Annals of Mathematics and Computer Science
ISSN: 2789-7206
Vol 16 (2023) 21-44

FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS IN J-NEUTROSOPHIC METRIC SPACE

M. JEYARAMAN[1*], M. SUGANTHI[2]

Abstract. The aim of this work is to construct and prove unique common fixed point theorems for six weakly compatible mappings and coupled coincidence point theorem for w-compatible mappings in the setting of J-Neutrosophic Metric Spaces. This work also presents unique common coupled fixed point theorem for Junck's type and for three mapping in the same setting.

1. Introduction

The common fixed point and the coincidence point are the generalization of fixed point. The concept of common fixed points started to raise around the year 1954 whence Eldon Dyer, Allon Schiels and Lester Dubins have independently posted a question over two continuous commuting mappings defined from [0,1] to itself that is it possible for them to have a common fixed point. In 1957, Isbell[8] came up with an answer regarding this post, which attracted the research community and this is when researchers started to work common fixed points, though Isbell's answer was later independently proved to be wrong by W.M. Boyce[4] and Huneke[7].

Following Isbell, several extensions and generalizations of the commuting mappings are found and analyzed. During this period, Jungck[9] came up with a generalization of commuting maps, called compatible mappings. This finding has made a significant impact as it is less restrictive than commutativity. He also derived common fixed point theorems for two compatible pairs of mappings.

Moving onto the concept of coincidence points, it is first studied by Eilenberg and Montgomery[5] in the year 1946. The significant one among the coincidence theorems is the Lefschetz coincidence theorem which leads to several findings on coincidence theorems in various space settings. Lakshmikantham and Ćirić[11] introduced the concept of coupled coincidence point and proved coupled coincidence theorems for nonlinear contractive mappings. Later then several coupled coincidence point theorems were proved in various spaces.

Date: Received: May 18, 2023; Accepted: May 28, 2023.

* M. Jeyaraman.

2010 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. t-norm, t-conorm, Coupled coincidence point, Compatible mappings, Metric space, Symmetric.

Here, this work aims to construct and prove both common fixed point theorem and coupled coincidence point theorem in the setting of the metric space constructed on neutrosophic set[19]. Being an exclusive set, almost all the existing metric spaces can be comprised under some kind of neutrosophic metric space. The metric space, fuzzy metric space and intuitionistic metric are special cases of neutrosophic metric space. Hence it is worth to present this work in the setting of an Neutrosophic Metric Space. Here we are into the J-Neutrosophic Metric Space[23] to derive the proposed theorems.

2. Preliminaries

This section starts with the definitions of triangular norm and triangular conorm that are introduced by Schweitzer and Sklar[22] in the year 1960. These norms convert the triangle inequality of the metric space into an equivalent for while extending or generalizing the metric spaces via fuzzy sets, intuitionistic fuzzy sets and neutrosophic sets.

Definition 2.1. [22] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm [CTN] if it satisfies the following conditions:

(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $\varepsilon_1 * 1 = \varepsilon_1$ for all $\varepsilon_1 \in [0,1]$,
(iv) $\varepsilon_1 * \varepsilon_2 \leq \varepsilon_3 * \varepsilon_4$ whenever $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_4 \in [0,1]$.

Definition 2.2. [22] A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm [CTC] if it satisfies the following conditions:

(i) $\Diamond$ is commutative and associative,
(ii) $\Diamond$ is continuous,
(iii) $\varepsilon_1 \Diamond 0 = \varepsilon_1$ for all $\varepsilon_1 \in [0,1]$,
(iv) $\varepsilon_1 \Diamond \varepsilon_2 \leq \varepsilon_3 \Diamond \varepsilon_4$ whenever $\varepsilon_1 \leq \varepsilon_3$ and $\varepsilon_2 \leq \varepsilon_4$, for each $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_4 \in [0,1]$.

Definition 2.3. [23] A 6-tuple $(\sum, \Xi, \Theta, \Upsilon, *, \Diamond)$ is said to be a J-Neutrosophic Metric Space (shortly JNMS) whence $\sum$ is an arbitrary non empty set, $*$ is a CTN, $\Diamond$ is a CTC and Ξ, Θ and Υ are fuzzy sets on $\sum^3 \times \mathbb{R}^+$ satisfying the following conditions:
For all $\zeta, \eta, \delta, \omega \in \sum$, $\lambda \in \mathbb{R}^+$.

(1) $\Xi(\zeta, \eta, \delta, \lambda) + \Theta(\zeta, \eta, \delta, \lambda) + \Upsilon(\zeta, \eta, \delta, \lambda) \leq 3$;
(2) $\Xi(\zeta, \eta, \delta, \lambda) > 0$;
(3) $\Xi(\zeta, \eta, \delta, \lambda)$ is symmetric in ζ, η and δ;
(4) $\Xi(\zeta, \zeta, \eta, \lambda) \geq \Xi(\zeta, \eta, \delta, \lambda)$ if $\eta \neq \delta$;
(5) $\Xi(\zeta, \eta, \delta, \lambda) = 1$ if and only if $\zeta = \eta = \delta$;
(6) $\Xi(\zeta, \eta, \omega, \lambda) * \Xi(\omega, \delta, \delta, \mu) \leq \Xi(\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
(7) $\Xi(\zeta, \eta, \delta, \lambda)$ is continuous with respect to λ;
(8) Ξ is nondecreasing on $[0, +\infty]$, $\lim_{\lambda \to \infty} \Xi(\zeta, \eta, \delta, \lambda) = 1$ and $\lim_{\lambda \to 0} \Xi(\zeta, \eta, \delta, \lambda) = 0$;

(9) $\Theta(\zeta, \eta, \delta, \lambda) < 1$;
(10) $\Theta(\zeta, \eta, \delta, \lambda)$ is symmetric in ζ, η and δ;
(11) $\Theta(\zeta, \zeta, \eta, \lambda) \leq \Theta(\zeta, \eta, \delta, \lambda)if\eta \neq \delta$;
(12) $\Theta(\zeta, \eta, \delta, \lambda) = 0$ if and only if $\zeta = \eta = \delta$;
(13) $\Theta(\zeta, \eta, \omega, \lambda) \Diamond \Theta(\omega, \delta, \delta, \mu) \geq \Theta(\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
(14) $\Theta(\zeta, \eta, \delta, \lambda)$ is continuous with respect to λ;
(15) Θ is nonincreasing on $[0, +\infty]$, $\lim_{\lambda \to \infty} \Theta(\zeta, \eta, \delta, \lambda) = 0$ and $\lim_{\lambda \to 0} \Theta(\zeta, \eta, \delta, \lambda) = 1$;
(16) $\Upsilon(\zeta, \eta, \delta, \lambda) < 1$;
(17) $\Upsilon(\zeta, \eta, \delta, \lambda)$ is symmetric in ζ, η and δ;
(18) $\Upsilon(\zeta, \zeta, \eta, \lambda) \leq \Upsilon(\zeta, \eta, \delta, \lambda)$ if $\eta \neq \delta$;
(19) $\Upsilon(\zeta, \eta, \delta, \lambda) = 0$ if and only if $\zeta = \eta = \delta$;
(20) $\Upsilon(\zeta, \eta, \omega, \lambda) \Diamond \Upsilon(\omega, \delta, \delta, \mu) \geq \Upsilon(\zeta, \eta, \delta, \lambda + \mu)$, for all $\lambda, \mu > 0$;
(21) $\Upsilon(\zeta, \eta, \delta, \lambda)$ is continuous with respect to λ;
(22) Υ is nonincreasing on $[0, +\infty]$, $\lim_{\lambda \to \infty} \Upsilon(\zeta, \eta, \delta, \lambda) = 0$ and $\lim_{\lambda \to 0} \Upsilon(\zeta, \eta, \delta, \lambda) = 1$;

Then, (Ξ, Θ, Υ) is called J-Neutrosophic Metric on $\sum$. The functions Ξ, Θ and Υ denote, respectively, degree of closeness, neutrality and noncloseness between ζ, η and δ with respect to λ respectively.

Definition 2.4. [3] Let $\sum$ be a nonempty set. An element $(\zeta, \eta) \in \sum \times \sum$ is called a coupled fixed point of the mapping $\mathring{A} : \sum \times \sum \to \sum$ if $\zeta = \mathring{A}(\zeta, \eta)$ and $\eta = \mathring{A}(\eta, \zeta)$.

Definition 2.5. [11] Let $\sum$ be a nonempty set. An element $(\zeta, \eta) \in \sum \times \sum$ is called a coupled coincidence point of $\mathring{A} : \sum \times \sum \to \sum$ and $\rho : \sum \times \sum \to \sum$ if $\rho\zeta = \mathring{A}(\zeta, \eta)$ and $\rho\eta = \mathring{A}(\eta, \zeta)$; a common coupled fixed point of $\mathring{A} : \sum \times \sum \to \sum$ and $\rho : \sum \times \sum \to \sum$ if $\zeta = \rho\zeta = \mathring{A}(\zeta, \eta)$ and $\eta = \rho\eta = \mathring{A}(\eta, \zeta)$.

Lemma 2.6. [23] *Let $(\sum, \Xi, \Theta, \Upsilon, *, \Diamond)$ be a JNMS. Then Ξ, Θ and Υ are continuous functions on $\sum^3 \times \zeta(0, \infty)$.*

Now onwards we assume the following condition:
$\lim_{\lambda \to \infty} \Xi(\zeta, \eta, \delta, \lambda) = 1$, $\lim_{\lambda \to \infty} \Theta(\zeta, \eta, \delta, \lambda) = 0$ *and* $\lim_{\lambda \to \infty} \Upsilon(\zeta, \eta, \delta, \lambda) = 0$ *for all* $\zeta, \eta, \delta \in \sum$.

Lemma 2.7. [23] *Let $(\sum, \Xi, \Theta, \Upsilon, *, \Diamond)$ be an NMS. If there exists $\kappa \in (0, 1)$ such that*

$$\begin{aligned}
\min\{\Xi(\zeta, \eta, \delta, k\lambda), \Xi(u, v, w, k\lambda)\} &\geq \min\{\Xi(\zeta, \eta, \delta, \lambda), \Xi(u, v, w, \lambda)\}, \\
\max\{\Theta(\zeta, \eta, \delta, k\lambda), \Theta(u, v, w, k\lambda)\} &\leq \max\{\Theta(\zeta, \eta, \delta, \lambda), \Theta(u, v, w, \lambda)\} \textit{and} \\
\max\{\Upsilon(\zeta, \eta, \delta, k\lambda), \Upsilon(u, v, w, k\lambda)\} &\leq \max\{\Upsilon(\zeta, \eta, \delta, \lambda), \Upsilon(u, v, w, \lambda)\},
\end{aligned}$$

for all $\zeta, \eta, \delta, u, v, w \in \sum$ and $\lambda > 0$, then $\zeta = \eta = \delta$ and $u = v = w$.

Definition 2.8. [9] Let $\sum$ be a nonempty set. The mappings $\mathring{A} : \sum \times \sum \to \sum$ and $\rho : \sum \to \sum$ are called weakly compatible, if $\rho(\mathring{A}(\zeta)) = \mathring{A}(\rho(\zeta))$ whenever $\rho\zeta = \mathring{A}\zeta$ for $\zeta \in \sum$.

Definition 2.9. [1] Let $\sum$ be a nonempty set. The mappings $\mathring{A} : \sum \times \sum \to \sum$ and $\rho : \sum \to \sum$ are called w-compatible, if $\rho(\mathring{A}(\zeta,\eta)) = \mathring{A}(\rho\zeta, \rho\eta)$ whenever $\rho\zeta = \mathring{A}(\zeta,\eta)$ and $\rho\eta = \mathring{A}(\eta,\zeta)$ for all $\zeta, \eta \in \sum$.

3. Common Fixed Point Theorems for Six Weakly Compatible Mappings

To start with the folowing result, we need to bring here the following two classes of functions, namely, Φ and Ψ, which are essential to construct the contractive conditions.

$$\Phi = \{\phi : [0,1] \to [0,1] : \phi(t) > t,\ \forall\, t \in [0,1),\ \phi(1) = 1\};$$
$$\Psi = \{\psi : [0,1] \to [0,1] : \psi(t) < t,\ \forall\, t \in [0,1),\ \psi(0) = 0\}.$$

Theorem 3.1. *Let* $(\sum, \Xi, \Theta, \Upsilon, *, \Diamond)$ *be an NMS and* $\Im, \daleth, \Re, \mathscr{F}, \rho, \tau : \sum \to \sum$ *be satisfying*

(i)

$$\Im\left(\textstyle\sum\right) \subseteq \rho\left(\textstyle\sum\right), \daleth\left(\textstyle\sum\right) \subseteq \tau\left(\textstyle\sum\right) \quad \textit{and} \quad \Re\left(\textstyle\sum\right) \subseteq \mathscr{F}\left(\textstyle\sum\right), \tag{3.1}$$

(ii)

$$\textit{One of } \mathscr{F}\left(\textstyle\sum\right), \rho\left(\textstyle\sum\right) \textit{ and } \tau\left(\textstyle\sum\right) \textit{ is a complete subspace of } \textstyle\sum, \tag{3.2}$$

(iii)

$$\textit{The pairs } (\Im, \mathscr{F}), (\daleth, \rho) \textit{ and } (\Re, \tau) \textit{ are weakly compatible, and} \tag{3.3}$$

(iv)

$$\Xi(\Im\zeta, \daleth\eta, \Re\delta, \lambda) \geq \phi\left(\min\left\{\begin{array}{c} \Xi(\mathscr{F}\zeta, \rho\eta, \tau\delta, \lambda), \\ \frac{1}{3}\left[\begin{array}{c}\Xi(\mathscr{F}\zeta, \Im\zeta, \daleth\eta, \lambda)+ \\ \Xi(\rho\eta, \daleth\eta, \Re\delta, \lambda) + \Xi(\tau\delta, \Re\delta, \Im\zeta, \lambda)\end{array}\right], \\ \frac{1}{4}\left[\begin{array}{c}\Xi(\mathscr{F}\zeta, \daleth\eta, \tau\delta, \lambda)+ \\ \Xi(\Im\zeta, \rho\eta, \tau\delta, \lambda) + \Xi(\mathscr{F}\zeta, \rho\eta, \Re\delta, \lambda)\end{array}\right]\end{array}\right\}\right) \tag{3.4}$$

$$\Theta(\Im\zeta, \daleth\eta, \Re\delta, \lambda) \leq \psi\left(\max\left\{\begin{array}{c} \Theta(\mathscr{F}\zeta, \rho\eta, \tau\delta, \lambda), \\ \frac{1}{3}\left[\begin{array}{c}\Theta(\mathscr{F}\zeta, \Im\zeta, \daleth\eta, \lambda)+ \\ \Theta(\rho\eta, \daleth\eta, \Re\delta, \lambda) + \Theta(\tau\delta, \Re\delta, \Im\zeta, \lambda)\end{array}\right], \\ \frac{1}{4}\left[\begin{array}{c}\Theta(\mathscr{F}\zeta, \daleth\eta, \tau\delta, \lambda)+ \\ \Theta(\Im\zeta, \rho\eta, \tau\delta, \lambda) + \Theta(\mathscr{F}\zeta, \rho\eta, \Re\delta, \lambda)\end{array}\right]\end{array}\right\}\right) \tag{3.5}$$

$$\Upsilon(\Im\zeta, \urcorner\eta, \Re\delta, \lambda) \leq \kappa\left(\max\left\{\begin{array}{c} \Upsilon(\mathscr{F}\zeta, \rho\eta, \tau\delta, \lambda), \\ \frac{1}{3}\left[\begin{array}{c} \Upsilon(\mathscr{F}\zeta, \Im\zeta, \urcorner\eta, \lambda)+ \\ \Upsilon(\rho\eta, \urcorner\eta, \Re\delta, \lambda) + \Upsilon(\tau\delta, \Re\delta, \Im\zeta, \lambda) \end{array}\right], \\ \frac{1}{4}\left[\begin{array}{c} \Upsilon(\mathscr{F}\zeta, \urcorner\eta, \tau\delta, \lambda)+ \\ \Upsilon(\Im\zeta, \rho\eta, \tau\delta, \lambda) + \Upsilon(\mathscr{F}\zeta, \rho\eta, \Re\delta, \lambda) \end{array}\right] \end{array}\right\}\right) \tag{3.6}$$

for all $\zeta, \eta, \delta \in \sum$, *where* $\phi \in \Phi$, $\psi, \kappa \in \Psi$. *Then, either one of the pairs* $(\Im, \mathscr{F}), (\urcorner, \rho)$, *and* (R, τ) *has a coincidence point or the maps* $\Im, \urcorner, \Re, \mathscr{F}, \rho$ *and* τ *have a unique common fixed point in* $\sum$.

