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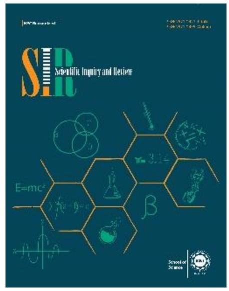

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Statistically Convergent Sequences in Neutrosophic Metric Spaces

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Abstract

In this study, the introduction of statistical convergence and statistical Cauchy sequences with respect to neutrosophic metric spaces is motivated by the notion of statistical convergence in fuzzy metric spaces. We offer useful characterizations for statistically convergent and statistically Cauchy sequences.

Keywords**:** neutrosophic metric spaces, statistically Cauchy sequence, statistical convergence, statistically convergent sequence

Introduction

Zadeh [1] established the concept of fuzzy sets. Afterwards, several authors explored it in various contexts, such as the fuzzy metric space [2]. George and Veeramani [3] established the concept of a fuzzy metric space, first presented by Kramosil and Michalek [2] utilizing continuous t-norms. Their work was extended sucessfully in [4, 5]. The concept of a fuzzy metric space presented in [3] was expanded into the concept of an intuitionistic fuzzy metric space by Park [6], utilizing the notion of intuitionistic fuzzy set given in [7]. Park used continuous t-norms and continuous t-conorms to describe the concept of an intuitionistic fuzzy metric space. Fuzzy metric spaces and intuitionistic fuzzy metric spaces have been used to study a variety of phenomena, including convergence and fixed-point theorems given in [8–11].

Park presented the concept of intuitionistic fuzzy metric spaces for dealing with both membership and non-membership functions, since fuzzy metric spaces have been primarily discussed in the literature in relation to membership functions, see [12]. The method of neutrosophic metric spaces (NMSs), which deals with membership, non-membership, and naturalness functions alike, was proposed by Kirişci and Simsek [13]. Keeping in view NMSs, Simsek and Kirişci [14] and Sowndrarajan et al. [15] provided certain fixed point (FP) results. Fast [16] and Steinhous [17] independently developed the notion of statistical convergence (s-convergence) in 1951. Since then, both pure and applied mathematicians have been interested in this concept. Li et al. [18] explored statistical convergence in cone metric spaces, while Maio et al. [19] worked on statistical convergence in topology. Statistically convergent sequences in fuzzy metric spaces were first introduced in 2020 by Li et al. [20]. Schweizer and Sklar [21] investigated the metrization of statistical metric spaces. Similarly, Varol [22] proved statistical convergence in intuitionistic fuzzy metric spaces. We capitalize on these results using the concept of neutrosophic metric space. Hence, in this manuscript,

- We focus on statistical convergence (s-convergence) in NMSs.
- We examine the correlations between convergence and
- s-convergence.
- We investigate the notions of statistical Cauchy sequences (SCS) and statistical completeness (s-completeness).

This article has two parts. The first part comprises the preliminaries (it has some basic definitions). The second part elaborates the notions of s-completeness and s-convergence in neutrosophic metric spaces (NMSs).

2. Preliminaries

In this section, we provide some basic terminologies and concepts to explain the key findings of previous researches. The terms $\mathbb{R}$ and $\mathbb{N}$ are used to refer to the sets of all real numbers and all positive integer numbers, respectively, throughout this work.

Definition 2.1 [21] A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-norm if $*$ satisfies the following:

(1) $a * 1 = a, \forall a \in [0,1]$;
(2) $a * b = b * a$ and $a * (b * c) = (a * b) * c \; \forall a, b, c \in [0,1]$;
(3) if $a \leq c$ and $b \leq d$ then $a * b \leq c * d, \forall a, b, c, d \in [0,1]$;
(4) $*$ is continuous.

Definition 2.2 A binary operation $\diamond : [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-norm if $\diamond$ satisfies the following:

(1) $a \diamond 0 = a, \; \forall a \in [0,1]$;
(2) $a \diamond b = b \diamond a$ and $a \diamond (b \diamond c) = (a \diamond b) \diamond c \; \forall a, b, c \in [0,1]$;
(3) If $a \leq c, b \leq d$, then $a \diamond b \leq c \diamond d \; \forall a, b, c, d \in [0,1]$;
(4) $\diamond$ is continuous.

Note that $a * b = \min\{a, b\}, a \diamond b = \max\{a, b\}, a * b = ab$ and $a \diamond b = \min\{a + b, 1\}$ are basic examples of continuous t-norms and continuous t-conorms for all $a, b \in [0,1]$. From the previous two definitions, we see that if $r_1 > r_2$ then there exists $r_3, r_4 \in (0,1)$, such that $r_1 * r_2 \geq r_2$ and $r_2 \diamond r_4 \leq r_1$.

Definition 2.3 [7] Let Ψ and ψ be fuzzy sets on $\Sigma^2 \times (0, \infty), *$ be a continuous t-norm, and $\diamond$ be a continuous t-conorm. If Ψ and ψ satisfy the following conditions, we say that (Ψ, ψ) is an intuitionistic fuzzy metric (IFM) on Σ,

(IFM1) $\Psi(\vartheta, \beta, t) + \psi(\vartheta, \beta, t) \leq 1$;
(IFM2) $\Psi(\vartheta, \beta, t) > 0$;
(IFM3) $\Psi(\vartheta, \beta, t) = 1$ if and only if $\vartheta = \beta$;
(IFM4) $\Psi(\vartheta, \beta, t) = \Psi(\beta, \vartheta, t)$;
(IFM5) $\Psi(\vartheta, \beta, t) * \Psi(\beta, \gamma, s) \leq \Psi(\vartheta, \gamma, t + s)$;
(IFM6) $\Psi(\vartheta, \beta, \; .) : (0, \infty) \rightarrow (0,1]$ is continuous;
(IFM7) $\psi(\vartheta, \beta, t) > 0$;
(IFM8) $\psi(\vartheta, \beta, t) = 0$ if and only if $\vartheta = \beta$;
(IFM9) $\psi(\vartheta, \beta, t) = \psi(\beta, \vartheta, t)$;
(IFM10) $\psi(\vartheta, \beta, t) \diamond \psi(\beta, \gamma, s) \geq \psi(\vartheta, \gamma, t + s)$;
(IFM11) $\psi(\vartheta, \beta, \; .) : (0, \infty) \rightarrow (0,1]$ is continuous.

