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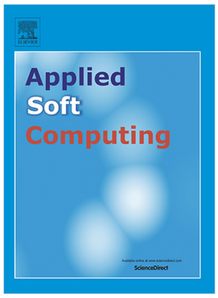

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Interval neutrosophic stochastic dynamical systems driven by Brownian motion

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Abstract

Stochastic and neutrosophic theory are effective tools for modeling real-world phenomena and natural dynamical systems, where inputs are often affected by stochastic noises and outputs often contain both randomness and indeterminacy. In this work, we present a new type of stochastic differential equations (SDE) driven by an one-dimensional Brownian motion that can be considered as an efficient tool to describe the uncertain behavior of dynamical systems operating in interval neutrosophic environments with stochastic noises. After introducing some basic foundations on neutrosophic arithmetic, neutrosophic calculus and neutrosophic stochastic process, we define the new form of interval neutrosophic stochastic differential equations taking values in neutrosophic environment. Under some suitable conditions, the unique existence result of stochastic solution is obtained based on the use of Picard successive approximation. We also introduce an efficient numerical algorithm, namely Euler - Maruyama method, to solve the numerical solution of proposed problem and further demonstrate the effectiveness of the numerical method by solving some examples in stochastic biological systems such as stochastic logistic growth model, stochastic Lotka-Volterra predator-prey model, and stochastic SARS model, respectively.

Keywords: Interval neutrosophic numbers; Interval neutrosophic calculus; Interval neutrosophic stochastic process; Euler - Maruyama method.

1. Introduction

Neutrosophic sets introduced by Smarandache [28] are known as the general framework for unification of many existing sets such as fuzzy sets, intuitionistic fuzzy sets, paraconsistent sets, etc. The main role of neutrosophic sets is to characterize each logical statement and set in a 3D-neutrosophic space, where each dimension of the space represents respectively the truth (T), the falsehood (F) and the indeterminacy (I) of the statement under consideration respectively. Compared to the concepts of intuitionistic fuzzy sets introduced by Atanassov [4], fuzzy sets introduced by Zadeh [47] or other types of uncertainty, it may be asserted that in case of a neutrosophic set, it is possible to determine not only the truth of information but also its indeterminacy/neutrality and falsity degree, which may be defined as new independent components of fuzzy sets. Recently, neutrosophic sets and systems have been rapidly developing and manifesting their importance in many fields of science and technology. Indeed, due to the indeterminacy parameter can help a more detailed definition of membership functions, the usage of neutrosophic sets in decision making can produce better results. On the other hand, by the definition of neutrosophic set, we can assert that it will be very useful in distinguishing absolute values from relative values. In fact, neutrosophic sets may also be used to find the differences between absolute truth and relative truth, absolute falsehood and relative falsehood in logic and, absolute membership/non-membership,

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and relative membership/non-membership. Therefore, many fields in science and engineering have been paid more attention to explore research topics that underpin neutrosophic set and logic, neutrosophic probability and statistics, neutrosophic dynamical system, and modeling. For instance, neutrosophic precalculus and neutrosophic calculus were introduced by Smarandache [31] based on preceding results of interval analysis, while Çevik et. al. [10] studied various results on the algebraic structures of neutrosophic sets. In addition, some other related topics such as neutrosophic measure, neutrosophic probability and and statistics are yet to be explored [29, 30, 42]. Especially, neutrosophic sets and systems have great applicability in various fields of engineering, see for example [3, 11, 19, 27, 38, 37, 44].

Recently, neutrosophic sets and related problems have been gaining popularity and attraction from researchers. In the real-world situations, it is the fact that the degree of truth, indeterminacy and falsity of an arbitrary statement sometimes can not be defined by an exactly value, but it can be given in the form of possible interval value. Thus, it leads to the requirement of studying the problems in the interval neutrosophic set (INS) environment. After the pioneering work of Smarandache, Wang et al. [43] proposed the concept of interval neutrosophic set which is a particular case of neutrosophic sets. Here, each interval neutrosophic set can be described by a membership interval, a non-membership interval and an indeterminate interval. It is worth mentioning that interval-valued neutrosophic set along with single-valued neutrosophic set are two great applicable objects, that have achieved great success in various areas of real-world processes such as medical diagnosis [1], database [2], image processing [49], decision making problem [16, 18, 24] and so on. Some recent considerable literature related to interval neutrosophic sets is reflected by the contribution of Wang et al. [43] with the focus on interval neutrosophic sets and initial foundation on the set-theoretic operators. Another result on the operations of interval neutrosophic sets was proposed in [7, 20, 25]. Furthermore, [46] established the similarity measures between interval neutrosophic sets based on Hamming and Euclidean distance.

When real-world processes are modeled by dynamical systems, there always exist different kinds of uncertainty due to the imprecision of measurement equipment, imperfect human judgments, and opinions on parameters. In fact, differential systems with uncertainty such as fuzzy differential equations or set-valued differential equations have been significantly investigated. However, there are only few studies on intuitionistic fuzzy differential equation or neutrosophic differential equations. Ettoussi et al. [14] studied the existence and uniqueness of the intuitionistic fuzzy differential equation using the successive approximation method. The existence and uniqueness of solution to intuitionistic fuzzy differential equation was also discussed by Melliani [21] by using intuitionistic fuzzy semigroup and the contraction mapping principle. The differential system with initial value as triangular intuitionistic fuzzy number was solved by Modal and Roy [22]. Son et al. [34] introduced the calculus of single-valued neutrosophic functions under granular computing which was further used to analyze classes of neutrosophic differential equations. In the work [26], the authors applied fuzzy set theory to design an adaptive robust control based on a creative uncertainty decomposition for the optimal control design problem of underactuated dynamical systems with uncertainty. Recently, Campagner et al. [8] studied the application of rough set theory and proposed an abstract knowledge representation formalism to represent the uncertainty and partial knowledge in dynamical systems. Furthermore, the probability theory defined on the space of intuitionistic fuzzy numbers has also obtained a great achievement. Indeed, Szmidt and Kacprzyk [41] proposed the concept of an intuitionistic fuzzy event and the probability of an intuitionistic fuzzy event. Here, the proposed probability is such that when considered an intuitionistic fuzzy event becomes a fuzzy event, the proposed interval reduces to the probability of a fuzzy event defined by Zadeh [48]. After that, in [13], the authors extended the notion of fuzzy probabilities by representing probabilities through the intuitionistic fuzzy numbers in sense of Atanassov. Next, Wang et.al. discussed in [45] some characteristic properties of intuitionistic fuzzy sets and introduced the concept of intuitionistic fuzzy random variables. In addition, the expected value of intuitionistic fuzzy random variables was defined and the law of large numbers of intuitionistic fuzzy random variables was also given. However, it can be seen that there is not any work discussed the interval intuitionistic stochastic differential equations.