Proof:
Choose $\zeta_0 \in \sum$. By (3.1), there exist $\zeta_1, \zeta_2, \zeta_3, \in \sum$ such that
$\Im\zeta_0 = \rho\zeta_1 = \eta_0$, $\urcorner\zeta_1 = \tau\zeta_2 = \eta_1$ and $\Re\zeta_2 = \mathscr{F}\zeta_3 = \eta_2$.
Inductively, there exist sequences $\{\zeta_n\}$ and $\{\eta_n\}$ in $\sum$ such that
$\eta_{3n} = \Im\zeta_{3n} = \rho\zeta_{3n+1}, \eta_{3n+1} = \urcorner\zeta_{3n+1} = \tau\zeta_{3n+2}$ and $\eta_{3n+2} = \Re\zeta_{3n+2} = \mathscr{F}\zeta_{3n+3}$, where $n = 0, 1, 2 \ldots$. If $\eta_{3n} = \eta_{3n+1}$, then ζ_{3n+1} is a coincidence point of ρ and $\urcorner$.
If $\eta_{3n+1} = \eta_{3n+2}$, then ζ_{3n+2} is a coincidence point of τ and $\Re$.
If $\eta_{3n+2} = \eta_{3n+3}$, then ζ_{3n+3} is a coincidence point of $\mathscr{F}$ and $\Im$.
Now, assume that $\eta_n \neq \eta_{n+1}$ for all n. Denote $d_n = \Xi(\eta_n, \eta_{n+1}, \eta_{n+2}, \lambda)$.
Putting $\zeta = \zeta_{3n}, \eta = \zeta_{3n+1}, \delta = \zeta_{3n+2}$ in (3.4), we get

$$\begin{aligned} d_{3n} &= \Xi(\eta_{3n}, \eta_{3n+1}, \eta_{3n+2}, \lambda) = \Xi(\Im\zeta_{3n}, \urcorner\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda) \\ &\geq \phi\left(\min\left\{\begin{array}{c} \Xi(\mathscr{F}\zeta_{3n}, \rho\zeta_{3n+1}, \tau\zeta_{3n+2}, \lambda), \\ \frac{1}{3}\left[\begin{array}{c} \Xi(\mathscr{F}\zeta_{3n}, \Im\zeta_{3n}, \urcorner\zeta_{3n+1}, \lambda)+ \\ \Xi(\rho\zeta_{3n+1}, \urcorner\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda) + \Xi(\tau\zeta_{3n+2}, \Re\zeta_{3n+2}, \Im\zeta_{3n}, \lambda) \end{array}\right] \\ \frac{1}{4}\left[\begin{array}{c} \Xi(\mathscr{F}\zeta_{3n}, \urcorner\zeta_{3n+1}, \tau\zeta_{3n+2}, \lambda)+ \\ \Xi(\Im\zeta_{3n}, \rho\zeta_{3n+1}, \tau\zeta_{3n+2}, \lambda) + \Xi(\mathscr{F}\zeta_{3n}, \rho\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda) \end{array}\right] \end{array}\right\}\right) \\ &\geq \phi\left(\min\left\{\begin{array}{c} \Xi(\eta_{3n-1}, \eta_{3n}, \eta_{3n+1}, \lambda), \\ \frac{1}{3}\left[\begin{array}{c} \Xi(\eta_{3n-1}, \eta_{3n}, \eta_{3n+1}, \lambda)+ \\ \Xi(\eta_{3n}, \eta_{3n+1}, \eta_{3n+2}, \lambda) + \Xi(\eta_{3n+1}, \eta_{3n+2}, \eta_{3n}, \lambda) \end{array}\right], \\ \frac{1}{4}\left[\begin{array}{c} \Xi(\eta_{3n-1}, \eta_{3n+1}, \eta_{3n+1}, \lambda)+ \\ \Xi(\eta_{3n}, \eta_{3n}, \eta_{3n+1}, \lambda) + \Xi(\eta_{3n-1}, \eta_{3n}, \eta_{3n+2}, \lambda) \end{array}\right] \end{array}\right\}\right) \\ &\geq \phi\left(\min\left\{\begin{array}{c} d_{3n-1}, \frac{1}{3}\left[d_{3n-1} + d_{3n} + d_{3n}\right], \\ \frac{1}{4}\left[d_{3n-1} + d_{3n} + (d_{3n-1} + d_{3n})\right] \end{array}\right\}\right) \end{aligned} \tag{3.7}$$

If $d_{3n} \leq d_{3n-1}$, then from (2.6), we have $d_{3n} \geq \phi(d_{3n}) > d_{3n}$.
It is a contradiction. Hence, $d_{3n} \geq d_{3n-1}$. Now, from (2.6), $d_{3n} \geq \phi(d_{3n-1})$.
Denote $\omega_n = \Theta(\eta_n, \eta_{n+1}, \eta_{n+2}, \lambda)$. Putting $\zeta = \zeta_{3n}, \eta = \zeta_{3n+1}, \delta = \zeta_{3n+2}$ in (3.5), we get

$$\begin{aligned} \omega_{3n} &= \Theta(\eta_{3n}, \eta_{3n+1}, \eta_{3n+2}, \lambda) = \Theta(\Im\zeta_{3n}, \urcorner\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda) \\ &\leq \psi\left(\max\left\{\begin{array}{c} \Theta(\mathscr{F}\zeta_{3n}, \omega\zeta_{3n+1}, \tau\zeta_{3n+2}, \lambda), \\ \frac{1}{3}\left[\begin{array}{c} \Theta(\mathscr{F}\zeta_n, \Im\zeta_{3n}, \urcorner\zeta_{3n+1}, \lambda)+ \\ \Theta(\omega\zeta_{3n+1}, \urcorner\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda) + \Theta(\tau\zeta_{3n+2}, \Re\zeta_{3n+2}, \Im\zeta_{3n}, \lambda) \end{array}\right] \\ \frac{1}{4}\left[\begin{array}{c} \Theta(\mathscr{F}\zeta_{3n}, \urcorner\zeta_{3n+1}, \tau\zeta_{3n+2}, \lambda)+ \\ \Theta(\Im\zeta_{3n}, \omega\zeta_{3n+1}, \tau\zeta_{3n+2}, \lambda) + \Theta(\mathscr{F}\zeta_{3n}, \omega\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda) \end{array}\right] \end{array}\right\}\right) \end{aligned}$$

$$\leq \psi\left(\max\left\{\begin{array}{c}\Theta(\eta_{3n-1},\eta_{3n},\eta_{3n+1},\lambda),\\ \frac{1}{3}\left[\begin{array}{c}\Theta(\eta_{3n-1},\eta_{3n},\eta_{3n+1},\lambda)+\\ \Theta(\eta_{3n},\eta_{3n+1},\eta_{3n+2},\lambda)+\Theta(\eta_{3n+1},\eta_{3n+2},\eta_{3n},\lambda)\end{array}\right],\\ \frac{1}{4}\left[\begin{array}{c}\Theta(\eta_{3n-1},\eta_{3n+1},\eta_{3n+1},\lambda)+\\ \Theta(\eta_{3n},\eta_{3n},\eta_{3n+1},\lambda)+\Theta(\eta_{3n-1},\eta_{3n},\eta_{3n+2},\lambda)\end{array}\right]\end{array}\right\}\right)$$

$$\leq \psi\left(\max\left\{\begin{array}{c}\omega_{3n-1},\frac{1}{3}\left[\omega_{3n-1}+\omega_{3n}+\omega_{3n}\right],\\ \frac{1}{4}\left[\omega_{3n-1}+\omega_{3n}+(\omega_{3n-1}+\omega_{3n})\right]\end{array}\right\}\right) \tag{3.8}$$

If $\omega_{3n} \geq \omega_{3n-1}$ then from (2.7), we have $d_{3n} \leq \psi(d_{3n}) < \omega_{3n}$.
It is a contradiction. Hence, $\omega_{3n} \leq \omega_{3n} - 1$. Now from (2.7), $\omega_{3n} \leq \psi(\omega_{3n-1})$.
Denote $\sigma_n = \Upsilon(\eta_n, \eta_{n+1}, \eta_{n+2}, \lambda)$. Putting $\zeta = \zeta_{3n}, \eta = \zeta_{3n+1}, \delta = \zeta_{3n+2}$ in (3.6), we get

$$\sigma_{3n} = \Upsilon(\eta_{3n},\eta_{3n+1},\eta_{3n+2},\lambda) = \Upsilon(\Im\zeta_{3n},\daleth\zeta_{3n+1},\Re\zeta_{3n+2},\lambda) \tag{3.9}$$

$$\leq \kappa\left(\max\left\{\begin{array}{c}\Upsilon(\mathscr{F}\zeta_{3n},\omega\zeta_{3n+1},\tau\zeta_{3n+2},\lambda),\\ \frac{1}{3}\left[\begin{array}{c}\Upsilon(\mathscr{F}\zeta_{n},\Im\zeta_{3n},\daleth\zeta_{3n+1},\lambda)+\\ \Upsilon(\omega\zeta_{3n+1},\daleth\zeta_{3n+1},\Re\zeta_{3n+2},\lambda)+\Upsilon(\tau\zeta_{3n+2},\Re\zeta_{3n+2},\Im\zeta_{3n},\lambda)\end{array}\right]\\ \frac{1}{4}\left[\begin{array}{c}\Upsilon(\mathscr{F}\zeta_{3n},\daleth\zeta_{3n+1},\tau\zeta_{3n+2},\lambda)+\\ \Upsilon(\Im\zeta_{3n},\omega\zeta_{3n+1},\tau\zeta_{3n+2},\lambda)+\Upsilon(\mathscr{F}\zeta_{3n},\omega\zeta_{3n+1},\Re\zeta_{3n+2},\lambda)\end{array}\right]\end{array}\right\}\right)$$

$$\leq \kappa\left(\max\left\{\begin{array}{c}\Upsilon(\eta_{3n-1},\eta_{3n},\eta_{3n+1},\lambda),\\ \frac{1}{3}\left[\begin{array}{c}\Upsilon(\eta_{3n-1},\eta_{3n},\eta_{3n+1},\lambda)+\\ \Upsilon(\eta_{3n},\eta_{3n+1},\eta_{3n+2},\lambda)+\Upsilon(\eta_{3n+1},\eta_{3n+2},\eta_{3n},\lambda)\end{array}\right],\\ \frac{1}{4}\left[\begin{array}{c}\Upsilon(\eta_{3n-1},\eta_{3n+1},\eta_{3n+1},\lambda)+\\ \Upsilon(\eta_{3n},\eta_{3n},\eta_{3n+1},\lambda)+\Upsilon(\eta_{3n-1},\eta_{3n},\eta_{3n+2},\lambda)\end{array}\right]\end{array}\right\}\right)$$

$$\leq \kappa\left(\max\left\{\begin{array}{c}\sigma_{3n-1},\frac{1}{3}\left[\ \sigma_{3n-1}+\sigma_{3n}+\sigma_{3n}\ \right],\\ \frac{1}{4}\left[\ \sigma_{3n-1}+\sigma_{3n}+(\sigma_{3n-1}+\sigma_{3n})\ \right]\end{array}\right\}\right) \tag{3.10}$$

If $\sigma_{3n} \geq \sigma_{3n-1}$, then from (2.7), we have $\sigma_{3n} \leq \kappa(\sigma_{3n}) < \sigma_{3n}$.
It is a contradiction. Hence, $\sigma_{3n} \leq \sigma_{3n-1}$. Now, from (2.7), $\sigma_{3n} \leq \kappa(\sigma_{3n-1})$.
Similarly, by putting $\zeta = \zeta_{3n+3}, \eta = \zeta_{3n+1}, \delta = \zeta_{3n+2}$ and
$\zeta = \zeta_{3n+3}, \eta = \zeta_{3n+4}, \delta = \zeta_{3n+2}$ in (3.4) to (3.6) we get

$$d_{3n+1} \geq \phi(d_{3n}), d_{3n+2} \geq \phi(d_{3n+1}) \tag{3.11}$$

$$\omega_{3n+1} \leq \psi(\omega_{3n}), \omega_{3n+2} \leq \psi(\omega_{3n+1}) \text{ and} \tag{3.12}$$

$$\sigma_{3n+1} \leq \kappa(\sigma_{3n}), \sigma_{3n+2} \leq \kappa(\sigma_{3n+1}). \tag{3.13}$$

Thus, from (3.7) to (3.13), we have

$$\begin{aligned}\Xi(\eta_n,\eta_{n+1},\eta_{n+2},\lambda) &\geq \phi(\Xi(\eta_{n-1},\eta_n,\eta_{n+1},\lambda)) \geq \phi^2(\Xi(\eta_{n-2},\eta_{n-1},\eta_n,\lambda))\\ &\geq \cdots \geq \phi^n(\Xi(\eta_0,\eta_1,\eta_2,\lambda)),\\ \Xi(\eta_n,\eta_n,\eta_{n+1},\lambda) &\geq \Xi(\eta_n,\eta_{n+1},\eta_{n+2},\lambda)\\ &\geq \phi^n(\Xi(\eta_0,\eta_1,\eta_2,\lambda)),\\ \Theta(\eta_n,\eta_{n+1},\eta_{n+2},\lambda) &\leq \psi(\Theta(\eta_{n-1},\eta_n,\eta_{n+1},\lambda)) \leq \psi^2(\Theta(\eta_{n-2},\eta_{n-1},\eta_n,\lambda))\\ &\leq \cdots \leq \psi^n(\Theta(\eta_0,\eta_1,\eta_2,\lambda)),\\ \Theta(\eta_n,\eta_n,\eta_{n+1},\lambda) &\leq \Theta(\eta_n,\eta_{n+1},\eta_{n+2},\lambda)\end{aligned}$$

$$
\begin{aligned}
&\leq \psi^n(\Theta(\eta_0,\eta_1,\eta_2,\lambda)),\\
\Upsilon(\eta_n,\eta_{n+1},\eta_{n+2},\lambda) &\leq \kappa(\Upsilon(\eta_{n-1},\eta_n,\eta_{n+1},\lambda)) \leq \kappa^2(\Upsilon(\eta_{n-2},\eta_{n-1},\eta_n,\lambda))\\
&\leq \cdots \leq \kappa^n(\Upsilon(\eta_0,\eta_1,\eta_2,\lambda)),\\
\Upsilon(\eta_n,\eta_n,\eta_{n+1},\lambda) &\leq \Upsilon(\eta_n,\eta_{n+1},\eta_{n+2},\lambda) \leq \kappa^n(\Upsilon(\eta_0,\eta_1,\eta_2,\lambda)).
\end{aligned}
$$

Now, for $m > n$ from (2.3) and (3.11) to (3.13),

$$
\begin{aligned}
\Xi(\eta_n,\eta_n,\eta_m,\lambda) &\geq \Xi(\eta_n,\eta_n,\eta_{n+1},\lambda) + \Xi(\eta_{n+1},\eta_{n+1},\eta_{n+2},\lambda) + \cdots + \Xi(\eta_{m-1},\eta_{m-1},\eta_m,\lambda)\\
&\geq \phi^n(\Xi(\eta_0,\eta_1,\eta_2,\lambda)) + \phi^{n+1}(\Xi(\eta_0,\eta_1,\eta_2,\lambda)) + \cdots + \phi^{m-1}(\Xi(\eta_0,\eta_1,\eta_2,\lambda))\\
&\to 1 \text{ as } n \to \infty,\\
\Theta(\eta_n,\eta_n,\eta_m,\lambda) &\leq \Theta(\eta_n,\eta_n,\eta_{n+1},\lambda) + \Theta(\eta_{n+1},\eta_{n+1},\eta_{n+2},\lambda) + \cdots + \Theta(\eta_{m-1},\eta_{m-1},\eta_m,\lambda)\\
&\leq \psi^n(\Theta(\eta_0,\eta_1,\eta_2,\lambda)) + \psi^{n+1}(\Theta(\eta_0,\eta_1,\eta_2,\lambda)) + \cdots + \psi^{m-1}(\Theta(\eta_0,\eta_1,\eta_2,\lambda))\\
&\to 0 \text{ as } n \to \infty,\\
\Upsilon(\eta_n,\eta_n,\eta_m,\lambda) &\leq \Upsilon(\eta_n,\eta_n,\eta_{n+1},\lambda) + \Upsilon(\eta_{n+1},\eta_{n+1},\eta_{n+2},\lambda) + \cdots + \Upsilon(\eta_{m-1},\eta_{m-1},\eta_m,\lambda)\\
&\leq \kappa^n(\Upsilon(\eta_0,\eta_1,\eta_2,\lambda)) + \kappa^{n+1}(\Upsilon(\eta_0,\eta_1,\eta_2,\lambda)) + \cdots + \kappa^{m-1}(\Upsilon(\eta_0,\eta_1,\eta_2,\lambda))\\
&\to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Since $\phi^n(\lambda) \to 1, \psi^n(\lambda) \to 0$ and $\kappa^n(\lambda) \to 0$ as $n \to \infty$, for all $\lambda > 0$.
Hence, $\{\eta_n\}$ is Ξ- Cauchy. Suppose $\mathscr{F}(\sum)$ is Ξ- complete.
Then, there exists $\wp, \hbar \in \sum$ such that $\eta_{3n+2} \to \wp = \mathscr{F}\hbar$.
Since $\{\eta_n\}$ is Ξ- Cauchy, it follows that $\eta_{3n} \to \wp$ and $\eta_{3n+1} \to \wp$ as $n \to \infty$.

