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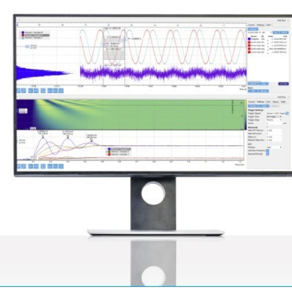

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Neutrosophic Positive Implicative $\mathcal{N}$-Ideals in KU-Algebras

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Abstract. The notion of a neutrosophic positive implicative $\mathcal{N}$-ideal in KU-algebras is introduced, and several properties are investigated. Relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic positive implicative $\mathcal{N}$-ideal are discussed. Characterizations of a neutrosophic positive implicative $\mathcal{N}$-ideal are considered. Conditions for a neutrosophic $\mathcal{N}$-ideal to be a neutrosophic positive implicative $\mathcal{N}$-ideal are provided. An extension property of a neutrosophic positive implicative $\mathcal{N}$-ideal based on the negative indeterminacy membership function is discussed.
Keywords and phrases: neutrosophic $\mathcal{N}$-structure, neutrosophic $\mathcal{N}$-ideal, neutrosophic positive implicative $\mathcal{N}$-ideal.
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Introduction

Prabpayak and Leerawat [7, 8] introduced a algebraic structure called KU-algebras. They studied ideals and congruences in KU-algebras. Additionally, they introduced the concept of homomorphism of KU-algebra and examined some related properties. In 2017 Mostafa et al. [5] introduced positive implicative ideals in KU-algebras. Jun et al. [2] introduced a new function, called a negative-valued function, and constructed $\mathcal{N}$-structures. Zadeh [11] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [1] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality as an independent component in 1995 [9, 10] and defined the neutrosophic set on three components (t, i, f) = (truth, indeterminacy, falsehood).

In this paper, we introduce the notion of a neutrosophic positive implicative $\mathcal{N}$-ideal in KU-algebras, and investigate several properties. We discuss relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic positive implicative $\mathcal{N}$-ideal, and provide conditions for a neutrosophic $\mathcal{N}$-ideal to be a neutrosophic positive implicative $\mathcal{N}$-ideal. We consider characterizations of a neutrsophic positive implicative $\mathcal{N}$-ideal. We establish an extension property of a neutrosophic positive implicative $\mathcal{N}$-ideals based on the negative interminacy membership function.

Preliminaries

We let $K(\tau)$ be the class of all algebras with type $\tau = (2, 0)$. A KU-algebra [7, 8] on a system $P = (P, *, 0) \in K(\tau)$ satisfies

(KU1) $(l_{11} * l_{22}) * ((l_{22} * l_{33}) * (l_{11} * l_{33})) = 0$,
(KU2) $l_{11} * 0 = 0$,
(KU3) $0 * l_{11} = l_{11}$,
(KU4) $l_{11} * l_{22} = 0$ & $l_{22} * l_{11} = 0$ implies $l_{11} = l_{11}$,
(KU5) $l_{11} * l_{11} = 0, \forall\, l_{11}, l_{22}, l_{33} \in P$.

Also a binary relation $\leq$ by putting $l_{11} \leq l_{22} \Leftrightarrow l_{22} * l_{11} = 0, \forall\, l_{11}, l_{22} \in P$.

In a KU-algebra P, the following hold:

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(KU1′) $(l_{22} * l_{33}) * (l_{11} * l_{33}) \leq (l_{11} * l_{22})$,

(KU2′) $0 \leq l_{11}$,

(KU3′) $l_{11} \leq l_{22}, l_{22} \leq l_{11}$ implies $l_{11} = l_{22}$,

(KU4′) $l_{22} * l_{11} \leq l_{11}$.

Theorem 2.1 [4] In a KU-algebra P, the following axioms are satisfied: $\forall\, l_{11}, l_{22}, l_{33} \in P$,

(i) $l_{11} \leq l_{22}$ imply $l_{22} * l_{33} \leq l_{11} * l_{33}$,

(ii) $l_{11} * (l_{22} * l_{33}) = l_{22} * (l_{11} * l_{33}), \forall\, l_{11}, l_{22}, l_{33} \in P$,

(iii) $((l_{22} * l_{11}) * l_{11}) \leq l_{22}$,

(iv) $(((l_{22} * l_{11}) * l_{11}) * l_{11}) = (l_{22} * l_{11})$.

A subset **I** of a KU-algebra P is called an KU-ideal [7, 8] of P if it satisfies

(I1) $0 \in \mathbf{I}$,

(I2) $(\forall\, l_{11}, l_{22} \in P)\,(l_{22} * l_{11} \in \mathbf{I}, l_{22} \in \mathbf{I} \Rightarrow l_{11} \in \mathbf{I})$.

Let I be a subset of a KU-algebra. Then I is called a positive implicative ideal [5] of P if the Condition (I1) holds and the following assertion is valid.

$$(\forall l_{11}, l_{22}, l_{33} \in P)\,(l_{33} * (l_{11} * l_{22}) \in I,\ l_{33} * l_{11} \in I \Rightarrow l_{11} * l_{22} \in I). \tag{1}$$

Any positive implicative ideal is an ideal, but the converse is not true [5].
Lemma 2.1 [5] A subset I of a KU-algebra P is a positive implicative ideal of P iff I is an ideal of P which satisfies the following condition.

$$(\forall l_{11}, l_{22} \in P)\,(l_{22} * (l_{22} * l_{11}) \in I \Rightarrow l_{22} * l_{11} \in I). \tag{2}$$

A non-empty subset S of a KU-algebra P is called a KU-subalgebra [7, 8] of P if $l_{11} * l_{22} \in S\ \forall\, l_{11}, l_{22} \in S$.
For any family $\{\lambda_j \mid j \in \Delta\}$ of real numbers, we define

$$\bigvee \left\{\lambda_j \mid j \in \Delta\right\} := \begin{cases} \max\left\{\lambda_j \mid j \in \Delta\right\} & \text{if } \Delta \text{ is finite} \\ \sup\left\{\lambda_j \mid j \in \Delta\right\} & \text{otherwise} \end{cases}$$
$$\bigwedge \left\{\lambda_j \mid j \in \Delta\right\} := \begin{cases} \min\left\{\lambda_j \mid j \in \Delta\right\} & \text{if } \Delta \text{ is finite} \\ \inf\left\{\lambda_j \mid j \in \Delta\right\} & \text{otherwise} \end{cases}$$

Let P denote the nonempty universe of discourse unless otherwise specified. The collection of functions $F(P, [-1, 0])$ from a set P to $[-1, 0]$. It is a negative-valued function from P to $[-1, 0]$ (briefly, $\mathcal{N}$-function on P). An $\mathcal{N}$-structure refers to an ordered pair (P, f) of P and an $\mathcal{N}$-function f on P ([2]).
A neutrosophic $\mathcal{N}$ (briefly, $N\mathcal{N}$)-structure over P ([3]) is defined as

$$P_N = \frac{P}{(\mathbb{T}_N, \mathbb{I}_N, \mathbb{F}_N)} = \left\{\frac{l}{(\mathbb{T}_N(l), \mathbb{I}_N(l), \mathbb{F}_N(l))} \mid l \in P\right\} \tag{3}$$

where $\mathbb{T}_N, \mathbb{I}_N$ & $\mathbb{F}_N$ are $\mathcal{N}$-functions called the negative truth (resp. indeterminacy & falsity) membership function on P.
We note that every $N\mathcal{N}$-structure P_N over P satisfies

$$(\forall\, l \in P)\,(-3 \leq \mathbb{T}_N(l) + \mathbb{I}_N(l) + \mathbb{F}_N(l) \leq 0)\,.$$