On the other hand, it is well-known that many phenomena and processes occurring in real-world always contain some types of uncertainty, simultaneously. For example, in structural dynamical systems in biology, the designed systems are constructed from a group of experiment and measurement processes in order to predict the long term behavior of a population. The mathematical modeling process of a desired dynamical system needs to introduce more than one type of uncertainty. Indeed, the random factors added into input data or input variables, such as the size of population at any certain time or the source of food and water, etc., create the random or stochastic dynamic of our considered dynamical systems and make them become stochastic dynamical systems while some other neutral factors, such as internal competition or mild illness, still remain in this population biology. This indeterminacy input causes the output of systems to not be a fixed value as in classical random dynamical systems. Hence, there is a need to model the real-world process by combining both random and neutral factors in a dynamical system instead of a classical random dynamical system. This leads to the foundation of an interesting concept of a dynamical system, namely a neutrosophic random dynamical system.

Motivated by aforesaid, we establish the analysis of random mappings taking values in the set of interval neutrosophic numbers and introduce some new concepts on interval neutrosophic stochastic process, interval neutrosophic stochastic differential equations. The contributions of this paper are as follows:

1. We propose a parametric representation, namely $(\alpha, \beta, \gamma)-$ cuts, of interval neutrosophic numbers. This representation proves its advantage when we define the parametric form of derivative and integral as well as building up numerical algorithms for interval neutrosophic dynamical systems. Next, we introduce the concept of $\rho_\infty -$ distance and some other useful properties. In addition, the concept of interval neutrosophic derivative plays a key role in considering interval neutrosophic differential equations and systems.

2. Secondly, we introduce a new notion of interval neutrosophic stochastic process. Some fundamental concepts and related properties such as random variables, Itô integral, stochastic process are defined in the environment of interval neutrosophic numbers. Then, we can study the Cauchy problem for interval neutrosophic stochastic differential equations. The unique existence of the stochastic solution is presented in Theorem 5.1 based on Picard's successive approximation. After that, we propose the Euler - Maruyama method for numerically solving the considered problem. The convergent of this method is presented in Theorem 5.2.

3. Finally, in order to demonstrate the significance and effectiveness of our theoretical results, we investigate some real-world biological and medical problems modeled by stochastic interval neutrosophic differential equations such as stochastic Malthusian growth model, stochastic Lotka-Volterra predator-prey model and stochastic SARS model in the neutrosophic environment. Some analytical and numerical methods are conducted to solve the exact and approximate stochastic interval neutrosophic solutions of the considered problems. It should be noted that this research directly employs the concepts of derivative and integral calculus as essential tools to investigate dynamical systems of neutrosophic objects, that will open up a valuable approach in studying applied science in an uncertain environment.

The organizational structure of this paper is as follows: In Section 2, we recall some preliminaries on the space of interval neutrosophic sets and interval neutrosophic numbers. Especially, we introduce here the notion of $(\alpha, \beta, \gamma)-$ cuts of interval neutrosophic number as a bridge connected interval neutrosophic arithmetic with set-valued arithmetic. Next, we introduce the calculus of interval neutrosophic mappings in Section 3 with some basic concepts of calculus such as metric, the continuity, differentiability and integrability of interval neutrosophic mappings. In Section 4, we present basic concepts about interval neutrosophic random variables, interval neutrosophic stochastic processes and some of their interesting properties. The next section is devoted to introducing a new class of differential equation, namely, interval neutrosophic stochastic differential equations. The existence and uniqueness theorem as well as the numerical method are also presented. Next, Section 6 introduces some biological real-world problems to illustrate theoretical results. At last, the Conclusions and Appendix are discussed.

2. Interval neutrosophic number

In this section, we recall from [4, 5, 28, 33, 44] some essential concepts of generalized fuzzy sets such as intuitionistic fuzzy sets, interval-valued intuitionistic fuzzy sets, inconsistent intuitionistic fuzzy sets and neutrosophic sets. In addition, with the introduction of the level sets form of neutrosophic numbers, we define the arithmetic operations on the space of neutrosophic sets.

Definition 2.1 ([4]). An intuitionistic fuzzy set (IFS) A, defined on the universe of discourse X, is of the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\},$$

in which the functions $\mu_A : X \to [0,1]$, $\nu_A : X \to [0,1]$ presents the degree of membership and the degree of non-membership of an element $x \in X$ to A. These membership functions are constrained by inequalities

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

A generalization of the notion of intuitionistic fuzzy set was given in the spirit of ordinary interval valued fuzzy sets, namely interval intuitionistic fuzzy set, see [5].

Definition 2.2 ([5]). An interval valued intutionistic fuzzy set (IVIFS) A over X has the form

$$A = \{\langle x, M_A(x), N_A(x)\rangle : x \in X\},$$

where $M_A(x) \subset [0,1]$ and $N_A(x) \subset [0,1]$ are intervals and for every $x \in X$, we have $\sup M_A(x) + \sup N_A(x) \leq 1$.

Definition 2.3 ([33]). An inconsistent intuitionistic fuzzy set (IIFS) A over the universe of discourse X can be represented in following form

$$A = \{\langle x, \mu_A(x), \nu_A(x), \iota_A(x)\rangle : x \in X\},$$

where $\mu_A(x), \nu_A(x), \iota_A(x) \in [0,1]$ are the membership degree, neutral degree, non-membership degree of the element x, respectively, and $0 \leq \mu_A(x) + \nu_A(x) + \iota_A(x) \leq 1$. In addition, the function $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) - \iota_A(x)$ denotes for the refusal degree of the element x.

Smarandache presented some types of advanced fuzzy sets, namely neutrosophic sets, interval neutrosophic sets and interval neutrosophic numbers, which will be considered throughout the paper.

Definition 2.4 ([28]). A neutrosophic set (NS) A, defined on the universe of discourse X and denoted generally by x, can be represented in following form

$$A = \{\langle x, \mathtt{T}_A(x), \mathtt{I}_A(x), \mathtt{F}_A(x)\rangle : x \in X\},$$

in which $\mathtt{T}_A : X \to ^{-}]0,1[^{+}$ presents the degree of confidence which will be called by truth membership function; $\mathtt{I}_A : X \to ^{-}]0,1[^{+}$ presents the degree of uncertainty, namely, indeterminacy membership function; $\mathtt{F}_A : X \to ^{-}]0,1[^{+}$ is the falsity membership function representing the degree of scepticism; and these membership functions are constrained by inequalities $0^{-} \leq \mathtt{T}_A(x) + \mathtt{I}_A(x) + \mathtt{F}_A(x) \leq 3^{+}$.

Remark 2.1. According to Definition 2.2 and Definition 2.4, we can see that three components of a neutrosophic set A are independent and their sum may be up to 3, that shows the importance of independence of the neutrosophic components among themselves, while the IIFS's components $\mu_A(x), \nu_A(x), \iota_A(x), \pi_A(x)$ are dependent concerning each other and their sum is constrained in $[0,1]$.