$$
\Xi(\Im\hbar, \daleth\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda) \geq \phi\left(\min\left\{\begin{array}{c}
\Xi(\mathscr{F}\hbar, \omega\zeta_{3n}, \tau\zeta_{3n+2}, \lambda),\\
\frac{1}{3}\left[\begin{array}{c}
\Xi(\mathscr{F}\hbar, \Im\hbar, \daleth\zeta_{3n+1}, \lambda)+\\
\Xi(\omega\zeta_{3n+1}, \daleth\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda)+\\
\Xi(\tau\zeta_{3n+2}, \Re\zeta_{3n+2}, \Im\hbar, \lambda)
\end{array}\right],\\
\frac{1}{4}\left[\begin{array}{c}
\Xi(\mathscr{F}\hbar, \daleth\zeta_{3n+1}, \tau\zeta_{3n+2}, \lambda)+\\
\Xi(\Im\hbar, \omega\zeta_{3n+1}, \tau\zeta_{3n+2}, \lambda)+\\
\Xi(\mathscr{F}\hbar, \omega\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda)
\end{array}\right]
\end{array}\right\}\right),
$$

$$
\Theta(\Im\hbar, \daleth\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda) \leq \psi\left(\max\left\{\begin{array}{c}
\Theta(\mathscr{F}\hbar, \omega\zeta_{3n}, \tau\zeta_{3n+2}, \lambda),\\
\frac{1}{3}\left[\begin{array}{c}
\Theta(\mathscr{F}\hbar, \Im\hbar, \daleth\zeta_{3n+1}, \lambda)+\\
\Theta(\omega\zeta_{3n+1}, \daleth\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda)+\\
\Theta(\tau\zeta_{3n+2}, \Re\zeta_{3n+2}, \Im\hbar, \lambda)
\end{array}\right],\\
\frac{1}{4}\left[\begin{array}{c}
\Theta(\mathscr{F}\hbar, \daleth\zeta_{3n+1}, \tau\zeta_{3n+2}, \lambda)+\\
\Theta(\Im\hbar, \omega\zeta_{3n+1}, \tau\zeta_{3n+2}, \lambda)+\\
\Theta(\mathscr{F}\hbar, \omega\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda)
\end{array}\right]
\end{array}\right\}\right) \quad \text{and}
$$

$$
\Upsilon(\Im\hbar, \daleth\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda) \leq \kappa\left(\max\left\{\begin{array}{c}
\Upsilon(\mathscr{F}\hbar, \omega\zeta_{3n}, \tau\zeta_{3n+2}, \lambda),\\
\frac{1}{3}\left[\begin{array}{c}
\Upsilon(\mathscr{F}\hbar, \Im\hbar, \daleth\zeta_{3n+1}, \lambda)+\\
\Upsilon(\omega\zeta_{3n+1}, \daleth\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda)+\\
\Upsilon(\tau\zeta_{3n+2}, \Re\zeta_{3n+2}, \Im\hbar, \lambda)
\end{array}\right],\\
\frac{1}{4}\left[\begin{array}{c}
\Upsilon(\mathscr{F}\hbar, \daleth\zeta_{3n+1}, \tau\zeta_{3n+2}, \lambda)+\\
\Upsilon(\Im\hbar, \omega\zeta_{3n+1}, \tau\zeta_{3n+2}, \lambda)+\\
\Upsilon(\mathscr{F}\hbar, \omega\zeta_{3n+1}, \Re\zeta_{3n+2}, \lambda)
\end{array}\right]
\end{array}\right\}\right).
$$

Letting $n \to \infty$, we get

$$\Xi(\Im\wp,\wp,\wp,\lambda) \geq \phi\left(\min\left\{\begin{array}{c} 1, \frac{1}{3}\left[\ \Xi(\wp,\Im\hbar,\wp,\lambda)+1+\Xi(\wp,\wp,\Im\hbar,\lambda)\ \right] \\ \frac{1}{4}\left[\ 1+\Xi(\Im\hbar,\wp,\wp,\lambda)+1\ \right] \end{array}\right\}\right)$$

$$\Xi(\Im\hbar,\wp,\wp,\lambda) \geq \phi\left(\Xi\Im\hbar,\wp,\wp,\lambda\right),$$

$$\Theta(\Im\wp,\wp,\wp,\lambda) \leq \psi\left(\max\left\{\begin{array}{c} 0, \frac{1}{3}\left[\ \Theta(\wp,\Im\hbar,\wp,\lambda)+0+\Theta(\wp,\wp,\Im\hbar,\lambda)\ \right] \\ \frac{1}{4}\left[\ 0+\Theta(\Im\hbar,\wp,\wp,\lambda)+0\ \right] \end{array}\right\}\right)$$

$$\Theta(\Im\hbar,\wp,\wp,\lambda) \leq \psi(\Theta(\Im\hbar,\wp,\wp,\lambda) \text{ and}$$

$$\Upsilon(\Im\wp,\wp,\wp,\lambda) \leq \kappa\left(\max\left\{\begin{array}{c} 0, \frac{1}{3}\left[\ \Upsilon(\wp,\Im\hbar,\wp,\lambda)+0+\Upsilon(\wp,\wp,\Im\hbar,\lambda)\ \right] \\ \frac{1}{4}\left[\ 0+\Upsilon(\Im\hbar,\wp,\wp,\lambda)+0\ \right] \end{array}\right\}\right)$$

$$\Upsilon(\Im\hbar,\wp,\wp,\lambda) \leq \kappa(\Upsilon(\Im\hbar,\wp,\wp,\lambda).$$

Since ϕ is nondecreasing and ψ, κ are nonincreasing, we have that $\Im\hbar = \wp$. Hence $\wp = \mathscr{F}\hbar = \Im\hbar$. Since the pair $(\Im, \mathscr{F})$ is weakly compatible, we have $\mathscr{F}\wp = \Im\wp$. Putting $\zeta = \wp, \eta = \zeta_{3n+1}, \delta = \zeta_{3n+2}$ in (3.4), (3.5) and (3.6) we get

$$\Xi(\Im\wp,\urcorner\zeta_{3n+1},\Re\zeta_{3n+2},\lambda) \geq \phi\left(\min\left\{\begin{array}{c} \Xi(\mathscr{F}\wp,\omega\zeta_{3n+1},\tau\zeta_{3n+2},\lambda), \\ \frac{1}{3}\left[\begin{array}{c} \Xi(\mathscr{F}\wp,\Im\wp,\urcorner\zeta_{3n+1},\lambda)+ \\ \Xi(\omega\zeta_{3n+1},\urcorner\zeta_{3n+1},\Re\zeta_{3n+2},\lambda)+ \\ \Xi(\tau\zeta_{3n+2},\Re\zeta_{3n+2},\Im\wp,\lambda) \end{array}\right], \\ \frac{1}{4}\left[\begin{array}{c} \Xi(\mathscr{F}\wp,\urcorner\zeta_{3n+1},\tau\zeta_{3n+2},\lambda)+ \\ \Xi(\Im\wp,\omega\zeta_{3n+1},\tau\zeta_{3n+2},\lambda)+ \\ \Xi(\mathscr{F}\wp,\omega\zeta_{3n+1},\Re\zeta_{3n+2},\lambda) \end{array}\right] \end{array}\right\}\right),$$

$$\Theta(\Im\wp,\urcorner\zeta_{3n+1},\Re\zeta_{3n+2},\lambda) \leq \psi\left(\max\left\{\begin{array}{c} \Theta(\mathscr{F}\wp,\omega\zeta_{3n+1},\tau\zeta_{3n+2},\lambda), \\ \frac{1}{3}\left[\begin{array}{c} \Theta(\mathscr{F}\wp,\Im\wp,\urcorner\zeta_{3n+1},\lambda)+ \\ \Theta(\omega\zeta_{3n+1},\urcorner\zeta_{3n+1},\Re\zeta_{3n+2},\lambda)+ \\ \Theta(\tau\zeta_{3n+2},\Re\zeta_{3n+2},\Im\wp,\lambda) \end{array}\right], \\ \frac{1}{4}\left[\begin{array}{c} \Theta(\mathscr{F}\wp,\urcorner\zeta_{3n+1},\tau\zeta_{3n+2},\lambda)+ \\ \Theta(\Im\wp,\omega\zeta_{3n+1},\tau\zeta_{3n+2},\lambda)+ \\ \Theta(\mathscr{F}\wp,\omega\zeta_{3n+1},\Re\zeta_{3n+2},\lambda) \end{array}\right] \end{array}\right\}\right) \quad \text{and}$$

$$\Upsilon(\Im\wp,\urcorner\zeta_{3n+1},\Re\zeta_{3n+2},\lambda) \leq \kappa\left(\max\left\{\begin{array}{c} \Upsilon(\mathscr{F}\wp,\omega\zeta_{3n+1},\tau\zeta_{3n+2},\lambda), \\ \frac{1}{3}\left[\begin{array}{c} \Upsilon(\mathscr{F}\wp,\Im\wp,\urcorner\zeta_{3n+1},\lambda)+ \\ \Upsilon(\omega\zeta_{3n+1},\urcorner\zeta_{3n+1},\Re\zeta_{3n+2},\lambda)+ \\ \Upsilon(\tau\zeta_{3n+2},\Re\zeta_{3n+2},\Im\wp,\lambda) \end{array}\right], \\ \frac{1}{4}\left[\begin{array}{c} \Upsilon(\mathscr{F}\wp,\urcorner\zeta_{3n+1},\tau\zeta_{3n+2},\lambda)+ \\ \Upsilon(\Im\wp,\omega\zeta_{3n+1},\tau\zeta_{3n+2},\lambda)+ \\ \Upsilon(\mathscr{F}\wp,\omega\zeta_{3n+1},\Re\zeta_{3n+2},\lambda) \end{array}\right] \end{array}\right\}\right).$$

Letting $n \to \infty$, we have

$$\Xi(\Im\wp,\wp,\wp,\lambda) \geq \phi\left(\min\left\{\begin{array}{c} \Xi(\Im\wp,\wp,\wp,\lambda), \\ \frac{1}{3}\left[\ \Xi(\Im\wp,\Im\wp,\wp,\lambda)+1+\Xi(\wp,\wp,\Im\wp,\lambda)\ \right], \\ \frac{1}{4}\left[\ \Xi(\Im\wp,\wp,\wp,\lambda)+\Xi(\Im\wp,\wp,\wp,\lambda)+\Xi(\Im\wp,\wp,\wp,\lambda)\ \right] \end{array}\right\}\right),$$

$$\Theta(\Im\wp,\wp,\wp,\lambda) \leq \left(\max\left\{\begin{array}{c} \Theta(\Im\wp,\wp,\wp,\lambda), \\ \frac{1}{3}\left[\ \Theta(\Im\wp,\Im\wp,\wp,\lambda)+0+\Theta(\wp,\wp,\Im\wp,\lambda)\ \right], \\ \frac{1}{4}\left[\ \Theta(\Im\wp,\wp,\wp,\lambda)+\Theta(\Im\wp,\wp,\wp,\lambda)+\Theta(\Im\wp,\wp,\wp,\lambda)\ \right] \end{array}\right\}\right),$$

$$\Upsilon(\Im\wp,\wp,\wp,\lambda)\leq\kappa\left(\max\left\{\begin{array}{c}\Upsilon(\Im\wp,\wp,\wp,\lambda),\\ \frac{1}{3}\left[\ \Upsilon(\Im\wp,\Im\wp,\wp,\lambda)+0+\Upsilon(\wp,\wp,\Im\wp,\lambda)\ \right],\\ \frac{1}{4}\left[\ \Upsilon(\Im\wp,\wp,\wp,\lambda)+\Upsilon(\Im\wp,\wp,\wp,\lambda)+\Upsilon(\Im\wp,\wp,\wp,\lambda)\ \right]\end{array}\right\}\right).$$

Since, $\Xi(\Im\wp,\Im\wp,\wp,\lambda)\geq 2\Xi(\Im\wp,\wp,\wp,\lambda),\Theta(\Im\wp,\Im\wp,\wp,\lambda)\leq 2\Theta(\Im\wp,\wp,\wp,\lambda)$ and $\Upsilon(\Im\wp,\Im\wp,\wp,\lambda)\leq 2\Upsilon(\Im\wp,\wp,\wp,\lambda)$.
$\Xi(\Im\wp,\wp,\wp,\lambda)\geq\phi(\Xi(\Im\wp,\wp,\wp,\lambda)),\Theta(\Im\wp,\wp,\wp,\lambda)\leq\psi(\Theta(\Im\wp,\wp,\wp,\lambda))$ and $\Upsilon(\Im\wp,\wp,\wp,\lambda)\leq\kappa(\Upsilon(\Im\wp,\wp,\wp,\lambda))$.
Thus $\Im\wp=\wp$. Hence $\mathscr{F}\wp=\Im\wp=\wp$. Putting $\zeta=\wp,\eta=v_{3n+1},\delta=\zeta_{3n+2}$ in (3.4), (3.5) and (3.6) we get

$$\Xi(\Im\wp,\daleth v,\Re\zeta_{3n+2},\lambda)\geq\phi\left(\min\left\{\begin{array}{c}\Xi(\mathscr{F}\wp,\omega v,\tau\zeta_{3n+2},\lambda),\\ \frac{1}{3}\left[\begin{array}{c}\Xi(\mathscr{F}\wp,\Im\wp,\daleth v,\lambda)+\Xi(\omega v,\daleth v,\Re\zeta_{3n+2},\lambda)+\\ \Xi(\tau\zeta_{3n+2},\Re\zeta_{3n+2},\Im\wp,\lambda)\end{array}\right],\\ \frac{1}{4}\left[\begin{array}{c}\Xi(\mathscr{F}\wp,\daleth v,\tau\zeta_{3n+2},\lambda)+\Xi(\Im\wp,\omega v,\tau\zeta_{3n+2},\lambda)+\\ \Xi(\mathscr{F}\wp,\omega v,\Re\zeta_{3n+2},\lambda)\end{array}\right]\end{array}\right\}\right),$$

$$\Theta(\Im\wp,\daleth v,\Re\zeta_{n+2},\lambda)\leq\psi\left(\max\left\{\begin{array}{c}\Theta(\mathscr{F}\wp,\omega v,\tau\zeta_{3n+2},\lambda),\\ \frac{1}{3}\left[\begin{array}{c}\Theta(\mathscr{F}\wp,\Im\wp,\daleth v,\lambda)+\Theta(\omega v,\daleth v,\Re\zeta_{3n+2},\lambda)+\\ \Theta(\tau\zeta_{3n+2},\Re\zeta_{3n+2},\Im\wp,\lambda)\end{array}\right],\\ \frac{1}{4}\left[\begin{array}{c}\Theta(\mathscr{F}\wp,\daleth v,\tau\zeta_{3n+2},\lambda)+\\ \Theta(\Im\wp,\omega v,\tau\zeta_{3n+2},\lambda)+\Theta(\mathscr{F}\wp,\omega v,\Re\zeta_{3n+2},\lambda)\end{array}\right]\end{array}\right\}\right),$$

$$\Upsilon(\Im\wp,\daleth v,\Re\zeta_{n+2},\lambda)\leq\kappa\left(\max\left\{\begin{array}{c}\Upsilon(\mathscr{F}\wp,\omega v,\tau\zeta_{3n+2},\lambda),\\ \frac{1}{3}\left[\begin{array}{c}\wp(\mathscr{F}\wp,\Im\wp,\daleth v,\lambda)+\\ \Upsilon(\omega v,\daleth v,\Re\zeta_{3n+2},\lambda)+\Upsilon(\tau\zeta_{3n+2},\Re\zeta_{3n+2},\Im\wp,\lambda)\end{array}\right],\\ \frac{1}{4}\left[\begin{array}{c}\Upsilon(\mathscr{F}\wp,\daleth v,\tau\zeta_{3n+2},\lambda)+\\ \Upsilon(\Im\wp,\omega v,\tau\zeta_{3n+2},\lambda)+\Upsilon(\mathscr{F}\wp,\omega v,\Re\zeta_{3n+2},\lambda)\end{array}\right]\end{array}\right\}\right).$$

Letting $n\to\infty$, we deduce that

$$\begin{aligned}\Xi(\wp,\daleth v,\wp,\lambda)&\geq\phi\left(\min\left\{\begin{array}{c}1,\frac{1}{3}\left[\ \Xi(\wp,\wp,\daleth v,\lambda)+\Xi(\wp,\daleth v,\wp,\lambda)+1\ \right],\\ \frac{1}{4}\left[\ \Xi(\wp,\daleth v,\wp,\lambda)+1+1\ \right]\end{array}\right\}\right)\\ &\geq\phi(\Xi(\wp,\daleth v,\wp,\lambda)),\\ \Theta(\wp,\daleth v,\wp,\lambda)&\leq\psi\left(\max\left\{\begin{array}{c}0,\frac{1}{3}\left[\ \Theta(\wp,\wp,\daleth v,\lambda)+\Theta(\wp,\daleth v,\wp,\lambda)+0\ \right],\\ \frac{1}{4}\left[\ \Theta(\wp,\daleth v,\wp,\lambda)+0+0\ \right]\end{array}\right\}\right)\\ &\leq\psi(\Theta(\wp,\daleth v,\wp,\lambda)),\\ \Upsilon(\wp,\daleth v,\wp,\lambda)&\leq\kappa\left(\max\left\{\begin{array}{c}0,\frac{1}{3}\left[\ \Upsilon(\wp,\wp,\daleth v,\lambda)+\Upsilon(\wp,\daleth v,\wp,\lambda)+0\ \right],\\ \frac{1}{4}\left[\ \Upsilon(\wp,\daleth v,\wp,\lambda)+0+0\ \right]\end{array}\right\}\right)\\ &\leq\kappa(\Upsilon(\wp,\daleth v,\wp,\lambda)).\end{aligned}$$

Since ϕ is nondecreasing and ψ, κ are nonincreasing, we have that $\daleth v=\wp$ and so that $\wp=\daleth v=\omega v$.

