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NEUTROSOPHIC FUZZY I-CONVERGENT FIBONACCI DIFFERENCE SEQUENCE SPACES

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Abstract. In this work, we analyze the Fibonacci difference matrix and use it to develop new sequence spaces in the neutrosophic normed spaces (NNS). This study also includes a detailed analysis of several important properties of these spaces, including their linearity property, first countability and Hausdorffness.

1. **Introduction.** The theory of fuzzy sets was first introduced by [23] which is a generalization of classical set theory in situations where data exhibit ambiguity or imprecision in their membership. Fuzzy sets are valuable in various fields, including artificial intelligence, control systems and decision-making. Subsequently, numerous authors [9, 7] investigated the study of fuzzy topology to define fuzzy metric space. Later, [1] extended the idea of fuzzy sets to introduced intuitionistic fuzzy sets (IFS), which deal with both the degree of membership as well as non-membership functions of an element within a set. In the field of fuzzy theory, recent work has focused on intuitionistic fuzzy normed space (IFNS) [18] and 2-normed space [16].

The statistical convergence proposed by [8] is the most intriguing generalization of the notion of classical convergence of sequences. The idea was subsequently explored further by [19, 20], as well as by [22], among several other researchers.

Recently, [21] introduced a novel extension of intuitionistic fuzzy sets known as neutrosophic sets (NS). Subsequently, this concept was further utilised to define neutrosophic metric space [15] and neutrosophic soft linear spaces [3]. [2] introduced the notion of "Neutrosophic Norm" in their paper, extending the concepts of norms to neutrosophic spaces. Additionally, they defined sequential concepts such as convergence, Cauchy sequence and convexity within these neutrosophic spaces. We represent the vector space of all real sequences by ω. The classes of all bounded, convergent, and null sequences are denoted by l_∞, c and c_0, respectively, throughout the work with norm $\|b\|_\infty = \sup_{j\in N} |b_j|$. Kizmaz introduced the concept of "difference sequence spaces," which is defined by:

$$\vartheta(\Delta) = \{b = b_j \in \omega : (b_j - b_{j+1}) \in \vartheta\}, \qquad \text{for } \vartheta \in \{l_\infty,\ c,\ c_0\}.$$

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Recently, numerous authors have adopted a novel approach for constructing sequence spaces based on matrix domains [4, 5]. Recently, there has been an investigation into the difference sequence space by [10].

$l_\infty(\mathbf{F}) = \{b = b_j \in \omega : \sup_{j\in N} |\frac{f_j}{f_{j+1}} b_j - \frac{f_{j+1}}{f_j} b_{j-1}| < \infty\}$,
where $\mathbf{F} = (f_{jn})$ is Fibonacci difference matrix which is defined by :

$$f_{jn} = \begin{cases} -\frac{f_{j+1}}{f_j} & n = j-1, \\ \frac{f_j}{f_{j+1}} & n = j, \\ 0 & 0 \leq n < j-1 \text{ or } n > j, \end{cases}$$

Where f_j, $j \in \mathbb{N}$ represent the sequence of Fibonacci numbers, which is defined by the linear recurrence relation as $f_0 = 1 = f_1$ and $f_{j-1} + f_{j-2} = f_j$ for $j \geqslant 2$. [13] has recently introduced the concept of I-convergent of Fibonacci difference sequence spaces denoted as $c_0^I(\mathbf{F})$, $c^I(\mathbf{F})$ and $l_\infty^I(\mathbf{F})$. Fibonacci numbers find practical applications in various domains, including the arts, sciences and architecture. The following is a reference for more details: [6, 11].

The primary objective of this paper is to introduce novel sequence spaces in NNS using the Fibonacci difference matrix. Additionally, we explore and analyze their properties, including linearity property and Hausdorffness.

2. **Preliminaries.** First, we go over a few definitions that are essential in this work. The sets of natural numbers and real numbers, respectively, are indicated throughout the work by the symbols $\mathbb{N}$ and $\mathbb{R}$.

Definition 2.1. "[3] *A continuous triangular norm (t-norm) is a binary operation on the interval $I_1 = [0, 1]$ satisfying the following conditions:*

(1) $\oslash$ is continuous, associative, commutative,
(2) $z \oslash 1 = z$ for every $z \in I_1$,
(3) $z_1 \oslash z_2 \leq w_1 \oslash w_2$ whenever $z_1 \leq w_1$ and $z_2 \leq w_2$ for each z_1, z_2, w_1 and $w_2 \in I_1$."

Example.

(i) $z_1 \oslash z_2 = z_1.z_2$ *(the ordinary product of real numbers)*
(ii) $z_1 \oslash z_2 = min(z_1, z_2)$

where z_1, $z_2 \in I_1$.