Definition 2.5 ([44]). Suppose that X is a universal space of points and $x \in X \subset \mathbb{R}$. An interval neutrosophic set A in X can be characterized by three quantities: the truth membership function $\mathtt{T}_A(x)$, indeterminacy membership function $\mathtt{I}_A(x)$ and falsity membership function $\mathtt{F}_A(x)$. In particular, for each $x \in X$, $\mathtt{T}_A(x)$, $\mathtt{I}_A(x)$, $\mathtt{F}_A(x) \in [0,1]$ and we can present the set A as follow

$$A = \left\{\langle x; \left[\mathtt{T}^l_A(x), \mathtt{T}^u_A(x)\right], \left[\mathtt{I}^l_A(x), \mathtt{I}^u_A(x)\right], \left[\mathtt{F}^l_A(x), \mathtt{F}^u_A(x)\right]\rangle : x \in X\right\},$$

where $0 \leq \mathtt{T}^u_A(x) + \mathtt{I}^u_A(x) + \mathtt{F}^u_A(x) \leq 3$. Throughout this work, we will consider a class of interval neutrosophic sets on the real line $\mathbb{R}$ and denote it by $\mathscr{U}$. Such interval neutrosophic set is then called *interval neutrosophic number.*

Example 2.1. Let A be an interval triangular neutrosophic number whose graphical representation is given in Figure 1.

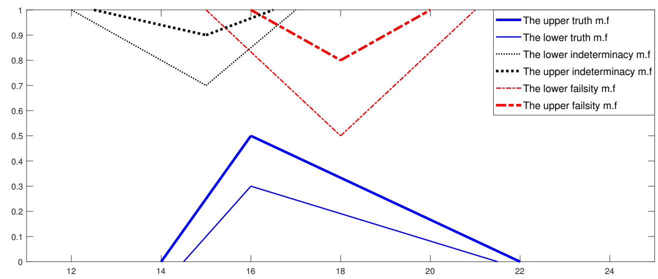

Figure 1: An illustration of interval triangular neutrosophic number

Here, the three characterized membership functions of the interval neutrosophic triangular number A are given in the forms of triangular fuzzy numbers.

$$\mathtt{T}^l_A(x)=\begin{cases}\frac{x}{4}-\frac{7}{2} & x\in[14,16],\\ \frac{1}{2} & x=16,\\ \frac{11}{6}-\frac{x}{12} & x\in[16,22],\\ 0 & \text{otherwise,}\end{cases}\qquad \mathtt{T}^u_A(x)=\begin{cases}\frac{x}{5}-\frac{29}{10} & x\in[14.5,16],\\ \frac{3}{10} & x=16,\\ \frac{129}{110}-\frac{3x}{55} & x\in[16,21.5],\\ 0 & \text{otherwise,}\end{cases}$$

$$\mathtt{I}^l_A(x)=\begin{cases}-\frac{x}{10}-\frac{11}{5} & x\in[12,15],\\ \frac{7}{10} & x=15,\\ \frac{3x}{20}-\frac{31}{20} & x\in[15,17],\\ 1 & \text{otherwise,}\end{cases}\qquad \mathtt{I}^u_A(x)=\begin{cases}-\frac{x}{25}+\frac{3}{2} & x\in[12.5,15],\\ \frac{9}{10} & x=15,\\ \frac{x}{15}-\frac{1}{10} & x\in[15,16.5],\\ 1 & \text{otherwise,}\end{cases}$$

$$\mathtt{F}^l_A(x)=\begin{cases}-\frac{x}{6}+\frac{7}{2} & x\in[15,18],\\ \frac{1}{2} & x=18,\\ \frac{x}{6}-\frac{5}{2} & x\in[18,21],\\ 1 & \text{otherwise,}\end{cases}\qquad \mathtt{F}^u_A(x)=\begin{cases}-\frac{x}{10}+\frac{13}{5} & x\in[16,18],\\ \frac{4}{5} & x=18,\\ \frac{x}{10}-1 & x\in[18,20],\\ 1 & \text{otherwise,}\end{cases}$$

Definition 2.6. Let A be an interval triangular neutrosophic number. Then, the neutrosophic number A can be rewritten in following canonical form

$$\begin{aligned}A=\big\langle\, &\big\{(a_1,b_1,c_1;\lambda_1),(\overline{a}_1,\overline{b}_1,\overline{c}_1;\overline{\lambda}_1)\big\},\big\{(a_2,b_2,c_2;\lambda_2),(\overline{a}_2,\overline{b}_2,\overline{c}_2;\overline{\lambda}_2)\big\},\\ &\big\{(a_3,b_3,c_3;\lambda_3),(\overline{a}_3,\overline{b}_3,\overline{c}_3;\overline{\lambda}_3)\big\}\,\big\rangle,\end{aligned}\tag{1}$$

where $\overline{\lambda}_i\le\lambda_i$, and λ_i, $\overline{\lambda}_i\in[0,1]$ for all $i=\overline{1,3}$. Here, $(a,b,c;\lambda)$ is denoted for a triangular function with parameters shown in Figure 2.

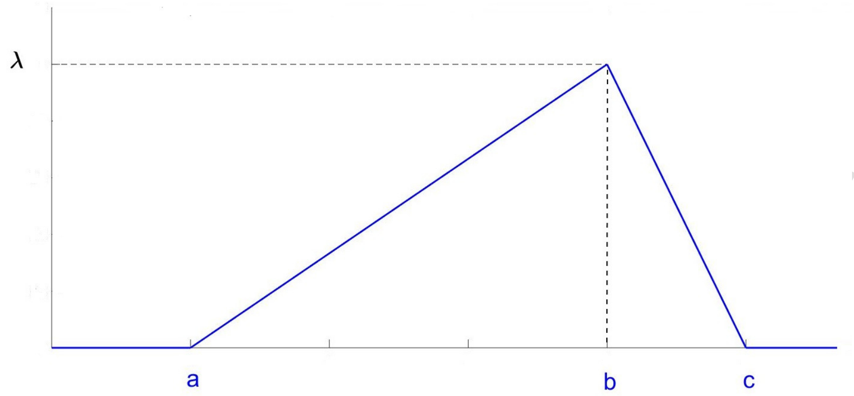

Figure 2: The triangular function $(a, b, c; \lambda)$

The canonical form of the interval triangular neutrosophic number A can be given as follows:

$$A = \Big\langle \{(14, 16, 22; 0.5), (14.5, 16, 21.5; 0.3)\}, \ \{(12, 15, 17; 0.7), (12.5, 15, 16.5; 0.9)\}, \\ \{(15, 18, 21; 0.5), (16, 18, 20; 0.8)\} \Big\rangle.$$

