 = (1,0,0,0,-2),$$

$$x_3 = (1,1,0,0,-1),$$

$$x_4 = (1,0,1,0,-1),$$

$$x_5 = (1,0,0,1,-1),$$

$$x_6 = (1,-1,0,0,-1),$$

$$x_7 = (1,0,-1,0,-1),$$

$$x_8 = (1,0,0,-1,-1),$$

$$-x_1 = (-1,0,0,0,0),$$

$$-x_2 = (-1,0,0,0,-2),$$

$$-x_3 = (-1,-1,0,0,1),$$

$$-x_4 = (-1,0,-1,0,1),$$

$$-x_5 = (-1,0,0,-1,1),$$

$$-x_6 = (-1,1,0,0,1),$$

$$-x_7 = (-1,0,1,0,1),$$

$$-x_8 = (-1,0,0,1,1)$$

.

The group of units $U(Z_4(I))$ can be classified as follows:

$U(Z_4(I)) \cong Z_2 \times Z_2 \times Z_4$.

Doi: https://doi.org/10.54216/IJNS.210414
Received: February 06, 2023 Revised: May 12, 2023 Accepted: July 28, 2023

5. Conclusion

In this work, we have proved that first Von Shtawzen's conjecture is true, where we showed that the group of units of the 3-cyclic refined ring of integers is finite with order 12. On the other hand, we determined the all 12 solutions of first Von Shtawzen's Diophantine equation $(a_0 + a_3)^3 + (a_1)^3 + (a_2)^3 - 3a_1a_2(a_0 + a_3) = 1\ or - 1$. With $a_0 \in \{1, -1\}$. In addition, we have proved that second Von Shtawzen's conjecture is true, where we showed that the group of units of the 4-cyclic refined ring of integers is finite with order 16.

On the other hand, we determined the all 16 solutions of second Von Shtawzen's Diophantine equation.

References

[1] D. G. Northcott, Lessons on Rings, Modules, and Multiplicities, Cambridge Univ. Press, Cambridge, 1968.

[2] Sadiq. B., " A Contribution To The group Of Units Problem In Some 2-Cyclic Refined Neutrosophic Rings ", International Journal Of Neutrosophic Science, 2022.

[3] Von Shtawzen, O., " Conjectures For Invertible Diophantine Equations Of 3-Cyclic and 4-Cyclic Refined Integers", Journal Of Neutrosophic And Fuzzy Systems, Vol.3, 2022.

[4] Basheer, A., Ahmad, K., and Ali, R., " A Short Contribution To Von Shtawzen's Abelian Group In n-Cyclic Refined Neutrosophic Rings", Journal Of Neutrosophic And Fuzzy Systems, 2022.

[5] Von Shtawzen, O., " On A Novel Group Derived From A Generalization Of Integer Exponents and Open Problems", Galoitica journal Of Mathematical Structures and Applications, Vol 1, 2022.

[6] Basheer, A., Ahmad, K., and Ali, R., " On Some Open Problems About n-Cyclic Refined Neutrosophic Rings and Number Theory", Journal Of Neutrosophic And Fuzzy Systems,, 2022.

[7] A. Alrida Basheer , Katy D. Ahmad , Rozina Ali., "Examples on Some Novel Diophantine Equations Derived from the Group of Units Problem in n-Cyclic Refined Neutrosophic Rings of Integers", Galoitica Journal Of Mathematical Structures And Applications, Vol.3, 2022.

[8] Sarkis, M., " On The Solutions Of Fermat's Diophantine Equation In 3-refined Neutrosophic Ring of Integers", Neoma Journal of Mathematics and Computer Science, 2023.

Doi: https://doi.org/10.54216/IJNS.210414
Received: February 06, 2023 Revised: May 12, 2023 Accepted: July 28, 2023