Definition 2.2. "[3] *A continuous triangular conorm (t-conorm) is a binary operation on the interval $I_1 = [0, 1]$ satisfying the following conditions:*

(1) $\otimes$ is continuous, associative, commutative,
(2) $z \otimes 0 = z$ for every $z \in I_1$,
(3) $z_1 \otimes z_2 \leq w_1 \otimes w_2$ whenever $z_1 \leq w_1$ and $z_2 \leq w_2$ for each z_1, z_2, w_1 and $w_2 \in I_1$.

Example.

(i) $z_1 \otimes z_2 = max(z_1, z_2)$
(ii) $z_1 \otimes z_2 = min(z_1 + z_2, 1)$

where $z_1, z_2 \in I_1$."

Remark 1. 1. [(i)]
2. For $a_1, a_2 \in (0, 1)$ with $a_1 > a_2$, there exists $b_1, b_2 \in (0, 1)$ such that $a_1 \oslash b_1 \geqslant a_2$ and $a_1 \geqslant b_2 \otimes a_2$.
3. For any $c \in (0, 1)$, there exist $d, e \in (0, 1)$ such that $d \oslash d \geqslant c$ and $c \geqslant e \otimes e$.

Definition 2.3. "[14] Given that $\mathbb{N}$ represents the set of natural numbers and $A_\epsilon \subset \mathbb{N}$ then,

$d(A_\epsilon) = \lim_{n\to\infty} \left(\frac{1}{n}\right) |\{j \leq n : j \in A_\epsilon\}|$ is defined as the natural density of the set A_ϵ, whenever the limit exists, where the vertical bar signifies the cardinality of the set A_ϵ.

Definition 2.4. "[13] For each $\epsilon > 0$, a sequence $b = (b_j) \in \omega$ is said to be statistically convergent to $\eta \in \mathbb{R}$, if the set T has natural density zero where,

$T = \{j \leq n : |b_j - \eta| \geqslant \epsilon\}$. That is $d(T) = 0$

The convergent in this case is denoted by $S_T - \lim b = \eta$. If $\eta = 0$, then $b = (b_j)$ is called $S_T - null$."

Example. Define a sequence as,

$$b_j = \begin{cases} j & \text{if } j = 4^n, n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

and $\eta = 0$ then,

$$\lim_{n\to\infty} \left(\frac{1}{n}\right) |\{j \leq n : |b_j| \geqslant \epsilon\}| \leqslant \lim_{n\to\infty} \left(\frac{1}{n}\right) \{\frac{\log n}{\log 4}\} = 0, \quad for\ every\ \epsilon > 0.$$

This implies that the sequence $b = (b_j)$ is statistically convergent to 0. But the sequence $b = (b_j)$ does not converge to 0 in ordinary sense.

Definition 2.5. "[13] A sequence $b = (b_j) \in \omega$ is known as statistically Cauchy sequence if for every $\epsilon > 0$, $\exists$ a number $N = N(\epsilon)$ such that the set T_c has natural density zero where ,

$T_c = |\{j \leq n : |b_j - b_N| \geq \epsilon\}|$. That is $d(T_c) = 0$."

Definition 2.6. "[19] Let $S \neq \phi$. A class $I \subset 2^S$ of subsets of S is said to be an ideal in S if

(a) $\emptyset \in I$,
(b) $B_1, B_2 \in I$ imply $B_1 \cup B_2 \in I$,
(c) $B_1 \in I$, $B_2 \subset B_1$ imply $B_2 \in I$.

Definition 2.7. "[13] Let $S \neq \phi$. A non-empty class $\varphi(I) \subset 2^S$ of subsets of S is said to be filter in S if

(a) $\emptyset \notin \varphi(I)$,
(b) $B_1, B_2 \in \varphi(I)$ imply $B_1 \cap B_2 \in \varphi(I)$,
(c) $B_1 \in \varphi(I)$, $B_2 \supset B_1$ imply $B_2 \in \varphi(I)$.

An ideal I is called non-trivial if $I \neq \emptyset$ and $S \notin I$. A non-trivial ideal $I \subset P(S)$ is known as admissible ideal in S if and only if it contains all singletons, i.e., if it contains $\{\{x\} : x \in S\}$.

Definition 2.8. "[13] A sequence $b = (b_j) \in \omega$ is said to be I- convergent to $\eta \in \mathbb{R}$ if for any $\epsilon > 0$, the set $\{j \leq n : |b_j - \eta| \geqslant \epsilon\} \in I$. The convergent in this case is denoted by I-lim $b = \eta$. If $\eta = 0$, then $b = (b_j)$ is said to be I-null."

