Novel Applications Of Neutrosophic AH-Isometries To The Group Of Units Problem In Neutrosophic Rings and Refined Neutrosophic Rings

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Novel Applications Of Neutrosophic AH-Isometries To The Group Of Units Problem In Neutrosophic Rings and Refined Neutrosophic Rings

Katy D. Ahmad, Islamic University Of Gaza, Palestine

Mikail Bal, Gaziantep University, Turkey

Arwa A. Hajjari, Cairo University, Egypt.

Rozina Ali, Cairo University, Egypt

Co-: Katyon765@gmail.com

Rozyyy123n@gmail.om

Abstract: The Objective of this paper is to study the group of units problem in two different kinds of neutrosophic structures (neutrosophic rings and refined neutrosophic rings), where we use the concept of AH-isometry to classify neutrosophic rings\refined neutrosophic rings as direct products of classical rings with itself. Also, this classification will lead to the algebraic structure of the corresponding group of units.

Keywords: Group of units, AH-isometry, neutrosophic ring, refined neutrosophic ring

1.Introduction

After the arrival of neutrosophic ideas by Smarandache [1], many authors used this ideas to study algebra and algebraic structures, where we find many neutrosophical algebraic structures as generalizations of the classical ones, such as neutrosophic rings [4-5] and refined neutrosophic rings [7], and neutrosophic matrix theory [8-9].

In classical ring theory, the group of units problem is one of the most famous open problems [6]. This problem can be considered as a question about the classification of the group of units corresponded with a given ring. We can find many previous works and contributions about this subject such as [18].

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In the literature, we find many essential studies in the field of neutrosophic functions or functions between neutrosophic structures such as neutrosophic inner products [2], AH-geometrical isometry [14].

In neutrosophic ring theory, the group of units problem [21] is still uncovered in general. Through this paper, we will study the group of units problem in refined neutrosophic rings\ neutrosophic rings by using the concept of AH-isometry (algebraic projections between rings and spaces). This combination between neutrosophic functions and neutrosophic structures will lead to a classification property of these rings, then we can solve the group of units problem in these rings easily.

All rings through this paper are taken with unity 1.

Main discussion

Group of Units Problem In Neutrosophic Rings

First of all, we recall some helpful definitions and concepts about neutrosophic rings.

Definition 1:

Let R be a ring, I be the indeterminacy with property $I^2 = I$, then the neutrosophic ring is defined as follows:

$R(I) = \{a + bI; a, b \in R\}$.

Definition 2:

Define the one-dimensional isometry between the ring R(I) and the Cartesian product R×R as follows:

$g: R(I) \to R \times R; g(a + bI) = (a, a + b)$.

Remark: It is an algebraic isomorphism between R(I) and $R \times R$

Example 3:

For The ring $R = (Z, +, .)$, the corresponding neutrosophic ring is $Z(I) = \{m + nI;\ m, n \in Z\}$.

Definition 4:

Let $R(I) = \{a + bI;\ a, b \in R\}$ be a neutrosophic ring with unity 1, $A = x + yI$ be an arbitrary element in R(I). It is called invertible or (unit) if there exists $B = c + dI \in R(I)$ such that $A.B = 1$.

Example 5:

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Consider the neutrosophic ring of integers Z(I)=$\{a+bI;\ a,b \in Z\}$. We can see that $1-2I$ is a unit, that is because $(1-2I)(1-2I)=1$.

The following theorem determines the criteria of invertibility.

Theorem 6:

Let R be any ring with unity 1, R(I) be its corresponding neutrosophic ring. An arbitrary element $z=a+bI \in R(I)$ is invertible if and only if $a, a+b\ are\ invertible$ in R.

Proof:

Let $z=a+bI \in R(I)$ be an invertible element in R(I). There is $m=x+yI \in R(I); z.m=1$.

Thus $(a.x)+(a.y+b.x+b.y)I=1$, this means that $x=a^{-1},\ a.y+b(a^{-1})+by=0$, hence

$y=-ba^{-1}(a+b)^{-1}$, this implies that $a, a+b\ are\ invertible$ elements in R.