Then, $(\Sigma, \Psi, \psi, *, \diamond)$ is an IFM. The functions $\Psi(\vartheta, \beta, t)$ and $\psi(\vartheta, \beta, t)$ denote the degree of nearness and the degree of non-nearness between ϑ and β with respect to t, respectively.

Remark 2.1 [7] Let $(\Sigma, \Psi, \psi, *, \diamondsuit)$ be an IFM, then $(\Sigma, \Psi, *)$ is a fuzzy metric space. Conversely, if $(\Sigma, \Psi, *)$ is a fuzzy metric space, then $(\Sigma, \Psi, 1-\Psi, *, \diamondsuit)$ is an IFM, where $a \diamondsuit b = 1 - \big((1-a) * (1-b)\big), \forall a, b \in [0,1]$.

Definition 2.4 [13] Suppose $\Sigma \neq \emptyset$. Given a six tuple $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$ where * is a CTN, $\diamondsuit$ is a CTCN, Ψ, ψ and ϕ are NSs on $\Sigma \times \Sigma \times (0, \infty)$. If $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$ meets the below circumstances for all $\vartheta, \beta, \gamma, \in \Sigma$ and $t, s > 0$:

1. $\Psi(\vartheta, \beta, t) + \psi(\vartheta, \beta, t) + \phi(\vartheta, \beta, t) \leq 3$;
2. $0 \leq \Psi(\vartheta, \beta, t) \leq 1$;
3. $\Psi(\vartheta, \beta, t) = 1$ if and only if $\vartheta = \beta$;
4. $\Psi(\vartheta, \beta, t) = \Psi(\beta, \vartheta, t)$;
5. $\Psi\big(\vartheta, \gamma(t+s)\big) \geq \Psi(\vartheta, \beta, t) * \Psi(\beta, \gamma, s)$;
6. $\Psi(\vartheta, \beta, \cdot): [0, \infty) \to [0,1]$ is continuous;
7. $\lim_{t\to\infty} \Psi(\vartheta, \beta, t) = 1$;
8. $0 \leq \psi(\vartheta, \beta, t) \leq 1$;
9. $\psi(\vartheta, \beta, t) = 0$ if and only if $\vartheta = \beta$;
10. $\psi(\vartheta, \beta, t) = \psi(\beta, \vartheta, t)$;
11. $\psi\big(\vartheta, \gamma, (t+s)\big) \leq \psi(\vartheta, \beta, t) \diamondsuit \psi(\beta, \gamma, s)$;
12. $\psi(\vartheta, \beta, \cdot): [0, \infty) \to [0,1]$ is continuous;
13. $\lim_{t\to\infty} \psi(\vartheta, \beta, t) = 0$;
14. $0 \leq \phi(\vartheta, \beta, t) \leq 1$;
15. $\phi(\vartheta, \beta, t) = 0$ if and only if $\vartheta = \beta$;
16. $\phi(\vartheta, \beta, t) = \phi(\beta, \vartheta, t)$;
17. $\phi\big(\vartheta, \gamma, (t+s)\big) \leq \phi(\vartheta, \beta, t) \diamondsuit \phi(\beta, \gamma, s)$;
18. $\phi(\vartheta, \beta, \cdot): [0, \infty) \to [0,1]$ is a continuous;
19. $\lim_{t\to\infty} \phi(\vartheta, \beta, t) = 0$;
20. if $t \leq 0$ then $\Psi(\vartheta, \beta, t) = 0, \psi(\vartheta, \beta, t) = 1, \phi(\vartheta, \beta, t) = 1$;

where (Ψ, ψ, ϕ) is a neutrosophic metric space and $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$ is an NMS. The functions $\Psi(\vartheta, \beta, t), \psi(\vartheta, \beta, t)$ and $\phi(\vartheta, \beta, t)$ represent the degree of nearness, non-nearness, and naturalness between ϑ and β with respect to t, respectively.

Definition 2.5 [13] Let $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$ be an NMS, $t > 0, r \in (0,1)$, and $\vartheta \in \Sigma$. The set $B_\vartheta(r,t) = \{\beta \in \Sigma: \Psi(\vartheta, \beta, t) > 1 - r, \psi(\vartheta, \beta, t) < r$ and $\phi(\vartheta, \beta, t) < r\}$ is said to be an open ball with center ϑ and radius r with respect to t.

$$B_\vartheta(r,t) : \vartheta \in \Sigma, r \in (0,1), t > 0,$$

generates a topology $\tau_{(\Psi,\psi,\phi)}$, known as the (Ψ, ψ, ϕ) topology.

Definition 2.6 [14] Let $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$ be an NMS. Then,

1. (ϑ_n) is convergent to ϑ if for all $t > 0$ and $r \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that $\Psi(\vartheta_n, \vartheta, t) > 1 - r$, $\psi(\vartheta_n, \vartheta, t) < r$, and $\phi(\vartheta_n, \vartheta, t) < r$ for all $n \geq n_0$. It is denoted by $\vartheta_n \to \vartheta$ as $n \to \infty$. $\Psi(\vartheta_n, \vartheta, t) \to 1$, $\psi(\vartheta_n, \vartheta, t) \to 0$, and $\phi(\vartheta_n, \vartheta, t) \to 0$, as $n \to \infty$ for each $t > 0$.
2. (ϑ_n) is a Cauchy sequence if for $t > 0$ and $r \in (0,1)$. Then, there exists $n_0 \in \mathbb{N}$ such that $\Psi(\vartheta_n, \vartheta_{m,t}) > 1 - r, \psi(\vartheta_n, \vartheta_{m,t}) < r$, and $\phi(\vartheta_n, \vartheta_{m,t}) < r$ for all $n, m \geq n_0$.
3. An NMS $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$ is said to be complete if every Cauchy sequence is convergent and to a point of Σ.