Neutrosophic positive implicative $\mathcal{N}$-ideals

Definition 3.1 Let P_N be a $N\mathcal{N}$-structure over P. Then P_N is called a neutrosophic $\mathcal{N}$-ideal [6] (briefly, $N\mathcal{N}$-I) of P if the following condition holds.

$$(\forall\, l, m \in P)\left(\begin{array}{l} \mathbb{T}_N(0) \leq \mathbb{T}_N(l) \leq \bigvee\{\mathbb{T}_N(m * l),\ \mathbb{T}_N(m)\} \\ \mathbb{I}_N(0) \geq \mathbb{I}_N(l) \geq \bigwedge\{\mathbb{I}_N(m * l),\ \mathbb{I}_N(m)\} \\ \mathbb{F}_N(0) \leq \mathbb{F}_N(l) \leq \bigvee\{\mathbb{F}_N(m * l),\ \mathbb{F}_N(m)\} \end{array}\right). \tag{4}$$

Definition 3.2 A $N\mathcal{N}$-structure P_N over P is called a neutrosophic positive implicative $\mathcal{N}$-ideal (briefly, $NPi\mathcal{N}$-I) of P if the following assertions are valid.

$$(\forall\, l_{11} \in P)\,(\mathbb{T}_N(0) \leq \mathbb{T}_N(l_{11}),\ \mathbb{I}_N(0) \geq \mathbb{I}_N(l_{11}),\ \mathbb{F}_N(0) \leq \mathbb{F}_N(l_{11}))\,. \tag{5}$$

$$(\forall\, l_{11}, m_{11}, n_{11} \in P)\left(\begin{array}{l} \mathbb{T}_N(n_{11} * m_{11}) \leq \bigvee\{\mathbb{T}_N(n_{11} * (l_{11} * m_{11})),\ \mathbb{T}_N(n_{11} * l_{11})\} \\ \mathbb{I}_N(n_{11} * m_{11}) \geq \bigwedge\{\mathbb{I}_N(n_{11} * (l_{11} * m_{11})),\ \mathbb{I}_N(n_{11} * l_{11})\} \\ \mathbb{F}_N(n_{11} * m_{11}) \leq \bigvee\{\mathbb{F}_N(n_{11} * (l_{11} * m_{11})),\ \mathbb{F}_N(n_{11} * l_{11})\} \end{array}\right) \tag{6}$$

Example 3.1 Consider a KU-algebra $P = \{0_k, 1_k, 2_k, 3_k, 4_k\}$ with the following Cayley table.

$*$	0_k	1_k	2_k	3_k	4_k
0_k	0_k	1_k	2_k	3_k	4_k
1_k	0_k	0_k	1_k	3_k	4_k
2_k	0_k	0_k	0_k	3_k	4_k
3_k	0_k	0_k	0_k	0_k	4_k
4_k	0_k	0_k	0_k	0_k	0_k

The $N\mathcal{N}$-structure $P_N = \left\{\frac{0_k}{(-0.7,-0.2,-0.6)}, \frac{1_k}{(-0.5,-0.3,-0.4)}, \frac{2_k}{(-0.5,-0.3,-0.4)}, \frac{3_k}{(-0.3,-0.8,-0.5)}, \frac{4_k}{(-0.3,-0.8,-0.5)}\right\}$ be a $N\mathcal{N}$-structure over P. Then P_N is a $NPi\mathcal{N}$-I of P.
If we take $n = 0$ in (6) and use (KU3), then we have the following theorem.
Theorem 3.1 Every $NPi\mathcal{N}$-I is a $N\mathcal{N}$-I.
The following example shows that the converse of Theorem 3.1 does not holds.
Example 3.2 Consider a KU-algebra $P = \{0_5, a_5, b_5, c_5, d_5\}$ with the following Cayley table.

$*$	0_5	a_5	b_5	c_5	d_5
0_5	0_5	a_5	b_5	c_5	d_5
a_5	0_5	0_5	a_5	a_5	b_5
b_5	0_5	0_5	0_5	a_5	a_5
c_5	0_5	0_5	a_5	0_5	b_5
d_5	0_5	0_5	0_5	0_5	0_5

The $N\mathcal{N}$-structure $P_N = \left\{\frac{0_5}{(-0.7,-0.2,-0.6)}, \frac{a_5}{(-0.5,-0.3,-0.4)}, \frac{b_5}{(-0.5,-0.3,-0.4)}, \frac{c_5}{(-0.3,-0.8,-0.5)}, \frac{d_5}{(-0.3,-0.8,-0.5)}\right\}$. Then P_N is a $N\mathcal{N}$-I of P but not a $NPi\mathcal{N}$-I of P since $\mathbb{T}_N(c_5 * b_5) = \mathbb{T}_N(a_5) = -0.5 \nleq -0.7 = \bigvee\{\mathbb{T}_N(c_5 * (a_5 * b_5)),\ \mathbb{T}_N(c_5 * a_5)\}$.
Given a $N\mathcal{N}$-structure P_N over P and $\lambda, \mu, \delta \in [-1, 0]$ with $-3 \leq \lambda + \mu + \delta \leq 0$, we define the following sets.

$$\begin{array}{l} \mathbb{T}_N^{\lambda} := \{l \in P \mid \mathbb{T}_N(l) \leq \lambda\}\,, \\ \mathbb{I}_N^{\mu} := \{l \in P \mid \mathbb{I}_N(l) \geq \mu\}\,, \\ \mathbb{F}_N^{\delta} := \{l \in P \mid \mathbb{F}_N(l) \leq \delta\}\,. \end{array}$$

Then we say that the set

$$P_N(\lambda, \mu, \delta) := \{l \in P \mid \mathbb{T}_N(l) \leq \lambda,\ \mathbb{I}_N(l) \geq \mu,\ \mathbb{F}_N(l) \leq \delta\}$$

is the (λ, μ, δ)-level set of P_N (see [6]). Obviously, we have

$$P_N(\lambda, \mu, \delta) = \mathbb{T}_N^\lambda \cap \mathbb{I}_N^\mu \cap \mathbb{F}_N^\delta.$$

Theorem 3.2 If P_N is a *NPi𝒩-I* of P, then $\mathbb{T}_N^\lambda, \mathbb{I}_N^\mu$ and $\mathbb{F}_N^\delta$ are positive implicative ideals of P for all $\lambda, \mu, \delta \in [-1, 0]$ with $-3 \le \lambda + \mu + \delta \le 0$ whenever they are nonempty.

