A Study of Derivative and Integration a Neutrosophic Functions

Ahmed Salamah[1], Malath F Alaswad[2], Rasha Dallah[3]

[1] Departement of Mathematics, port said, Egypt,
[2,3] Departement of Mathematics, Albaath University,Homs, Syria

Emails: drsalama44@gmail.com; Malaz.Aswad@yahoo.com; rasha.dallah20@gmail.com

Abstract

The objective of this paper is to study and define the neutrosophic real functions with one neutrosophic variable depending on the geometric isometry (AH-Isometry), with a lot of concepts from real analysis including continuality, differentiability, derivativility, integrability. We have presented the formal forms of different popular functions in neutrosophic environment like logarithmic function, exponential function, trigonometric functions. Rising neutrosophic derivative, indefinite integral, and definite integral well defined including rising to neutrosophic functions.

Keywords: Neutrosophic real analysis; AH-isometry; Integration; Derivative neutrosophic function; Indefinite integral neutrosophic function; Definite integral neutrosophic function.

1.Introduction

Neutrosophy is a new branch of philosophy concerns with the indeterminacy in all areas of life and science. It has become a useful tool in generalizing many classical systems such as equations [1,9], number theory [2,3], topology [4,5], linear spaces [6,10], modules [4,5], and ring of matrices [7,8].

In the literature, we find many studies about neutrosophic calculus, where some definitions and properties were presented about neutrosophic real functions and numbers [10].

The neutrosophic real functions with one variable were defined only in a special case [11], as follows:

$\boldsymbol{f(x) = g(x) + h(x)I}$ where $\boldsymbol{I}$ takes an interval value defining what is called by neutrosophic thick functions . For example$\boldsymbol{f(x) = 2x + 5xI, I \in [0, 0.01]}$ is a neutrosophic real thick function.

The problem with this definition, that it does not consider the general case $\boldsymbol{f: R(I) \rightarrow R(I); f = f(X)}$

And $\boldsymbol{X = x + yI \in R(I)}$.

Recently, Abobala et.al, have presented the concept of two-dimensional AH-isometry to study the correspondence between neutrosophic plane $\boldsymbol{R(I) \times R(I)}$ and the classical module $\boldsymbol{R^2 \times R^2}$. Also, the one-dimensional AH-isometry between R(I) and $\boldsymbol{R \times R}$. This isometry was useful in defining inner products and norms [10], ordering [9], and neutrosophic geometrical shapes [10].

DOI: https://doi.org/10.54216/JNFS.050203
Received: August 02, 2022 Accepted: January 27, 2023

In this work, we use the one-dimensional AH-isometry to turn the general case of neutrosophic real functions with one variable into two classical real functions so we will go from $\boldsymbol{R}(\boldsymbol{I})$ space into $\boldsymbol{R}\times\boldsymbol{R}$ space, we study the properties of our functions then we go back to $\boldsymbol{R}(\boldsymbol{I})$ space using AH-isometry.

2 Definitions.

Definition 2.1. Neutrosophic Real Number: [12]

Suppose that w is a neutrosophic number, then it takes the following standard form: $w = a + bI$ where a, b are real coefficients, and I represents the indeterminacy, where $0.I = 0$ and $I^n = I$ for all positive integers n.

For example:

$w = 1 + 2I,\ w = 3 = 3 + 0I.$

Definition 2.2. Division of neutrosophic real numbers: [13]

Suppose that w_1, w_2 are two neutrosophic number,where

$$w_1 = a_1 + b_1 I, w_2 = a_2 + b_2 I$$

Then:

$$\frac{w_1}{w_2} = \frac{a_1 + b_1 I}{a_2 + b_2 I} = \frac{a_1}{a_2} + \frac{a_2 b_1 - a_1 b_2}{a_2(a_2 + b_2)} I$$

3 Neutrosophic Functions on $\boldsymbol{R}(\boldsymbol{I})$.

Definition 3.1 [14]