Definition 2.9. "[13] If there is a number $N = N(\epsilon)$ such that for any $\epsilon > 0$, the set $\{j \in N : |b_j - b_N| \geq \epsilon\} \in I$, then the sequence $b = (b_j) \in \omega$ is known as I-Cauchy."

Definition 2.10. "[12] Let $\mathbb{Y}$ be a NNS, the sequence $b = (b_j)$ in $\mathbb{Y}$ is said to be convergent to η iff there exists $m \in \mathbb{N}$, with respect NN if for every $0 < \epsilon < 1$ and $\gamma > 0$ such that

$$\sigma(b_j - \eta, \gamma) > 1 - \epsilon, \Upsilon(b_j - \eta, \gamma) < \epsilon \ and \ \Lambda(b_j - \eta, \gamma) < \epsilon \ for \ all \ j \geq m, i.e.,$$

$$\lim_{j\to\infty} \sigma(b_j - \eta, \gamma) = 1, \lim_{j\to\infty} \Upsilon(b_j - \eta, \gamma) = 0 \ and \lim_{j\to\infty} \Lambda(b_j - \eta, \gamma) = 0 \ for \ all \ \gamma > 0.$$

In such a case, the convergence in this space is denoted by $N - \lim_{j\to\infty} b_j = \eta$."

Definition 2.11. "[12] Let $\mathbb{Y}$ be a NNS, the sequence $b = (b_j)$ is said to be a Cauchy sequence with respect to NN if for each $0 < \epsilon < 1$ and $s > 0$ there exist $m \in \mathbb{N}$ such that $\sigma(b_j - b_l, \gamma) > 1 - \epsilon, \Upsilon(b_j - b_l, \gamma) < \epsilon$ and $\Lambda(b_j - b_l, \gamma) < \epsilon$ for all $j, l \geq m$."

Definition 2.12. "[14] Consider a NNS $\mathbb{Y}$. A sequence $b = (b_j)$ is said to be I-convergent to η with respect to NN (IC-NN) if, for every $\epsilon > 0$ and $\gamma > 0$, the following holds:

$\{j \in N : \sigma(b_j - \eta, \gamma) \leqslant 1 - \epsilon$ or $\Upsilon(b_j - \eta, \gamma) \geqslant \epsilon, \Lambda(b_j - \eta, \gamma) \geqslant \epsilon\} \in I$.

In this instance, we express the limit of the sequence of IC-NN as $I_N - \lim(b_j) = \eta$. The notation I_N will represnt the set of IC-NN."

Definition 2.13. "[13] Let I be an admissible ideal. A sequence $b = (b_j) \in \omega$ is said to be Fibonacci I-convergent to $\eta \in \mathbb{R}$, if for every $\epsilon > 0$, the set $\{j \in N : |\mathbf{F}_j(b) - \eta| \geq \epsilon\} \in I$."

Definition 2.14. "[13] Let I be an admissible ideal. The Fibonacci sequence $b = (b_j) \in \omega$ is said to be I-cauchy if for every $\epsilon > 0$, $\exists N = N(\epsilon)$ such that $\{j \in N : |\mathbf{F}_j(b) - \mathbf{F}_N(b)| \geq \epsilon\} \in I$."

3. Main work.

The $\mathbf{F}$ transform of a new class of sequence spaces that we introduce in the this section is I-convergent with regard to the neutrosophic norm. We also demonstrate several of these areas' characteristics, such as their Hausdorffness and first countability. In this work, notation I refers to the admissible ideal. We define

$$\begin{aligned}\chi^I_{0(\sigma,\Upsilon,\Lambda)}(\mathbf{F}) =& \{b = (b_j) \in l_\infty : \{j \in \mathbb{N} : \sigma(\mathbf{F}_j(b), t) \\ &\leq 1 - \epsilon \text{ or } \Upsilon(\mathbf{F}_j(b), t) \geq \epsilon, \ \Lambda(\mathbf{F}_j(b), t) \geq \epsilon\} \in I\}\end{aligned}$$

$$\begin{aligned}\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F}) =& \{b = (b_j) \in l_\infty : \{j \in \mathbb{N} : \sigma(\mathbf{F}_j(b) - \eta, t) \\ &\leq 1 - \epsilon \text{ or } \Upsilon(\mathbf{F}_j(b) - \eta, t) \geq \epsilon, \ \Lambda(\mathbf{F}_j(b) - \eta, t) \geq \epsilon\} \in I\}\end{aligned}$$