Definition 2.7. Let A be an interval neutrosophic number defined by

$$A = \left\{ \langle x; \left[\mathtt{T}^l_A(x), \mathtt{T}^u_A(x) \right], \left[\mathtt{I}^l_A(x), \mathtt{I}^u_A(x) \right], \left[\mathtt{F}^l_A(x), \mathtt{F}^u_A(x) \right] \rangle : x \in \mathbb{R} \right\}.$$

We define the $(\alpha, \beta, \gamma)-$ cut of A by level sets of each component as follows:

$$[A]_{(\alpha,\beta,\gamma)} = \{x \in \mathbb{R} : \mathtt{T}_A(x) \geq \alpha, \mathtt{I}_A(x) \leq \beta, \mathtt{F}_A(x) \leq \gamma\},$$

where α, β, $\gamma \in [0, 1]$ and $\alpha + \beta + \gamma \leq 3$. In addition, the parametric form of A can be given by

$$[A]_{(\alpha,\beta,\gamma)} = \Bigg[\Big\{ [A^l_-(\alpha), A^l_+(\alpha)]; [A^u_-(\alpha), A^u_+(\alpha)] \Big\}, \Big\{ [A^l_-(\beta), A^l_+(\beta)]; [A^u_-(\beta), A^u_+(\beta)] \Big\}, \\ \Big\{ [A^l_-(\gamma), A^l_+(\gamma)]; [A^u_-(\gamma), A^u_+(\gamma)] \Big\} \Bigg]. \quad (2)$$

Remark 2.2. We can see that the canonical form (1) is equivalent to the parametric form (2). However, in practical terms, the parametric form (2) is more convenient for computing and applicable numerical algorithms than canonical form (1).

Remark 2.3. Since the upper and lower membership functions can be written as triangular forms in fuzzy numbers, an interval neutrosophic number A can be represented in parametric form (2). Then by using Negoita-Ralescu characterization theorem [23], we can obtain the canonical form of the interval neutrosophic number A.

Example 2.2. We apply the formula (2) to determine the parametric form of the interval triangular neutrosophic number A defined in Example 2.1. For this aim, we need to determine the parametric form of each component of the number A. For example, the parametric form of the first component of the number A, consisting of two triangular functions $(14, 16, 22; 0.5)$ and $(14.5, 16, 21.5; 0.3)$, is shown in Figure 3.

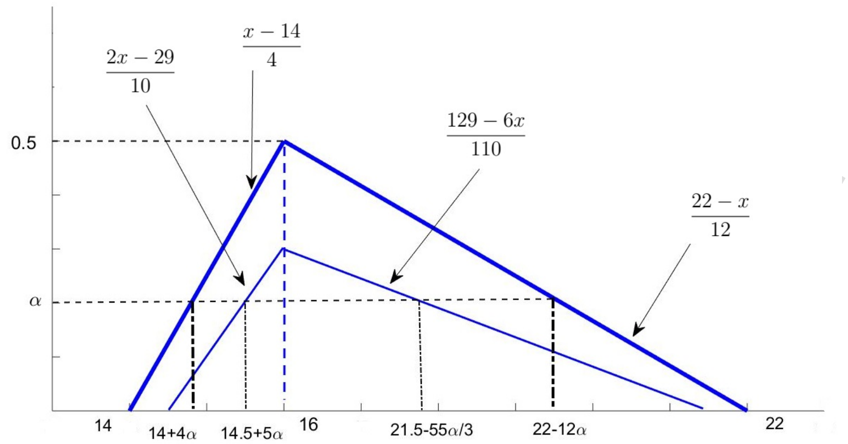

Figure 3: The parametric form of the first component of interval neutrosophic number A

Here, for the $\alpha-$ cuts of the truth membership function $\mathtt{T}_A(x)$, we will find the set $\{x \in \mathbb{R} : \mathtt{T}_A(x) \geq \alpha\}$, which means $\{x \in \mathbb{R} : \mathtt{T}^l_A(x) \geq \alpha\}$ and $\{x \in \mathbb{R} : \mathtt{T}^u_A(x) \geq \alpha\}$. In addition, according to Example 2.1, we have the truth membership function $\mathtt{T}_A(x)$ is

$$\mathtt{T}^l_A(x) = \begin{cases} \frac{x}{4} - \frac{7}{2} & x \in [14, 16], \\ \frac{1}{2} & x = 16, \\ \frac{11}{6} - \frac{x}{12} & x \in [16, 22], \\ 0 & \text{otherwise}, \end{cases} \qquad \mathtt{T}^u_A(x) = \begin{cases} \frac{x}{5} - \frac{29}{10} & x \in [14.5, 16], \\ \frac{3}{10} & x = 16, \\ \frac{129}{110} - \frac{3x}{55} & x \in [16, 21.5], \\ 0 & \text{otherwise}, \end{cases}$$

Therefore, we immediately obtain

$$\left\{x \in \mathbb{R} : \mathtt{T}^l_A(x) \geq \alpha\right\} = [A^l_-(\alpha), A^l_+(\alpha)] = \left[14.5 + 5\alpha, 21.5 - \frac{55}{3}\alpha\right],$$
$$\{x \in \mathbb{R} : \mathtt{T}^u_A(x) \geq \alpha\} = [A^u_-(\alpha), A^u_+(\alpha)] = [14 + 4\alpha, 22 - 12\alpha].$$

Then, the parametric form of the first component is

$$\left\{\left[14.5 + 5\alpha, 21.5 - \frac{55}{3}\alpha\right]; [14 + 4\alpha, 22 - 12\alpha]\right\}.$$

Similarly, it follows that the $(\alpha, \beta, \gamma)-$ cuts of the interval triangular neutrosophic number A is given by

$$[A]_{(\alpha,\beta,\gamma)} = \left[\left\{\left[14.5 + 5\alpha, 21.5 - \frac{55}{3}\alpha\right]; [14 + 4\alpha, 22 - 12\alpha]\right\},\right.$$
$$\left\{\left[15 - \frac{25}{9}\beta, 15 + \frac{5}{3}\beta\right]; \left[15 - \frac{30}{7}\beta, 15 + \frac{20}{7}\beta\right]\right\},$$
$$\left.\{[18 - 2.5\gamma, 18 + 2.5\gamma]; [18 - 6\gamma, 18 + 6\gamma]\}\right],$$

for each $\alpha \in [0, 0.5]$, $\beta \in [0.7, 1]$, $\gamma \in [0.5, 1]$.