As the pair $(\daleth, \omega)$ is weakly compatible, we have $\daleth\wp = \omega\wp$ and

$$\Xi(\Im\wp, \daleth\wp, \Re\zeta_{3n+2}, \lambda) \geq \phi\left(\min\left\{\begin{array}{c}\Xi(\mathscr{F}\wp, \omega\wp, \tau\zeta_{3n+2}, \lambda), \\ \frac{1}{3}\left[\begin{array}{c}\Xi(\mathscr{F}\wp, \Im\wp, \daleth\wp, \lambda)+ \\ \Xi(\omega\wp, \daleth\wp, \Re\zeta_{3n+2}, \lambda) + \Xi(\tau\zeta_{3n+2}, \Re\zeta_{3n+2}, \Im\wp, \lambda)\end{array}\right], \\ \frac{1}{4}\left[\begin{array}{c}\Xi(\mathscr{F}\wp, \daleth\wp, \tau\zeta_{3n+2}, \lambda)+ \\ \Xi(\Im\wp, \omega\wp, \tau\zeta_{3n+2}, \lambda) + \Xi(\mathscr{F}\wp, \omega\wp, \Re\zeta_{3n+2}, \lambda)\end{array}\right]\end{array}\right\}\right),$$

$$\Theta(\Im\wp, \daleth\wp, \Re\zeta_{3n+2}, \lambda) \leq \left(\max\left\{\begin{array}{c}\Theta(\mathscr{F}\wp, \omega\wp, \tau\zeta_{3n+2}, \lambda), \\ \frac{1}{3}\left[\begin{array}{c}\Theta(\mathscr{F}\wp, \Im\wp, \daleth\wp, \lambda)+ \\ \Theta(\omega\wp, \daleth\wp, \Re\zeta_{3n+2}, \lambda) + \Theta(\tau\zeta_{3n+2}, \Re\zeta_{3n+2}, \Im\wp, \lambda)\end{array}\right], \\ \frac{1}{4}\left[\begin{array}{c}\Theta(\mathscr{F}\wp, \daleth\wp, \tau\zeta_{3n+2}, \lambda)+ \\ \Theta(\Im\wp, \omega\wp, \tau\zeta_{3n+2}, \lambda) + \Theta(\mathscr{F}\wp, \omega\wp, \Re\zeta_{3n+2}, \lambda)\end{array}\right]\end{array}\right\}\right),$$

$$\Upsilon(\Im\wp, \daleth\wp, \Re\zeta_{3n+2}, \lambda) \leq \kappa\left(\max\left\{\begin{array}{c}\Upsilon(\mathscr{F}\wp, \omega\wp, \tau\zeta_{3n+2}, \lambda), \\ \frac{1}{3}\left[\begin{array}{c}\Upsilon(\mathscr{F}\wp, \Im\wp, T\wp, \lambda)+ \\ \Upsilon(\omega\wp, \daleth\wp, \Re\zeta_{3n+2}, \lambda) + \Upsilon(\tau\zeta_{3n+2}, \Re\zeta_{3n+2}, \Im\wp, \lambda)\end{array}\right], \\ \frac{1}{4}\left[\begin{array}{c}\Upsilon(\mathscr{F}\wp, \daleth\wp, \tau\zeta_{3n+2}, \lambda)+ \\ \Upsilon(\Im\wp, \omega\wp, \tau\zeta_{3n+2}, \lambda) + \Upsilon(\mathscr{F}\wp, \omega\wp, \Re\zeta_{3n+2}, \lambda)\end{array}\right]\end{array}\right\}\right).$$

Letting $n \to \infty$, we have

$$\Xi(\wp, \daleth\wp, \wp, \lambda) \geq \phi\left(\min\left\{\begin{array}{c}\Xi(\wp, \daleth\wp, \wp, \lambda), \\ \frac{1}{3}\left[\ \Xi(\wp, \wp, \daleth\wp, \lambda) + \Xi(\daleth\wp, \daleth\wp, \wp, \lambda) + 1\ \right], \\ \frac{1}{4}\left[\ \Xi(\wp, \daleth\wp, \wp, \lambda) + \Xi(\wp, \daleth\wp, \wp, \lambda) + \Xi(\wp, \daleth\wp, \wp, \lambda)\ \right]\end{array}\right\}\right),$$

$$\Theta(\wp, \daleth\wp, \wp, \lambda) \leq \psi\left(\max\left\{\begin{array}{c}\Theta(\wp, \daleth\wp, \wp, \lambda), \\ \frac{1}{3}\left[\ \Theta(\wp, \wp, \daleth\wp, \lambda) + \Theta(\daleth\wp, \daleth\wp, \wp, \lambda) + 0\ \right], \\ \frac{1}{4}\left[\ \Theta(\wp, \daleth\wp, \wp, \lambda) + \Theta(\wp, \daleth\wp, \wp, \lambda) + \Theta(\wp, \daleth\wp, \wp, \lambda)\ \right]\end{array}\right\}\right),$$

$$\Upsilon(\wp, \daleth\wp, \wp, \lambda) \leq \kappa\left(\max\left\{\begin{array}{c}\Upsilon(\wp, \daleth\wp, \wp, \lambda), \\ \frac{1}{3}\left[\ \Upsilon(\wp, \wp, \daleth\wp, \lambda) + \Upsilon(\daleth\wp, \daleth\wp, \wp, \lambda) + 0\ \right], \\ \frac{1}{4}\left[\ \Upsilon(\wp, \daleth\wp, \wp, \lambda) + \Upsilon(\wp, \daleth\wp, \wp, \lambda) + \Upsilon(\wp, \daleth\wp, \wp, \lambda)\ \right]\end{array}\right\}\right).$$

Since $\Xi(\daleth\wp, \daleth\wp, \wp, \lambda) \geq 2\Xi(\daleth\wp, \wp, \wp, \lambda)$, we have, $\Xi(\wp, \daleth\wp, \wp, \lambda) \geq \phi(\Xi(\wp, \daleth\wp, \wp, \lambda))$, $\Theta(\daleth\wp, \daleth\wp, \wp, \lambda) \leq 2\Theta(\daleth\wp, \wp, \wp, \lambda)$, we have, $\Theta(\wp, \daleth\wp, \wp, \lambda) \leq \psi\Theta(\wp, \daleth\wp, \wp, \lambda))$ and
$\Upsilon(\daleth\wp, \daleth\wp, \wp, \lambda) \leq 2\Upsilon(\daleth\wp, \wp, \wp, \lambda)$, we have, $\Upsilon(\wp, \daleth\wp, \wp, \lambda) \leq \kappa\Upsilon(\wp, \daleth\wp, \wp, \lambda))$.
Thus, $\daleth\wp = \wp$. Hence $\omega\wp = \daleth\wp = \wp$. Since $\wp = \daleth\wp \in \tau(\sum)$, there exists $w \in \sum$ such that $\wp = \tau w$. Putting $\zeta = \wp, \eta = \wp, \delta = w$ in (3.4), (3.5) and (3.6),we get

$$\Xi(\Im\wp, \daleth\wp, \Re w, \lambda) \geq \phi\left(\min\left\{\begin{array}{c}\Xi(\mathscr{F}\wp, \omega\wp, \tau w, \lambda), \\ \frac{1}{3}\left[\begin{array}{c}\Xi(\mathscr{F}\wp, \Im\wp, \daleth\wp, \lambda)+ \\ \Xi(\omega\wp, \daleth\wp, \Re w, \lambda) + \Xi(\tau w, \Re w, \Im\wp, \lambda)\end{array}\right], \\ \frac{1}{4}\left[\begin{array}{c}\Xi(\mathscr{F}\wp, \daleth\wp, \tau w, \lambda)+ \\ \Xi(\Im\wp, \omega\wp, \tau w, \lambda) + \Xi(\mathscr{F}\wp, \omega\wp, \Re w, \lambda)\end{array}\right]\end{array}\right\}\right)$$

$$\begin{aligned}\Xi(\wp, \wp, \Re w, \lambda) &\geq \phi\left(\min\left\{\begin{array}{c}1, \frac{1}{3}\left[\ (1 + \Xi(\wp, \wp, \Re w, \lambda) + \Xi(\wp, \Re w, \wp, \lambda))\ \right], \\ \frac{1}{4}\left[\ 1 + 1 + \Xi(\wp, \wp, \Re w, \lambda)\ \right]\end{array}\right\}\right) \\ &\geq \phi(\Xi(\wp, \wp, \Re w, \lambda))\end{aligned}$$

$$\Theta(\Im\wp,\daleth\wp,\Re w,\lambda)\leq\psi\left(\max\left\{\begin{array}{c}\Theta(\mathscr{F}\wp,\omega\wp,\tau w,\lambda),\\ \frac{1}{3}\left[\begin{array}{c}\Theta(\mathscr{F}\wp,\Im\wp,\daleth\wp,\lambda)+\\ \Theta(\omega\wp,\daleth\wp,\Re w,\lambda)+\Theta(\tau w,\Re w,\Im\wp,\lambda)\end{array}\right],\\ \frac{1}{4}\left[\begin{array}{c}\Theta(\mathscr{F}\wp,\daleth\wp,\tau w,\lambda)+\\ \Theta(\Im\wp,\omega\wp,\tau w,\lambda)+\Theta(\mathscr{F}\wp,\omega\wp,\Re w,\lambda)\end{array}\right]\end{array}\right\}\right).$$

$$\begin{aligned}\Theta(\wp,\wp,\Re w,\lambda)&\leq\psi\left(\max\left\{\begin{array}{c}0,\frac{1}{3}\left[\ (0+\Theta(\wp,\wp,\Re w,\lambda)+\Theta(\wp,\Re w,\wp,\lambda))\ \right],\\ \frac{1}{4}\left[\ 0+0+\Theta(\wp,\wp,\Re w,\lambda)\ \right]\end{array}\right\}\right)\\ &\leq\psi(\Theta(\wp,\wp,\Re w,\lambda).\end{aligned}$$

$$\Upsilon(\Im\wp,\daleth\wp,\Re w,\lambda)\leq\kappa\left(\max\left\{\begin{array}{c}\Upsilon(\mathscr{F}\wp,\omega\wp,\tau w,\lambda),\\ \frac{1}{3}\left[\begin{array}{c}\Upsilon(\mathscr{F}\wp,\Im\wp,\daleth\wp,\lambda)+\\ \Upsilon(\omega\wp,\daleth\wp,\Re w,\lambda)+\Upsilon(\tau w,Rw,I\wp,\lambda)\end{array}\right],\\ \frac{1}{4}\left[\begin{array}{c}\Upsilon(f\wp,\daleth\wp,\tau w,\lambda)+\\ \Upsilon(\Im\wp,\omega\wp,\tau w,\lambda)+\Upsilon(\mathscr{F}\wp,\omega\wp,\Re w,\lambda)\end{array}\right]\end{array}\right\}\right).$$

$$\begin{aligned}\Upsilon(\wp,\wp,\Re w,\lambda)&\leq\kappa\left(\max\left\{\begin{array}{c}0,\frac{1}{3}\left[\ (0+\Upsilon(\wp,\wp,\Re w,\lambda)+\Upsilon(\wp,\Re w,\wp,\lambda))\ \right],\\ \frac{1}{4}\left[\ 0+0+\Upsilon(\wp,\wp,\Re w,\lambda)\ \right]\end{array}\right\}\right)\\ &\leq\kappa(\Upsilon(\wp,\wp,\Re w,\lambda).\end{aligned}$$

Since ϕ is nondecreasing and ψ, κ are nondecreasing. we have that $\Re w=\wp$. So that $\wp=\tau w=\Re w$. Since the pair $(\Re,\tau)$ is weakly compatible, we have $\Re\wp=\tau\wp$.
Putting $\zeta=\wp,\eta=\wp,\delta=w$ in (3.4), (3.5) and (3.6), we get

$$\begin{aligned}\Xi(\wp,\wp,\Re\wp,\lambda)&=\Xi(\Im\wp,\daleth\wp,\Re\wp,\lambda)\\ &\geq\phi\left(\min\left\{\begin{array}{c}\Xi(\mathscr{F}\wp,\omega\wp,\Re\wp,\lambda),\\ \frac{1}{3}\left[\ 1+\Xi(\wp,\wp\Re\wp,\lambda)+\Xi(\Re\wp,\Re\wp,\wp,\lambda)\ \right],\\ \frac{1}{4}\left[\ \Xi(\wp,\wp,\Re\wp,\lambda)+\Xi(\wp,\wp,\Re\wp,\lambda)+\Xi(\wp,\wp,\Re\wp,\lambda)\ \right]\end{array}\right\}\right),\\ \Theta(\wp,\wp,\Re\wp,\lambda)&=\Theta(\Im\wp,\daleth\wp,\Re\wp,\lambda)\\ &\leq\psi\left(\max\left\{\begin{array}{c}\Theta(\mathscr{F}\wp,\omega\wp,\Re\wp,\lambda),\\ \frac{1}{3}\left[\ 0+\Theta(\wp,\wp\Re\wp,\lambda)+\Theta(\Re\wp,\Re\wp,\wp,\lambda)\ \right],\\ \frac{1}{4}\left[\ \Theta(\wp,\wp,\Re\wp,\lambda)+\Theta(\wp,\wp,\Re\wp,\lambda)+\Theta(\wp,\wp,\Re\wp,\lambda)\ \right]\end{array}\right\}\right),\\ \Upsilon(\wp,\wp,\Re\wp,\lambda)&=\Upsilon(\Im\wp,\daleth\wp,\Re\wp,\lambda)\\ &\leq\kappa\left(\max\left\{\begin{array}{c}\Upsilon(\mathscr{F}\wp,\omega\wp,\Re\wp,\lambda),\\ \frac{1}{3}\left[\ 0+\Upsilon(\wp,\wp,\Re\wp,\lambda)+\Upsilon(\Re\wp,\Re\wp,\wp,\lambda)\ \right],\\ \frac{1}{4}\left[\ \Upsilon(\wp,\wp,\Re\wp,\lambda)+\Upsilon(\wp,\wp,\Re\wp,\lambda)+\Upsilon(\wp,\wp,\Re\wp,\lambda)\ \right]\end{array}\right\}\right).\end{aligned}$$

Since $\Xi(\Re\wp,\Re\wp,\wp,\lambda)\geq 2\Xi(\wp,\wp,\Re\wp,\lambda)$, $\Theta(\Re\wp,\Re\wp,\wp,\lambda)\leq 2\Theta(\wp,\wp,\Re\wp,\lambda)$ and $\Upsilon(\Re\wp,\Re\wp,\wp,\lambda)\leq 2\Upsilon(\wp,\wp,\Re\wp,\lambda)$.
We have $\Xi(\wp,\wp,\Re\wp,\lambda)\geq\phi(\Xi(\wp,\wp,\Re\wp,\lambda))$, $\Theta(\wp,\wp,\Re\wp,\lambda)\leq\psi\Theta(\wp,\wp,\Re\wp,\lambda))$ and $\Upsilon(\wp,\wp,\Re\wp,\lambda)\leq\kappa\Upsilon(\wp,\wp,\Re\wp,\lambda))$.
Thus, $\Re\wp=\wp$, so that $\Re\wp=\tau\wp=\wp$.
It follows that $\wp$ is a common fixed point of $\Im,\daleth,\Re,\mathscr{F},\omega$ and τ.
Uniqueness of common fixed point follows easily from (3.4), (3.5) and (3.6).
The proof, when $\omega(\sum)$ or $\tau(\sum)$ is a complete subspace of $\sum$, is similar.

Corollary 3.2. *Let $(\sum, \Xi, \Theta, \Upsilon, *, \Diamond)$ be a NMS and $\Im, \daleth, \Re, \mathscr{F}, \omega, \tau : \sum \to \sum$ be satisfying*

(i) $\Im(\sum) \subseteq \omega(\sum), \daleth(\sum)$ *and* $\Re(\sum) \subseteq \mathscr{F}(\sum)$
(ii) *One of* $\mathscr{F}(\sum), \omega(\sum)$ *and* $\tau(\sum)$ *is a complete subspace of* $\sum$
(iii) *The pairs* $(\Im, \mathscr{F}), (\daleth, \omega)$ *and* $(\Re, \tau)$ *are weakly compatible and*

$$\Xi(\Im\zeta, \daleth\eta, \Re\delta, \lambda) \geq \phi(\Xi(\mathscr{F}\zeta, \omega\eta, \tau\delta, \lambda)) \text{ for all } \zeta, \eta, \delta \in \sum, \text{ where } \phi \in \Delta,$$
$$\Theta(\Im\zeta, \daleth\eta, \Re\delta, \lambda) \leq \psi(\Theta(\mathscr{F}\zeta, \omega\eta, \tau\delta, \lambda)) \text{ for all } \zeta, \eta, \delta \in \sum, \text{ where } \psi \in \psi,$$
$$\Upsilon(\Im\zeta, \daleth\eta, \Re\delta, \lambda) \leq \kappa(\Upsilon(\mathscr{F}\zeta, \omega\eta, \tau\delta, \lambda)) \text{ for all } \zeta, \eta, \delta \in \sum, \text{ where } \kappa \in \Delta.$$

Then, the maps $\Im, \daleth, \Re, \mathscr{F}, \omega$ and τ have a unique fixed point in $\sum$.

Corollary 3.3. *Let $(\sum, \Xi, \Theta, \Upsilon, *, \Diamond)$ be a complete NMS and $\Im, \daleth, \Re : \sum \to \sum$ be satisfying*
$\Xi(\Im\zeta, \daleth\eta, \Re\delta, \lambda) \geq \phi(\Xi(\zeta, \eta, \delta, \lambda)), \Theta(\Im\zeta, \daleth\eta, \Re\delta, \lambda) \leq \psi(\Theta(\zeta, \eta, \delta, \lambda))$ *and*
$\Upsilon(\Im\zeta, \daleth\eta, \Re\delta, \lambda) \leq \kappa(\Upsilon(\zeta, \eta, \delta, \lambda))$,
for all $\zeta, \eta, \delta \in \sum$, where $\phi \in \Delta, \psi \in \Psi$ and $\kappa \in \Delta$. Then, the maps $\Im, \daleth$ and $\Re$ have a unique common fixed point $\wp \in \sum$ and $\Im, \daleth$ and $\Re$ are Ξ, Θ and Υ are continuous at $\wp$.