Definition 2.7 [20] Let $(\Sigma, \Psi, *)$ be a fuzzy metric space. Then,

1. A sequence $(\vartheta_n) \subset \Sigma$ is s-convergent to $\vartheta_0 \in \Sigma$ if $\delta(\{n \in \mathbb{N}: \Psi((\vartheta_n, \vartheta_0, t) > 1 - r\}) = 1$ for every $r \in (0,1)$ and $t > 0$.
2. A sequence $(\vartheta_n) \subset \Sigma$ is SCS if for every $r \in (0,1)$ and $t > 0$ there exists $m \in \mathbb{N}$ such that $\delta(\{n \in \mathbb{N}: \Psi(\vartheta_n, \vartheta_m, t) > 1 - r\}) = 1$.

3. S-Completeness and S-Convergence in an NMS

In this section, we examine s-convergent sequences in an NMS. We also introduce the notion of an SCS on NMS and examine its characterization.

Definition 3.1 Let $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$ be an NMS. A sequence $\vartheta_n \subset \Sigma$ is an SCS if for every $r \in (0,1)$ and $t > 0$ there exists $m \in \mathbb{N}$ such that

$$\delta(\{\, n \in \mathbb{N}: \Psi(\vartheta_n, \vartheta_m, t) > 1 - r\, , \psi(\vartheta_n, \vartheta_m, t) < r \text{ and } \psi(\vartheta_n, \vartheta_m, t) < r\}) = 1.$$

Definition 3.2 Let $(\Sigma, \Psi, \psi, \phi, *, \diamond)$ be an NMS. A sequence $(\vartheta_n) \subset \Sigma$ is s-convergent to $\vartheta_0 \in \Sigma$ with an NMS provided that for every $r \in (0,1)$ and $t > 0$,

$$\delta(\{n \in \mathbb{N}: \Psi(\vartheta_n, \vartheta_0, t) > 1 - r, \psi(\vartheta_n, \vartheta_0, t) < r, \phi(\vartheta_n, \vartheta_0, t) < r\}) = 1.$$

The sequence (ϑ_n) is s-convergent to ϑ_0. We see that

$$\delta(\{n \in \mathbb{N}: \Psi(\vartheta_n, \vartheta_0, t) > 1 - r, \psi(\vartheta_n, \vartheta, t) < r\, , \phi\, (\vartheta_n, \vartheta, t) < r\}) = 1$$
$$\Leftrightarrow \lim_{n\to\infty} \frac{|\{k \leq n: \Psi(\vartheta_k, \vartheta_0, t) > 1 - r, \psi(\vartheta_k, \vartheta_0, t) < r\, , \phi(\vartheta_n, \vartheta, t) < r\}|}{n} = 1.$$

Example 3.1 Let $\Sigma = \mathbb{R}, a * b = ab$ and

$$a \diamond b = \min\{a + b, 1\} \text{ for all } a, b \in [0,1].$$

Define Ψ, ψ and

$$\phi\ b\ \beta\ \Psi(\vartheta, \beta, t) = \frac{t}{t+|\vartheta-\beta|}, \psi(\vartheta, \beta, t) = \frac{|\vartheta-\beta|}{t+|\vartheta-\beta|}, \text{and } \phi(\vartheta, \beta, t) = \frac{|\vartheta-\beta|}{t},$$

for all $\vartheta, \beta \in \Sigma$ *and* $t > 0$. Then, $(\Sigma, \Psi, \psi, \phi, *, \diamond)$ is an NMS.

New define a sequence (ϑ_n) by

$$\vartheta_n = \begin{cases} 1, & n = k^2, k \in \mathbb{N}, \\ 0, & \text{Otherwise.} \end{cases}$$

Then, for every $r \in (0,1)$ and for any $t > 0$, let

$$k = \{n \leq m: \Psi(\vartheta_n, 0, t) \leq 1 - r, \psi(\vartheta_n, 0, t) \geq r,$$
$$\phi(\vartheta_n, 0, t) \geq r\} = \{n \leq m: \frac{t}{t + |\vartheta_n|} \leq 1 - r, \frac{|\vartheta_n|}{t + |\vartheta_n|} \geq r,$$
$$\frac{|\vartheta_n|}{t} \geq r\} = \{n \leq m: |\vartheta_n| \geq \frac{rt}{1 - r} > 0\} = \{n \leq m: \vartheta_n = 1\}$$
$$= \{n \leq m: n = k^2, k \in \mathbb{N}\}.$$

Now, we obtain

$$\frac{1}{m}|k| \leq \frac{1}{m}|\{n \leq m: n = k^2, \quad n \in \mathbb{N}| \leq \frac{\sqrt{m}}{m} \to 0, m \to \infty.$$

Hence, we conclude that (ϑ_n) is s-convergent to 0 with respect to the NMS $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$.

Lemma 3.1 Let $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$ be an NMS. Then, for every $r \in (0,1)$ and $t > 0$ the following are equivalent

(i) (ϑ_n) is s-convergent to ϑ_0;

(ii) $\delta\{n \in \mathbb{N}: \Psi(\vartheta_n, \vartheta_0, t) \leq 1 - r\}) = \delta(\{\psi(\vartheta_n, \vartheta_0, t) \geq r\}) = \delta(\{\phi(\vartheta_n, \vartheta_0, t) \geq r\}) = 0$;

(iii) $\delta\{n \in \mathbb{N}: \Psi(\vartheta_n, \vartheta_0, t) > 1 - r\}) = \delta(\{\psi(\vartheta_n, \vartheta_0, t) < r\}) = \delta(\{\phi(\vartheta_n, \vartheta_0, t) < r\} = 1$.

Proof: Using definition 2.1 and the properties of density, we have the lemma.

Theorem 3.1 Let $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$be an NMS. If a sequence (ϑ_n) is s-convergent with respect to the above NMS, then the s-convergent limit is unique.

Proof: Suppose that (ϑ_n) is s-convergent to ϑ_1 and ϑ_2 for a given $r \in (0,1)$, choose $t > 0$ such that $(1-t) * (1-t) > 1 - r$ and $t \diamondsuit\ t < r$.