Let $R(I) = \{a + bI\ ; a, b \in R\}$ where $I^2 = I$ be the neutrosophic field of reals. The one-dimensional isometry (AH-Isometry) is defined as follows: [49]

$$T: R(I) \to R \times R\ ;\ T(a + bI) = (a, a + b)$$

Remark 3.1. [14]

T is an algebraic isomorphism between two rings, it has the following properties:

1) T is bijective.
2) T preserves addition and multiplication, i.e.:

$$T[(a + bI) + (c + dI)] = T(a + bI) + T(c + dI)$$

And:

$$T[(a + bI) \cdot (c + dI)] = T(a + bI) \cdot T(c + dI)$$

3) Since T is bijective, then it is invertible by:

$$T^{-1}: R \times R \to R(I)\ ;\ T^{-1}(a, b) = a + (b - a)I$$

4) T preserves distances, i.e.:

The distance on $R(I)$ can be defined as follows:

Let $A = a + bI, B = c + dI$ be two neutrosophic real numbers, then $L = \|\overrightarrow{AB}\| = d[(a + bI, c + dI)] = |a + bI - (c + dI)| = |(a - c) + I(b - d)| = |a - c| + I[|a + b - c - d| - |a - c|]$.

On the other hand, we have:

$T(\|\overrightarrow{AB}\|) = (|a - c|, |(a + b) - (c + d)|) = \big(d(a, c), d(a + b, c + d)\big) = d[(a, a + b), (c, c + d)] = d\big(T(a + bI), T(c + dI)\big)$

$= \|T(\overrightarrow{AB})\|$.

DOI: https://doi.org/10.54216/JNFS.050203
Received: August 02, 2022 Accepted: January 27, 2023

This implies that the distance is preserved up to isometry. i.e. $\|T(AB)\| = T(\|AB\|)$

Definition 3.2. [15]

Let $f: R(I) \to R(I); f = f(X)$ and $X = x + yI \in R(I)$ the f is called a neutrosophic real function with one neutrosophic variable.

a neutrosophic real function $f(X)$ written as follows:

$$f(X) = f(x + yI) = f(x) + I[f(x + y) - f(x)]$$

Theorem3.1. any neutrosophic real function into two classical real functions, i.e., to the classical Euclidean plane $R \times R$.

Proof.

Let $f(X) = f(x + yI) = f(x) + I[f(x + y) - f(x)]$ a neutrosophic real function.

Now, Using the one-dimensional AH-isometry, we have.

$T(f(X)) = T(f(x) + I[f(x + y) - f(x)])$, then.

$(f_1, f_2) = (f(x), f(x + y))$, then, we have.

$$\begin{cases} f_1 = f(x) \\ f_2 = f(x + y) \end{cases}$$

the functions $f(x), f(x + y)$ are a real functions.

Definition 3.3. Neutrosophic exponential and logarithmic functions. [14]

Let $R(I)$ be the neutrosophic field of reals, we have:

1. $f(X) = e^X = e^{x+yI} = e^x + I(e^{x+y} - e^x)$
2. $f(X) = \ln(X) = \ln(x + yI) = \ln x + I(\ln(x + y) - \ln(x))$, where $x + yI > 0$.

Definition 3.4. [14]

Let $R(I)$ be the neutrosophic field of reals, we have:

$$f(X) = \sqrt{x + yI} = \sqrt{x} + I(\sqrt{x + y} - \sqrt{x})$$

Definition 3.5.