Proof. Assume that $\mathbb{T}_N^\lambda, \mathbb{I}_N^\mu$ and $\mathbb{F}_N^\delta$ are nonempty for all $\lambda, \mu, \delta \in [-1, 0]$ with $-3 \le \lambda+\mu+\delta \le 0$. Then $l_{11} \in \mathbb{T}_N^\lambda$, $m_{11} \in \mathbb{I}_N^\mu$ and $n_{11} \in \mathbb{F}_N^\delta$ for some $l_{11}, m_{11}, n_{11} \in P$. Thus $\mathbb{T}_N(0) \le \mathbb{T}_N(l_{11}) \le \lambda$, $\mathbb{I}_N(0) \ge \mathbb{I}_N(m_{11}) \ge \mu$, and $\mathbb{F}_N(0) \le \mathbb{F}_N(n_{11}) \le \delta$, that is, $0 \in \mathbb{T}_N^\lambda \cap \mathbb{I}_N^\mu \cap \mathbb{F}_N^\delta$. Let $n_{11} * (l_{11} * m_{11}) \in \mathbb{T}_N^\lambda$ and $n_{11} * l_{11} \in \mathbb{T}_N^\lambda$. Then $\mathbb{T}_N(n_{11} * (l_{11} * m_{11})) \le \lambda$ and $\mathbb{T}_N(n_{11} * l_{11}) \le \lambda$, which imply that

$$\mathbb{T}_N(n_{11} * m_{11}) \le \bigvee \{\mathbb{T}_N(n_{11} * (l_{11} * m_{11})),\ \mathbb{T}_N(n_{11} * l_{11})\} \le \lambda,$$

that is, $n_{11} * m_{11} \in \mathbb{T}_N^\lambda$. If $c_{11} * (a_{11} * b_{11}) \in \mathbb{I}_N^\mu$ and $c_{11} * a_{11} \in \mathbb{I}_N^\mu$, then $\mathbb{I}_N(c_{11} * (a_{11} * b_{11})) \ge \mu$ and $\mathbb{I}_N(c_{11} * a_{11}) \ge \mu$. Thus

$$\mathbb{I}_N(c_{11} * b_{11}) \ge \bigwedge \{\mathbb{I}_N(c_{11} * (a_{11} * b_{11})), \mathbb{I}_N(c_{11} * a_{11})\} \ge \mu,$$

and so $c_{11} * b_{11} \in \mathbb{I}_N^\mu$. Finally, suppose that $w_{11} * (u_{11} * v_{11}) \in \mathbb{F}_N^\delta$ and $w_{11} * u_{11} \in \mathbb{F}_N^\delta$. Then $\mathbb{F}_N(w_{11} * (u_{11} * v_{11})) \le \delta$ and $\mathbb{F}_N(w_{11} * u_{11}) \le \delta$. Thus

$$\mathbb{F}_N(w_{11} * v_{11}) \le \bigvee \{\mathbb{F}_N(w_{11} * (u_{11} * v_{11})),\ \mathbb{F}_N(w_{11} * u_{11})\} \le \delta,$$

that is, $w_{11} * v_{11} \in \mathbb{F}_N^\delta$. Therefore $\mathbb{T}_N^\lambda, \mathbb{I}_N^\mu$ and $\mathbb{F}_N^\delta$ are positive implicative ideals of P.

Corollary 3.1 Let P_N be a *𝒩𝒩*-structure over P and let $\lambda, \mu, \delta \in [-1, 0]$ be such that $-3 \le \lambda + \mu + \delta \le 0$. If P_N is a *NPi𝒩-I* of P, then the nonempty (λ, μ, δ)-level set of P_N is a positive implicative ideal of P.

The following example illustrates Theorem 3.2.

Example 3.3 Consider a KU-algebra $P = \{0_k, 1_k, 2_k, 3_k, 4_k\}$ with the following Cayley table.

$*$	0_k	1_k	2_k	3_k	4_k
0_k	0_k	1_k	2_k	3_k	4_k
1_k	0_k	0_k	1_k	3_k	4_k
2_k	0_k	0_k	0_k	3_k	4_k
3_k	0_k	0_k	0_k	0_k	4_k
4_k	0_k	0_k	0_k	0_k	0_k

The *𝒩𝒩*-structure $P_N = \left\{ \frac{0_k}{(-0.7,-0.2,-0.6)}, \frac{1_k}{(-0.5,-0.3,-0.4)}, \frac{2_k}{(-0.5,-0.3,-0.4)}, \frac{3_k}{(-0.3,-0.8,-0.5)}, \frac{4_k}{(-0.3,-0.8,-0.5)} \right\}$ be a *NPi𝒩-I* of P. Then

$$\mathbb{T}_N^\lambda = \begin{cases} \emptyset & \text{if } \lambda \in [-1, -0.7) \\ \{0_k\} & \text{if } \lambda \in [-0.7, -0.5) \\ \{0_k, 1_k, 2_k\} & \text{if } \lambda \in [-0.5, -0.3) \\ P & \text{if } \lambda \in [-0.3, 0] \end{cases}$$

$$\mathbb{I}_N^\mu = \begin{cases} \emptyset & \text{if } \mu \in (-0.2, 0] \\ \{0_k\} & \text{if } \mu \in (-0.3, -0.2] \\ \{0_k, 1_k, 2_k\} & \text{if } \mu \in (-0.8, -0.3] \\ P & \text{if } \mu \in [-1, -0.8] \end{cases}$$

and

$$\mathbb{F}_N^\delta = \begin{cases} \emptyset & \text{if } \delta \in [-1, -0.6) \\ \{0_k\} & \text{if } \delta \in [-0.6, -0.5) \\ \{0_k, 3_k, 4_k\} & \text{if } \delta \in [-0.5, -0.4) \\ P & \text{if } \delta \in [-0.4, 0] \end{cases}$$

which are positive implicative ideals of P.

Lemma 3.1 [6] Every *𝒩𝒩-I* P_N of P satisfies the following assertions:

$$(l_{11}, m_{11} \in P)\,(l_{11} \le m_{11} \Rightarrow \mathbb{T}_N(l_{11}) \le \mathbb{T}_N(m_{11}),\ \mathbb{I}_N(l_{11}) \ge \mathbb{I}_N(m_{11}),\ \mathbb{F}_N(l_{11}) \le \mathbb{F}_N(m_{11}))\,. \quad (7)$$

We discuss conditions for a $\mathcal{NN}$-I to be a $NPi\mathcal{N}$-I.