We shall now introduce the definition of an open ball with centre b and radius ρ with respect to t in the following manner:

$$\begin{aligned}\beta_b(\rho, t)(\mathbf{F}) = \{c = (c_j) \in l_\infty : \{j \in \mathbb{N} : &\sigma(\mathbf{F}_j(b) - \mathbf{F}_j(c), t) \\ > 1 - \rho \text{ or } &\Upsilon(\mathbf{F}_j(b) - \mathbf{F}_j(c), t) < \rho, \\ &\Lambda(\mathbf{F}_j(b) - \mathbf{F}_j(c), t) < \rho\}.\end{aligned}$$

Theorem 3.1. *The spaces $\chi^I_{0(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ and $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ are linear spaces over $\mathbb{R}$.*

Proof. The proof for another space will continue similarly after we show the result for $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$. Let $b = (b_j)$ and $c = (c_j) \in \chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$. Thus, by definition, η_1 and η_2 exist, and for each ϵ, $t > 0$, we have $A = \{j \in \mathbb{N} : \sigma(\mathbf{F}_j(b) - \eta_1, \frac{t}{2|\alpha_1|}) \leq$

$1-\epsilon$ or $\Upsilon(\mathbf{F}_j(b)-\eta_1, \frac{t}{2|\alpha_1|}) \geq \epsilon,\ \Lambda(\mathbf{F}_j(b)-\eta_1, \frac{t}{2|\alpha_1|}) \geq \epsilon\} \in I$,
$B = \{j \in \mathbb{N} : \sigma(\mathbf{F}_j(c)-\eta_2, \frac{t}{2|\alpha_2|}) \leq 1-\epsilon$ or $\Upsilon(\mathbf{F}_j(c)-\eta_2, \frac{t}{2|\alpha_2|}) \geq \epsilon,\ \Lambda(\mathbf{F}_j(c)-\eta_2, \frac{t}{2|\alpha_2|}) \geq \epsilon\} \in I$
where α_1 and α_2 are scalars.
$A^c = \{j \in \mathbb{N} : \sigma(\mathbf{F}_j(b)-\eta_1, \frac{t}{2|\alpha_1|}) > 1-\epsilon$ or $\Upsilon(\mathbf{F}_j(b)-\eta_1, \frac{t}{2|\alpha_1|}) < \epsilon,\ \Lambda(\mathbf{F}_j(b)-\eta_1, \frac{t}{2|\alpha_1|}) < \epsilon\} \in \varphi(I)$,
$B^c = \{j \in \mathbb{N} : \sigma(\mathbf{F}_j(c)-\eta_2, \frac{t}{2|\alpha_2|}) > 1-\epsilon$ or $\Upsilon(\mathbf{F}_j(c)-\eta_2, \frac{t}{2|\alpha_2|}) < \epsilon,\ \Lambda(\mathbf{F}_j(c)-\eta_2, \frac{t}{2|\alpha_2|}) < \epsilon\} \in \varphi(I)$

Therefore $L \in I$ where $L = A \cup B$. As a result $L^c \in \varphi(I)$ and isn't empty. We are going to show $L^c \subset \{j \in \mathbb{N} : \sigma(\alpha_1\mathbf{F}_j(b)+\alpha_2\mathbf{F}_j(c)-(\alpha_1\eta_1+\alpha_2\eta_2), t) > 1-\epsilon$ or $\Upsilon(\alpha_1\mathbf{F}_j(b)+\alpha_2\mathbf{F}_j(c)-(\alpha_1\eta_1+\alpha_2\eta_2), t) < \epsilon,\ \Lambda(\alpha_1\mathbf{F}_j(b)+\alpha_2\mathbf{F}_j(c)-(\alpha_1\eta_1+\alpha_2\eta_2), t) < \epsilon\}$.

Let $n \in L^c$. Then

$$\sigma(\mathbf{F}_j(b)-\eta_1, \frac{t}{2|\alpha_1|}) > 1-\epsilon \text{ or } \Upsilon(\mathbf{F}_j(b)-\eta_1, \frac{t}{2|\alpha_1|}) < \epsilon,\ \Lambda(\mathbf{F}_j(b)-\eta_1, \frac{t}{2|\alpha_1|}) < \epsilon,$$

$$\sigma(\mathbf{F}_j(c)-\eta_2, \frac{t}{2|\alpha_2|}) > 1-\epsilon \text{ or } \Upsilon(\mathbf{F}_j(c)-\eta_2, \frac{t}{2|\alpha_2|}) < \epsilon,\ \Lambda(\mathbf{F}_j(c)-\eta_2, \frac{t}{2|\alpha_2|}) < \epsilon$$