Let $R(I)$ be the neutrosophic field of reals, we have:

$$\begin{aligned} f(X) = A_n X^n + A_{n-1} X^{n-1} + \cdots + A_1 X + A_0 \\ = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\ + I[(b_n (x + y)^n + b_{n-1} (x + y)^{n-1} + \cdots + b_1 (x + y) + b_0 \\ - (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0))] \end{aligned}$$

Definition 3.6. Neutrosophic Trigonometric Functions:

Let $R(I)$ be the neutrosophic field of reals, we have:

1. $\sin(x + yI) = \sin x + I[\sin(x + y) - \sin x\ ]$
2. $\cos(x + yI) = \cos x + I[\cos(x + y) - \cos x]$
3. $\tan(x + yI) = \tan x + I[\tan(x + y) - \tan x]$

4. derivative a Neutrosophic Functions on $R(I)$.

Definition 4.1.

Let $f(X) = f(x + yI) = f(x) + I[f(x + y) - f(x)]$ a neutrosophic function on $R(I)$, the we define a derivative of a neutrosophic function $f(X)$ as follows:

$$f'(X) = f'(x + yI) = f'(x) + I[f'(x + y) - f'(x)]$$

DOI: https://doi.org/10.54216/JNFS.050203
Received: August 02, 2022 Accepted: January 27, 2023

Examples 4.1.

1. $f(X) = e^X = e^x + I(e^{x+y} - e^x)$. We have.
$f'(X) = (e^x)' + I[(e^{x+y})' - (e^x)'] = e^x + I(e^{x+y} - e^x) = e^X$.
2. $f(X) = \ln(X) = \ln x + I[\ln(x+y) - \ln(x)]$. We have.
$$f'(X) = (\ln x)' + I[(\ln(x+y))' - (\ln x)'] = \frac{1}{x} + I\left[\frac{1}{(x+y)} - \frac{1}{x}\right] = \frac{1}{x} + I\left[\frac{x-x-y}{x(x+y)}\right] = \frac{1}{x} + I\left[\frac{-y}{x(x+y)}\right] = \frac{1+0I}{x+yI} = \frac{1}{X}$$
3. $f(X) = Xe^X = xe^x + I((x+y)e^{x+y} - xe^x)$. We have.
$f'(X) = (xe^x)' + I[((x+y)e^{x+y})' - (xe^x)'] = e^x + xe^x + I\left(e^{x+y} + (x+y)e^{x+y} - (e^x + xe^x)\right) = e^x + I(e^{x+y} - e^x) + \left[xe^x + I\left((x+y)e^{x+y} - xe^x\right)\right] = e^X + Xe^X$.
4. $f(X) = \sin X = \sin(x+yI) = \sin x + I[\sin(x+y) - \sin x\ ]$.
We have.
$f'(X) = (\sin x)' + I[(\sin(x+y))' - (\sin x)'] = \cos x + I[[\cos(x+y) - \cos x] = \cos X$.
5. $f(X) = \cos X = \cos(x+yI) = \cos x + I[\cos(x+y) - \cos x]$. We have.
$f'(X) = (\cos x)' + I[(\cos(x+y))' - (\cos x)'] = -\sin x + I[[-\sin(x+y) + \sin x] = -(\sin x + I[[\sin(x+y) - \sin x]) = -\sin X$.
6. $f(X) = e^X \ln(X) + X^2 = e^x \ln(x) + x^2 + I\left[e^{(x+y)} \ln(x+y) + (x+y)^2 - (e^x \ln(x) + x^2)\right]$. We have.
$f'(X) = (e^x \ln(x) + x^2)' + I\left[\left(e^{(x+y)} \ln(x+y) + (x+y)^2\right)' - (e^x \ln(x) + x^2)'\right] = \left(e^x \ln(x) + \frac{e^x}{x} + 2x\right)' + I\left[e^{(x+y)} \ln\left((x+y)\right) + \frac{e^{(x+y)}}{x+y} + 2(x+y) - \left(e^x \ln(x) + \frac{e^x}{x} + 2x\right)\right] = e^X \ln(X) + \frac{e^X}{X} + 2X$.
7. $f(X) = sin^2X\,.\,cosX = sin^2x\,.\,cosx + I[(sin^2(x+y)\,.\,cos(x+y) - sin^2x\,.\,cosx)]$. We have.
$f'(X) = (sin^2x\,.\,cosx)' + I\left[\left(sin^2(x+y)\,.\,cos(x+y)\right)' - (sin^2x\,.\,cosx)'\right] = 2sinx.\,cos^2x - sin^3x + I[2sin(x+y).\,cos^2(x+y) - sin^3(x+y) - (2sinx.\,cos^2x - sin^3x)] = 2sinX.\,cos^2X - sin^3X$.