Theorem 3.3 Let P_N be a $\mathcal{NN}$-I of P. Then P_N is a $NPi\mathcal{N}$-I of P iff the following assertion is valid.

$$(\forall\, l_{11}, m_{11} \in P)\left(\begin{array}{l} \mathbb{T}_N(n_{11} * m_{11}) \leq \mathbb{T}_N(n_{11} * (n_{11} * m_{11})) \\ \mathbb{I}_N(n_{11} * m_{11}) \geq \mathbb{I}_N(n_{11} * (n_{11} * m_{11})) \\ \mathbb{F}_N(n_{11} * m_{11}) \leq \mathbb{F}_N(n_{11} * (n_{11} * m_{11})) \end{array}\right) \tag{8}$$

Proof. Assume that P_N is a $NPi\mathcal{N}$-I of P. If l_{11} is replaced by n_{11} in (6) then

$$\begin{aligned} \mathbb{T}_N(n_{11} * m_{11}) &\leq \bigvee \{\mathbb{T}_N(n_{11} * (n_{11} * m_{11})),\ \mathbb{T}_N(n_{11} * n_{11})\} \\ &= \bigvee \{\mathbb{T}_N(n_{11} * (n_{11} * m_{11})),\ \mathbb{T}_N(0)\} = \mathbb{T}_N(n_{11} * (n_{11} * m_{11})) \\ \mathbb{I}_N(n_{11} * m_{11}) &\geq \bigwedge \{\mathbb{I}_N(n_{11} * (n_{11} * m_{11})),\ \mathbb{I}_N(n_{11} * n_{11})\} \\ &= \bigwedge \{\mathbb{I}_N(n_{11} * (n_{11} * m_{11})),\ \mathbb{I}_N(0)\} = \mathbb{I}_N(n_{11} * (n_{11} * m_{11})) \end{aligned}$$

and

$$\begin{aligned} \mathbb{F}_N(n_{11} * m_{11}) &\leq \bigvee \{\mathbb{F}_N(n_{11} * (n_{11} * m_{11})),\ \mathbb{F}_N(n_{11} * n_{11})\} \\ &= \bigvee \{\mathbb{F}_N(n_{11} * (n_{11} * m_{11})), \mathbb{F}_N(0)\} = \mathbb{F}_N(n_{11} * (n_{11} * m_{11})) \end{aligned}$$

by (KU5) and (5).

Conversely, let P_N be a $\mathcal{NN}$-I of P satisfying (8). Since

$$(n_{11} * l_{11}) * (n_{11} * (n_{11} * m_{11})) \leq l_{11} * (n_{11} * m_{11}) = n_{11} * (l_{11} * m_{11})$$

for all $l_{11}, m_{11}, n_{11} \in P$, we have

$$(\forall\, l_{11}, m_{11}, n_{11} \in P)\left(\begin{array}{l} \mathbb{T}_N((n_{11} * l_{11}) * (n_{11} * (n_{11} * m_{11}))) \leq \mathbb{T}_N(n_{11} * (l_{11} * m_{11})) \\ \mathbb{I}_N((n_{11} * l_{11}) * (n_{11} * (n_{11} * m_{11}))) \geq \mathbb{I}_N(n_{11} * (l_{11} * m_{11})) \\ \mathbb{F}_N((n_{11} * l_{11}) * (n_{11} * (n_{11} * m_{11}))) \leq \mathbb{F}_N(n_{11} * (l_{11} * m_{11})) \end{array}\right)$$

by Lemma 3.1. It follows from (4) and (8) that

$$\begin{aligned} \mathbb{T}_N(n_{11} * m_{11}) &\leq \mathbb{T}_N(n_{11} * (n_{11} * m_{11})) \\ &\leq \bigvee \{\mathbb{T}_N((n_{11} * l_{11}) * (n_{11} * (n_{11} * m_{11}))),\ \mathbb{T}_N(n_{11} * l_{11})\} \\ &\leq \bigvee \{\mathbb{T}_N(n_{11} * (l_{11} * m_{11})),\ \mathbb{T}_N(n_{11} * l_{11})\} \\ \mathbb{I}_N(n_{11} * m_{11}) &\geq \mathbb{I}_N(n_{11} * (n_{11} * m_{11})) \\ &\geq \bigwedge \{\mathbb{I}_N((n_{11} * l_{11}) * (n_{11} * (n_{11} * m_{11}))),\ \mathbb{I}_N(n_{11} * l_{11})\} \\ &\geq \bigwedge \{\mathbb{I}_N(n_{11} * (l_{11} * m_{11})),\ \mathbb{I}_N(n_{11} * l_{11})\} \end{aligned}$$

and

$$\begin{aligned} \mathbb{F}_N(n_{11} * m_{11}) &\leq \mathbb{F}_N(n_{11} * (n_{11} * m_{11})) \\ &\leq \bigvee \{\mathbb{F}_N((n_{11} * l_{11}) * (n_{11} * (n_{11} * m_{11}))),\ \mathbb{F}_N(n_{11} * l_{11})\} \\ &\leq \bigvee \{\mathbb{F}_N(n_{11} * (l_{11} * m_{11})),\ \mathbb{F}_N(n_{11} * l_{11})\} \end{aligned}$$

Therefore P_N is a $NPi\mathcal{N}$-I of P.

Lemma 3.2 [6] For any $\mathcal{NN}$-I P_N of P, we have

$$(\forall\, l_{11}, m_{11}, n_{11} \in P)\left(m_{11} * l_{11} \leq n_{11} \Rightarrow \left\{\begin{array}{l} \mathbb{T}_N(l_{11}) \leq \bigvee \{\mathbb{T}_N(m_{11}),\ \mathbb{T}_N(n_{11})\} \\ \mathbb{I}_N(l_{11}) \geq \bigwedge \{\mathbb{I}_N(m_{11}),\ \mathbb{I}_N(n_{11})\} \\ \mathbb{F}_N(l_{11}) \leq \bigvee \{\mathbb{F}_N(m_{11}),\ \mathbb{F}_N(n_{11})\} \end{array}\right.\right) \tag{9}$$

Lemma 3.3 If a $\mathcal{NN}$-structure P_N over P satisfies the condition (9), then P_N is a $\mathcal{NN}$-I of P.
Proof. Since $l_{11} * 0 \leq l_{11}$ for all $l_{11} \in P$, we have $\mathbb{T}_N(0) \leq \mathbb{T}_N(l_{11})$, $\mathbb{I}_N(0) \geq \mathbb{I}_N(l_{11})$ and $\mathbb{F}_N(0) \leq \mathbb{F}_N(l_{11})$ for all $l_{11} \in P$ by (9). Note that $l_{11} * (m_{11} * l_{11}) \leq m_{11}$ for all $l_{11}, m_{11} \in P$. It follows from (9) that

$$\mathbb{T}_N(l_{11}) \leq \bigvee \{\mathbb{T}_N(m_{11} * l_{11}),\ \mathbb{T}_N(m_{11})\},\ \mathbb{I}_N(l_{11}) \geq \bigwedge \{\mathbb{I}_N(m_{11} * l_{11}),\ \mathbb{I}_N(m_{11})\},$$

and $\mathbb{F}_N(l_{11}) \leq \bigvee \{\mathbb{F}_N(m_{11} * l_{11}),\ \mathbb{F}_N(m_{11})\}$ for all $l_{11}, m_{11} \in P$. Therefore P_N is a $\mathcal{NN}$-I of P.
Theorem 3.4 For any $\mathcal{NN}$-structure P_N over P, the following assertions are equivalent.