Consider

$$\begin{aligned}\sigma(\alpha_1\mathbf{F}_j(b)+\alpha_2\mathbf{F}_j(c)-(\alpha_1\eta_1+\alpha_2\eta_2), t) &\geqslant \sigma(\alpha_1\mathbf{F}_j(b)-\alpha_1\eta_1, \frac{t}{2}) \oslash \sigma(\alpha_2\mathbf{F}_j(c)-\alpha_2\eta_2, \frac{t}{2}) \\ &= \sigma(\mathbf{F}_j(b)-\eta_1, \frac{t}{2|\alpha_1|}) \oslash \sigma(\mathbf{F}_j(c)-\eta_2, \frac{t}{2|\alpha_2|}) \\ &> (1-\epsilon) \oslash (1-\epsilon) = (1-\epsilon)\end{aligned}$$

$$\begin{aligned}\Upsilon(\alpha_1\mathbf{F}_j(b)+\alpha_2\mathbf{F}_j(c)-(\alpha_1\eta_1+\alpha_2\eta_2), t) &\leqslant \Upsilon(\alpha_1\mathbf{F}_j(b)-\alpha_1\eta_1, \frac{t}{2}) \otimes \Upsilon(\alpha_2\mathbf{F}_j(c)-\alpha_2\eta_2, \frac{t}{2}) \\ &= \Upsilon(\mathbf{F}_j(b)-\eta_1, \frac{t}{2|\alpha_1|}) \otimes \Upsilon(\mathbf{F}_j(c)-\eta_2, \frac{t}{2|\alpha_2|}) \\ &< \epsilon \otimes \epsilon = \epsilon\end{aligned}$$

$$\begin{aligned}\Lambda(\alpha_1\mathbf{F}_j(b)+\alpha_2\mathbf{F}_j(c)-(\alpha_1\eta_1+\alpha_2\eta_2), t) &\leqslant \Lambda(\alpha_1\mathbf{F}_j(b)-\alpha_1\eta_1, \frac{t}{2}) \otimes \Upsilon(\alpha_2\mathbf{F}_j(c)-\alpha_2\eta_2, \frac{t}{2}) \\ &= \Lambda(\mathbf{F}_j(b)-\eta_1, \frac{t}{2|\alpha_1|}) \otimes \Upsilon(\mathbf{F}_j(c)-\eta_2, \frac{t}{2|\alpha_2|}) \\ &< \epsilon \otimes \epsilon = \epsilon\end{aligned}$$

$L^c \subset \{j \in \mathbb{N} : \sigma(\alpha_1\mathbf{F}_j(b)+\alpha_2\mathbf{F}_j(c)-(\alpha_1\eta_1+\alpha_2\eta_2), t) > 1-\epsilon$ or $\Upsilon(\alpha_1\mathbf{F}_j(b)+\alpha_2\mathbf{F}_j(c)-(\alpha_1\eta_1+\alpha_2\eta_2), t) < \epsilon,\ \Lambda(\alpha_1\mathbf{F}_j(b)+\alpha_2\mathbf{F}_j(c)-(\alpha_1\eta_1+\alpha_2\eta_2), t) < \epsilon\}$. $L^c \in \varphi(I)$ so its compliment belongs to I. $\Rightarrow (\alpha_1 b+\alpha_2 c) \in \chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$. Hence $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ is a linear space over $\mathbb{R}$. □

Theorem 3.2. *Every open ball $\beta_b(\rho,t)(\mathbf{F})$ is an open set in $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$.*

Proof. The open ball has been defined in the following manner:

$$\begin{aligned}\beta_b(\rho,t)(\mathbf{F}) = \{c = (c_j) \in l_\infty : \{&j \in \mathbb{N} : \sigma(\mathbf{F}_j(b)-\mathbf{F}_j(c), t) \\ &> 1-\rho \text{ or } \Upsilon(\mathbf{F}_j(b)-\mathbf{F}_j(c), t) < \rho,\ \Lambda(\mathbf{F}_j(b)-\mathbf{F}_j(c), t) < \rho\}\end{aligned}$$

$$\begin{aligned}
&\textit{Let } d = (d_j) \in \beta_b(\rho, t)(\mathbf{F}) \text{ so that } \sigma(\mathbf{F}_j(b) - \mathbf{F}_j(d), t) \\
&\quad > 1 - \rho \text{ or } \Upsilon(\mathbf{F}_j(b) - \mathbf{F}_j(d), t) \\
&\quad < \rho, \Lambda(\mathbf{F}_j(b) - \mathbf{F}_j(d), t) < \rho
\end{aligned}$$

Then there exists $t_1 \in (0, t)$ with $\sigma(\mathbf{F}_j(b) - \mathbf{F}_j(d), t_1) > 1 - \rho$ or $\Upsilon(\mathbf{F}_j(b) - \mathbf{F}_j(d), t_1) < \rho, \Lambda(\mathbf{F}_j(b) - \mathbf{F}_j(d), t_1) < \rho$

Put $l_0 = \sigma(\mathbf{F}_j(b) - \mathbf{F}_j(d), t_1)$, so we have $l_0 > 1 - \rho$, then there exists $h \in (0, 1)$ such that $l_0 > 1 - h > 1 - \rho$. Using remark 1(2), given $l_0 > 1 - h$, we can find $h_1, h_2, h_3 \in (0, 1)$ with $l_0 \oslash h_1 > 1 - h$, $(1 - l_0) \otimes (1 - h_2) < h$ and $(1 - l_0) \otimes (1 - h_3) < h$. Let $h_4 = max(h_1, h_2, h_3)$.