5. integral a Neutrosophic Functions on $R(I)$.

Definition 5.1.

Let $f(X) = f(x+yI) = f(x) + I[f(x+y) - f(x)]$ a neutrosophic function on $R(I)$, the we define a integration of a neutrosophic function $f(X)$ as follows:
$$\int f(X)\,dX = \int f(x)\,dx + I\left[\int f(x+y)\,d(x+y) - \int f(x)\,dx\right] + (a+bI)$$
Where $(a+bI)$ is a neutrosophic constant number, and $\int f(X)\,dX = F(X) = F(x) + I[F(x+y) - F(x)]$.

Examples 5.1.

1. $f(X) = e^X = e^x + I(e^{x+y} - e^x)$. We have.
$\int e^X dX = \int e^x dx + I[\int e^{x+y} d(x+y) - \int e^x dx] = e^x + I(e^{x+y} - e^x) + (a+bI) = e^X + (a+bI)$.
2. $f(X) = X^2e^X = x^2e^x + I((x+y)^2e^{x+y} - x^2e^x)$. We have.
$\int X^2e^X dX = \int x^2e^x dx + I[\int (x+y)^2e^{x+y} d(x+y) - \int x^2e^x dx] = (x^2 - 2x + 2)e^x + I([(x+y)^2 - 2(x+y) + 2]e^{x+y} - (x^2 - 2x + 2)e^x) + (a+bI) = (X^2 - 2X + 2)e^X + (a+bI)$.
3. $f(X) = Xe^{X^2} = xe^{x^2} + I((x+y)e^{(x+y)^2} - xe^{x^2})$. We have.
$\int Xe^{X^2} dX = \int xe^{x^2} dx + I\left[\int (x+y)e^{(x+y)^2} d(x+y) - \int xe^{x^2} dx\right] = \frac{1}{2}e^{x^2} + I\left(\frac{1}{2}e^{(x+y)^2} - \frac{1}{2}e^{x^2}\right) + (a+bI) = \frac{1}{2}\left[e^{x^2} + I\left(e^{(x+y)^2} - e^{x^2}\right)\right] + (a+bI) = \frac{1}{2}e^{X^2} + (a+bI)$.
4. $f(X) = \sin^3 X = \sin^3 x + I[\sin^3(x+y) - \sin^3 x\ ]$We have.
$\int Xe^{X^2} dX = \int \sin^3 x\,dx + I[\int \sin^3(x+y)\,d(x+y) - \int \sin^3 x\,dx] = \frac{\cos^3 x}{3} - cosx + I\left[\frac{\cos^3(x+y)}{3} - cos(x+y) - \left(\frac{\cos^3 x}{3} - cosx\right)\right] + (a+bI) = \frac{\cos^3 X}{3} - cosX + (a+bI)$.
5. $f(X) = X^3 \ln(X) = x^3 \ln x + I[(x+y)^3 \ln(x+y) - x^3 \ln(x)]$. We have.