Then, define the following sets for any $\in > 0$:

$$K_{\Psi 1}(t, \in) = \{n \in \mathbb{N}: \Psi(\vartheta_n, \vartheta_1, \in) > 1 - t\};$$

$$K_{\Psi 2}(t, \in) = \{n \in \mathbb{N}: \Psi(\vartheta_n, \vartheta_2, \in) > 1 - t\};$$

$$K_{\psi 1}(t, \in) = \{n \in \mathbb{N}: \psi(\vartheta_n, \vartheta_1, \in) < 1 - t\};$$

$$K_{\psi 2}(t, \in) = \{n \in \mathbb{N}: \psi(\vartheta_n, \vartheta_2, \in) < 1 - t\};$$

$$K_{\phi 1}(t, \in) = \{n \in \mathbb{N}: \phi(\vartheta_n, \vartheta_1, \in) < 1 - t\}$$

$$K_{\phi 2}(t, \in) = \{n \in \mathbb{N}: \phi(\vartheta_n, \vartheta_2, \in) < 1 - t\}.$$

Since, ϑ_n is s-convergent with respect to ϑ_1 and ϑ_2, we obtain

$$\delta\{\, K_{\Psi 1}(t, \in)\} = \delta\left\{\, K_{\psi 1}(t, \in)\right\} = \delta\left\{\, K_{\phi 1}(t, \in)\right\} = 1$$

and

$$\delta\{\, K_{\Psi 2}(t, \in)\} = \delta\left\{\, K_{\psi 2}(t, \in)\right\} = \delta\left\{\, K_{\phi 2}(t, \in)\right\} = 1 \;\; \forall \; \in > 0.$$

Let

$$K_{\Psi\psi\phi}(t,\in) := \{K_{\Psi 1}(t,\in) \cup K_{\Psi 2}(t,\in)\} \cap \{K_{\psi 1}(t,\in) \cup K_{\psi 2}(t,\in) \cap \left\{K_{\phi 1}(t,\in) \cup K_{\phi 2}(t,\in)\right\}.$$

Hence,

$$\delta\left\{K_{\Psi\psi\phi}(t,\in) = 1\right\} \text{ which implies that } \delta\left\{\mathbb{N}\backslash K_{\Psi\psi\phi}(t,\in) = 0\right\}.$$

If $n \in \mathbb{N}\backslash K_{\Psi\psi\phi}(t,\in)$ then we have

$$n \in \mathbb{N}\ \{K_{\Psi 1}(t,\in) \cup K_{\Psi 2}(t,\in)\} \text{ or } n \in \mathbb{N}\ \{K_{\psi 1}(t,\in) \cup K_{\psi 2}(t,\in)\} \text{ or}$$
$$n \in \mathbb{N}\backslash\ \{K_{\phi 1}(t,\in) \cup K_{\phi 2}(t,\in)\}.$$

Let us consider $n \in \mathbb{N}\backslash\ \{K_{\Psi 1}(t,\in) \cup K_{\Psi 2}(t,\in)\}$, then we obtain

$$\Psi(\vartheta_1,\vartheta_2,\in) \geq \Psi\left(\vartheta_1,\vartheta_n,\frac{\in}{2}\right) * \Psi\left(\vartheta_n,\vartheta_2,\frac{\in}{2}\right) > (1-t) * (1-t) > 1-r.$$

Therefore, $\Psi(\vartheta_1,\vartheta_2,\in) > 1-r$. Since $r > 0$ is arbitrary, we obtain $\Psi(\vartheta_1,\vartheta_2,\in) = 1$ for all $\in > 0$, which implies that $\vartheta_1 = \vartheta_2$. If

$n \in \mathbb{N}\ \{K_{\psi 1}(t,\in) \cup K_{\psi 2}(t,\in)\}$,

Then

$$\psi(\vartheta_1,\vartheta_2,\in) \leq \psi(\vartheta_1,\vartheta_n,\in) \diamond \psi(\vartheta_n,\vartheta_2,\in) < t \diamond t < r.$$

Since $r > 0$ is arbitrary, we obtain $\psi(\vartheta_1,\vartheta_2,\in) = 0$ for all $\in > 0$, which implies $\vartheta_1 = \vartheta_2$.

If we consider $n \in \mathbb{N}\backslash\ \{K_{\phi 1}(t,\in) \cup K_{\phi 2}(t,\in)\}$, then

$$\phi(\vartheta_1,\vartheta_2,\in) \leq \phi(\vartheta_1,\vartheta_n,\in) \diamond \phi(\vartheta_n,\vartheta_2,\in) < t \diamond t < r.$$

Since $r > 0$ is arbitrary, then we get $\phi(\vartheta_1,\vartheta_2,\in) = 0$ for all $\in > 0$, which implies $\vartheta_1 = \vartheta_2$. This completes the proof.

Theorem 3.2 Let ϑ_n be a sequence in an NMS $(\Sigma,\Psi,\psi,\phi,*,\ \ )$. If ϑ_n is convergent to ϑ_0 with respect to the above NMS, then ϑ_n is s-convergent to ϑ_0 with respect to the said NMS.

Proof: Let ϑ_n be convergent to ϑ_0. Then, for every $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $\Psi(\vartheta_n,\vartheta_0,t) > 1-r$, $\psi(\vartheta_n,\vartheta_0,t) < r$, and $\phi(\vartheta_n,\vartheta_0,t) < r$. We have $|\{k \leq n: \Psi(\vartheta_n,\vartheta_0,\ t) > 1 -$

r , $\psi(\vartheta_n,\ \vartheta_0, t) < r$ and $\phi(\vartheta_n,\ \vartheta_0, t) < r| \geq n - n_0$. Hence, the set $\{k \leq n: \Psi(\vartheta_n, \vartheta_0,\ t) > 1 - r,\ \psi(\vartheta_n,\ \vartheta_0, t) < r$ and $\phi(\vartheta_n,\ \vartheta_0, t) < r$ has a finite number of terms. Then,

$$\lim_{n\to\infty} \frac{|\{k \leq n: \Psi(\vartheta_n, \vartheta_0,\ t) > 1 - r,\ \psi(\vartheta_n,\ \vartheta_0, t) < r, \phi(\vartheta_n,\ \vartheta_0, t) < r|}{n}$$
$$\geq \lim_{n\to\infty} \frac{n - n_0}{n} = 1.$$

Consequently,

$$\delta\{n \in \mathbb{N} : \Psi(\vartheta_n, \vartheta_0, t) > 1 - r, \psi(\vartheta_n, \vartheta_0\ , t) < r, \phi(\vartheta_n, \vartheta_0\ , t) < r\} = 1.$$

The converse of the theorem need not hold.