Now we consider the open ball $\beta_x(1 - h_4, t - t_1)(\mathbf{F})$. We will show that $\beta_x(1 - h_4, t - t_1)(\mathbf{F}) \subset \beta_b(\rho, t)(\mathbf{F})$.

Let $y = (y_j) \in \beta_x(1 - h_4,\ t - t_1)(\mathbf{F})$. Hence

$$\begin{aligned}
\sigma(\mathbf{F}_j(b) - \mathbf{F}_j(y), t) &\geqslant \sigma(\mathbf{F}_j(b) - \mathbf{F}_j(x), t_1) \oslash \sigma(\mathbf{F}_j(x) - \mathbf{F}_j(y), t - t_1) \\
&> l_0 \oslash h_4 \geqslant l_0 \oslash h_1 > 1 - h > 1 - \rho \\
&\Rightarrow \{j \in \mathbb{N} : \sigma(\mathbf{F}_j(b) - \mathbf{F}_j(y), t) > 1 - \rho\}
\end{aligned}$$

$$\begin{aligned}
\Upsilon(\mathbf{F}_j(b) - \mathbf{F}_j(y), t) &\leqslant \Upsilon(\mathbf{F}_j(b) - \mathbf{F}_j(x), t_1) \otimes \Upsilon(\mathbf{F}_j(x) - \mathbf{F}_j(y), t - t_1) \\
&\leqslant (1 - l_0) \otimes (1 - h_4) \leqslant (1 - l_0) \otimes (1 - h_2) \\
&\leqslant h < \rho \\
&\Rightarrow \{j \in \mathbb{N} : \Upsilon(\mathbf{F}_j(b) - \mathbf{F}_j(y), t) < \rho\}
\end{aligned}$$

and

$$\begin{aligned}
\Lambda(\mathbf{F}_j(b) - \mathbf{F}_j(y), t) &\leqslant \Lambda(\mathbf{F}_j(b) - \mathbf{F}_j(x), t_1) \otimes \Upsilon(\mathbf{F}_j(x) - \mathbf{F}_j(y), t - t_1) \\
&\leqslant (1 - l_0) \otimes (1 - h_4) \leqslant (1 - l_0) \otimes (1 - h_3) \\
&\leqslant h < \rho \\
&\Rightarrow \{j \in \mathbb{N} : \Lambda(\mathbf{F}_j(b) - \mathbf{F}_j(y), t) < \rho\}
\end{aligned}$$

Hence $y \in \beta_b(\rho, t)(\mathbf{F})$ and therefore $\beta_x(1 - h_4, t - t_1)(\mathbf{F}) \subset \beta_b(\rho, t)(\mathbf{F})$. □

Remark 2. Let $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ be NNS. Define $\varsigma^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F}) = \{G \subset \chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$: for given $b \in G$, we can find $t > 0$ and $0 < \rho < 1$ such that $\beta_b(\rho, t)(\mathbf{F}) \subset G\}$. Then $\varsigma^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ is a topology on $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$.

Remark 3. Since $\{\beta_b(\frac{1}{n}, \frac{1}{n})(\mathbf{F}) : n \in \mathbb{N}\}$ is a local base at b, the topology $\varsigma^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ is first countable.

Theorem 3.3. *The spaces $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ and $\chi^I_{0(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ are Hausdorff.*

Proof. We are going to show the results for $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$, and another will do the same. Let $b = (b_j)$ and $c = (c_j) \in \chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ such that $b \neq c$. Put $0 < \sigma(\mathbf{F}_j(b) - \mathbf{F}_j(c), t) < 1$, $0 < \Upsilon(\mathbf{F}_j(b) - \mathbf{F}_j(c), t) < 1$ and $0 < \Lambda(\mathbf{F}_j(b) - \mathbf{F}_j(c), t) < 1$. Put $h_1 = \sigma(\mathbf{F}_j(b) - \mathbf{F}_j(c), t)$, $h_2 = \Upsilon(\mathbf{F}_j(b) - \mathbf{F}_j(c), t)$, $h_3 = \Lambda(\mathbf{F}_j(b) - \mathbf{F}_j(c), t)$ and $h = max\{h_1, 1 - h_2, 1 - h_3\}$. Using remark 1(3), for each $h_0 > h$ there exists $h_4, h_5, h_6 \in (0, 1)$ such that $h_4 \oslash h_4 \geqslant h_0$, $(1 - h_5) \otimes (1 - h_5) \leqslant (1 - h_0)$, $(1 - h_6) \otimes (1 - h_6) \leqslant (1 - h_0)$.