DOI: https://doi.org/10.54216/JNFS.050203
Received: August 02, 2022 Accepted: January 27, 2023

$\int X^3\ln(X)\,dX=\int x^3\ln x\,dx+I[\int(x+y)^3\ln(x+y)\,d(x+y)-\int x^3\ln x\,dx]=\frac{x^4}{4}\ln x-\frac{1}{16}x^4+I\left[\frac{(x+y)^4}{4}\ln(x+y)-\frac{1}{16}(x+y)^4-\left(\frac{x^4}{4}\ln x-\frac{1}{16}x^4\right)\right]+(a+bI)=\frac{X^4}{4}\ln X-\frac{1}{16}X^4+(a+bI).$

6. $f(X)=\frac{X^2}{1-X^2}=\frac{x^2}{1-x^2}+I(\frac{(x+y)^2}{1-(x+y)^2}-\frac{x^2}{1-x^2})$. We have.

$\int\frac{X^2}{1-X^2}dX=\int\frac{x^2}{1-x^2}dx+I\left[\int\frac{(x+y)^2}{1-(x+y)^2}d(x+y)-\int\frac{x^2}{1-x^2}dx\right]=-x+\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|+I\left[-(x+y)+\frac{1}{2}\ln\left|\frac{1+(x+y)}{1-(x+y)}\right|-\left(-x+\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right)\right]+(a+bI)=-X+\frac{1}{2}\ln\left|\frac{1+X}{1-X}\right|+(a+bI).$

7. $f(X)=\frac{\sqrt{1+X^2}+\sqrt{1-X^2}}{\sqrt{1-X^4}}=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}+I(\frac{\sqrt{1+(x+y)^2}+\sqrt{1-(x+y)^2}}{\sqrt{1-(x+y)^4}}-\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}})$. We have.

$\int\frac{\sqrt{1+X^2}+\sqrt{1-X^2}}{\sqrt{1-X^4}}dX=\int\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}dx+I\left[\int\frac{\sqrt{1+(x+y)^2}+\sqrt{1-(x+y)^2}}{\sqrt{1-(x+y)^4}}d(x+y)-\int\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}dx\right]=\int\left(\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1+x^2}}\right)dx+I\left[\int\left(\frac{1}{\sqrt{1-(x+y)^2}}+\frac{1}{\sqrt{1+(x+y)^2}}\right)d(x+y)-\int\left(\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1+x^2}}\right)dx\right]=arc\ sinx+\ln\left|x+\sqrt{1+x^2}\right|+I\left[arc\ sin(x+y)+\ln\left|(x+y)+\sqrt{1+(x+y)^2}\right|-\left(arc\ sinx+\ln\left|x+\sqrt{1+x^2}\right|\right)\right]+(a+bI)=arc\ sinX+\ln\left|X+\sqrt{1+X^2}\right|+(a+bI).$

6. The definite integration a Neutrosophic Functions on $R(I)$.

Definition 6.1.

Let $a+bI, c+dI\in R(I)$, the we say $a+bI\le c+dI$ if $a\le c$ aand $a+b\le c+d$.

Definition 6.2.

Let $f(X)=f(x+yI)=f(x)+I[f(x+y)-f(x)]$ a neutrosophic function on $R(I)$, we define the definite integration of a neutrosophic function $f(X)$ as follows:

$$\int_{a+bI}^{c+dI} f(X)dX=\int_a^c f(x)dx+I\left[\int_{a+b}^{c+d} f(x+y)d(x+y)-\int_a^c f(x)dx\right]$$

Examples 5.1.

1. $J=\int_{0+0I}^{1+I}e^XdX=\int_0^1 e^xdx+I\left[\int_0^2 e^{(x+y)}d(x+y)-\int_0^1 e^xdx\right]=[e^x]_0^1+I\left[\left[e^{(x+y)}\right]_0^2-[e^x]_0^1\right]$

$J=(e-e^0)+I[(e^2-e^0)-(e-e^0)]=(e-1)+I[(e^2-1)-(e-1)]=e-1+I[e^2-e]$

2. $J=\int_{\frac{\pi}{6}+\frac{\pi}{6}I}^{\frac{\pi}{2}+\frac{\pi}{2}I}sinX\,dX=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}sinx\,dx+I\left[\int_{\frac{\pi}{2}}^{\pi}sin(x+y)d(x+y)-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}sinx\,dx\right]=[-cosx]_{\frac{\pi}{6}}^{\frac{\pi}{2}}+I\left[[-cos(x+y)]_{\frac{\pi}{2}}^{\pi}+[cosx]_{\frac{\pi}{6}}^{\frac{\pi}{2}}\right]$