Example 3.2 Let $\Sigma = [1,3], a * b = ab,$ and $a \diamond b = \min\{a + b, 1\}\ \forall\ a, b \in [0,1]$.

Define Ψ, ψ and ϕ by

$$\Psi(\vartheta, \beta, t) = \frac{t}{t + |\vartheta - \beta|}, \psi(\vartheta, \beta, t) = \frac{|\vartheta - \beta|}{t + |\vartheta - \beta|}, \text{and } \phi(\vartheta, \beta, t) = \frac{|\vartheta - \beta|}{t}$$

for all $\vartheta, \beta \in \Sigma$ and $t > 0$.

Then $(\Sigma, \Psi, \psi, \phi, *, \diamond)$ is an NMS. Now, define a sequence ϑ_n by

$$\vartheta_n = \begin{cases} 2, & n = k^2, k \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

We can see that ϑ_n is not convergent to 1. We need to show that ϑ_n is s-convergent to 1. Let $r \in (0,1)$ and $t > 0$. Then,

$$K = \{n \in \mathbb{N} : \Psi(\vartheta_n, 1, t) > 1 - r, \psi(\vartheta_n, 1, t) < r, \phi(\vartheta_n, 1, t) < r\}.$$

Case 1: $r \in (0, \frac{1}{t+1}]$ if $n \neq k^2$ for all $k \in \mathbb{N}$, then

$$\Psi(\vartheta_n, 1, t) = 1 > 1 - r, \psi(\vartheta_n, 1, t) = 0 < r, \text{and } \phi(\vartheta_n, 1, t) = 0 < r.$$

If $n = k^2$ for some $k \in \mathbb{N}$ then

$$\Psi(\vartheta_n, 1, t) = \frac{t}{1 + t} = 1 - \frac{1}{1 + t} \leq 1 - r,$$
$$\psi(\vartheta_n, 1, t) = \frac{1}{1 + t} \geq r,$$

$$\phi(\vartheta_n, 1, t) = \frac{1}{t} \geq r.$$

Now, let $n \in \mathbb{N}$, if $n = k_0^2$ for all $k_0 \in \mathbb{N}$ then
$\lim_{n\to\infty} \frac{|k(n)|}{n} = \lim_{k_0\to\infty} \frac{k_0^2 - k_0}{k_0^2} = 1$. If $n \neq k^2$ for all $k \in \mathbb{N}$ then we obtain

$k_1 \in \mathbb{N}$ such that $n = k_1^2 - l$ with $l \in \mathbb{N}$ and $1 \leq l \leq k_1$.

$$\lim_{n\to\infty} \frac{|k(n)|}{n} = \lim_{k_1\to\infty} \frac{k_1^2 - l - (k_1 - 1)}{k_1^2 - l} = \lim_{k_1\to\infty} \frac{k_1^2 - k_1 - l + 1)}{k_1^2 - l} = 1.$$

Case 2: $r \in (\frac{1}{t+1}, 1)$ if $n \neq k^2$ for all $k \in \mathbb{N}$ then

$\Psi(\vartheta_n, 1, t) = 1 > 1 - r$, $\psi(\vartheta_n, 1, t) = 0 < r$ and $\phi(\vartheta_n, 1, t) = 0 < r$.

If $n = k^2$ for some $k \in \mathbb{N}$ then

$$\Psi(\vartheta_n, 1, t) = \frac{t}{1+t} = 1 - \frac{1}{1+t} > 1 - r,$$

$$\psi(\vartheta_n, 1, t) = \frac{1}{1+t} < r,$$

$$\phi(\vartheta_n, 1, t) = \frac{1}{1+t} < r.$$

Hence,

$\Psi(\vartheta_n, 1, t) > 1 - r, \psi(\vartheta_n, 1, t) < r$ and $\phi(\vartheta_n, 1, t) < r$ for all $n \in \mathbb{N}$.

Therefore, $\lim_{n\to\infty} \frac{|k(n)|}{n} = \lim_{n\to\infty} \frac{n}{n} = 1$.

$\delta\{n \in \mathbb{N}: \Psi(\vartheta_n, 1, t) > 1 - r, \psi(\vartheta_n, 1, t) < r, \phi(\vartheta_n, 1, t) < r\} =$
1 for all $r \in (0,1)$ and $t > 0$.

Theorem 3.3 Let ϑ_n be a sequence in an NMS $(\Sigma, \Psi, \psi, \phi, *, \diamond)$. Then, ϑ_n is s-convergent to ϑ_0 if and only if there exists an increasing index sequence $A = \{n_i\}_{i\in\mathbb{N}}$ of natural numbers such that ϑ_{n_i} converges to ϑ_0 and $\delta(A) = 1$.

Proof: Assume that ϑ_n s-converges to ϑ_0. Let

$$K_{\Psi\psi\phi}(j,t) := \left\{n \in \mathbb{N} : \Psi(\vartheta_n, \vartheta_0, t) > 1 - \frac{1}{j}, \psi(\vartheta_n, \vartheta_0, t) < \frac{1}{j} \text{ and } \phi(\vartheta_n, \vartheta_0, t) < \frac{1}{j}\right\}$$

for any $t > 0$ and $j \in \mathbb{N}$.