(i) P_N is a $NPi\mathcal{N}$-I of P.
(ii) P_N satisfies the following condition.

$$a_{11} * (n_{11} * (n_{11} * m_{11})) \leq b_{11} \Rightarrow \begin{cases} \mathbb{T}_N(n_{11} * m_{11}) \leq \bigvee \{\mathbb{T}_N(a_{11}), \mathbb{T}_N(b_{11})\} \\ \mathbb{I}_N(n_{11} * m_{11}) \geq \bigwedge \{\mathbb{I}_N(a_{11}), \mathbb{I}_N(b_{11})\} \\ \mathbb{F}_N(n_{11} * m_{11}) \leq \bigvee \{\mathbb{F}_N(a_{11}), \mathbb{F}_N(b_{11})\} \end{cases} \tag{10}$$

for all $l_{11}, m_{11}, a_{11}, b_{11} \in P$.

Proof. Suppose that P_N is a $NPi\mathcal{N}$-I of P. Then P_N is a $\mathcal{NN}$-I of P by Theorem 3.1. Let $l_{11}, m_{11}, a_{11}, b_{11} \in P$ be such that $a_{11} * (n_{11} * (n_{11} * m_{11})) \leq b_{11}$. Then

$$\begin{aligned} &\mathbb{T}_N(n_{11} * m_{11}) \leq \mathbb{T}_N(n_{11} * (n_{11} * m_{11})) \leq \bigvee \{\mathbb{T}_N(a_{11}),\ \mathbb{T}_N(b_{11})\}, \\ &\mathbb{I}_N(n_{11} * m_{11}) \geq \mathbb{I}_N(n_{11} * (n_{11} * m_{11})) \geq \bigwedge \{\mathbb{I}_N(a_{11}),\ \mathbb{I}_N(b_{11})\}, \\ &\mathbb{F}_N(n_{11} * m_{11}) \leq \mathbb{F}_N(n_{11} * (n_{11} * m_{11})) \leq \bigvee \{\mathbb{F}_N(a_{11}),\ \mathbb{F}_N(b_{11})\}, \end{aligned}$$

by Theorem 3.3 and Lemma 3.2.

Conversely, let P_N be a $\mathcal{NN}$-structure over P that satisfies (10). Let $l_{11}, a_{11}, b_{11} \in P$ be such that $a_{11} * l_{11} \leq b_{11}$. Then $(0 * (0 * a_{11})) * l_{11} \leq b_{11}$, and so

$$\begin{aligned} &\mathbb{T}_N(l_{11}) = \mathbb{T}_N(0 * l_{11}) \leq \bigvee \{\mathbb{T}_N(a_{11}),\ \mathbb{T}_N(b_{11})\}, \\ &\mathbb{I}_N(l_{11}) = \mathbb{I}_N(0 * l_{11}) \geq \bigwedge \{\mathbb{I}_N(a_{11}),\ \mathbb{I}_N(b_{11})\}, \\ &\mathbb{F}_N(l_{11}) = \mathbb{F}_N(0 * l_{11}) \leq \bigvee \{\mathbb{F}_N(a_{11}),\ \mathbb{F}_N(b_{11})\}. \end{aligned}$$

Hence P_N is a $\mathcal{NN}$-I of P by Lemma 3.3. Since $(n_{11} * (l_{11} * m_{11}) * (n_{11} * (l_{11} * m_{11})) \leq 0$, it follows from (10) and (5) that

$$\begin{aligned} &\mathbb{T}_N(n_{11} * m_{11}) \leq \bigvee \{\mathbb{T}_N(n_{11} * (n_{11} * m_{11})),\ \mathbb{T}_N(0)\} = \mathbb{T}_N(n_{11} * (n_{11} * m_{11})), \\ &\mathbb{I}_N(n_{11} * m_{11}) \geq \bigwedge \{\mathbb{I}_N(n_{11} * (n_{11} * m_{11})),\ \mathbb{I}_N(0)\} = \mathbb{I}_N(n_{11} * (n_{11} * m_{11})), \\ &\mathbb{F}_N(n_{11} * m_{11}) \leq \bigvee \{\mathbb{F}_N(n_{11} * (n_{11} * m_{11})),\ \mathbb{F}_N(0)\} = \mathbb{F}_N(n_{11} * (n_{11} * m_{11})). \end{aligned}$$

for all $l_{11}, m_{11} \in P$. Therefore P_N is a $NPi\mathcal{N}$-I of P by Theorem 3.3.
Lemma 3.4 [6] Let P_N be a $\mathcal{NN}$-structure over P and assume that $\mathbb{T}_N^{\lambda}$, $\mathbb{I}_N^{\mu}$ and $\mathbb{F}_N^{\delta}$ are ideals of P for all $\lambda, \mu, \delta \in [-1, 0]$ with $-3 \leq \lambda + \mu + \delta \leq 0$. Then P_N is a $\mathcal{NN}$-I of P.
Theorem 3.5 Let P_N be a $\mathcal{NN}$-structure over P and assume that $\mathbb{T}_N^{\lambda}$, $\mathbb{I}_N^{\mu}$ and $\mathbb{F}_N^{\delta}$ are positive implicative ideals of P for all $\lambda, \mu, \delta \in [-1, 0]$ with $-3 \leq \lambda + \mu + \delta \leq 0$. Then P_N is a $NPi\mathcal{N}$-I of P.
Proof. If $\mathbb{T}_N^{\lambda}, \mathbb{I}_N^{\mu}$ and $\mathbb{F}_N^{\delta}$ are positive implicative ideals of P, then $\mathbb{T}_N^{\lambda}$, $\mathbb{I}_N^{\mu}$ and $\mathbb{F}_N^{\delta}$ are ideals of P. Thus P_N is a $\mathcal{NN}$-I of P by Lemma 3.4. Let $l_{11}, m_{11} \in P$ and $\lambda, \mu, \delta \in [-1, 0]$ with $-3 \leq \lambda + \mu + \delta \leq 0$ such that $\mathbb{T}_N(m_{11} * (m_{11} * l_{11})) = \lambda$, $\mathbb{I}_N(m_{11} * (m_{11} * l_{11})) = \mu$ and $\mathbb{F}_N(m_{11} * (m_{11} * l_{11})) = \delta$. Then $m_{11} * (m_{11} * l_{11}) \in \mathbb{T}_N^{\lambda} \cap \mathbb{I}_N^{\mu} \cap \mathbb{F}_N^{\delta}$. Since $\mathbb{T}_N^{\lambda} \cap \mathbb{I}_N^{\mu} \cap \mathbb{F}_N^{\delta}$ is a positive implicative ideal of P, it follows from Lemma 2.1 that $m_{11} * l_{11} \in \mathbb{T}_N^{\lambda} \cap \mathbb{I}_N^{\mu} \cap \mathbb{F}_N^{\delta}$. Hence

$$\begin{aligned} &\mathbb{T}_N(m_{11} * l_{11}) \leq \lambda = \mathbb{T}_N(m_{11} * (m_{11} * l_{11})), \\ &\mathbb{I}_N(m_{11} * l_{11}) \geq \mu = \mathbb{I}_N(m_{11} * (m_{11} * l_{11})), \\ &\mathbb{F}_N(m_{11} * l_{11}) \leq \delta = \mathbb{F}_N(m_{11} * (m_{11} * l_{11})). \end{aligned}$$

Therefore P_N is a $NPi\mathcal{N}$-I of P by Theorem 3.3.
Lemma 3.5 [6] Let P_N be a $\mathcal{NN}$-I of P. Then P_N satisfies the condition (8) iff it satisfies the following condition.