Next, we introduce the concepts of equality and four basic arithmetic operations on the set $\mathscr{U}$ such as addition, multiplication, scalar multiplication and division. Firstly, we assume that

$$A = \left\{\langle x; \left[\mathtt{T}^l_A(x), \mathtt{T}^u_A(x)\right], \left[\mathtt{I}^l_A(x), \mathtt{I}^u_A(x)\right], \left[\mathtt{F}^l_A(x), \mathtt{F}^u_A(x)\right]\rangle : x \in \mathbb{R}\right\},$$
$$B = \left\{\langle x; \left[\mathtt{T}^l_B(x), \mathtt{T}^u_B(x)\right], \left[\mathtt{I}^l_B(x), \mathtt{I}^u_B(x)\right], \left[\mathtt{F}^l_B(x), \mathtt{F}^u_B(x)\right]\rangle : x \in \mathbb{R}\right\}.$$

are two interval neutrosophic numbers whose parametric representations are given by

$$[A]_{(\alpha,\beta,\gamma)} = \Big[\Big\{[A^l_-(\alpha), A^l_+(\alpha)]; [A^u_-(\alpha), A^u_+(\alpha)]\Big\}, \Big\{[A^l_-(\beta), A^l_+(\beta)]; [A^u_-(\beta), A^u_+(\beta)]\Big\}, \Big\{[A^l_-(\gamma), A^l_+(\gamma)]; [A^u_-(\gamma), A^u_+(\gamma)]\Big\} \Big],$$

$$[B]_{(\alpha,\beta,\gamma)} = \Big[\Big\{[B^l_-(\alpha), B^l_+(\alpha)]; [B^u_-(\alpha), B^u_+(\alpha)]\Big\}, \Big\{[B^l_-(\beta), B^l_+(\beta)]; [B^u_-(\beta), B^u_+(\beta)]\Big\}, \Big\{[B^l_-(\gamma), B^l_+(\gamma)]; [B^u_-(\gamma), B^u_+(\gamma)]\Big\} \Big],$$

respectively for all parameters $\alpha,\ \beta,\ \gamma \in [0,1]$.

Remark 2.4. In [43], the authors defined some mathematical algorithms of interval neutrosophic sets via truth-membership, indeterminacy-membership and falsity-membership functions. However, the calculation of these operations is not simple. In this paper, we will introduce the basic operators of interval neutrosophic sets via their parametric forms. Let us note that we consider interval neutrosophic numbers on the universal $X \subset \mathbb{R}$, so the level-sets functions of $T^l_A(x),\ T^u_A(x),\ I^l_A(x),\ I^u_A(x)$ and $F^l_A(x),\ F^u_A(x)$ are the closed intervals of $\mathbb{R}$.

Definition 2.8. Two interval neutrosophic numbers A and B are called equal, denoted by $A = B$, if and only if

$$\begin{cases} \mathtt{T}^l_A(x) = \mathtt{T}^l_B(x),\ \mathtt{T}^u_A(x) = \mathtt{T}^u_B(x), \\ \mathtt{I}^l_A(x) = \mathtt{I}^l_B(x),\ \mathtt{I}^u_A(x) = \mathtt{I}^u_B(x), \\ \mathtt{F}^l_A(x) = \mathtt{F}^l_B(x),\ \mathtt{F}^u_A(x) = \mathtt{F}^u_B(x), \end{cases}$$

for each $x \in \mathbb{R}$.

Definition 2.9 (Addition)**.** Let A, B be two interval neutrosophic numbers. Then, the sum of two interval neutrosophic numbers A and B, denoted by $A+B$, is an interval neutrosophic number C

$$C = \Big\{ \langle x; \Big[\mathtt{T}^l_C(x), \mathtt{T}^u_C(x)\Big], \Big[\mathtt{I}^l_C(x), \mathtt{I}^u_C(x)\Big], \Big[\mathtt{F}^l_C(x), \mathtt{F}^u_C(x)\Big] \rangle : x \in \mathbb{R} \Big\},$$

whose parametric representation is given by

$$\begin{aligned}[C]_{(\alpha,\beta,\gamma)} = \Big[& \Big\{[A^l_-(\alpha) + B^l_-(\alpha), A^l_+(\alpha) + B^l_+(\alpha)]; [A^u_-(\alpha) + B^u_-(\alpha), A^u_+(\alpha) + B^u_+(\alpha)]\Big\}, \\ & \Big\{[A^l_-(\beta) + B^l_-(\beta), A^l_+(\beta) + B^l_+(\beta)]; [A^u_-(\beta) + B^u_-(\beta), A^u_+(\beta) + B^u_+(\beta)]\Big\}, \\ & \Big\{[A^l_-(\gamma) + B^l_-(\gamma), A^l_+(\gamma) + B^l_+(\gamma)]; [A^u_-(\gamma) + B^u_-(\gamma), A^u_+(\gamma) + B^u_+(\gamma)]\Big\} \Big].\end{aligned}$$

Remark 2.5. The level-sets of the additional operator $A + B$ is obtained as the sum of $(\alpha,\ \beta,\gamma) -$ cuts of A and B in the sense of Minskovskii addition. Moreover, the addition of interval neutrosophic numbers are well-defined. In fact, by doing the same argument as in Proposition 1.5.2. [17], we can see that $[C]_{(\alpha,\beta,\gamma)}$ is the level-sets of an interval neutrosophic number and by using Negoita-Ralescu characterization theorem [23], we receive the canonical form of the interval neutrosophic number C. This implies $A + B$ is also an interval neutrosophic number.

Example 2.3. Let A be the interval triangular neutrosophic number defined in Example 2.2 and B be an interval triangular neutrosophic number given by

$$\begin{aligned} B = \langle\ & \{(8, 9, 11; 0.5), (8.5, 9, 10; 0.3)\}, \{(6, 7, 8; 0.7), (6.5, 7, 7.5; 0.9)\} \\ & \{(2, 4, 5; 0.5), (3, 4, 4.5; 0.8)\} \rangle. \end{aligned}$$

According to the formula (2), the $(\alpha,\ \beta,\gamma)-$ cuts of the number B is defined by

$$[B]_{(\alpha,\beta,\gamma)} = \left[\left\{\left[8.5+\frac{5}{3}\alpha, 10-\frac{10}{3}\alpha\right]; [8+2\alpha, 10-4\alpha]\right\}, \left\{\left[7-\frac{5}{9}\beta, 7+\frac{5}{9}\beta\right]; \left[7-\frac{10}{7}\beta, 7+\frac{10}{7}\beta\right]\right\}, \left\{\left[4-\frac{5}{4}\gamma, 4+\frac{5}{4}\gamma\right]; [4-4\gamma, 4+2\gamma]\right\}\right],$$

for all $\alpha,\beta,\gamma\in[0,1]$. Now, for each $\alpha,\beta,\gamma\in[0,1]$, we have

$$\begin{aligned}[C]_{(\alpha,\beta,\gamma)} =\Big[&\left\{\left[23+\frac{20}{3}\alpha, 31.5-\frac{65}{3}\alpha\right]; [22+6\alpha, 32-16\alpha]\right\},\\ &\left\{\left[22-\frac{10}{3}\beta, 22+\frac{20}{9}\beta\right]; \left[22-\frac{40}{7}\beta, 22+\frac{30}{7}\beta\right]\right\},\\ &\left\{\left[22-\frac{15}{4}\gamma, 22+\frac{15}{4}\gamma\right]; [22-10\gamma, 22+8\gamma]\right\}\Big].\end{aligned}$$