Conversely, suppose that $a, a+b\ are\ invertible$ elements in R, then there is $m=x+yI \in R(I)$, where

$x=a^{-1},\ y=-ba^{-1}(a+b)^{-1}$ with $z.m=1$.

Remark 7:

In a neutrosophic field K(I). The invertibility condition becomes as follows:

An arbitrary element $z=a+bI \in K(I)$ is invertible if and only if $a \neq 0\ and\ a \neq -b$.

That is because all nonzero elements in the field K are invertible.

Example 8:

Let $Z_4(I)=\{a+bI; a,b \in Z_4\}$ be the neutrosophic ring of integers modulo 4.

$x=3+2I$ is invertible because $a=3, a+b=1$ are invertible in $R=Z_4$. The inverse of x is

$y=3^{-1}+[-2.3^{-1}(2+3)^{-1}]I=3+(-2.3.1)I=3-6I=3+2I.$

Theorem 9:

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Let R be any finite ring with unity 1, R(I) be its corresponding neutrosophic ring. The order of the group of units $o(U(R(I)))$ is equal to $o(U(R)) \times o(U(R))$.

Proof:

Let U(R(I)) be the group of units of the neutrosophic ring R(I). For any element $a + bI$ in $U(R(I))$, we have $a, a + b$ are invertible in R, hence we can choose (a) by $O(U(R))$ ways. Also, $a + b$ can be chosen by $O(U(R))$ ways, hence we get $O(U(R))$ way to choose (b). This implies that the number of elements in $U(R(I))$ is $O(U(R)) \times O(U(R))$

The previous theorem gives us an aspect about the algebraic structure of $U(R(I))$ in general. For example if we consider the neutrosophic field $Z_5(I)$ (which is a ring by classical meaning). It will has a group of units of order $4 \times 4 = 16$. It is logical to think that $U(R(I)) \cong U(R) \times U(R)$. So it is natural to think about a classification property between R(I) and $R \times R$.

This goal can be achieved by regarding the properties of the one-dimensional AH-isometry. See [66].

We check it by the following theorem.

Theorem 10:

Let R be any ring with unity 1, R(I) be its corresponding neutrosophic ring. Then $R(I) \cong R \times R$.

Proof:

Consider the one dimensional AH-isometry

$f: R(I) \to R \times R; f(a + bI) = (a, a + b)$, it is an isomorphism between the rings R(I), and $R \times R$.

This means that $R(I) \cong R \times R$.

Now, we are able to determine the algebraic structure of $U(R(I))$.

Theorem 11:

Let R be any ring with unity 1, R(I) be its corresponding neutrosophic ring. Then $U(R(I)) \cong U(R) \times U(R)$.

Proof:

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The proof holds directly from the fact that $R(I) \cong R \times R$ in the previous theorem, so that $U(R(I)) \cong U(R) \times U(R)$.

Example 12:

Let $Z_4(I) = \{a + bI; a, b \in Z_4\}$ be the neutrosophic ring of integers modulo 4. It is known that the group of units of the ring Z_4 is $\{1,3\}$and isomorphic to Z_2.

The group of units of is U($Z_4(I)$)={$1,3,1 + 2I, 3 + 2I$}. We can see that $U(Z_4(I)) \cong U(Z_4) \times U(Z_4) \cong Z_2 \times Z_2$, i.e. the order f any element is 2.

Group Of Units Problem In Refined Neutrosophic Rings

Definition 13:

The element I can be split into two indeterminacies I_1 , I_2 with conditions:

$I_1^2 = I_1$, $I_2^2 = I_2$, $I_1 I_2 = I_2 I_1 = I_1$.

Definition 14:

Let (R,+, ×) be a ring, (R(I_1, I_2) , + ,×) is called a refined neutrosophic ring generated by R , I_1, I_2.