$$(\forall\ l_{11}, m_{11}, n_{11} \in P) \begin{pmatrix} \mathbb{T}_N((l_{11} * m_{11}) * (l_{11} * n_{11})) \leq \mathbb{T}_N(l_{11} * (m_{11} * n_{11})) \\ \mathbb{I}_N((l_{11} * m_{11}) * (l_{11} * n_{11})) \geq \mathbb{I}_N(l_{11} * (m_{11} * n_{11})) \\ \mathbb{F}_N((l_{11} * m_{11}) * (l_{11} * n_{11})) \leq \mathbb{F}_N(l_{11} * (m_{11} * n_{11})) \end{pmatrix}. \tag{11}$$

Corollary 3.2 Let P_N be a $N\mathcal{N}$-I of P. Then P_N is a $NPi\mathcal{N}$-I of P iff P_N satisfies (11).
Proof. It follows from Theorem 3.3 and Lemma 3.5.
Theorem 3.6 For any $N\mathcal{N}$-structure P_N over P, then the assertions

(i) P_N is a $NPi\mathcal{N}$-I of P.
(ii) P_N satisfies the following condition.

$$a_{11} * (l_{11} * (m_{11} * n_{11})) \leq b_{11} \Rightarrow \begin{cases} \mathbb{T}_N((l_{11} * m_{11}) * (l_{11} * n_{11})) \leq \bigvee \{\mathbb{T}_N(a_{11}), \mathbb{T}_N(b_{11})\}, \\ \mathbb{I}_N((l_{11} * m_{11}) * (l_{11} * n_{11})) \geq \bigwedge \{\mathbb{I}_N(a_{11}), \mathbb{I}_N(b_{11})\}, \\ \mathbb{F}_N((l_{11} * m_{11}) * (l_{11} * n_{11})) \leq \bigvee \{\mathbb{F}_N(a_{11}), \mathbb{F}_N(b_{11})\}, \end{cases} \tag{12}$$

for all $l_{11}, m_{11}, n_{11}, a_{11}, b_{11} \in P$

are equivalent.
Proof. Suppose that P_N is a $NPi\mathcal{N}$-I of P. Then P_N is a $N\mathcal{N}$-I of P by Theorem 3.1. Let $l_{11}, m_{11}, n_{11}, a_{11}, b_{11} \in P$ be such that $a_{11} * (l_{11} * (m_{11} * n_{11})) \leq b_{11}$. Using Corollary 3.2 and Lemma 3.2, we have

$$\begin{aligned} &\mathbb{T}_N((l_{11} * m_{11}) * (l_{11} * n_{11})) \leq \mathbb{T}_N(l_{11} * (m_{11} * n_{11})) \leq \bigvee \{\mathbb{T}_N(a_{11}),\ \mathbb{T}_N(b_{11})\}, \\ &\mathbb{I}_N((l_{11} * m_{11}) * (l_{11} * n_{11})) \geq \mathbb{I}_N(l_{11} * (m_{11} * n_{11})) \geq \bigwedge \{\mathbb{I}_N(a_{11}),\ \mathbb{I}_N(b_{11})\}, \\ &\mathbb{F}_N((l_{11} * m_{11}) * (l_{11} * n_{11})) \leq \mathbb{F}_N(l_{11} * (m_{11} * n_{11})) \leq \bigvee \{\mathbb{F}_N(a_{11}),\ \mathbb{F}_N(b_{11})\}, \end{aligned}$$

for all $l_{11}, m_{11}, n_{11}, a_{11}, b_{11} \in P$.

Conversely, let P_N be a $N\mathcal{N}$-structure over P that satisfies (12). Let $l_{11}, m_{11}, a_{11}, b_{11} \in P$ be such that $a_{11} * (n_{11} * (n_{11} * m_{11})) \leq b_{11}$. Then

$$\begin{aligned} &\mathbb{T}_N(n_{11} * m_{11}) = \mathbb{T}_N((n_{11} * m_{11}) * (n_{11} * n_{11})) \leq \bigvee \{\mathbb{T}_N(a_{11}),\ \mathbb{T}_N(b_{11})\}, \\ &\mathbb{I}_N(n_{11} * m_{11}) = \mathbb{I}_N((n_{11} * m_{11}) * (n_{11} * n_{11})) \geq \bigwedge \{\mathbb{I}_N(a_{11}),\ \mathbb{I}_N(b_{11})\}, \\ &\mathbb{F}_N(n_{11} * m_{11}) = \mathbb{F}_N((n_{11} * m_{11}) * (n_{11} * n_{11})) \leq \bigvee \{\mathbb{F}_N(a_{11}),\ \mathbb{F}_N(b_{11})\}, \end{aligned}$$

by (KU3), (KU5) and (12). It follows from Theorem 3.4 that P_N is a $NPi\mathcal{N}$-I of P.
Theorem 3.7 Let P_N be a $N\mathcal{N}$-structure over P. Then P_N is a $NPi\mathcal{N}$-I of P iff P_N satisfies (5) and

$$(\forall l_{11}, m_{11}, n_{11} \in P) \begin{pmatrix} \mathbb{T}_N(l_{11} * m_{11}) \leq \bigvee \{\mathbb{T}_N(n_{11} * (l_{11} * (l_{11} * m_{11}))),\ \mathbb{T}_N(n_{11})\}, \\ \mathbb{I}_N(l_{11} * m_{11}) \geq \bigwedge \{\mathbb{I}_N(n_{11} * (l_{11} * (l_{11} * m_{11}))),\ \mathbb{I}_N(n_{11})\}, \\ \mathbb{F}_N(l_{11} * m_{11}) \leq \bigvee \{\mathbb{F}_N(n_{11} * (l_{11} * (l_{11} * m_{11}))),\ \mathbb{F}_N(n_{11})\}. \end{pmatrix} \tag{13}$$

Proof. Assume that P_N is a $NPi\mathcal{N}$-I of P. Then P_N is a $N\mathcal{N}$-I of P by Theorem 3.1, and so the condition (5) is valid. Using (4), (KU3), (KU5), Theorem 2.1 and (11), we have