Again, putting $h_7 = max\{h_4, h_5, h_6\}$. Consider the open ball $\beta_b(1-h_7, \frac{t}{2})(\mathbf{F})$ and $\beta_c(1-h_7, \frac{t}{2})(\mathbf{F})$ centered at b and c respectively. We show that $\beta_b(1-h_7, \frac{t}{2})(\mathbf{F}) \cap \beta_c(1-h_7, \frac{t}{2})(\mathbf{F}) = \emptyset$.

If possible, let $d = (d_j) \in \beta_b(1-h_7, \frac{t}{2})(\mathbf{F}) \cap \beta_c(1-h_7, \frac{t}{2})(\mathbf{F})$.

Then, for each the set $\{j \in \mathbb{N}\}$, we have

$$\begin{aligned} h_1 &= \sigma(\mathbf{F}_j(b) - \mathbf{F}_j(c), t) \\ &\geqslant \sigma(\mathbf{F}_j(b) - \mathbf{F}_j(d), \frac{t}{2}) \oslash \sigma(\mathbf{F}_j(d) - \mathbf{F}_j(c), \frac{t}{2}) \\ &\geq h_7 \oslash h_7 \geqslant h_4 \oslash h_4 \geqslant h_0 > h > h_1 \end{aligned} \tag{1}$$

$$\begin{aligned} h_2 &= \Upsilon(\mathbf{F}_j(b) - \mathbf{F}_j(c), t) \\ &\leqslant \Upsilon(\mathbf{F}_j(b) - \mathbf{F}_j(d), \frac{t}{2}) \otimes \Upsilon(\mathbf{F}_j(d) - \mathbf{F}_j(c), \frac{t}{2}) \\ &< (1-h_7) \otimes (1-h_7) \leq (1-h_5) \otimes (1-h_5) \\ &< (1-h_0) < h_2. \end{aligned} \tag{2}$$

and

$$\begin{aligned} h_3 &= \Lambda(\mathbf{F}_j(b) - \mathbf{F}_j(c), t) \\ &\leqslant \Lambda(\mathbf{F}_j(b) - \mathbf{F}_j(d), \frac{t}{2}) \otimes \Lambda(\mathbf{F}_j(d) - \mathbf{F}_j(c), \frac{t}{2}) \\ &< (1-h_7) \otimes (1-h_7) \leq (1-h_6) \otimes (1-h_6) \\ &< (1-h_0) < h_3. \end{aligned} \tag{3}$$

From equations $(1), (2)$ and (3) we get a contradiction. Therefore space $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ is a Hausdorff space. □

Theorem 3.4. *Let $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ be NNS and $\varsigma^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ be a topology on $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$. A sequence $(b_j) \in \chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ converges to η iff $\sigma(\mathbf{F}_j(b) - \eta, t) \to 1$ or $\Upsilon(\mathbf{F}_j(b) - \eta, t) \to 0$, $\Lambda(\mathbf{F}_j(b) - \eta, t) \to 0$ as $j \to \infty$.*

Proof. Let $b = (b_j) \to \eta$ in $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$, then given $0 < \rho < 1$ $\exists$ $l_0 \in \mathbb{N}$ such that $b = (b_j) \in \beta_b(\rho, t)(\mathbf{F})$ for all $j \geqslant j_0$ given $t > 0$. As a result, we have $1 - \sigma(\mathbf{F}_j(b) - \eta, t) < \rho$ or $\Upsilon(\mathbf{F}_j(b) - \eta, t) < \rho, \Lambda(\mathbf{F}_j(b) - \eta, t) < \rho$.

Therefore $\sigma(\mathbf{F}_j(b) - \eta, t) \to 1$ or $\Upsilon(\mathbf{F}_j(b) - \eta, t) \to 0, \Lambda(\mathbf{F}_j(b) - \eta, t) \to 0$.