Remark 15:

For the operations (addition and multiplication) on the refined neutrosophic ring. See [15].

Definition 16:

Let R(I_1, I_2) $= \{(a, bI_1, cI_2);\ a, b, c \in R\}$ be a refined neutrosophic ring with unity (1,0,0), $A = (x, yI_1, zI_2)$ be an arbitrary element in R(I_1, I_2). It is called invertible or (unit) if there is $B = (c, dI_1, tI_2) \in$ R(I_1, I_2), such that $A.B = (1,0,0)$.

Example 17:

$Z(I_1, I_2) = \{(a, bI_1, cI_2);\ a, b, c \in Z\}$ is the refined neutrosophic ring of integers.

The following theorem determines the criteria of invertibility.

Theorem 18:

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Let $R(I_1, I_2)$ be a refined neutrosophic ring. An arbitrary element $t = (a, bI_1, cI_2)$ is invertible if and only if $a, a + c, a + b + c$ $are\ invertible\ in\ R$.

Proof:

Suppose that $t = (a, bI_1, cI_2)$ is invertible, then there is $m = (x, yI_1, zI_2); m.t = (1,0,0)$.

$m.t = (a.x, [a.y + b.x + b.z + b.y + c.y]I_1, [a.z + c.x + c.z]I_2)$ = (1,0,0), this means

$x = a^{-1}, z = -ca^{-1}(a + c)^{-1}, y = (a + b + c)^{-1}.(-b.a^{-1} + bca^{-1}(a + c)^{-1})$, which implies that

$a, a + c, a + b + c\ are\ invertible\ in\ R$.

Conversely, if $a, a + c, a + b + c\ are\ invertible\ in\ R$, then there is $m = (x, yI_1, zI_2); m.t = (1,0,0)$ where

$x = a^{-1}, z = -ca^{-1}(a + c)^{-1}, y = (a + b + c)^{-1}.(-b.a^{-1} + bca^{-1}(a + c)^{-1})$. Thus t is invertible.

Remark 19:

In a refined neutrosophic field $K(I_1, I_2)$. The invertibility condition becomes as follows:

An arbitrary element An arbitrary element $t = (a, bI_1, cI_2)$ is invertible if and only if $a \neq 0, a + c \neq 0, a + b + c \neq 0$.

That is because all nonzero elements in the field K are invertible.

Example 20:

Let $Z_4(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in Z_4\}$ be the refined neutrosophic ring of integers modulo 4.

$x = (3,2I_1, 2I_2)$ is invertible because $a = 3, a + c = 1$, $a + b + c = 3$ are invertible in $R = Z_4$. The inverse of x is

$y = (3^{-1}, [(3 + 2 + 2)^{-1}(-2.3^{-1} + 2.2.3^{-1}(3 + 2)^{-1}]I_1, [-2.3^{-1}(2 + 3)^{-1}]I_2)$=

$(3, [3.(-6 + 0)]I_1, [-6.1]I_2) = (3,2I_1, 2I_2)$.

Theorem 21:

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Let R be any ring with unity 1, $R(I_1, I_2)$ be its corresponding neutrosophic ring. Then $R(I_1, I_2) \cong R \times R \times R$.

Proof:

We can define the refined one-dimensional AH-isometry between $R(I_1, I_2)$ and the ring $R \times R \times R$ as follows:

$g: R(I_1, I_2) \to R^3; g(a_0 + a_1 I_1 + a_2 I_2) = (a_0, a_0 + a_1 + a_2, a_0 + a_2),$

Let $w_1 = a_0 + a_1 I_1 + a_2 I_2, w_2 = b_0 + b_1 I_1 + b_2 I_2$ be two refined neutrosophic elements, then