$$\begin{aligned} \mathbb{T}_N(l_{11} * m_{11}) &\leq \bigvee \{\mathbb{T}_N(n_{11} * (l_{11} * m_{11})),\ \mathbb{T}_N(n_{11})\} \\ &= \bigvee \{\mathbb{T}_N(((l_{11} * l_{11}) * (l_{11} * (n_{11} * m_{11})),\ \mathbb{T}_N(n_{11})\} \\ &\leq \bigvee \{\mathbb{T}_N(l_{11} * (l_{11} * (n_{11} * m_{11}))),\ \mathbb{T}_N(n_{11})\} \\ &= \bigvee \{\mathbb{T}_N(n_{11} * (l_{11} * (l_{11} * m_{11}))),\ \mathbb{T}_N(n_{11})\}, \\ \mathbb{I}_N(l_{11} * m_{11}) &\geq \bigwedge \{\mathbb{I}_N(n_{11} * (l_{11} * m_{11})),\ \mathbb{I}_N(n_{11})\} \\ &= \bigwedge \{\mathbb{I}_N((l_{11} * l_{11}) * (l_{11} * (n_{11} * m_{11}))),\ \mathbb{I}_N(n_{11})\} \\ &\geq \bigwedge \{\mathbb{I}_N(l_{11} * (l_{11} * (n_{11} * m_{11}))),\ \mathbb{I}_N(n_{11})\} \\ &= \bigwedge \{\mathbb{I}_N(n_{11} * (l_{11} * (l_{11} * m_{11}))),\ \mathbb{I}_N(n_{11})\}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{F}_N(l_{11} * m_{11}) &\leq \bigvee \{\mathbb{F}_N(n_{11} * (l_{11} * m_{11})),\ \mathbb{F}_N(n_{11})\} \\ &= \bigvee \{\mathbb{F}_N((l_{11} * l_{11}) * (l_{11} * (n_{11} * m_{11}))),\ \mathbb{F}_N(n_{11})\} \\ &\leq \bigvee \{\mathbb{F}_N(l_{11} * (l_{11} * (n_{11} * m_{11}))),\ \mathbb{F}_N(n_{11})\} \\ &= \bigvee \{\mathbb{F}_N(n_{11} * (l_{11} * (l_{11} * m_{11}))),\ \mathbb{F}_N(n_{11})\}, \end{aligned}$$

for all $l_{11}, m_{11}, n_{11} \in P$. Therefore (13) is valid.

Conversely, if P_N is a $\mathcal{NN}$-structure over P satisfying two Conditions (5) and (13), then

$$\begin{aligned}
\mathbb{T}_N(l_{11}) &= \mathbb{T}_N(0 * l_{11}) \leq \bigvee\{\mathbb{T}_N(n_{11} * (0 * (0 * l_{11}))), \mathbb{T}_N(n_{11})\} = \bigvee\{\mathbb{T}_N(n_{11} * l_{11}), \mathbb{T}_N(n_{11})\} \\
\mathbb{I}_N(l_{11}) &= \mathbb{I}_N(0 * l_{11}) \geq \bigwedge\{\mathbb{I}_N(n_{11} * (0 * (0 * l_{11}))), \mathbb{I}_N(n_{11})\} = \bigwedge\{\mathbb{I}_N(n_{11} * l_{11}), \mathbb{I}_N(n_{11})\} \\
\mathbb{F}_N(l_{11}) &= \mathbb{F}_N(0 * l_{11}) \leq \bigvee\{\mathbb{F}_N(n_{11} * (0 * (0 * l_{11}))), \mathbb{F}_N(n_{11})\} = \bigvee\{\mathbb{F}_N(n_{11} * l_{11}), \mathbb{F}_N(n_{11})\}
\end{aligned}$$

for all $l_{11}, n_{11} \in P$. Hence P_N is a $\mathcal{NN}$-I of P. Now, if we take $n = 0$ in (13) and use (KU3), then

$$\begin{aligned}
\mathbb{T}_N(l_{11} * m_{11}) &\leq \bigvee\{\mathbb{T}_N(0 * (l_{11} * (l_{11} * m_{11}))),\ \mathbb{T}_N(0)\} \\
&= \bigvee\{\mathbb{T}_N(l_{11} * (l_{11} * m_{11})),\ \mathbb{T}_N(0)\} = \mathbb{T}_N(l_{11} * (l_{11} * m_{11})) \\
\mathbb{I}_N(l_{11} * m_{11}) &\geq \bigwedge\{\mathbb{I}_N(0 * (l_{11} * (l_{11} * m_{11}))),\ \mathbb{I}_N(0)\} \\
&= \bigwedge\{\mathbb{I}_N(l_{11} * (l_{11} * m_{11})),\ \mathbb{I}_N(0)\} = \mathbb{I}_N(l_{11} * (l_{11} * m_{11}))
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{F}_N(l_{11} * m_{11}) &\leq \bigvee\{\mathbb{F}_N(0 * (l_{11} * (l_{11} * m_{11}))),\ \mathbb{F}_N(0)\} \\
&= \bigvee\{\mathbb{F}_N(l_{11} * (l_{11} * m_{11})),\ \mathbb{F}_N(0)\} = \mathbb{F}_N(l_{11} * (l_{11} * m_{11}))
\end{aligned}$$

for all $l_{11}, m_{11} \in P$. It follows from Theorem 3.3 that P_N is a $NPi\mathcal{N}$-I of P.
Summarizing the above results, we have a characterization of a $NPi\mathcal{N}$-I.

Theorem 3.8 For a $\mathcal{NN}$-structure P_N over P, the following assertions are equivalent.

(i) P_N is a $NPi\mathcal{N}$-I of P.
(ii) P_N is a $\mathcal{NN}$-I of P satisfying the condition (8).
(iii) P_N is a $\mathcal{NN}$-I of P satisfying the condition (11).
(iv) P_N satisfies two conditions (5) and (13)
(v) P_N satisfies the condition (10)
(vi) P_N satisfies the condition (I1)

For any fixed numbers $\zeta_T, \zeta_F \in [-1, 0), \zeta_1 \in (-1, 0]$ and a nonempty subset G of P, a $\mathcal{NN}$-structure P_N^G over P is defined to be the structure

$$P_N^G := \frac{P}{\left(\mathbb{T}_N^G, \mathbb{I}_N^G,\ \mathbb{F}_N^G\right)} = \left\{\frac{l}{\left(\mathbb{T}_N^G(l_{11}),\ \mathbb{I}_N^G(l_{11}), \mathbb{F}_N^G(l_{11})\right)} \mid l \in P\right\} \tag{14}$$

where $\mathbb{T}_N^G$, $\mathbb{I}_N^G$ and $\mathbb{F}_N^G$ are $\mathcal{N}$-functions on P which are given as follows:

$$\mathbb{T}_N^G : P \to [-1, 0],\ l \mapsto \begin{cases} \zeta_T & \text{if } l \in G \\ 0 & \text{otherwise,} \end{cases}$$
$$\mathbb{I}_N^G : P \to [-1, 0],\ l \mapsto \begin{cases} \zeta_I & \text{if } l \in G \\ -1 & \text{otherwise} \end{cases}$$

and

$$\mathbb{F}_N^G : P \to [-1, 0],\ l \mapsto \begin{cases} \zeta_F & \text{if } l \in G \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3.9 Given a nonempty subset H of P, a $\mathcal{NN}$-structure P_N^H over P is a $NPi\mathcal{N}$-I of P iff H is a positive implicative ideal of P.