Proof:
There exists $\wp \in \sum$ such that $\wp$ is the unique common fixed point of $\Im, \daleth$ and $\Re$ as in Theorem (3.1). Let $\{\eta_n\}$ be any sequence in $\sum$ which converges to $\wp$. Then,

$$\Xi(\Im\eta_n, \Im\wp, \Im\wp, \lambda) = \Xi(\Im\eta_n, \daleth\wp, \Re\wp, \lambda) \geq \phi(\Xi(\eta_n, \wp, \wp, \lambda)) \to 1,$$
$$\Theta(\Im\eta_n, \Im\wp, \Im\wp, \lambda) = \Theta(\Im\eta_n, \daleth\wp, \Re\wp, \lambda) \leq \psi(\Theta(\eta_n, \wp, \wp, \lambda)) \to 0 \text{ and}$$
$$\Upsilon(\Im\eta_n, \Im\wp, \Im\wp, \lambda) = \Upsilon(\Im\eta_n, \daleth\wp, \Re\wp, \lambda) \leq \kappa(\Upsilon(\eta_n, \wp, \wp, \lambda)) \to 0 \text{ as } n \to \infty.$$

Hence, $\Im$ is Ξ, Θ and Υ continuous at $\wp$. Similarly, it can be shown that $\daleth$ and $\Re$ are also Ξ, Θ and Υ continuous at $\wp$.

4. Common Coupled Fixed Point Theorems for w-Compatible Mappings

Theorem 4.1. *Let $(\sum, \Xi, \Theta, \Upsilon, *, \Diamond)$ be an NMS with $a * b = \min\{a, b\}$ and $a \Diamond b = \max\{a, b\}$, for all $a, b \in [0, 1]$ and $\Im : \sum \times \sum \to \sum$ and $\mathscr{F} : \sum \to \sum$ be mappings satisfying*

$$\Xi(\Im(\zeta, \eta), \Im(u, v), \Im(u, v), \kappa\lambda) \geq \min\left\{\Xi(\mathscr{F}\zeta, \mathscr{F}u, \mathscr{F}u, \lambda), \Xi(\mathscr{F}\eta, \mathscr{F}v, \mathscr{F}v, \lambda)\right\}, \tag{4.1}$$
$$\Theta(\Im(\zeta, \eta), \Im(u, v), \Im(u, v), \kappa\lambda) \leq \max\left\{\Theta(\mathscr{F}\zeta, \mathscr{F}u, \mathscr{F}u, \lambda), \Theta(\mathscr{F}\eta, \mathscr{F}v, \mathscr{F}v, \lambda)\right\}, \tag{4.2}$$
$$\Upsilon(\Im(\zeta, \eta), \Im(u, v), \Im(u`, v), \kappa\lambda) \leq \max\left\{\Upsilon(\mathscr{F}\zeta, \mathscr{F}u, \mathscr{F}u, \lambda), \Upsilon(\mathscr{F}\eta, \mathscr{F}v, \mathscr{F}v, \lambda)\right\}. \tag{4.3}$$

For all $\zeta, \eta, u, v \in \sum$ where $0 \leq \kappa < 1$.$\Im(\sum \times \sum) \subseteq \mathscr{F}(\sum)$ and if $\mathscr{F}(\sum)$ is a complete subspace of $\sum$, the pair $(\mathscr{F}, \Im)$ is w-compatible. Then $\Im$ and $\mathscr{F}$ have a unique common coupled fixed point of the form (α, α) in $\sum \times \sum$.

Proof:
Let $\zeta_0, \eta_0 \in \sum$ and denote $z_n = \Im(\zeta_n, \eta_n) = \mathscr{F}\zeta_{n+1}, \wp_n = \Im(\eta_n, \zeta_n) = \mathscr{F}\eta_{n+1}, n = 0, 1, 2 \ldots$
Let $d_n(\lambda) = \Xi(\delta_n, \delta_{n+1}, \delta_{n+1}, \lambda), \omega_n(\lambda) = \Theta(\delta_n, \delta_{n+1}, \delta_{n+1}, \lambda)$ and $\sigma_n(\lambda) = \Upsilon(\delta_n, \delta_{n+1}, \delta_{n+1}, \lambda)$. $e_n(\lambda) = \Xi(\wp_n, \wp_{n+1}, \wp_{n+1}, \lambda), r_n(\lambda) = \Theta(\wp_n, \wp_{n+1}, \wp_{n+1}, \lambda)$ and $b_n(\lambda) = \Upsilon(\wp_n, \wp_{n+1}, \wp_{n+1}, \lambda)$. From (4.1) we have

$$\begin{aligned}
d_{n+1}(\kappa\lambda) &= \Xi(\delta_{n+1}, \delta_{n+2}, \delta_{n+2}, \kappa\lambda) \\
&= \Xi(\Im(\zeta_{n+1}, \eta_{n+1}), \Im(\zeta_{n+2}, \eta_{n+2}), \Im(\zeta_{n+2}, \eta_{n+2}), \kappa\lambda)) \\
&\geq \min\{\Xi(\delta_n, \delta_{n+1}, \delta_{n+1}, \lambda), \Xi(\wp_n, \wp_{n+1}, \wp_{n+1}, \lambda)\} \\
&\geq \min\{d_n(\lambda), e_n(\lambda)\}, \\
\omega_{n+1}(\kappa\lambda) &= \Theta(\delta_{n+1}, \delta_{n+2}, \delta_{n+2}, \kappa\lambda) \\
&= \Theta(\Im(\zeta_{n+1}, \eta_{n+1}), \Im(\zeta_{n+2}, \eta_{n+2}), \Im(\zeta_{n+2}, \eta_{n+2}), \kappa\lambda) \\
&\leq \max\{\Theta(\delta_n, \delta_{n+1}, \delta_{n+1}, \lambda), \Theta(\wp_n, \wp_{n+1}, \wp_{n+1}, \lambda)\} \\
&\leq \max\{\omega_n(\lambda), r_n(\lambda)\}, \\
\sigma_{n+1}(\kappa\lambda) &= \Upsilon(\delta_{n+1}, \delta_{n+2}, \delta_{n+2}, \kappa\lambda) \\
&= \Upsilon(\Im(\zeta_{n+1}, \eta_{n+1}), \Im(\zeta_{n+2}, \eta_{n+2}), \Im(\zeta_{n+2}, \eta_{n+2}), \kappa\lambda) \\
&\leq \max\{\Upsilon(\delta_n, \delta_{n+1}, \delta_{n+1}, \lambda), \Upsilon(\wp_n, \wp_{n+1}, \wp_{n+1}, \lambda)\} \\
&\leq \max\{\sigma_n(\lambda), b_n(\lambda)\}, \\
e_{n+1}(\kappa\lambda) &= \Xi(\wp_{n+1}, \wp_{n+2}, \wp_{n+2}, \kappa\lambda) \\
&= \Xi(\Im(\eta_{n+1}, \zeta_{n+1}), \Im(\eta_{n+2}, \zeta_{n+2}), \Im(\eta_{n+2}, \zeta_{n+2}), \kappa\lambda) \\
&\geq \min\{\Xi(\wp_n, \wp_{n+1}, \wp_{n+1}, \lambda), \Xi(\delta_n, \delta_{n+1}, \delta_{n+1}, \lambda)\} \\
&\geq \min\{e_n(\lambda), d_n(\lambda)\} \\
r_{n+1}(\kappa\lambda) &= \Theta(\wp_{n+1}, \wp_{n+2}, \wp_{n+2}, \kappa\lambda) \\
&= \Theta(\Im(\eta_{n+1}, \zeta_{n+1}), \Im(\eta_{n+2}, \zeta_{n+2}), \Im(\eta_{n+2}, \zeta_{n+2}), \kappa\lambda) \\
&\leq \max\{\Theta(\wp_n, \wp_{n+1}, \wp_{n+1}, \lambda), \Theta(\delta_n, \delta_{n+1}, \delta_{n+1}, \lambda)\} \\
&\leq \max\{r_n(\lambda), \omega_n(\lambda)\} \\
b_{n+1}(\kappa\lambda) &= \Upsilon(\wp_{n+1}, \wp_{n+2}, \wp_{n+2}, \kappa\lambda) \\
&= \Upsilon(\Im(\eta_{n+1}, \zeta_{n+1}), \Im(\eta_{n+2}, \zeta_{n+2}), \Im(\eta_{n+2}, \zeta_{n+2}), \kappa\lambda) \\
&\leq \max\{\Upsilon(\wp_n, \wp_{n+1}, \wp_{n+1}, \lambda), \Upsilon(\delta_n, \delta_{n+1}, \delta_{n+1}, \lambda)\} \\
&\leq \max\{b_n(\lambda), \sigma_n(\lambda)\}.
\end{aligned}$$

Thus, $\min\{d_{n+1}(\kappa\lambda), e_{n+1}(\kappa\lambda)\} \geq \min\{d_n(\lambda), e_n(\lambda)\}$.
Hence,

$$\min\{d_n(\lambda), e_n(\lambda)\} \geq \min\left\{d_{n-1}(\lambda/\kappa), e_{n-1}(\lambda/\kappa)\right\}$$

$$
\begin{aligned}
&\geq \min\left\{d_{n-2}\left(\frac{\lambda}{\kappa^2}\right), e_{n-2}\left(\frac{\lambda}{\kappa^2}\right)\right\}\\
&\geq \cdots \geq \left\{d_0\left(\frac{\lambda}{\kappa^n}\right), e_0\left(\frac{\lambda}{\kappa^n}\right)\right\}\\
&= \min\left\{\Xi\left(\delta_0, \delta_1, \delta_1, \frac{\lambda}{\kappa^n}\right), \Xi\left(\wp_0, \wp_1, \wp_1, \frac{\lambda}{\kappa^n}\right)\right\},\\
\max\left\{\omega_{n+1}(\kappa\lambda), r_{n+1}(\kappa\lambda)\right\} &\leq \max\left\{\omega_n(\lambda), r_n(\lambda)\right\}, \max\left\{\omega_n(\lambda), r_n(\lambda)\right\}\\
&\leq \max\left\{\omega_{n-1}\left(\frac{\lambda}{\kappa}\right), r_{n-1}\left(\frac{\lambda}{\kappa}\right)\right\}\\
&\leq \max\left\{\omega_{n-2}\left(\frac{\lambda}{\kappa^2}\right), r_{n-2}\left(\frac{\lambda}{\kappa^2}\right)\right\}\dots\\
&\leq \left\{\omega_0\left(\frac{\lambda}{\kappa^n}\right), r_0\left(\frac{\lambda}{\kappa^n}\right)\right\}\\
&= \max\left\{\Theta\left(\delta_0, \delta_1, \delta_1, \frac{\lambda}{\kappa^n}\right), \Theta\left(\wp_0, \wp_1, \wp_1, \frac{\lambda}{\kappa^n}\right)\right\}.\\
\max\left\{\sigma_{n+1}(\kappa\lambda), b_{n+1}(\kappa\lambda)\right\} &\leq \max\left\{\omega_n(\lambda), b_n(\lambda)\right\},\\
\max\left\{\sigma_n(\lambda), b_n(\lambda)\right\} &\leq \max\left\{\sigma_{n-1}\left(\frac{\lambda}{\kappa}\right), b_{n-1}\left(\frac{\lambda}{\kappa}\right)\right\}\\
&\leq \max\left\{\sigma_{n-2}\left(\frac{\lambda}{\kappa^2}\right), b_{n-2}\left(\frac{\lambda}{\kappa^2}\right)\right\}\dots\\
&\leq \left\{\sigma_0\left(\frac{\lambda}{\kappa^n}\right), b_0\left(\frac{\lambda}{\kappa^n}\right)\right\}\\
&= \max\left\{\Upsilon\left(\delta_0, \delta_1, \delta_1, \frac{\lambda}{\kappa^n}\right), \Upsilon\left(\wp_0, \wp_1, \wp_1, \frac{\lambda}{\kappa^n}\right)\right\}.
\end{aligned}
$$

For any positive integer n and fixed positive integer p, we have

$$
\begin{aligned}
\Xi(\delta_n, \delta_{n+p}, \delta_{n+p}, \lambda) &\geq \Xi\left(\delta_{n+p-1}, \delta_{n+p}, \delta_{n+p}, \frac{\lambda}{p}\right) * \Xi\left(\delta_{n+p-2}, \delta_{n+p-1}, \delta_{n+p-1}, \frac{\lambda}{p}\right)\\
&* \cdots * \Xi\left(\delta_n, \delta_{n+1}, \delta_{n+1}, \frac{\lambda}{p}\right),\\
\Xi(\delta_n, \delta_{n+p}, \delta_{n+p}, \lambda) &\geq \min\left\{\Xi\left(\delta_0, \delta_1, \delta_1, \frac{\lambda}{p\kappa^{n+p-1}}\right), \Xi\left(\wp_0, \wp_1, \wp_1, \frac{\lambda}{p\kappa^{n+p-1}}\right)\right\}\\
&* \min\left\{\Xi\left(\delta_0, \delta_1, \delta_1, \frac{\lambda}{p\kappa^{n+p-2}}\right), \Xi\left(\wp_0, \wp_1, \wp_1, \frac{\lambda}{p\kappa^{n+p-2}}\right)\right\}\\
&* \cdots * \min\left\{\Xi\left(\delta_0, \delta_1, \delta_1, \frac{\lambda}{p\kappa^n}\right), \Xi\left(\wp_0, \wp_1, \wp_1, \frac{\lambda}{p\kappa^n}\right)\right\},\\
\Theta(\delta_n, \delta_{n+p}, \delta_{n+p}, \lambda) &\leq \Theta\left(\delta_{n+p-1}, \delta_{n+p}, \delta_{n+p}, \frac{\lambda}{p}\right) \Diamond \Theta\left(\delta_{n+p-2}, \delta_{n+p-1}, \delta_{n+p-1}, \frac{\lambda}{p}\right)
\end{aligned}
$$

$$\begin{aligned}
&\Diamond \dots \Diamond \Theta\left(\delta_n, \delta_{n+1}, \delta_{n+1}, \frac{\lambda}{p}\right), \\
\Theta(\delta_n, \delta_{n+p}, \delta_{n+p}, \lambda) \le{}& \max\left\{\Theta\left(\delta_0, \delta_1, \delta_1, \frac{\lambda}{p\kappa^{n+p-1}}\right), \Theta\left(\wp_0, \wp_1, \wp_1, \frac{\lambda}{p\kappa^{n+p-1}}\right)\right\} \\
&\Diamond \max\left\{\Theta\left(\delta_0, \delta_1, \delta_1, \frac{\lambda}{p\kappa^{n+p-2}}\right), \Theta\left(\wp_0, \wp_1, \wp_1, \frac{\lambda}{p\kappa^{n+p-2}}\right)\right\} \\
&\Diamond \dots \Diamond \max\left\{\Theta\left(\delta_0, \delta_1, \delta_1, \frac{\lambda}{p\kappa^{n}}\right), \Theta\left(\wp_0, \wp_1, \wp_1, \frac{\lambda}{p\kappa^{n}}\right)\right\} \text{ and} \\
\Upsilon(\delta_n, \delta_{n+p}, \delta_{n+p}, \lambda) \le{}& \Upsilon\left(\delta_{n+p-1}, \delta_{n+p}, \delta_{n+p}, \frac{\lambda}{p}\right) \Diamond \Upsilon\left(\delta_{n+p-2}, \delta_{n+p-1}, \delta_{n+p-1}, \frac{\lambda}{p}\right) \\
&\Diamond \dots \Diamond \Upsilon\left(\delta_n, \delta_{n+1}, \delta_{n+1}, \frac{\lambda}{p}\right) \\
\Upsilon(\delta_n, \delta_{n+p}, \delta_{n+p}, \lambda) \le{}& \max\left\{\Upsilon\left(\delta_0, \delta_1, \delta_1, \frac{\lambda}{p\kappa^{n+p-1}}\right), \Upsilon\left(\wp_0, \wp_1, \wp_1, \frac{\lambda}{p\kappa^{n+p-1}}\right)\right\} \\
&\Diamond \max\left\{\Upsilon\left(\delta_0, \delta_1, \delta_1, \frac{\lambda}{p\kappa^{n+p-2}}\right), \Upsilon\left(\wp_0, \wp_1, \wp_1, \frac{\lambda}{p\kappa^{n+p-2}}\right)\right\} \\
&\Diamond \dots \Diamond \max\left\{\Upsilon\left(\delta_0, \delta_1, \delta_1, \frac{\lambda}{p\kappa^{n}}\right), \Upsilon\left(\wp_0, \wp_1, \wp_1, \frac{\lambda}{p\kappa^{n}}\right)\right\}
\end{aligned}$$

Letting $n \to \infty$, and using (2.5), we get

$$\begin{aligned}
&\lim_{n\to\infty} \Xi(\delta_n, \delta_{n+p}, \delta_{n+p}, \lambda) \ge 1 * \dots * 1 = 1, \\
&\lim_{n\to\infty} \Theta(\delta_n, \delta_{n+p}, \delta_{n+p}, \lambda) \le 0\Diamond \dots \Diamond 0 = 0 \text{ and} \\
&\lim_{n\to\infty} \Upsilon(\delta_n, \delta_{n+p}, \delta_{n+p}, \lambda) \le 0\Diamond \dots \Diamond 0 = 0.
\end{aligned}$$

Hence, $\lim_{n\to\infty} \Xi(\delta_n, \delta_{n+p}, \delta_{n+p}, \lambda) = 1$, $\lim_{n\to\infty} \Theta(\delta_n, \delta_{n+p}, \delta_{n+p}, \lambda) = 0$ and $\lim_{n\to\infty} \Upsilon(\delta_n, \delta_{n+p}, \delta_{n+p}, \lambda) = 0$.