We show that $K_{\Psi\psi\phi}\ (j+1,t) \subset K_{\Psi\psi\phi}(j,t)$ for $t > 0, j \in \mathbb{N}$. Since ϑ_n is s-convergent to ϑ_0,

$$\delta\left(K_{\Psi\psi\phi}(j,t)\right) = 1 \tag{1}$$

Take $s_1 \in K_{\Psi\psi\phi}\ (1,t)$, since $\delta\left(K_{\Psi\psi\phi}(2,t)\right) = 1$ (by equation 1 we have a number $s_2 \in \left(K_{\Psi\psi\phi}(2,t)(s_2 > s_1)\right)$ such that

$$\frac{\left|k \leq n: \Psi(\vartheta_k, \vartheta_0, t) > 1 - \frac{1}{2},\ \psi(\vartheta_k, \vartheta_0, t) < \frac{1}{2}, \phi(\vartheta_k, \vartheta_0, t) < \frac{1}{2}\right|}{n} > \frac{1}{2},$$

for all $n \geq s_2$.

Again, by equation (1), $\delta(K_{\Psi\psi\phi}(3,t) = 1$. We can choose $s_3 \in K_{\Psi\psi\phi}(3,t)(s_3 > s_2)$ such that

$$\frac{\left|k \leq n: \Psi(\vartheta_k, \vartheta_0, t) > 1 - \frac{1}{3},\ \psi(\vartheta_k, \vartheta_0, t) < \frac{1}{3}, \phi(\vartheta_k, \vartheta_0, t) < \frac{1}{3}\right|}{n} > \frac{2}{3},$$

for all $n \geq s_3$.

If we continue like this, we obtain an increasing index sequence $\{s_j\}_{j\in\mathbb{N}}$ of natural numbers such that $s_j \in \left(K_{\Psi\psi\phi}(j,t)\right)$. We also have the following:

$$\frac{\left|k \leq n: \Psi(\vartheta_k, \vartheta_0, t) > 1 - \frac{1}{j},\ \psi(\vartheta_k, \vartheta_0, t) < \frac{1}{j},\ \phi(\vartheta_k, \vartheta_0, t) < \frac{1}{j}\right|}{n} > \frac{j-1}{j},$$

for all $n \geq s_j, j \in \mathbb{N}$ (2)

We obtain an increasing index sequence A as

$A = \{n \in \mathbb{N} : 1 < n < s_1\} \cup \{\cup_{j\in\mathbb{N}} \left\{ K_{\Psi\psi\phi}(j,t): s_j \leq n < s_{j+1}\right\}$.

On the basis of equation (2) and $K_{\Psi\psi\phi}(j+1,t) \subset K_{\Psi\psi\phi}(j,t)$, we write

$$\frac{|k \le n: k \in A|}{n} \ge \frac{\left|k \le n: \Psi(\vartheta_k, \vartheta_0, t) > 1 - \frac{1}{j}, \psi(\vartheta_k, \vartheta_0, t) < \frac{1}{j}, \phi(\vartheta_k, \vartheta_0, t) < \frac{1}{j}\right|}{n}$$
$$> \frac{j-1}{j}$$

for all $n, (s_j \le n < s_{j+1})$.

Since $j \to \infty$, when $n \to \infty$ we have $\lim_{n\to\infty} \frac{|k \le n: k \in A|}{n} = 1$ i.e., $\delta(A) = 1$. Now, we show that ϑ_{n_i} converges to ϑ_0. Let $r \in (0,1)$ and $t > 0$. Take $\psi_0 > s_2$ large enough that for some $l_0 \in \mathbb{N}$, $s_{l_0} \le \psi_0 < s_{l_0+1}$ with $\frac{1}{l_0} < r$. Assume that $n_m \ge \psi_0$ with $n_m \in A$. With the definition of A there exists $l \in \mathbb{N}$ such that $s_l \le n_m < s_{(j+1)}$ with $n_m \in K_{\Psi\psi\phi}(l, t), (l \ge l_0)$. Then, we obtain

$$\Psi(\vartheta_{n_m}, \vartheta_0, t) \ge \Psi\left(\vartheta_{n_m}, \vartheta_0, \frac{1}{l_0}\right) \ge \Psi\left(\vartheta_{n_m}, \vartheta_0, \frac{1}{l}\right) > 1 - \frac{1}{l} \ge 1 - \frac{1}{l_0}$$
$$> 1 - r,$$

$$\psi(\vartheta_{n_m}, \vartheta_0, t) > \frac{1}{l_0} < r,$$

$$\phi(\vartheta_{n_m}, \vartheta_0, t) > \frac{1}{l_0} < r.$$

Therefore, ϑ_{n_i} converges to ϑ_0.

Conversely, assume that there exists an increasing index sequence $A = \{n_i\}_{j\in\mathbb{N}}$ of the natural numbers such that $\delta(A) = 1$ and ϑ_{n_i} converges to ϑ_0. Let $r \in (0,1)$ and $t > 0$. Then, there exists a natural number $n_0 \in \mathbb{N}$ such that for each $n \ge n_0$ the inequalities $\Psi(\vartheta_n, \vartheta_0, t) > 1 - r, \psi(\vartheta_n, \vartheta_0, t) < r$ and $\phi(\vartheta_n, \vartheta_0, t) < r$ are satisfied. Let us define

$$K_{\Psi\psi\phi}(r, t) \coloneqq \{n \in \mathbb{N} : \Psi(\vartheta_{n_i}, \vartheta_0, t) \le 1 - r \text{ or } \psi(\vartheta_{n_i}, \vartheta_0, t)$$
$$\ge r \text{ or } \phi(\vartheta_{n_i}, \vartheta_0, t) \ge r\}.$$

We have

$$K_{\Psi\psi\phi}(r, t) \subset \mathbb{N}\{n_{n_0}, n_{n_0+1}, n_{n_0+2}, \dots\}.$$

Since $\delta(A) = 1$, we have $\delta(\mathbb{N}\{n_{n_0}, n_{n_0+1}, n_{n_0+2}, \dots\}) = 0$. So, we deduce $\delta\left(K_{\Psi\psi\phi}(r,t)\right) = 0$.

Hence, $\delta\{n \in \mathbb{N} : \Psi(\vartheta_{n_i}, \vartheta_0, t) < 1 - r,\ \psi(\vartheta_{n_i}, \vartheta_0, t) \geq r$ and $\phi(\vartheta_{n_i}, \vartheta_0, t) \geq r\} = 1$. Therefore, ϑ_n s-converges to ϑ_0.