Finally, we can convert the above parametric form to the canonical form of C

$$\begin{aligned}C = \langle\ &\{(22,25,32;0.5),(23,25,31.5;0.3)\}, \{(18,22,25;0.7),(19,22,24;0.9)\}\\ &\{(17,22,26;0.5),(19,22,25;0.8)\}\ \rangle.\end{aligned}$$

Definition 2.10 (Scalar multiplication)**.** Let A be an interval neutrosophic number whose parametric representation is defined by the formula (2). Then, we define the scalar multiplication of the interval neutrosophic number A with a constant $\lambda\in\mathbb{R}$ corresponding to two following cases:

(i) If $\lambda\in\mathbb{R}^+$ then the scalar multiplication of the interval neutrosophic number A with λ is an interval neutrosophic number, denoted by λA, whose parametric representation is given by

$$[\lambda A]_{(\alpha,\beta,\gamma)} = \Big[\left\{[\lambda A^l_-(\alpha), \lambda A^l_+(\alpha)]; [\lambda A^u_-(\alpha), \lambda A^u_+(\alpha)]\right\}, \left\{[\lambda A^l_-(\beta), \lambda A^l_+(\beta)]; [\lambda A^u_-(\beta), \lambda A^u_+(\beta)]\right\}, \left\{[\lambda A^l_-(\gamma), \lambda A^l_+(\gamma)]; [\lambda A^u_-(\gamma), \lambda A^u_+(\gamma)]\right\}\Big].$$

(ii) If $\lambda\in\mathbb{R}^-$ then the scalar multiplication of the interval neutrosophic number A with λ is an interval neutrosophic number, denoted by λA, whose parametric representation is given by

$$[\lambda A]_{(\alpha,\beta,\gamma)} = \Big[\left\{[\lambda A^l_+(\alpha), \lambda A^l_-(\alpha)]; [\lambda A^u_+(\alpha), \lambda A^u_-(\alpha)]\right\}, \left\{[\lambda A^l_+(\beta), \lambda A^l_-(\beta)]; [\lambda A^u_+(\beta), \lambda A^u_-(\beta)]\right\}, \left\{[\lambda A^l_+(\gamma), \lambda A^l_-(\gamma)]; [\lambda A^u_+(\gamma), \lambda A^u_-(\gamma)]\right\}\Big].$$

Example 2.4. Let A be the number defined in Example 2.2. Now, we consider the multiplication of A by a scalar via its $(\alpha,\ \beta,\gamma)-$ cuts

(i) Multiplying by a positive crisp number $\lambda=2$:

$$\begin{aligned}[2A]_{(\alpha,\beta,\gamma)} =\Big[&\left\{\left[29+10\alpha, 43-\frac{110}{3}\alpha\right]; [28+8\alpha, 44-24\alpha]\right\},\\ &\left\{\left[30-\frac{50}{9}\beta, 30+\frac{10}{3}\beta\right]; \left[30-\frac{60}{7}\beta, 30+\frac{40}{7}\beta\right]\right\},\\ &\{[36-5\gamma, 36+5\gamma]; [36-12\gamma, 36+12\gamma]\}\Big].\end{aligned}$$

By converting the above parametric form to the canonical form, we obtain

$$\begin{aligned}2A = \langle\ &\{(28,32,44;0.5),(29,32,43;0.3)\}, \{(24,30,34;0.7),(25,30,33;0.9)\}\\ &\{(30,36,42;0.5),(32,36,40;0.8)\}\ \rangle.\end{aligned}$$

(ii) Multiplying by a negative crisp number $\lambda = -2$:

$$\begin{aligned}[-2A]_{(\alpha,\beta,\gamma)} =\Big[& \left\{ \left[-43 + \frac{110}{3}\alpha, -29 - 10\alpha\right]; [-44 + 24\alpha, -28 - 8\alpha,] \right\}, \\ & \left\{ \left[-30 - \frac{10}{3}\beta, -30 + \frac{50}{9}\beta\right]; \left[-30 - \frac{40}{7}\beta, -30 + \frac{60}{7}\beta\right] \right\}, \\ & \{[-36 - 5\gamma, -36 + 5\gamma]; [-36 + 12\gamma, -36 - 12\gamma]\} \Big].\end{aligned}$$

By converting the above parametric form to the canonical form, we obtain

$$\begin{aligned}(-2)A = \big\langle & \{(-44, -32, -28; 0.5), (-43, -32, -29; 0.3)\}, \{(-34, -30, -24; 0.7), \\ & (-33, -30, -25; 0.9)\}, \{(-42, -36, -30; 0.5), (-40, -36, -32; 0.8)\} \big\rangle.\end{aligned}$$

Definition 2.11 (Difference)**.** Let A, B be two interval neutrosophic numbers. Then, the difference of A and B, denoted by $A \ominus_{neu} B$, is an interval neutrosophic number C of the form

$$C = \left\{ \langle x; \left[\mathtt{T}^l_C(x), \mathtt{T}^u_C(x)\right], \left[\mathtt{I}^l_C(x), \mathtt{I}^u_C(x)\right], \left[\mathtt{F}^l_C(x), \mathtt{F}^u_C(x)\right] \rangle : x \in \mathbb{R} \right\},$$

whose parametric representation is given by

$$\begin{aligned}[C]_{(\alpha,\beta,\gamma)} = \Big[& \left\{ [C^l_-(\alpha), C^l_+(\alpha)]; [C^u_-(\alpha), C^u_+(\alpha)] \right\}, \left\{ [C^l_-(\beta), C^l_+(\beta)]; [C^u_-(\beta), C^u_+(\beta)] \right\}, \\ & \left\{ [C^l_-(\gamma), C^l_+(\gamma)]; [C^u_-(\gamma), C^u_+(\gamma)] \right\} \Big],\end{aligned}$$

for all α, β, $\gamma \in [0, 1]$. In particular, we have

$$\begin{aligned}[C^l_-(\mu), C^l_+(\mu)] &= \Big[\min\left\{ A^l_-(\mu) - B^l_-(\mu), A^l_+(\mu) - B^l_+(\mu) \right\}, \\ &\quad \max\left\{ A^l_-(\mu) - B^l_-(\mu), A^l_+(\mu) - B^l_+(\mu) \right\}\Big] \\ [C^u_-(\mu), C^u_+(\mu)] &= \big[\min\left\{ A^u_-(\mu) - B^u_-(\mu), A^u_+(\mu) - B^u_+(\mu) \right\}, \\ &\quad \max\left\{ A^u_-(\mu) - B^u_-(\mu), A^u_+(\mu) - B^u_+(\mu) \right\}\big].\end{aligned}$$

Here, the parameter $\mu \in [0, 1]$ represents for α, β or γ.