$J=\left(-0+\frac{\sqrt{3}}{2}\right)+I\left[(1-0)-\left(-0+\frac{\sqrt{3}}{2}\right)\right]=\frac{\sqrt{3}}{2}+I\left[1-\frac{\sqrt{3}}{2}\right]$

3. Conclusion

In this paper, we have defined the concept of a neutrosophic functions. Also, we have discussed some of their elementary properties such as derivative, indefinite integral, definite integral.

References

[1] Abobala, M., "AH-Subspaces in Neutrosophic Vector Spaces", International Journal of Neutrosophic Science, Vol. 6 , pp. 80-86. 2020.

DOI: https://doi.org/10.54216/JNFS.050203
Received: August 02, 2022 Accepted: January 27, 2023

[2] Abobala, M.,. "A Study of AH-Substructures in *n*-Refined Neutrosophic Vector Spaces", International Journal of Neutrosophic Science", Vol. 9, pp.74-85. 2020.

[3] Sankari, H., and Abobala, M., "Neutrosophic Linear Diophantine Equations With two Variables", Neutrosophic Sets and Systems, Vol. 38, pp. 22-30, 2020.

[4] Sankari, H., and Abobala, M." *n*-Refined Neutrosophic Modules", Neutrosophic Sets and Systems, Vol. 36, pp. 1-11. 2020.

[5] Alhamido, R., and Abobala, M., "AH-Substructures in Neutrosophic Modules", International Journal of Neutrosophic Science, Vol. 7, pp. 79-86 . 2020.

[6] Smarandache, F., " A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability", American Research Press. Rehoboth, 2003.

[7] Suresh, R., and S. Palaniammal,. "Neutrosophic Weakly Generalized open and Closed Sets", Neutrosophic Sets and Systems, Vol. 33, pp. 67-77,. 2020.

[8] Olgun, N., and Hatip, A., "The Effect Of The Neutrosophic Logic On The Decision Making, in Quadruple Neutrosophic Theory And Applications", Belgium, EU, Pons Editions Brussels,pp. 238-253. 2020.

[9] Hatip, A., Alhamido, R., and Abobala, M., "A Contribution to Neutrosophic Groups", International Journal of Neutrosophic Science", Vol. 0, pp. 67-76 . 2019.

[10] Abobala, M., " *n*-Refined Neutrosophic Groups I", International Journal of Neutrosophic Science, Vol. 0, pp. 27-34. 2020.

[11] Abobala, M., "Classical Homomorphisms Between n-refined Neutrosophic Rings", International Journal of Neutrosophic Science", Vol. 7, pp. 74-78. 2020.

[12] Smarandache, F., and Abobala, M., n-Refined neutrosophic Rings, International Journal of Neutrosophic Science, Vol. 5 , pp. 83-90, 2020.

[13] Abobala, M., On Some Special Substructures of Neutrosophic Rings and Their Properties, International Journal of Neutrosophic Science", Vol. 4 , pp. 72-81, 2020.

[14] Abobala, M., "On Some Special Substructures of Refined Neutrosophic Rings", International Journal of Neutrosophic Science, Vol. 5, pp. 59-66. 2020.

[15] Sankari, H., and Abobala, M.," AH-Homomorphisms In neutrosophic Rings and Refined Neutrosophic Rings", Neutrosophic Sets and Systems, Vol. 38, pp. 101-112, 2020.

DOI: https://doi.org/10.54216/JNFS.050203
Received: August 02, 2022 Accepted: January 27, 2023