Corollary 3.1 Let ϑ_n be a sequence in a $(\Sigma, \Psi, \psi, \phi, *, \diamond)$. If ϑ_n is s-convergent to ϑ_0 and it is convergent, then $\vartheta_n$converges to ϑ_0.

Definition 3.3 Let $(\Sigma_1, \Psi_1, \psi_1, \phi_1, *_1, \diamond_1)$ and $(\Sigma_2, \Psi_2, \psi_2, \phi_2, *_2, \diamond_2)$ be two NMSs.

1. A mapping $f: \Sigma_1 \to \Sigma_2$ is called an isometry if for each $\vartheta, \beta \in \Sigma_1$ and $t > 0$
$$\Psi_1(\vartheta, \beta, t) = \Psi_2(f(\vartheta), f(\beta), t),\ \psi_1(\vartheta, \beta, t) = \psi_2(f(\vartheta), f(\beta), t) \text{ and } \phi_1(\vartheta, \beta, t) = \phi_2(f(\vartheta), f(\beta), t).$$
2. $(\Sigma_1, \Psi_1, \psi_1, \phi_1, *_1, \diamond_1)$ and $(\Sigma_2, \Psi_2, \psi_2, \phi_2\ *_2, \diamond_2)$ are called isometric if there exists an isometry from Σ_1 onto Σ_2.
3. A neutrosophic completion of $(\Sigma_1, \Psi_1, \psi_1, \phi_1, *_1, \diamond_1)$ is a complete NMS $(\Sigma_2, \Psi_2, \psi_2, \phi_2, *_2, \diamond_2)$ such that $(\Sigma_1, \Psi_1, \psi_1, \phi_1, *_1, \diamond_1)$ is isometric to a dense subspace of Σ_2.
4. $(\Sigma_1, \Psi_1, \psi_1, \phi_1\ *_1, \diamond_1)$ is completable if it leads to a neutrosophic completion.

Proposition 3.1 Let ϑ_n be a sequence in a completable NMS $(\Sigma, \Psi, \psi, \phi, *, \diamond)$. If ϑ_n is a Cauchy sequence in Σ and it s-converges to $\vartheta_{0,}$ then ϑ_n converges to ϑ_0.
Proof: Let $(\Sigma_1, \Psi_1, \psi_1, \phi_1, *_1, \diamond_1)$ be the completion of $(\Sigma, \Psi, \psi, \phi, *, \diamond)$.
Then, there exists $\vartheta_1 \in \Sigma_1 : \vartheta_n$ which converges to $\vartheta_{1.}$ We have $\Psi_1(\vartheta_n, \vartheta_0, t) = \Psi(\vartheta_n, \vartheta_0, t)$, $\psi_1(\vartheta_n, \vartheta_0, t) = \psi(\vartheta_n, \vartheta_0, t)$ and $\phi_1(\vartheta_n, \vartheta_0, t) = \phi(\vartheta_n, \vartheta_0, t)\ \forall t > 0$ and $n \in \mathbb{N}$. Let $r \in (0,1)$ and $t > 0$. Since $\delta(\{n \in \mathbb{N} : \Psi(\vartheta_n, \vartheta_0, t) > 1 - r, \psi(\vartheta_n, \vartheta_0, t) < r$ and $\psi(\vartheta_n, \vartheta_0, t) < r\}) = 1$, we obtain

$$\delta(\{n \in \mathbb{N} : \Psi_1(\vartheta_n, \vartheta_0, t) > 1 - r, \psi_1(\vartheta_n, \vartheta_0, t) < r \text{ and } \phi_1(\vartheta_n, \vartheta_0, t) < r\}) = 1.$$

Hence, we see that ϑ_n s-converges to $\vartheta_0 \in \Sigma_1$ with respect to (Ψ_1, ψ_1, ϕ_1). By corollary 1, we have $\vartheta_1 = \vartheta_0$.

Theorem 3.4 Let ϑ_n be a sequence in an NMS $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$. Then, the following are equivalent:

1. ϑ_n is an SCS.
2. There exists an increasing index sequence $K = \{n_i\}_{j\in\mathbb{N}}$ of natural numbers such that ϑ_{n_i} is a Cauchy sequence and $\delta(K) = 1$.

Proof: Straightforward.

Theorem 3.5 Let ϑ_n be a sequence in an NMS $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$. If ϑ_n s-converges with respect to the selected NMS, then ϑ_n is an SCS with respect to the said NMS.

Proof: Let ϑ_n s-converge to ϑ_0 and $r \in (0,1), t > 0$. Then, their exists $r_1 \in (0,1)$: $(1 - r_1) * (1 - r_1) > 1 - r$ and $r_1 \diamondsuit r_2 < r$. Hence, we have

$$\delta(\{ n \in \mathbb{N}: \Psi(\vartheta_n, \vartheta_0, t) > 1 - r, \psi(\vartheta_n, \vartheta_0, t) < r \text{ and } \phi(\vartheta_n, \vartheta_0, t) < r\}) = 1.$$

According to theorem 2.1, there exists an increasing index sequence $\{n_i\}_{i\in\mathbb{N}}$ such that ϑ_{n_i} converges to ϑ_0. Hence, there exists

$$n_{0_i} \in \{n_i\}_{i\in\mathbb{N}} : \Psi\left(\vartheta_{n_i}, \vartheta_0, \frac{t}{2}\right) > 1 - r_1, \psi\left(\vartheta_{n_i}, \vartheta_0, \frac{t}{2}\right) < r_1 \text{ and } \phi\left(\vartheta_{n_i}, \vartheta_0, \frac{t}{2}\right) < r_1 \quad \text{for all } n_i \geq n_{i_0}.$$