Example 2.5. Let A and B be numbers defined in Example 2.2 and Example 2.3. Then, by Definition 2.11, we directly obtain that the difference $A \ominus_{neu} B$ is an interval neutrosophic number C, where the parametric representation of C is

$$\begin{aligned}[C]_{(\alpha,\beta,\gamma)} = \Bigg[& \left\{ \left[6 + \frac{10\alpha}{3}, \frac{23}{2} - 15\alpha\right]; [6 + 2\alpha, 12 - 8\alpha] \right\}, \left\{ \left[8 - \frac{20\beta}{9}, 8 + \frac{10\beta}{9}\right]; \left[8 - \frac{20\beta}{7}, 8 + \frac{10\beta}{7}\right] \right\}, \\ & \left\{ \left[14 - \frac{5\gamma}{4}, 14 + \frac{5\gamma}{4}\right]; [14 - 2\gamma, 14 + 4\gamma] \right\} \Bigg],\end{aligned}$$

where α, β, $\gamma \in [0, 1]$. Moreover, we have the canonical form of the number C as follows:

$$C = \big\langle \{(6, 7, 8, 12; 0.5), (6, 7, 11.5; 0.3)\}, \{(6, 8, 9; 0.7), (6, 8, 9; 0.9)\}, \{(13, 14, 16; 0.5), (13, 14, 15; 0.8)\} \big\rangle.$$

Based on three above arithmetic operations, we introduce some properties of arithmetic on $\mathscr{U}$.

Proposition 2.1. *Let A, B, C be interval neutrosophic numbers and $\lambda_1, \lambda_2 \in \mathbb{R}$. Then, the following assertions are fulfilled:*

(i) $(A + B) + C = A + (B + C)$*;*

(ii) $\lambda_1(A + B) = \lambda_1 A + \lambda_1 B$*;*

(iii) $(\lambda_1+\lambda_2)A=\lambda_1 A+\lambda_2 A$;

(iv) $(\lambda_1\lambda_2)A=\lambda_1(\lambda_2 A)$;

(v) *If the difference* $A\ominus_{neu} B$ *exists and* $\lambda_1>0$ *then* $\lambda_1(A\ominus_{neu} B)=\lambda_1 A\ominus_{neu}\lambda_1 B$;

(vi) *If the difference* $A\ominus_{neu} B$ *exists and* $\lambda_1<0$ *then* $\lambda_1(A\ominus_{neu} B)=(-1)[(-\lambda_1 A)\ominus_{neu}(-\lambda_1 B)]$;

Proof. See Appendix. □

3. Interval neutrosophic calculus

This section is devoted to establishing some fundamental analysis concepts of interval neutrosophic space and interval neutrosophic functions such as the metric space, the continuity, the interval neutrosophic derivative or the integral. For simplicity in representation, we denote

$$\left[A^l\right]^{\mu_1}=[A^l_-(\mu_1),A^l_+(\mu_1)],\qquad [A^u]^{\mu_2}=[A^u_-(\mu_2),A^u_+(\mu_2)].$$

Definition 3.1 ($\rho_\infty-$ distance)**.** Let A and B be two interval neutrosophic numbers whose parametric representations are given by

$$[A]_{(\alpha,\beta,\gamma)}=\left[\left\{\left[A^l\right]^\alpha;[A^u]^\alpha\right\},\left\{\left[A^l\right]^\beta;[A^u]^\beta\right\},\left\{\left[A^l\right]^\gamma;[A^u]^\gamma\right\}\right]$$
$$[B]_{(\alpha,\beta,\gamma)}=\left[\left\{\left[B^l\right]^\alpha;[B^u]^\alpha\right\},\left\{\left[B^l\right]^\beta;[B^u]^\beta\right\},\left\{\left[B^l\right]^\gamma;[B^u]^\gamma\right\}\right]$$

respectively, for each $\alpha,\ \beta,\ \gamma\in[0,1]$. Then, we define the distance $\rho_\infty(\cdot,\cdot)$ on the space $\mathscr{U}$ by

$$\rho_\infty(A,B)=\sum_{i=1}^{3}\sup_{\mu_i\in[0,1]}\left\{d_H\left(\left[A^l\right]^{\mu_i},\left[B^l\right]^{\mu_i}\right)+d_H\left([A^u]^{\mu_i},[B^u]^{\mu_i}\right)\right\},$$

where d_H is the Hausdorff distance and the indices μ_i are given by $\mu_1=\alpha,\ \mu_2=\beta,\ \mu_3=\gamma$.

Theorem 3.1. *The* $\rho_\infty-$ *distance is a metric on the space* $\mathscr{U}$ *and it satisfies some following essential properties:*

(i) $\rho_\infty(A+C,B+C)=\rho_\infty(A,B)$,

(ii) $\rho_\infty(A+C,B+D)\le\rho_\infty(A,B)+\rho_\infty(C,D)$,

(iii) $\rho_\infty(\lambda A,\lambda B)=|\lambda|\,\rho_\infty(A,B)$,

(iv) $\rho_\infty(A\ominus_{neu}C,B\ominus_{neu}D)\le\rho_\infty(A,B)+\rho_\infty(C,D)$

for all $A,\ B,\ C,\ D\in\mathscr{U}$ *and* $\lambda\in\mathbb{R}$.

Proof. See Appendix. □

Definition 3.2. An interval neutrosophic function (or IN−function for short) $f:[a,b]\subset\mathbb{R}\to\mathscr{U}$ is defined by level-setwise

$$f(t)=\left\{\left\langle\left[\mathtt{T}^-_{f(t)}(x),\mathtt{T}^+_{f(t)}(x)\right],\left[\mathtt{I}^-_{f(t)}(x),\mathtt{I}^+_{f(t)}(x)\right],\left[\mathtt{F}^-_{f(t)}(x),\mathtt{F}^+_{f(t)}(x)\right]\right\rangle:x\in\mathbb{R}\right\},$$

whose parametric form is given as follows

$$\begin{aligned}[f(t)]_{(\alpha,\beta,\gamma)}&=\left[\left\{\left[f(t)^l\right]^\alpha;[f(t)^u]^\alpha\right\},\left\{\left[f(t)^l\right]^\beta;[f(t)^u]^\beta\right\},\left\{\left[f(t)^l\right]^\gamma;[f(t)^u]^\gamma\right\}\right]\\&=\Big[\left\{[f(t)^l_-(\alpha),f(t)^l_+(\alpha)];[f(t)^u_-(\alpha),f(t)^u_+(\alpha)]\right\},\left\{[f(t)^l_-(\beta),f(t)^l_+(\beta)];[f(t)^u_-(\beta),f(t)^u_+(\beta)]\right\},\\&\qquad\left\{[f(t)^l_-(\gamma),f(t)^l_+(\gamma)];[f(t)^u_-(\gamma),f(t)^u_+(\gamma)]\right\}\Big].\end{aligned}$$

Definition 3.3 (The differentiability)**.** Let $f:(a,b)\subset\mathbb{R}\to\mathscr{U}$ be an interval neutrosophic function and $t_0\in(a,b)$. Then, we say that the function f is differentiable at the point t_0 if there exists an element $f'(t_0)\in\mathscr{U}$ such that the following limit

$$\lim_{h\to 0}\frac{1}{h}\left[f(t_0+h)\ominus_{neu}f(t_0)\right]=f'(t_0),$$

holds for all $h>0$ sufficiently small. The interval neutrosophic number $f'(t_0)$ is then called interval neutrosophic derivative (or IN$-$derivative) of the function f at the point t_0. The interval neutrosophic function f is differentiable on (a,b) if and only if the IN$-$derivative $f'(t)$ exists for all $t\in(a,b)$. In this case, the mapping $f'(t)$ is called the IN$-$derivative of the function f on (a,b).