Proof. Assume that H is a positive implicative ideal of P. Since $0 \in H$, it follows that $\mathbb{T}_N^H(0) = \zeta_T \leq \mathbb{T}_N^H(l_{11})$, $\mathbb{I}_N^H(0) = \zeta_I \geq \mathbb{I}_N^H(l_{11})$, and $\mathbb{F}_N^H(0) = \zeta_F \leq \mathbb{F}_N^H(l_{11})$ for all $l_{11} \in P$. For any $l_{11}, m_{11}, n_{11} \in P$, we consider four cases:

Case 1. $n_{11} * (l_{11} * m_{11}) \in H$ and $n_{11} * l_{11} \in H$,
Case 2. $n_{11} * (l_{11} * m_{11}) \in H$ and $n_{11} * l_{11} \notin H$,
Case 3. $n_{11} * (l_{11} * m_{11}) \notin H$ and $n_{11} * l_{11} \in H$,
Case 4. $n_{11} * (l_{11} * m_{11}) \notin H$ and $n_{11} * l_{11} \notin H$,

Case 1 implies that $n_{11} * m_{11} \in H$, and thus

$$\mathbb{T}_N^H(n_{11} * m_{11}) = \mathbb{T}_N^H(n_{11} * (l_{11} * m_{11})) = \mathbb{T}_N^H(n_{11} * l_{11}) = \zeta_T,$$
$$\mathbb{I}_N^H(n_{11} * m_{11}) = \mathbb{I}_N^H(n_{11} * (l_{11} * m_{11})) = \mathbb{I}_N^H(n_{11} * l_{11}) = \zeta_I,$$
$$\mathbb{F}_N^H(n_{11} * m_{11}) = \mathbb{F}_N^H(n_{11} * (l_{11} * m_{11})) = \mathbb{F}_N^H(n_{11} * l_{11}) = \zeta_F.$$

Hence

$$\mathbb{T}_N^H(n_{11} * m_{11}) \leq \bigvee \left\{\mathbb{T}_N^H(n_{11} * (l_{11} * m_{11})), \mathbb{T}_N^H(n_{11} * l_{11})\right\},$$
$$\mathbb{I}_N^H(n_{11} * m_{11}) \geq \bigwedge \left\{\mathbb{I}_N^H(n_{11} * (l_{11} * m_{11})),\ \mathbb{I}_N^H(n_{11} * l_{11})\right\},$$
$$\mathbb{F}_N^H(n_{11} * m_{11}) \leq \bigvee \left\{\mathbb{F}_N^H(n_{11} * (l_{11} * m_{11})),\ \mathbb{F}_N^H(n_{11} * l_{11})\right\}.$$

If Case 2 is valid, then $\mathbb{T}_N^H(n_{11} * l_{11}) = 0,\ \mathbb{I}_N^H(n_{11} * l_{11}) = -1$ and $\mathbb{F}_N^H(n_{11} * l_{11}) = 0$. Thus

$$\mathbb{T}_N^H(n_{11} * m_{11}) \leq 0 = \bigvee \left\{\mathbb{T}_N^H(n_{11} * (l_{11} * m_{11})),\ \mathbb{T}_N^H(n_{11} * l_{11})\right\},$$
$$\mathbb{I}_N^H(n_{11} * m_{11}) \geq -1 = \bigwedge \left\{\mathbb{I}_N^H(n_{11} * (l_{11} * m_{11})),\ \mathbb{I}_N^H(n_{11} * l_{11})\right\},$$
$$\mathbb{F}_N^H(n_{11} * m_{11}) \leq 0 = \bigvee \left\{\mathbb{F}_N^H(n_{11} * (l_{11} * m_{11})),\ \mathbb{F}_N^H(n_{11} * l_{11})\right\}.$$

For the Case 3, it is similar to the Case 2 . For the Case 4, it is clear that

$$\mathbb{T}_N^H(n_{11} * m_{11}) \leq \bigvee \left\{\mathbb{T}_N^H(n_{11} * (l_{11} * m_{11})), \mathbb{T}_N^H(n_{11} * l_{11})\right\},$$
$$\mathbb{I}_N^H(n_{11} * m_{11}) \geq \bigwedge \left\{\mathbb{I}_N^H(n_{11} * (l_{11} * m_{11})), \mathbb{I}_N^H(n_{11} * l_{11})\right\},$$
$$\mathbb{F}_N^H(n_{11} * m_{11}) \leq \bigvee \left\{\mathbb{F}_N^H(n_{11} * (l_{11} * m_{11})), \mathbb{F}_N^H(n_{11} * l_{11})\right\}.$$

Therefore P_N^H is a $NPi\mathcal{N}$-I of P.

Conversely, suppose that P_N^H is a $NPi\mathcal{N}$-I of P. Then $\left(\mathbb{T}_N^H\right)^{\frac{\zeta_T}{2}} = H,\ \left(\mathbb{I}_N^H\right)^{\frac{\zeta_I}{2}} = H$ and $\left(\mathbb{F}_N^H\right)^{\frac{\zeta_F}{2}} = H$ are positive implicative ideals of P by Theorem 3.2.

We consider an extension property of a $NPi\mathcal{N}$-I based on the negative indeterminacy membership function.

Lemma 3.6 Let A_{11} and B_{11} be ideals of $P \ni A_{11} \subseteq B_{11}$. If A_{11} is a positive implicative ideal of P, then so is B_{11}.

Theorem 3.10 Let

$$P_N := \frac{P}{(\mathbb{T}_N, \mathbb{I}_N,\ \mathbb{F}_N)} = \left\{\frac{l}{(\mathbb{T}_N(l_{11}), \mathbb{I}_N(l_{11}),\ \mathbb{F}_N(l_{11}))} \mid l \in P\right\}$$

and

$$P_M := \frac{P}{(T_M, I_M,\ F_M)} = \left\{\frac{l}{(T_M(l_{11}), I_M(l_{11}),\ F_M(l_{11}))} \mid l \in P\right\}$$

be $N\mathcal{N}$-Is of P such that $P_N(=, \leq, =)P_M$, that is, $\mathbb{T}_N(l_{11}) = T_M(l_{11}),\ \mathbb{I}_N(l_{11}) \leq I_M(l_{11})$ and $\mathbb{F}_N(l_{11}) = F_M(l_{11})$ for all $l_{11} \in P$. If P_N is a $NPi\mathcal{N}$-I of P, then so is P_M.

Proof. Assume that P_N is a $NPi\mathcal{N}$-I of P. Then $\mathbb{T}_N^\lambda, \mathbb{I}_N^\mu$ and $\mathbb{F}_N^\delta$ are positive implicative ideals of P for all $\lambda, \mu, \delta \in [-1, 0]$ by Theorem 3.2. The condition $P_N(=,\ \leq,\ =)P_M$ implies that $\mathbb{T}_N^{\zeta_T} = T_M^{\zeta_T},\ \mathbb{I}_N^{\zeta_I} \subseteq I_M^{\zeta_I}$ and $\mathbb{F}_N^{\zeta_F} = F_M^{\zeta_F}$. It follows from Lemma 3.6 that T_M^λ, I_M^μ and F_M^δ are positive implicative ideals of P for all $\lambda, \mu, \delta \in [-1, 0]$. Therefore P_M is a $NPi\mathcal{N}$-I of P by Theorem 3.5.

Conclusions

In this paper, we have discussed the notion of a $NPi\mathcal{N}$-I in KU-algebras, and investigated several properties. We have considered relations between a $N\mathcal{N}$-I and a $NPi\mathcal{N}$-I. We have provided conditions for a $N\mathcal{N}$-I to be a $NPi\mathcal{N}$-I, and considered characterizations of a $NPi\mathcal{N}$-I. We have established an extension property of a $NPi\mathcal{N}$-I based on the negative indeterminacy membership function.

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