$g(w_1 + w_2) = g([a_0 + b_0] + [a_1 + b_1]I_1 + [a_2 + b_2]I_2)$

$= (a_0 + b_0, a_0 + a_1 + a_2 + b_0 + b_1 + b_2, a_0 + a_2 + b_0 + b_2)$

$= (a_0, a_0 + a_1 + a_2, a_0 + a_2) + (b_0, b_0 + b_1 + b_2, b_0 + b_2)$

$= g(a_0 + a_1 I_1 + a_2 I_2) + g(b_0 + b_1 I_1 + b_2 I_2) = g(w_1) + g(w_2).$

$g(w_1 . w_2) = g\big((a_0 + a_1 I_1 + a_2 I_2).(b_0 + b_1 I_1 + b_2 I_2)\big)$

$= g(a_0 b_0 + [a_1 b_1 + a_1 b_0 + a_1 b_1 + a_1 b_2 + a_2 b_1]I_1 + [a_0 b_2 + a_2 b_0 + a_2 b_2]I_2)$

$$= (a_0 b_0, a_0 b_0 + a_1 b_1 + a_1 b_0 + a_1 b_1 + a_1 b_2 + a_2 b_1 + a_0 b_2 + a_2 b_0 + a_2 b_2, a_0 b_0 + a_0 b_2 + a_2 b_0 + a_2 b_2)$$

$= (a_0, a_0 + a_1 + a_2, a_0 + a_2).(b_0, b_0 + b_1 + b_2, b_0 + b_2)$

$= g(a_0 + a_1 I_1 + a_2 I_2).g(b_0 + b_1 I_1 + b_2 I_2) = g(w_1).g(w_2).$

g is a correspondence one-to-one, that is because $ker(g) = \{0\}$, and for every

$(a_0, a_1, a_2) \in R \times R \times R$,there exists $x = a_0 + ([a_1 - a_2])I_1 + (a_2 - a_0)I_2 \in R(I_1, I_2)$such that $g(x) = (a_0, a_1, a_2)$.

Thus, g is isomorphism.

The inverse isomorphism of g is $g^{-1}: R \times R \times R \to R(I_1, I_2); g^{-1}(a, b, c) = (a, [b - a]I_1, [c - a]I_2)$.

Now, we are able to determine the algebraic structure of $U(R(I_1, I_2))$.

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Theorem 22:

Let R be any ring with unity 1, $R(I_1, I_2)$ be its corresponding refined neutrosophic ring. Then $U(R(I_1, I_2)) \cong U(R) \times U(R) \times U(R)$.

Proof:

The proof holds directly from the previous theorem.

Example 23:

The following example will show an effective algorithm to compute the units of any refined neutrosophic ring .

Let $Z_4(I_1, I_2) = \{(a, bI_1, cI_2); a, b, c \in Z_4\}$ be the refined neutrosophic ring of integers modulo 4.

By the last theorem, the algebraic structure of the group of units of $Z_4(I_1, I_2)$ is:

$U(Z_4) \times U(Z_4) \times U(Z_4) = Z_2 \times \mathrm{Z}_2 \times \mathrm{Z}_2$.

Hence it has exactly 8 units. To find all units, we can use the inverse isomorphism for the one dimensional refined AH-isometry, as follows:

The units in the ring $Z_4 \times Z_4 \times Z_4$ are the triples $\{(1,1,1), (1,3,3), (3,1,3), (3,3,1), (3,3,3), (3,1,1), (1,1,3), (1,3,1)\}$.

By taking the inverse image with the inverse isomorphism, we get all units in $Z_4(I_1, I_2)$.

The units are $\{(1,0,0), (1,2I_1, 2I_2), (3,2I_1, 2I_2), (1,2I_1, 0), (3,2I_1, 0), (3,0,0), (3,0,2I_2), (1,0,2I_2)\}$.

Conclusion

In this paper, we have studied the group of units problem in neutrosophic rings and refined neutrosophic rings. We have used the AH-algebraic isometry to classify the previous rings into Cartesian products of classical rings with itself and to classify the corresponding groups of units.

On the other hand, we have presented an effective algorithm to compute units in these rings.

As a future research direction, we aim to find an AH-isometry to classify n-refined neutrosophic rings and to solve the group of units problem in such rings.

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