Conversely, if $\sigma(\mathbf{F}_j(b) - \eta, t) \to 1$ or $\Upsilon(\mathbf{F}_j(b) - \eta, t) \to 0, \Lambda(\mathbf{F}_j(b) - \eta, t) \to 0$ as $j \to \infty$ holds for all $t > 0$. For $0 < \rho < 1$, $\exists$ $l_0 \in \mathbb{N}$ such that $1 - \sigma(\mathbf{F}_j(b) - \eta, t) < \rho$ or $\Upsilon(\mathbf{F}_j(b) - \eta, t) < \rho$, $\Lambda(\mathbf{F}_j(b) - \eta, t) < \rho$ for all $j \geqslant l_0$. $\Rightarrow \sigma(\mathbf{F}_j(b) - \eta, t) > 1 - \rho$ or $\Upsilon(\mathbf{F}_j(b) - \eta, t) < \rho$, $\Lambda(\mathbf{F}_j(b) - \eta, t) < \rho$. Thus $b = (b_j) \in \beta_b(\rho, t)(\mathbf{F})$ for each $j \geqslant l_0$ and hence $b_j \to \eta$. □

Theorem 3.5. *Every finite subset of $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ is compact.*

Proof. Let $\aleph = \{b_1, b_2, b_3, ..., b_n\}$ be the finite subset of $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$. For $0 < \rho < 1$, $t > 0$, let us assume $\{\beta_b(\rho, t)(\mathbf{F}) : b \in \aleph\}$ is an open cover of $\aleph$. Then $\aleph \subseteq \cup_{b \in \aleph} \beta_b(\rho, t)(\mathbf{F})$. Now for all $b_i \in \aleph$, $i = 1, \dots n$, we have $b_i \in \cup_{b_i \in \aleph} \beta_{b_i}(\rho, t)(\mathbf{F})$. That implies $b_i \in \beta_{b_j}(\rho, t)(\mathbf{F})$ for some $j \in \{1, \dots n\}$. Then $\{\beta_{b_i}(\rho, t)(\mathbf{F}); i = 1, \dots n\}$ is a finite subcover of $\aleph$. □

Theorem 3.6. *Every compact subset of Hausdorff space $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ is closed.*

Proof. Let $\aleph$ be compact subset of Hausdorff space $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$. Then we have to show that $\aleph \subset \chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ is closed. For this, we will show that $\aleph'$ is open. Let $p \in \aleph'$. Since $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ is Hausdorff space, for every $q \in \aleph$ there exist open balls which is denoted by $\beta_p(\rho,t)(\mathbf{F})$ and $\beta_q(\rho,t)(\mathbf{F})$ with $p \in \beta_p(\rho,t)(\mathbf{F})$, $q \in \beta_q(\rho,t)(\mathbf{F})$ such that $\beta_p(\rho,t)(\mathbf{F}) \cap \beta_q(\rho,t)(\mathbf{F}) = \emptyset$. Now the collection $\{\beta_q(\rho,t)(\mathbf{F}) : q \in \aleph\}$ is an open cover of $\aleph$. Since $\aleph$ is compact then there exists finite number of points q_i; $i = 1,\dots n$ such that $\aleph \subseteq \bigcup_{i=1}^n \beta_{q_i}(\rho,t)(\mathbf{F})$. Now corresponding to each point q_i; $i = 1,\dots n$, let $\beta_{p_i}(\rho,t)(\mathbf{F})$ be the open ball containing p. Let

$$U = \cap_{i=1}^n \beta_{p_i}(\rho,t)(\mathbf{F}),\ V = \cup_{i=1}^n \beta_{q_i}(\rho,t)(\mathbf{F}).$$

Then U is open ball which contain p (being the intersection of a finite number of open set is open). We claim that $U \cap V = \emptyset$. We have,

$$x \in V \Rightarrow x \in \beta_{q_i}(\rho,t)(\mathbf{F})\ for\ some\ i$$

$\Rightarrow x \notin \beta_{p_i}(\rho,t)(\mathbf{F})[\because \beta_{p_i}(\rho,t)(\mathbf{F}) \cap \beta_{q_i}(\rho,t)(\mathbf{F}) = \emptyset]$
$\Rightarrow x \notin U$

Thus $\Rightarrow U \cap V = \emptyset$ and since $\aleph \subset V$, we have $\aleph \cap U = \emptyset$ which implies that $U \subset \aleph'$. This shows that $\aleph'$ contains a nbd of each of its points and so $\aleph'$ is open i.e. $\aleph$ is closed. □

4. Conclusion.

In this article, we have introduce new sequence spaces $\chi^I_{0(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ and $\chi^I_{(\sigma,\Upsilon,\Lambda)}(\mathbf{F})$ in neutrosophic normed space by using Fibonacci difference matrix $\mathbf{F}$ and studied some properties such as linearity property, first countability and Hausdorffness. Such results will provide a fresh way to deal with science and engineering-related issues.

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