Thus, $\{\delta_n\}$ is Ξ, Θ and Υ Cauchy in σ. Similarly, it can be shown that $\{\wp_n\}$ is Ξ, Θ and Υ Cauchy in σ. Since $\mathscr{F}(\sigma)$ is Ξ, Θ and Υ complete, $\{\delta_n\}$ and $\{\wp_n\}$ converge, respectively, to some α and β in $\mathscr{F}(\sigma)$. Hence, there exists ζ and η in σ such that $\alpha = \mathscr{F}\zeta, \beta = \mathscr{F}\eta$.

$$\begin{aligned}
\Xi\left(\delta_n, \Im(\zeta,\eta), \Im(\zeta,\eta), \kappa\lambda\right) &= \Xi(\Im(\zeta_n,\eta_n), \Im(\zeta,\eta), \Im(\zeta,\eta), \kappa\lambda) \\
&\ge \min\left\{\Xi\left(\delta_{n-1}, \mathscr{F}\zeta, \mathscr{F}\zeta, \lambda\right), \Xi(\wp_{n-1}, \mathscr{F}\eta, \mathscr{F}\eta, \lambda)\right\}, \\
\Theta\left(\delta_n, \Im(\zeta,\eta), \Im(\zeta,\eta), \kappa\lambda\right) &= \Theta(\Im(\zeta_n,\eta_n), \Im(\zeta,\eta), \Im(\zeta,\eta), \kappa\lambda) \\
&\le \max\left\{\mathscr{F}\Theta\left(\delta_{n-1}, \mathscr{F}\zeta, \mathscr{F}\zeta, \lambda\right), \Theta\left(\wp_{n-1}, \mathscr{F}\eta, \mathscr{F}\eta, \lambda\right)\right\} \text{ and} \\
\Upsilon\left(\delta_n, \Im(\zeta,\eta), \Im(\zeta,\eta), \kappa\lambda\right) &= \Upsilon\left(\Im(\zeta_n,\eta_n), \Im(\zeta,\eta), \Im(\zeta,\eta), \kappa\lambda\right) \\
&\le \max\left\{\Upsilon\left(\delta_{n-1}, \mathscr{F}\zeta, \mathscr{F}\zeta, \lambda\right), \Upsilon\left(\wp_{n-1}, \mathscr{F}\eta, \mathscr{F}\eta, \lambda\right)\right\}.
\end{aligned}$$

Letting $n \to \infty$, we get,

$$\begin{aligned}
&\Xi(\mathscr{F}\zeta, \Im(\zeta,\eta), \Im(\zeta,\eta), \kappa\lambda) \geq \min\{1,1\} = 1,\\
&\Theta\left(\mathscr{F}\zeta, \Im(\zeta,\eta), \Im(\zeta,\eta), \kappa\lambda\right) \leq \max\{0,0\} = 0 \text{ and}\\
&\Upsilon\left(\mathscr{F}\zeta, \Im(\zeta,\eta), \Im(\zeta,\eta), \kappa\lambda\right) \leq \max\{0,0\} = 0.
\end{aligned}$$

Hence, $\Im(\zeta,\eta) = \mathscr{F}\zeta$. Similarly, it can be shown that $\Im(\eta,\zeta) = \mathscr{F}\eta$.
Since $(\mathscr{F},\Im)$ is w-compatible, we have $\mathscr{F}\alpha = \mathscr{F}\zeta = \mathscr{F}(\Im(\zeta,\eta)) = \Im(\mathscr{F}\zeta,\mathscr{F}\eta) = \Im(\alpha,\beta)$, $\mathscr{F}\beta = \mathscr{F}\eta = \mathscr{F}(\Im(\eta,\zeta)) = \Im(\mathscr{F}\eta,\mathscr{F}\zeta) = \Im(\beta,\alpha)$,

$$\begin{aligned}
\Xi\left(\delta_n, \mathscr{F}\alpha, \mathscr{F}\alpha, \kappa\lambda\right) &= \Xi\left(\Im(\zeta_n,\eta_n), \Im(\alpha,\beta), \Im(\alpha,\beta), \kappa\lambda\right)\\
&\geq \min\left\{\Xi(\delta_{n-1}, \mathscr{F}\alpha, \mathscr{F}\alpha, \lambda), \Xi(\wp_{n-1}, \mathscr{F}\beta, \mathscr{F}\beta, \lambda)\right\},\\
\Theta\left(\delta_n, \mathscr{F}\alpha, \mathscr{F}\alpha, \kappa\lambda\right) &= \Theta\left(\Im(\zeta_n,\eta_n), \Im(\alpha,\beta), \Im(\alpha,\beta), \kappa\lambda\right)\\
&\leq \max\left\{\Theta(\delta_{n-1}, \mathscr{F}\alpha, \mathscr{F}\alpha, \lambda), \Theta(\wp_{n-1}, \mathscr{F}\beta, \mathscr{F}\beta, \lambda)\right\},\\
\Upsilon\left(\delta_n, \mathscr{F}\alpha, \mathscr{F}\alpha, \kappa\lambda\right) &= \Upsilon\left(\Im(\zeta_n,\eta_n), \Im(\alpha,\beta), \Im(\alpha,\beta), \kappa\lambda\right)\\
&\leq \max\left\{\Upsilon(\delta_{n-1}, \mathscr{F}\alpha, \mathscr{F}\alpha, \lambda), \Upsilon(\wp_{n-1}, \mathscr{F}\beta, \mathscr{F}\beta, \lambda)\right\}.
\end{aligned}$$

Letting $n \to \infty$ we get,

$$\begin{aligned}
&\Xi\left(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \kappa\lambda\right) \geq \min\left\{\Xi(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \lambda), \Xi(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \lambda)\right\},\\
&\Theta\left(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \kappa\lambda\right) \leq \max\left\{\Theta(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \lambda), \Theta\left(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \lambda\right)\right\} \text{ and}\\
&\Upsilon\left(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \kappa\lambda\right) \leq \max\left\{\Upsilon\left(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \lambda\right), \Upsilon\left(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \lambda\right)\right\}.
\end{aligned}$$

Similarly, it can be shown that,

$$\begin{aligned}
&\Xi\left(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \kappa\lambda\right) \geq \min\left\{\Xi(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \lambda), \Xi(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \lambda)\right\},\\
&\Theta\left(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \kappa\lambda\right) \leq \max\left\{\Theta\left(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \lambda\right), \Theta\left(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \lambda\right)\right\} \text{ and}\\
&\Upsilon\left(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \kappa\lambda\right) \leq \max\left\{\Upsilon\left(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \lambda\right), \Upsilon\left(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \lambda\right)\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\min\left\{\Xi(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \kappa\lambda), \Xi(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \kappa\lambda)\right\} \geq \min\left\{\Xi(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \lambda), \Xi(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \lambda)\right\},\\
&\max\left\{\Theta(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \kappa\lambda), \Theta(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \kappa\lambda)\right\} \leq \max\left\{\Theta(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \lambda), \Theta(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \lambda)\right\} \quad \text{and}\\
&\max\left\{\Upsilon(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \kappa\lambda), \Upsilon(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \kappa\lambda)\right\} \leq \max\left\{\Upsilon(\alpha, \mathscr{F}\alpha, \mathscr{F}\alpha, \lambda), \Upsilon(\beta, \mathscr{F}\beta, \mathscr{F}\beta, \lambda)\right\}.
\end{aligned}$$

From (2.6), we have $\mathscr{F}\alpha = \alpha$ and $\mathscr{F}\beta = \beta$. Thus, $\alpha = \mathscr{F}\alpha = \Im(\alpha,\beta)$ and $\beta = \mathscr{F}\beta = \Im(\alpha,\beta)$,
Hence, (α,β) is a common coupled fixed point of $\Im$ and $\mathscr{F}$. Suppose (α',β') is

another common coupled fixed point of $\Im$ and $\mathscr{F}$.

$$\begin{aligned}
\Xi(\alpha,\alpha',\alpha',\kappa\lambda) &= \Xi(\Im(\alpha,\beta),\Im(\alpha',\beta'),\Im(\alpha',\beta'),\kappa\lambda)\\
&\geq \min\{\Xi(\alpha,\alpha',\alpha',\lambda),\Xi(\beta,\beta',\beta',\lambda)\}\\
\Xi(\beta,\beta',\beta',\kappa\lambda) &= \Xi(\Im(\beta,\alpha),\Im(\beta',\alpha'),\Im(\beta',\alpha'),\kappa\lambda)\\
&\geq \min\{\Xi(\alpha,\alpha',\alpha',\lambda),\Xi(\beta,\beta',\beta',\lambda)\}\\
&\min\{\Xi(\alpha,\alpha',\alpha',\kappa\lambda),\Xi(\beta,\beta',\beta',\kappa\lambda)\}\\
&\geq \min\{\Xi(\alpha,\alpha',\alpha',\lambda),\Xi(\beta,\beta',\beta',\lambda)\}.\\
\Theta(\alpha,\alpha',\alpha',\kappa\lambda) &= \Theta(\Im(\alpha,\beta),\Im(\alpha',\beta'),\Im(\alpha',\beta'),\kappa\lambda)\\
&\leq \max\{\Theta(\alpha,\alpha',\alpha',\lambda),\Theta(\beta,\beta',\beta',\lambda)\}\\
\Theta(\beta,\beta',\beta',\kappa\lambda) &= \Theta(\Im(\beta,\alpha),\Im(\beta',\alpha'),\Im(\beta',\alpha'),\kappa\lambda)\\
&\leq \max\{\Theta(\alpha,\alpha',\alpha',\lambda),\Theta(\beta,\beta',\beta',\lambda)\}\\
&\max\{\Theta(\alpha,\alpha',\alpha',\kappa\lambda),\Theta(\beta,\beta',\beta',\kappa\lambda)\}\\
&\leq \max\{\Theta(\alpha,\alpha',\alpha',\lambda),\Theta(\beta,\beta',\beta',\lambda)\}.\\
\Upsilon(\alpha,\alpha',\alpha',\kappa\lambda) &= \Upsilon(\Im(\alpha,\beta),\Im(\alpha',\beta'),\Im(\alpha',\beta'),\kappa\lambda)\\
&\leq \max\{\Upsilon(\alpha,\alpha',\alpha',\lambda),\Upsilon(\beta,\beta',\beta',\lambda)\}\\
\Upsilon(\beta,\beta',\beta',\kappa\lambda) &= \Upsilon(\Im(\beta,\alpha),\Im(\beta',\alpha'),\Im(\beta',\alpha'),\kappa\lambda)\\
&\leq \max\{\Theta(\alpha,\alpha',\alpha',\lambda),\Theta(\beta,\beta',\beta',\lambda)\}\\
&\max\{\Upsilon(\alpha,\alpha',\alpha',\kappa\lambda),\Upsilon(\beta,\beta',\beta',\kappa\lambda)\}\\
&\leq \max\{\Upsilon(\alpha,\alpha',\alpha',\lambda),\Upsilon(\beta,\beta',\beta',\lambda)\}.
\end{aligned}$$

From Lemma (2.6), $\alpha' = \alpha$ and $\beta' = \beta$. Thus, (α,β) is the unique common coupled fixed point of $\Im$ and $\mathscr{F}$. Now, we will show that $\alpha = \beta$.

$$\begin{aligned}
\Xi(\alpha,\alpha,\beta,\kappa\lambda) &= \Xi(\Im(\alpha,\beta),\Im(\alpha,\beta),\Im(\beta,\alpha),\kappa\lambda)\\
&\geq \min\{\Xi(\alpha,\alpha,\beta,\lambda),\Xi(\beta,\beta,\alpha,\lambda)\}\\
\Xi(\alpha,\beta,\beta,\kappa\lambda) &= \Xi(\Im(\alpha,\beta),\Im(\beta,\alpha),\Im(\beta,\alpha),\kappa\lambda)\\
&\geq \min\{\Xi(\alpha,\beta,\beta,\lambda),\Xi(\beta,\alpha,\alpha,\lambda)\}\\
&\min\{\Xi(\alpha,\alpha,\beta,\kappa\lambda),\Xi(\alpha,\beta,\beta,\kappa\lambda)\}\\
&\geq \min\{\Xi(\alpha,\alpha,\beta,\lambda),\Xi(\alpha,\beta,\beta,\lambda)\}.\\
\Theta(\alpha,\alpha,\beta,\kappa\lambda) &= \Theta(\Im(\alpha,\beta),\Im(\alpha,\beta),\Im(\beta,\alpha),\kappa\lambda)\\
&\leq \max\{\Theta(\alpha,\alpha,\beta,\lambda),\Theta(\beta,\beta,\alpha,\lambda)\}\\
\Theta(\alpha,\beta,\beta,\kappa\lambda) &= \Theta(\Im(\alpha,\beta),\Im(\beta,\alpha),\Im(\beta,\alpha),\kappa\lambda)\\
&\leq \max\{\Theta(\alpha,\beta,\beta,\lambda),\Theta(\beta,\alpha,\alpha,\lambda)\}\\
&\leq \max\{\Theta(\alpha,\alpha,\beta,\kappa\lambda),\Theta(\alpha,\beta,\beta,\kappa\lambda)\}\\
&\leq \max\{\Theta(\alpha,\alpha,\beta,\lambda),\Theta(\alpha,\beta,\beta,\lambda)\}.\\
\Upsilon(\alpha,\alpha,\beta,\kappa\lambda) &= \Upsilon(\Im(\alpha,\beta),\Im(\alpha,\beta),\Im(\beta,\alpha),\kappa\lambda)\\
&\leq \max\{\Upsilon(\alpha,\alpha,\beta,\lambda),\Upsilon(\beta,\beta,\alpha,\lambda)\}
\end{aligned}$$

$$\begin{aligned}\Upsilon(\alpha,\beta,\beta,\kappa\lambda) &= \Upsilon(\Im(\alpha,\beta),\Im(\beta,\alpha),\Im(\beta,\alpha),\kappa\lambda)\\ &\leq \max\{\Upsilon(\alpha,\beta,\beta,\lambda),\Upsilon(\beta,\alpha,\alpha,\lambda)\}\\ &\max\{\Upsilon(\alpha,\alpha,\beta,\kappa\lambda),\Upsilon(\alpha,\beta,\beta,\kappa\lambda)\}\\ &\leq \max\{\Upsilon(\alpha,\alpha,\beta,\lambda),\Upsilon(\alpha,\beta,\beta,\lambda)\}\end{aligned}$$

Thus, we have $\alpha = \beta$. Thus, α is a common fixed point of S and $\mathscr{F}$. That is, $\alpha = \mathscr{F}\alpha = \Im(\alpha,\alpha)$. Suppose, α' is another common fixed point of $\Im$ and $\mathscr{F}$.

$$\begin{aligned}\Xi(\alpha',\alpha,\alpha,\lambda) &= \Xi(\Im(\alpha',\alpha'),\Im(\alpha,\alpha),\Im(\alpha,\alpha),\lambda)\\ &\geq \min\left\{\Xi(\alpha',\alpha,\alpha,\frac{\lambda}{\kappa}\right),\Xi\left(\alpha',\alpha,\alpha,\frac{\lambda}{\kappa}\right\}\\ &\geq \Xi\left(\alpha',\alpha,\alpha,\frac{\lambda}{\kappa^2}\right)\geq\dots\\ &\geq \Xi\left(\alpha',\alpha,\alpha,\frac{\lambda}{\kappa^n}\right)\to 1.\\ \Theta(\alpha',\alpha,\alpha,\lambda) &= \Theta(\Im(\alpha',\alpha'),\Im(\alpha,\alpha),\Im(\alpha,\alpha),\lambda)\\ &\leq \max\{\Theta(\alpha',\alpha,\alpha,\lambda/\kappa),\Theta(\alpha',\alpha,\alpha,\lambda/\kappa)\}\\ &\leq \Theta\left(\alpha',\alpha,\alpha,\frac{\lambda}{\kappa^2}\right)\\ &\leq\dots\leq \Theta\left(\alpha',\alpha,\alpha,\frac{\lambda}{\kappa^n}\right)\to 0 \text{ and}\\ \Upsilon(\alpha',\alpha,\alpha,\lambda) &= \Upsilon(\Im(\alpha',\alpha'),\Im(\alpha,\alpha),\Im(\alpha,\alpha),\lambda)\\ &\leq \max\left\{\Upsilon\left(\alpha',\alpha,\alpha,\frac{\lambda}{\kappa}\right),\Upsilon\left(\alpha',\alpha,\alpha,\frac{\lambda}{\kappa}\right)\right\}\\ &\leq \Upsilon\left(\alpha',\alpha,\alpha,\frac{\lambda}{\kappa^2}\right)\\ &\leq\dots\leq \Upsilon\left(\alpha',\alpha,\alpha,\frac{\lambda}{\kappa^n}\right)\to 0.\end{aligned}$$

Hence, $\alpha' = \alpha$. Thus $\Im$ and $\mathscr{F}$ have a unique common coupled fixed point of the form (α,α).