Since

$$\Psi\left(\vartheta_n, \vartheta_{n_{i_0}}, t\right) \geq \Psi\left(\vartheta_n, \vartheta_0, \frac{t}{2}\right) * \Psi\left(\vartheta_0, \vartheta_{n_{i_0}} \ \frac{t}{2}\right) \geq (1 - r_1) * (1 - r_1) > 1 - r,$$

$$\psi\left(\vartheta_n, \vartheta_{n_{i_0}}, t\right) \leq \psi\left(\vartheta_n, \vartheta_0, \frac{t}{2}\right) \diamondsuit \psi\left(\vartheta_0, \vartheta_{n_{i_0}} \ \frac{t}{2}\right) < (r_1) \diamondsuit (r_1) < r,$$

$$\phi\left(\vartheta_n, \vartheta_{n_{i_0}}, t\right) \leq \phi\left(\vartheta_n, \vartheta_0, \frac{t}{2}\right) \diamondsuit \phi\left(\vartheta_0, \vartheta_{n_{i_0}} \ \frac{t}{2}\right) < (r_1) \diamondsuit (r_1) < r.$$

Hence, we have

$$\delta\left(\left\{ n \in \mathbb{N}: \Psi\left(\vartheta_n, \vartheta_{n_{i_0}}, t\right) > 1 - r, \psi\left(\vartheta_n, \vartheta_{n_{i_0}}, t\right) < r \text{ and } \phi\left(\vartheta_n, \vartheta_{n_{i_0}}, t\right) < r\right\}\right) = 1.$$

Therefore, ϑ_n is an SCS with respect to the selected NMS.

Remark 3.1 If a sequence is Cauchy in an NMS, then it is an SCS.

Definition 3.4 The NMS $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$ is s-complete if every SCS in Σ is s-convergent.

Theorem 3.6 Let $(\Sigma, \Psi, \psi, \phi, *, \diamondsuit)$ be an NMS. If Σ is s-complete, then it is complete with respect to the above NMS.

Proof: The proof is similar to Theorem 2.5.

4. Conclusion

In 1951, Fast and Steinhaus independently presented the notion of s-convergence. Subsequently, numerous authors developed interest in this topic and explored its application in various branches of mathematics. The notion of s-convergence in fuzzy metric spaces was first suggested by Li et al. [20] in 2020. On its basis, we discussed in this paper extending s-convergence to neutrosophic metric spaces. Hence, the terms s-convergence, SCS, and s-completeness have been defined with reference to neutrosophic metric spaces in this study. Additionally, we have investigated the characterizations of SCS and convergent sequences.

References

[1] Zadeh LA. Fuzzy sets. *Inf Control*. 1965; 8:338-53.

[2] Kramosil O, Michalek J. Fuzzy metric and statistical metric spaces. *Kybernetika*. 1975;11(5):336-344.

[3] George A, Veeramani P. On some result in fuzzy metric spaces. *Fuzzy Sets Syst*. 1994;64(3):395-399.https://doi.org/10.1016/0165-0114(94)90162-7

[4] George A, Veeramani P. Some theorems in fuzzy metric spaces. *J Fuzzy Math.* 1995; 3:933-940.

[5] Sostak A. George-Veeramani Fuzzy metrics revised. *Axioms*. 2018;7(3):60. https://doi.org/10.3390/axioms7030060

[6] Park JH. Intuitionistic fuzzy metric spaces. *Chaos Solitons Fractals*. 2004;22(3):1039-1046. https://doi.org/10.1016/j.chaos.2004.02.051

[7] Atanassov KT. Intuitionistic fuzzy sets. *Fuzzy Sets Syst*. 1986; 20:87-96.

[8] Alaca C, Turkoglu D, Yildiz C. Fixed points in intuitionistic fuzzy metric spaces. *Chaos Solitons Fractals*. 2006;29(5):1073-1078. https://doi.org/10.1016/j.chaos.2005.08.066

[9] Gregori V, Lopez-Crevillen A. Morillas S, Sapena A. On convergence in fuzzy metic spaces. *Topol Appl*. 2009;156(18):3002-3006. https://doi.org/10.1016/j.topol.2008.12.043

[10] Gregori V, Miñana J-J, Miravet D. Extended fuzzy metrics and fixed-point theorems. *Mathematics*. 2019;7(3):303. https://doi.org/10.3390/math7030303

[11] Saadati R, Park JH. On the intuitionistic fuzzy topological spaces. *Chaos Solitons Fractals*. 2006;27(2):331-344. https://doi.org/10.1016/j.chaos.2005.03.019

[12] Park JH. Intuitionistic fuzzy metric spaces. *Chaos Solitons Fractals*. 2004;22(5):1039-1046. https://doi.org/10.1016/j.chaos.2004.02.051

[13] Kirişci M, Şimşek N. Neutrosophic metric spaces. *Math Sci*. 2020;14(3):241-248. https://doi.org/10.1007/s40096-020-00335-8

[14] Şimşek N, Kirişci M. fixed point theorems in Neutrosophic metric spaces, *Sigma J Eng Nat Sci*. 2019;10(2):221-230.

[15] Sowndrara S, Jeyarama M, Smarandache F. Fixed point results for contraction theorems in neutrosophic metric spaces. *Neutro Sets Sys*. 2020;36(1):310-318.

[16] Frid, JA. On statistical convergence. *Analysis.* 1985; 5:301-313. https://doi.org/10.1524/anly.1985.5.4.301

[17] Steinhaus H. Sur la convergence ordinarie et la convergence asymptotique. *Colloq Math*. 1951; 2:73-74.

[18] Li KD, Lin S, Ge Y. On statistical convergence in cone metric spaces. *Topol Appl*. 2015; 196:641-651. https://doi.org/10.1016/j.topol.2015.05.038

[19] Maio GD, Koˇcinac LDR. Statistical convergence in topology. *Topol Appl*. 2008;156(1):28-45. https://doi.org/10.1016/j.topol.2008.01.015

[20] Li C, Zhang Y, Zhang J. On statistical convergence in fuzzy metric spaces. *J Intell Fuzzy Syst*. 2020;39(3):3987-3993. https://doi.org/10.3233/JIFS-200148

[21] Schweizer B, Sklar A, Thorp E. The metrization of statistical metric spaces. *Pacific J Math*. 1960;10(2):313–334.

[22] Varol BP. Statistically convergent sequences in intuitionistic fuzzy metric spaces. *Axioms*. 2022:11(4):159. https://doi.org/10.3390/axioms11040159