Theorem 4.2. *Let $(\sigma,\Xi,\Theta,\Upsilon,*,\Diamond)$ be a symmetric complete NMS with $a*b = \min\{a,b\}$ and $a\Diamond b = \max\{a,b\}$ for all $a,b\in[0,1]$ and let $\Im,\daleth,\Re:\sigma\times\sigma\to\sigma$ be mappings satisfying*

$$\Xi(\Im(\zeta,\eta),\daleth(u,v),\Re(\wp,q),\kappa\lambda)\geq\min\left\{\begin{array}{c}\Xi(\zeta,u,\wp,\lambda),\Xi(\eta,v,q,\lambda),\Xi(\zeta,\zeta,\Im(\zeta,\eta),\lambda),\\ \Xi(u,u,\hbar(u,v),\lambda),\Xi(\wp,\wp,\Re(\wp,q),\lambda)\end{array}\right\},$$

$$\Theta(\Im(\zeta,\eta),\daleth(u,v),\Re(\wp,q),\kappa\lambda)\leq\max\left\{\begin{array}{c}\Theta(\zeta,u,\wp,\lambda),\Theta(\eta,v,q,\lambda),\Theta(\zeta,\zeta,\Im(\zeta,\eta),\lambda),\\ \Theta(u,u,\hbar(u,v),\lambda),\Theta(\wp,\wp,\Re(\wp,q),\lambda)\end{array}\right\},$$

$$\Upsilon(\Im(\zeta,\eta),\daleth(u,v),\Re(\wp,q),\kappa\lambda)\leq\max\left\{\begin{array}{c}\Upsilon(\zeta,u,\wp,\lambda),\Upsilon(\eta,v,q,\lambda),\Upsilon(\zeta,\zeta,\Im(\zeta,\eta),\lambda),\\ \Upsilon(u,u,\hbar(u,v),\lambda),\Upsilon(\wp,\wp,\Re(\wp,q),\lambda)\end{array}\right\}$$

for all $\zeta, \eta, u, v, \wp, q \in \sigma$ *where* $0 \leq \kappa < 1$. *Then there exists* $(\zeta, \eta) \in \sigma \times \sigma$ *such that* $\zeta = \Im(\zeta, \eta) = \daleth(\zeta, \eta) = \Re(\zeta, \eta)$, $\eta = \Im(\eta, \zeta) = \daleth(\eta, \zeta) = \Re(\eta, \zeta)$, *and* $\Im, \hbar$ *and* $\Re$ *have a unique common coupled fixed point of the form* (ζ, ζ) *in* $\sigma \times \sigma$.

Proof:
Let $\zeta_0, \eta_0 \in \sigma$. Define sequences ζ_n and η_n in σ as follows:

$$\zeta_{3n+1} = \Im(\zeta_{3n},\ \eta_{3n}),\ \eta_{3n+1} = \Im(\eta_{3n},\ \zeta_{3n}),\ \zeta_{3n+2} = \daleth(\zeta_{3n+1},\ \eta_{3n+1}),$$
$$\eta_{3n+2} = \daleth(\eta_{3n+1},\ \zeta_{3n+1}),\ \zeta_{3n+3} = \Re(\zeta_{3n+2},\ \eta_{3n+2}),\ \eta_{3n+3} = \Re(\eta_{3n+2},\ \zeta_{3n+2}),\ n = 0, 1, 2, ...$$

Suppose $\zeta_{3n+1} = \zeta_{3n}$, for some n. Then $\Im(\zeta, \eta) = \zeta$, where $\zeta = \zeta_{3n}, \eta = \eta_{3n}$.
Suppose $\daleth(\zeta, \eta) \neq \Re(\zeta, \eta)$. Then

$$\begin{aligned}
\Xi(\zeta, \daleth(\zeta, \eta), \Re(\zeta, \eta), \kappa\lambda) &= \Xi(\Im(\zeta, \eta), \daleth(\zeta, \eta), \Re(\zeta, \eta), \kappa\lambda) \\
&\geq \min\{1, 1, 1, \Xi(\zeta, \zeta, \daleth(\zeta, \eta), \lambda), \Xi(\zeta, \zeta, \Re(\zeta, \eta), \lambda)\} \\
&\geq \Xi(\zeta, \daleth(\zeta, \eta), \Re(\zeta, \eta), \lambda), \\
\Theta(\zeta, \daleth(\zeta, \eta), \Re(\zeta, \eta), \kappa\lambda) &= \Theta(\Im(\zeta, \eta), \daleth(\zeta, \eta), \Re(\zeta, \eta), \kappa\lambda) \\
&\leq \max\{0, 0, 0, \Theta(\zeta, \zeta, \daleth(\zeta, \eta), \lambda), \Theta(\zeta, \zeta, \Re(\zeta, \eta), \lambda)\} \\
&\leq \Theta(\zeta, \daleth(\zeta, \eta), \Re(\zeta, \eta), \lambda) \text{ and} \\
\Upsilon(\zeta, \daleth(\zeta, \eta), \Re(\zeta, \eta), \kappa\lambda) &= \Upsilon(\Im(\zeta, \eta), \daleth(\zeta, \eta), \Re(\zeta, \eta), \kappa\lambda) \\
&\leq \max\{0, 0, 0, \Upsilon(\zeta, \zeta, \daleth(\zeta, \eta), \lambda), \Upsilon(\zeta, \zeta, \Re(\zeta, \eta), \lambda)\} \\
&\leq \Upsilon(\zeta, \daleth(\zeta, \eta), \Re(\zeta, \eta), \lambda).
\end{aligned}$$

It is a contradiction. Hence, $\daleth(\zeta, \eta) = \Re(\zeta, \eta)$ and σ is symmetric.

$$\begin{aligned}
\Xi(\zeta, \daleth(\zeta, \eta), \daleth(\zeta, \eta), \kappa\lambda) &\geq \Xi(\zeta, \zeta, \daleth(\zeta, \eta), \lambda) = \Xi(\zeta, \daleth(\zeta, \eta), \daleth(\zeta, \eta), \lambda), \\
\Theta(\zeta, \daleth(\zeta, \eta), \daleth(\zeta, \eta), \kappa\lambda) &\leq \Theta(\zeta, \zeta, \daleth(\zeta, \eta), \lambda) = \Theta(\zeta, \daleth(\zeta, \eta), \daleth(\zeta, \eta), \lambda) \text{ and} \\
\Upsilon(\zeta, \daleth(\zeta, \eta), \daleth(\zeta, \eta), \kappa\lambda) &\leq \Upsilon(\zeta, \zeta, \daleth(\zeta, \eta), \lambda) = \Upsilon(\zeta, \daleth(\zeta, \eta), \daleth(\zeta, \eta), \lambda).
\end{aligned}$$

We have $\daleth(\zeta, \eta) = \zeta$. Thus, $\Im(\zeta, \eta) = \daleth(\zeta, \eta) = \Re(\zeta, \eta) = \zeta$.
Similarly, if $\zeta_{3n+1} = \zeta_{3n+2}$ or $\zeta_{3n+2} = \zeta_{3n+3}$, then also we can show that $\Im(\zeta, \eta) = \daleth(\zeta, \eta) = \Re(\zeta, \eta) = \zeta$ for some ζ, η in σ.
Similarly, it can be shown that if $\eta_{3n} = \eta_{3n+1}$ or $\eta_{3n+1} = \eta_{3n+2}$ or $\eta_{3n+2} = \eta_{3n+3}$, then there exists $(\zeta, \eta) \in \sigma \times \sigma$ such that $\Im(\eta, \zeta) = \daleth(\eta, \zeta) = \Re(\eta, \zeta) = \eta$.
Now, assume that $\zeta_n \neq \zeta_{n+1}$ and $\eta_n \neq \eta_{n+1}$ for all n, write

$$\begin{aligned}
d_n(\lambda) &= \Xi(\zeta_n, \zeta_{n+1}, \zeta_n + 2, \lambda), \\
\rho_n(\lambda) &= \Theta(\zeta_n, \zeta_{n+1}, \zeta_{n+2}, \lambda) \text{ and} \\
\sigma_n(\lambda) &= \Upsilon(\zeta_n, \zeta_{n+1}, \zeta_{n+2}, \lambda) \\
e_n(\lambda) &= \Xi(\eta_n, \eta_{n+1}, \eta_{n+2}, \lambda), \\
r_n(\lambda) &= \Theta(\eta_n, \eta_{n+1}, \eta_{n+2}, \lambda) \text{ and} \\
b_n(\lambda) &= \Upsilon(\eta_n, \eta_{n+1}, \eta_{n+2}, \lambda). \\
d_{3n}(kt) &= \Xi(\zeta_{3n}, \zeta_{3n+1}, \zeta_{3n+2}, \kappa\lambda) \\
&= \Xi(\Im(\zeta_{3n}, \eta_{3n}), \daleth(\zeta_{3n+1}, \eta_{3n+1}, \Re(\zeta_{3n-1}, \eta_{3n-1}, \kappa\lambda)
\end{aligned}$$

$$
\begin{aligned}
&\geq \min\left\{\begin{array}{c} d_{3n-1}(\lambda), e_{3n-1}(\lambda), \Xi(\zeta_{3n}, \zeta_{3n}, \zeta_{3n+1}, \lambda), \\ \Xi(\zeta_{3n+1}, x_{3n+1}, \zeta_{3n+2}, \lambda) \\ d_{3n-1}(\lambda), e_{3n-1}(\lambda), \Xi(\zeta_{3n}, \zeta_{3n}, \zeta_{3n+1}, \lambda) \end{array}\right\} \\
&\geq \min\left\{d_{3n-1}(\lambda), e_{3n-1}(\lambda), d_{3n}(\lambda), d_{3n}(\lambda), d_{3n-1}(\lambda)\right\}, \\
\rho_{3n}(kt) &= \Theta(\zeta_{3n}, \zeta_{3n+1}, \zeta_{3n+2}, \kappa\lambda) \\
&= \Theta(\Im(\zeta_{3n}, \eta_{3n}), \daleth(\zeta_{3n+1}, \eta_{3n+1}), \Re(\zeta_{3n-1}, \eta_{3n-1}, \kappa\lambda) \\
&\leq \max\left\{\begin{array}{c} \rho_{3n-1}(\lambda), r_{3n-1}(\lambda), \Theta(\zeta_{3n}, \zeta_{3n}, \zeta_{3n+1}, \lambda), \\ \Theta(\zeta_{3n+1}, \zeta_{3n+1}, \zeta_{3n+2}, \lambda) \\ \Theta(\zeta_{3n-1}, \zeta_{3n-1}, \zeta_{3n}, \lambda) \end{array}\right\} \\
&\leq \max\left\{\rho_{3n-1}(\lambda), r_{3n-1}(\lambda), \rho_{3n}(\lambda), \rho_{3n}(\lambda), \rho_{3n-1}(\lambda)\right\} \quad \text{and} \\
\sigma_{3n}(kt) &= \Upsilon(\zeta_{3n}, \zeta_{3n+1}, \zeta_{3n+2}, \kappa\lambda) \\
&= \Upsilon(\Im(\zeta_{3n}, \eta_{3n}), \daleth(\zeta_{3n+1}, \eta_{3n+1}, \Re(\zeta_{3n-1}, \eta_{3n-1}, \kappa\lambda) \\
&\leq \max\left\{\begin{array}{c} \sigma_{3n-1}(\lambda), r_{3n-1}(\lambda), \Upsilon(\zeta_{3n}, \zeta_{3n}, \zeta_{3n+1}, \lambda), \\ \Upsilon(\zeta_{3n+1}, \zeta_{3n+1}, \zeta_{3n+2}, \lambda), \Upsilon(\zeta_{3n-1}, \zeta_{3n-1}, \zeta_{3n}, \lambda) \end{array}\right\} \\
&\leq \max\left\{\sigma_{3n-1}(\lambda), b_{3n-1}(\lambda), \sigma_{3n}(\lambda), \sigma_{3n}(\lambda), \sigma_{3n-1}(\lambda)\right\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
d_{3n}(\kappa\lambda) &\geq \min\{d_{3n-1}(\lambda), e_{3n-1}(\lambda), \rho_{3n}(\kappa\lambda)\} \\
&\leq \max\{\rho_{3n-1}(\lambda), r_{3n-1}(\lambda)\} \text{ and} \\
\sigma_{3n}(\kappa\lambda) &\leq \max\{\sigma_{3n-1}(\lambda), b_{3n-1}(\lambda)\}.
\end{aligned}
$$

Similarly, we have $e_{3n}(\kappa\lambda) \geq \min\{d_{3n-1}(\kappa\lambda), e_{3n-1}(\kappa\lambda)\}$.
Thus,

$$
\begin{aligned}
\min\{d_{3n}(\kappa\lambda), e_{3n}(\kappa\lambda)\} &\geq \min\{d_{3n-1}(\kappa\lambda), e_{3n-1}(\kappa\lambda)\}, \\
\max\{\rho_{3n}(\kappa\lambda), r_{3n}(\kappa\lambda)\} &\leq \max\left\{\rho_{3n-1}(\kappa\lambda), r_{3n-1}(\kappa\lambda)\right\} \text{ and} \\
\max\{\sigma_{3n}(\kappa\lambda), r_{3n}(\kappa\lambda)\} &\leq \max\{\sigma_{3n-1}(\kappa\lambda), b_{3n-1}(\kappa\lambda)\}.
\end{aligned}
$$

Similarly, it can be shown that

$$
\begin{aligned}
\min\{d_{3n+1}(\kappa\lambda), e_{3n+1}(\kappa\lambda)\} &\geq \min\{d_{3n}(\lambda), e_{3n}(\lambda)\}, \\
\min\{d_{3n+2}(\kappa\lambda), e_{3n+2}(\kappa\lambda)\} &\geq \min\{d_{3n+1}(\lambda), e_{3n+1}(\lambda)\}, \\
\max\{\rho_{3n+1}(\kappa\lambda), r_{3n+1}(\kappa\lambda)\} &\leq \max\{\rho_{3n}(\lambda), r_{3n}(\lambda)\}, \\
\max\{\rho_{3n+2}(\kappa\lambda), r_{3n+2}(\kappa\lambda)\} &\geq \max\{\rho_{3n+1}(\lambda), r_{3n+1}(\lambda)\} \text{ and} \\
\max\{\sigma_{3n+1}(\kappa\lambda), b_{3n+1}(\kappa\lambda)\} &\leq \max\{\sigma_{3n}(\lambda), b_{3n}(\lambda)\}, \\
\max\left\{\sigma_{3n+2}(\kappa\lambda), br_{3n+2}(\kappa\lambda)\right\} &\geq \max\{\sigma_{3n+1}(\lambda), b_{3n+1}(\lambda)\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\min\{d_{n+1}(\kappa\lambda), e_{n+1}(\kappa\lambda)\} &\geq \min\{d_n(\lambda), e_n(\lambda)\},\\
\max\{\rho_{n+1}(\kappa\lambda), r_{n+1}(\kappa\lambda)\} &\leq \max\{\rho_n(\lambda), r_n(\lambda)\} \text{ and}\\
\max\{\sigma_{n+1}(\kappa\lambda), b_{n+1}(\kappa\lambda)\} &\leq \max\{\sigma_n(\lambda), b_n(\lambda)\}.\\
\min\{d_n(\lambda), e_n(\lambda)\} &\geq \min\left\{d_n\left(\frac{\lambda}{\kappa}\right), e_n\left(\frac{\lambda}{\kappa}\right)\right\}\\
&\geq \min\left\{d_{n-1}\left(\frac{\lambda}{\kappa^2}\right), e_{n-1}\left(\frac{\lambda}{\kappa^2}\right)\right\}\dots\\
&\geq \min\left\{d_n\left(\frac{\lambda}{\kappa^n}\right) e_n\left(\frac{\lambda}{\kappa^n}\right)\right\}\\
&= \min\left\{\Xi\left(\zeta_0, \zeta_1, \zeta_2, \frac{\lambda}{\kappa^n}\right), \Xi\left(\eta_0, \eta_1, \eta_2, \frac{\lambda}{\kappa^n}\right)\right\},\\
\max\left\{\rho_n(\lambda), r_n(\lambda)\right\} &\leq \max\left\{\rho_n\left(\frac{\lambda}{\kappa}\right), r_n\left(\frac{\lambda}{\kappa}\right)\right\}\\
&\leq \max\left\{\rho_{n-1}\left(\frac{\lambda}{\kappa^2}\right), r_{n-1}\left(\frac{\lambda}{\kappa^2}\right)\right\}\dots\\
&\leq \max\left\{\rho_0\left(\frac{\lambda}{\kappa^n}\right), r_0\left(\frac{\lambda}{\kappa^n}\right)\right\}\\
&= \max\left\{\Theta\left(\zeta_0, \zeta_1, \zeta_2, \frac{\lambda}{\kappa^n}\right), \Theta\left(\eta_0, \eta_1, \eta_2, \frac{\lambda}{\kappa^n}\right)\right\},\\
\max\left\{\sigma_n(\lambda), b_n(\lambda)\right\} &\leq \max\left\{\sigma_n\left(\frac{\lambda}{\kappa}\right), b_n\left(\frac{\lambda}{\kappa}\right)\right\}\\
&\leq \max\left\{\sigma_{n-1}\left(\frac{\lambda}{\kappa^2}\right), b_{n-1}\left(\frac{\lambda}{\kappa^2}\right)\right\}\dots\\
&\leq \max\left\{\sigma_0\left(\frac{\lambda}{\kappa^n}\right), b_0\left(\frac{\lambda}{\kappa^n}\right)\right\}\\
&= \max\left\{\Upsilon\left(\zeta_0, \zeta_1, \zeta_2, \frac{\lambda}{\kappa^n}\right), \Upsilon\left(\eta_0, \eta_1, \eta_2, \frac{\lambda}{\kappa^n}\right)\right\},
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Xi(\zeta_n, \zeta_{n+1}, \zeta_{n+2}, \lambda) &\geq \min\left\{\Xi\left(\zeta_0, \zeta_1, \zeta_2, \frac{\lambda}{\kappa^n}\right), \Xi\left(\eta_0, \eta_1, \eta_2, \frac{\lambda}{\kappa^n}\right)\right\},\\
\Theta(\zeta_n, \zeta_{n+1}, \zeta_{n+2}, \lambda) &\leq \max\left\{\Theta\left(\zeta_0, \zeta_1, \zeta_2, \frac{\lambda}{\kappa^n}\right), \Theta\left(\eta_0, \eta_1, \eta_2, \frac{\lambda}{\kappa^n}\right)\right\},\\
\Upsilon(\zeta_n, \zeta_{n+1}, \zeta_{n+2}, \lambda) &\leq \max\left\{\Upsilon\left(\zeta_0, \zeta_1, \zeta_2, \frac{\lambda}{\kappa^n}\right), \Upsilon\left(\eta_0, \eta_1, \eta_2, \frac{\lambda}{\kappa^n}\right)\right\}.
\end{aligned}
$$

We have

$$\Xi(\zeta_n,\zeta_n,\zeta_{n+1},\lambda) \geq \Xi(\zeta_n,\zeta_(n+1),\zeta_{n+2},\lambda) \geq \min\left\{\Xi\left(\zeta_0,\zeta_1,\zeta_2,\frac{\lambda}{\kappa^n}\right),\Xi\left(\eta_0,\eta_1,\eta_2,\frac{\lambda}{\kappa^n}\right)\right\},$$

$$\Theta(\zeta_n,\zeta_n,\zeta_{n+1},\lambda) \leq \Theta(\zeta_n,\zeta_(n+1),\zeta_{n+2},\lambda) \leq \max\left\{\Theta\left(\zeta_0,\zeta_1,\zeta_2,\frac{\lambda}{\kappa^n}\right),\Theta\left(\eta_0,\eta_1,\eta_2,\frac{\lambda}{\kappa^n}\right)\right\},$$

$$\Upsilon(\zeta_n,\zeta_n,\zeta_{n+1},\lambda) \leq \Upsilon(\zeta_n,\zeta_(n+1),\zeta_{n+2},\lambda) \leq \max\left\{\Upsilon\left(\zeta_0,\zeta_1,\zeta_2,\frac{\lambda}{\kappa^n}\right),\Upsilon\left(\eta_0,\eta_1,\eta_2,\frac{\lambda}{\kappa^n}\right)\right\}.$$

As in the previous theorem (4.1), it can be show that $\{\zeta_n\}$ and $\{\eta_n\}$ are Ξ,Θ and Υ Cauchy sequences in σ.
Since σ is Ξ and Θ complete, there exists $\zeta,\eta\in\sigma$ such that $\zeta_n\to\zeta$ and $\eta_n\to\eta$.

$$\begin{aligned}
\Xi(\Im(\zeta,\eta),\zeta_{3n+2},\zeta_{3n+3},\kappa\lambda) &= \Xi(\Im(\zeta,\eta),\daleth(\zeta_{3n+1},\eta_{3n+1}),\Re(\zeta_{3n+2},\eta_{3n+2}),\kappa\lambda)\\
&\geq \min\left\{\begin{array}{c}\Xi(\zeta,\zeta_{3n+1},\zeta_{3n+2},\lambda),\Xi(\eta,\eta_{3n+1},\eta_{3n+2},\lambda),\\ \Xi(\zeta,\zeta,\Im(\zeta,\eta),\lambda),\Xi(\zeta_{3n+1},\zeta_{3n+1},\zeta_{3n+2},\lambda),\\ \Xi(\zeta_{3n+2},\zeta_{3n+2},\zeta_{3n+3},\lambda)\end{array}\right\},\\
\Theta(\Im(\zeta,\eta),\zeta_{3n+2},\zeta_{3n+3},\kappa\lambda) &= \Theta\left(\Im(\zeta,\eta),\daleth(\zeta_{3n+1},\eta_{3n+1}),\Re(\zeta_{3n+2},\eta_{3n+2}),\kappa\lambda\right)\\
&\leq \max\left\{\begin{array}{c}\Theta(\zeta,\zeta_{3n+1},\zeta_{3n+2},\lambda),\Theta(\eta,\eta_{3n+1},\eta_{3n+2},\lambda),\\ \Theta(\zeta,\zeta,\Im(\zeta,\eta),\lambda),\Theta(\zeta_{3n+1},\zeta_{3n+1},\zeta_{3n+2},\lambda),\\ \Theta(\zeta_{3n+2},\zeta_{3n+2},\zeta_{3n+3},\lambda)\end{array}\right\} \text{ and}\\
\Upsilon(\Im(\zeta,\eta),\zeta_{3n+2},\zeta_{3n+3},\kappa\lambda) &= \Upsilon(\Im(\zeta,\eta),\daleth(\zeta_{3n+1},\eta_{3n+1}),\Re(\zeta_{3n+2},\eta_{3n+2}),\kappa\lambda)\\
&\leq \max\left\{\begin{array}{c}\Upsilon(\zeta,\zeta_{3n+1},\zeta_{3n+2},\lambda),\Upsilon(\eta,\eta_{3n+1},\eta_{3n+2},\lambda),\\ \Upsilon(\zeta,\zeta,\Im(\zeta,\eta),\lambda),\Upsilon(\zeta_{3n+1},\zeta_{3n+1},\zeta_{3n+2},\lambda),\\ \Upsilon(\zeta_{3n+2},\zeta_{3n+2},\zeta_{3n+3},\lambda)\end{array}\right\}.
\end{aligned}$$

Letting $n\to\infty$,

$$\begin{aligned}
&\Xi(\Im(\zeta,\eta),\zeta,\zeta,\kappa\lambda) \geq \min\{1,1,\Xi(\zeta,\zeta,\Im(\zeta,\eta),\lambda),1,1\} = \Xi(\zeta,\zeta,\Im(\zeta,\eta),\lambda),\\
&\Theta(\Im(\zeta,\eta),\zeta,\zeta,\kappa\lambda) \leq \max\{0,0,\Theta(\zeta,\zeta,\Im(\zeta,\eta),\lambda),0,0\} = \Theta(\zeta,\zeta,\Im(\zeta,\eta),\lambda) \text{ and}\\
&\Upsilon(\Im(\zeta,\eta),\zeta,\zeta,\kappa\lambda) \leq \max\{0,0,\Upsilon(\zeta,\zeta,\Im(\zeta,\eta),\lambda),0,0\} = \Upsilon(\zeta,\zeta,\Im(\zeta,\eta),\lambda).
\end{aligned}$$

From this, we have $\Im(\zeta,\eta)=\zeta$. As in the first part of the proof, it can be shown that $\Im(\zeta,\eta)=\daleth(\zeta,\eta)=\Re(\zeta,\eta)=\zeta$.
And, similarly, it can be shown that $\Im(\eta,\zeta)=\daleth(\eta,\zeta)=\Re(\eta,\zeta)=\eta$.
Thus, (ζ,η) is a common coupled fixed point of $\Im,\daleth$ and $\Re$. Suppose (ζ',η') is another common coupled fixed point of $\Im,\daleth$ and $\Re$. Consider,

$$\begin{aligned}
\Xi(\zeta,\zeta,\zeta',\kappa\lambda) &= \Xi(\Im(\zeta,\eta),\daleth(\zeta,\eta),\Re(\zeta',\eta'),\kappa\lambda) \geq \min\{\Xi(\zeta,\zeta,\zeta'\lambda),\Xi(\eta,\eta,\eta',\lambda),1,1.1\}\\
&= \min\{\Xi(\zeta,\zeta,\zeta'\lambda),\Xi(\eta,\eta,\eta'\lambda)\},\\
\Theta(\zeta,\zeta,\zeta',\kappa\lambda) &= \Theta(\Im(\zeta,\eta),\daleth(\zeta,\eta),\Re(\zeta',\eta'),\kappa\lambda) \leq \max\{\Theta(\zeta,\zeta,\zeta'\lambda),\Theta(\eta,\eta,\eta',\lambda),0,0,0\}\\
&= \max\{\Theta(\zeta,\zeta,\zeta'\lambda),\Theta(\eta,\eta,\eta'\lambda)\} \text{ and}\\
\Upsilon(\zeta,\zeta,\zeta',\kappa\lambda) &= \Upsilon(\Im(\zeta,\eta),\daleth(\zeta,\eta),\Re(\zeta',\eta'),\kappa\lambda) \leq \max\{\Upsilon(\zeta,\zeta,\zeta'\lambda),\Upsilon(\eta,\eta,\eta',\lambda),0,0,0\}\\
&= \max\{\Upsilon(\zeta,\zeta,\zeta'\lambda),\Upsilon(\eta,\eta,\eta'\lambda)\}.
\end{aligned}$$

Also,

$$
\begin{aligned}
\Xi(\eta,\eta,\eta',\kappa\lambda) &= \Xi(\Im(\eta,\zeta),\daleth(\eta,\zeta),\Re(\eta',\zeta'),\kappa\lambda) \geq \min\{\Xi(\zeta,\zeta,\zeta'\lambda),\Xi(\eta,\eta,\eta',\lambda)1,1,1\}\\
&= \min\{\Xi(\zeta,\zeta,\zeta'\lambda),\Xi(\eta,\eta,\eta'\lambda)\},\\
\Theta(\eta,\eta,\eta',\kappa\lambda) &= \Theta(\Im(\eta,\zeta),\daleth(\eta,\zeta),\Re(\eta',\zeta'),\kappa\lambda) \leq \max\{\Theta(\zeta,\zeta,\zeta'\lambda),\Theta(\eta,\eta,\eta',\lambda)0,0,0\}\\
&= \max\{\Theta(\zeta,\zeta,\zeta'\lambda),\Theta(\eta,\eta,\eta'\lambda)\} \text{ and}\\
\Upsilon(\eta,\eta,\eta',\kappa\lambda) &= \Upsilon(\Im(\eta,\zeta),\daleth(\eta,\zeta),\Re(\eta',\zeta'),\kappa\lambda) \leq \max\{\Upsilon(\zeta,\zeta,\zeta'\lambda),\Upsilon(\eta,\eta,\eta',\lambda)0,0,0\}\\
&= \max\{\Upsilon(\zeta,\zeta,\zeta'\lambda),\Upsilon(\eta,\eta,\eta'\lambda)\}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\min\{\Xi(\zeta,\zeta,\zeta'\kappa\lambda),\Xi(\eta,\eta,\eta'\kappa\lambda)\} \geq \min\{\Xi(\zeta,\zeta,\zeta'\lambda),\Xi(\eta,\eta,\eta'\lambda)\},\\
&\max\{\Theta(\zeta,\zeta,\zeta'\kappa\lambda),\Theta(\eta,\eta,\eta'\kappa\lambda)\} \leq \max\{\Theta(\zeta,\zeta,\zeta'\lambda),\Theta(\eta,\eta,\eta'\lambda)\} \text{ and}\\
&\max\{\Upsilon(\zeta,\zeta,\zeta'\kappa\lambda),\Upsilon(\eta,\eta,\eta'\kappa\lambda)\} \leq \max\{\Upsilon(\zeta,\zeta,\zeta'\lambda),\Upsilon(\eta,\eta,\eta'\lambda)\}.
\end{aligned}
$$

From lemma (2.6) we have $\zeta' = \zeta$ and $\eta' = \eta$.
Thus, (ζ,η) is the unique common coupled fixed point of $\Im$, $\daleth$ and $\Re$.
Now, we will show that $\zeta = \eta$. Consider

$$
\begin{aligned}
\Xi(\zeta,\zeta,\eta,\kappa\lambda) &= \Xi(\Im(\zeta,\eta),\daleth(\zeta,\eta),\Re(\eta,\zeta),\kappa\lambda) \geq \min\{\Xi(\zeta,\zeta,\eta,\lambda),\Xi(\eta,\eta,\zeta,\lambda),1,1,1\}\\
&= \Xi(\zeta,\zeta,\eta,\lambda),\\
\Theta(\zeta,\zeta,\eta,\kappa\lambda) &= \Theta(\Im(\zeta,\eta),\daleth(\zeta,\eta),\Re(\eta,\zeta),\kappa\lambda) \leq \max\{\Theta(\zeta,\zeta,\eta,\lambda),\Theta(\eta,\eta,\zeta,\lambda),0,0,0\}\\
&= \Theta(\zeta,\zeta,\eta,\lambda) \text{ and}\\
\Upsilon(\zeta,\zeta,\eta,\kappa\lambda) &= \Upsilon(\Im(\zeta,\eta),\daleth(\zeta,\eta),\Re(\eta,\zeta),\kappa\lambda) \leq \max\{\Upsilon(\zeta,\zeta,\eta,\lambda),\Upsilon(\eta,\eta,\zeta,\lambda),0,0,0\}\\
&= \Upsilon(\zeta,\zeta,\eta,\lambda).
\end{aligned}
$$

Hence $\zeta = \eta$. Thus, $\Im$, $\daleth$ and $\Re$ have a unique common coupled fixed point of the form (ζ,ζ).

References

1. Abbas, M., Ali Khan, M., S. Radenović, Common Coupled Fixed Point Theorems in Cone Metric Spaces for w-compatible Mappings, Applied Mathematics and Computation, 217(1)(2010).
2. Atanassov, K., Intuitionistic Fuzzy Sets, Fuzzy Sets and Systems, 20(1986), 87-96.
3. Bhaskar, T.G., Lakshmikantham, V., Fixed Point Theorems in Partially Ordered Metric Spaces and Applications, Nonlinear Anal., TMA 65(2006), 1379-1397.
4. Boyce, W. M., Commuting Functions with No Common Fixed Point. Transactions of the American Mathematical Society, 137(1969), 77–92.
5. Eilenberg, S., Montgomery, D., Fixed Point Theorems for Multi-Valued Transformations. American Journal of Mathematics, 68(2)(1946), 214.
6. Guo, D., Lakshmikantham, V., Coupled Fixed Points of Nonlinear Operators with Applications, Nonlinear Anal.Theory Methods Appl. 11(1987), 623–632.
7. J. P. Huneke, Two Counterexamples to a Conjecture on Commuting Continuous Functions of the Closed Unit Interval, Abstract no. 67T-2311, Notices Amer. Math. Soc. 14 (1967), 284.
8. Isbell, J. R., Commuting Mappings of Trees, Research Problem 7, Bull. Amer. Math. Soc. 63(1957), 419.

9. Jungck, G., Rhoades, B.E., Fixed Points for Set Valued Functions without Continuity. Indian Journal of Pure and Applied Mathematics, 29(1998), 227-238.
10. Kirişci, M., Şimşek, N. Neutrosophic Metric Spaces, Math Sci 14,(2020) 241–248.
11. Lakshmikantham, V., Ciric, L., Coupled Fixed Point Theorems for Nonlinear Contractions in Partially Ordered Metric Spaces, Nonlinear Analysis: Theory, Methods and Appl., 70(12)(2009), 4341-4349.
12. Mohiuddine, S.A., Alotaibi, A., Coupled Coincidence Point Theorems for Compatible Mappings in Partially Ordered Intuitionistic Generalized Fuzzy Metric Spaces, Fixed Point Theory and Applications, 2013(1), 265.
13. Mustafa, Z., Sims, B., A New Approach to Generalized Metric Spaces, J.nonlinear Convex Anal. 7(2)(2006), 289-297.
14. Mustafa, Z., Shatanawi, W., Bataineh, M, Existence of Fixed Point Results in G-metric spaces, International Journal of Mathematical Sciences, (2009), 10 pages.
15. Rao, K.P.R., Lakshmi, K.B., Mustafa, Z., A Unique Common Fixed Point Theorem for Six Maps in G-Metric Spaces, Int. J. nonlinear Anal. Appl. 3(2012) 17-23.
16. Rao. K.P.R., Altun, I., Hima Bindu, S., Common Coupled Fixed Point Theorems in Generalized Fuzzy Metric Spaces, Hindawi Publishing Corporation, Advances in Fuzzy systems, (2011), 6 pages.
17. Rashno, E., Akbari, A., Nasersharif, B., A Convolutional Neural Network Model Based on Neutrosophy for Noisy Speech Recognition, Infinite Study (2019).
18. Smarandache, F., Neutrosophy/Neutrosophic Probability, Set, and Logic, American Research Press, 1998.
19. Smarandache, F., Neutrosophic set, A Generalization of Intuitionistic Fuzzy Sets, Int. J. Pure Appl. Math., 24(2005), 287-297.
20. Smarandache, F., Neutrosophy, a new Branch of Philosophy, Infinite Study, 2002.
21. Sowndrarajan, S., Jeyaraman, M., Smarandache, F., Fixed Point Results for Contraction Theorems in Neutrosophic Metric Spaces, Neutrosophic Sets and Systems, 36(2020),308-318.
22. Schweitzer, B., Sklar, A., Statistical Metric Spaces, Pac. J. Math., 10(1960), 313-334.
23. Suganthi M., Jeyaraman M., A Generalized neutrosophic Metric Space and Coupled Coincidence Point Results, Neutrosophic Sets and Systems, 42(2021), 253-269.
24. Sun, G., Yang, K., Generalized Fuzzy Metric Spaces with Properties, Res. J. Appl. Sci., 2(2010), 673-678.
25. Zead Mustafa, Hamed Obiedat and Fadi Awawdeh, Some Fixed Point Theorem for Mapping on Complete G-Metric Spaces, Fixed Point Theory and Applications, Article ID 189870(2008), 12 pages.
26. Zadeh, L.A., Fuzzy sets, Inform and control, 8(1965), 338-353.

[1] PG and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamil nadu, India.

Email address: jeya.math@gmail.com, ORCID: orcid.org/0000-0002-0364-1845

[2] Department of Mathematics, Govt. Arts College, Melur, India.

Email address: vimalsugan@gmail.com