Proposition 3.1. *If f has IN$-$derivative $f'(t)$ then*

$$f'(t)=\left\{\left\langle\left[\mathtt{T}^-_{f'(t)}(x),\mathtt{T}^+_{f'(t)}(x)\right],\left[\mathtt{I}^-_{f'(t)}(x),\mathtt{I}^+_{f'(t)}(x)\right],\left[\mathtt{F}^-_{f'(t)}(x),\mathtt{F}^+_{f'(t)}(x)\right]\right\rangle : x\in\mathbb{R}\right\},$$

with respective parametric representation

$$\begin{aligned}\left[f'(t)\right]_{(\alpha,\beta,\gamma)}&=\left[\left\{\left[f'(t)^l\right]^\alpha;\left[f'(t)^r\right]^\alpha\right\},\left\{\left[f'(t)^l\right]^\beta;\left[f'(t)^r\right]^\beta\right\},\left\{\left[f'(t)^l\right]^\gamma;\left[f'(t)^r\right]^\gamma\right\}\right]\\ &=\Big[\left\{[f'(t)^l_-(\alpha),f'(t)^l_+(\alpha)];[f'(t)^u_-(\alpha),f'(t)^u_+(\alpha)]\right\},\left\{[f'(t)^l_-(\beta),f'(t)^l_+(\beta)];[f'(t)^u_-(\beta),f'(t)^u_+(\beta)]\right\},\\ &\quad\left\{[f'(t)^l_-(\gamma),f'(t)^l_+(\gamma)];[f'(t)^u_-(\gamma),f'(t)^u_+(\gamma)]\right\}\Big].\end{aligned}$$

Next, the concept of Lebesgue integral for IN$-$functions can be given as follows

Definition 3.4. Let $f:[a,b]\subset\mathbb{R}\to\mathscr{U}$ be an interval neutrosophic function whose $(\alpha,\beta,\gamma)-$ cuts is

$$[f(t)]_{(\alpha,\beta,\gamma)}=\left[\left\{\left[f(t)^l\right]^\alpha;[f(t)^u]^\alpha\right\},\left\{\left[f(t)^l\right]^\beta;[f(t)^u]^\beta\right\},\left\{\left[f(t)^l\right]^\gamma;[f(t)^u]^\gamma\right\}\right].$$

Then, the Lebesgue integral of interval neutrosophic function f, denoted by $\int_a^b f(\tau)d\tau$, is known as the Aumann integral of set-valued functions. The space of all Lebesgue integrable interval neutrosophic functions defined on $[a,b]$ is denoted by $L^1([a,b],\mathscr{U})$.

4. Interval neutrosophic stochastic process

Denote by $\mathscr{K}(\mathbb{R})$ the space of all nonempty, compact subsets of $\mathbb{R}$. Next, we introduce the concept of Hausdorff metric d_H by

$$d_H(A,B):=\max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|,\sup_{b\in B}\inf_{a\in A}\|a-b\|\right\},$$

where $A,\,B\in\mathscr{K}(\mathbb{R})$ and $\|\cdot\|$ is the usual norm in $\mathbb{R}$. From [17], it is well-known that the space $\mathscr{K}(\mathbb{R})$ endowed with the metric d_H is a complete metric space.In addition, some fundamental arithmetic operations in $\mathscr{K}(\mathbb{R})$ such addition and scalar multiplication, are defined as follows:

$$\begin{aligned}&A+B=\{a+b\,|\,a\in A,\,b\in B\},\\ &A+\{b\}=\{a+b\,|\,a\in A\},\\ &\lambda A=\{\lambda a\,|\,a\in A\},\end{aligned}$$

for all $A,\,B\in\mathscr{K}(\mathbb{R})$, $b\in\mathbb{R}$ and $\lambda\in\mathbb{R}\setminus\{0\}$. Thus, it is easy to see that the space $(\mathscr{K}(\mathbb{R}),+,\cdot)$ is a semi-linear metric space. Denote $(\Omega,\mathscr{A},\mathbb{P})$ by the complete probability space.

Definition 4.1. An interval neutrosophic random variable u is a mapping $u : \Omega \to \mathscr{U}$ such that

$$\{(\omega, [\mathtt{T}_{u(\omega)}(x)])\}, \{(\omega, [\mathtt{I}_{u(\omega)}(x)])\}, \{(\omega, [\mathtt{F}_{u(\omega)}(x)])\} \in \mathscr{A} \times \mathscr{B},$$

for every $x \in \mathbb{R}$, where $\mathscr{B}$ denotes for the Borel subsets of $\mathscr{K}(\mathbb{R})$.

Remark 4.1. The above definition is compatible with the concept of intuitionistic fuzzy random variable proposed by Wang et.al. [45]. The difference here is that each component of the interval neutrosophic random variable u is known as a set-valued random variable while each component of an intuitionistic random variable is a real-valued random variable.

Denote $\mathscr{M}(\Omega, \mathscr{A}, \mathscr{U})$ by the set of all interval neutrosophic random variables $u : \Omega \to \mathscr{U}$ such that the set-valued mappings $[\mathtt{T}_u(x)]$, $[\mathtt{I}_u(x)]$ and $[\mathtt{F}_u(x)] : \Omega \to \mathscr{K}(\mathbb{R})$ are $\mathscr{A}-$ measurable, see e.g. [9].

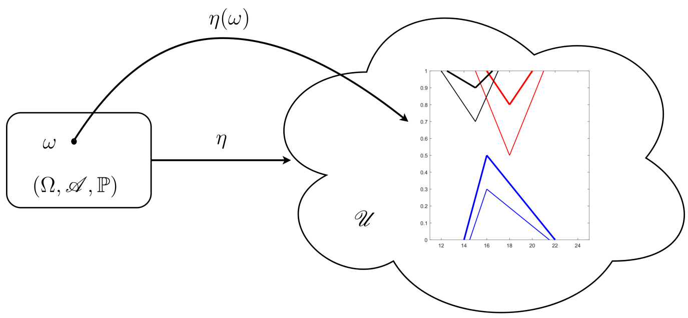

Figure 4: An illustration of interval triangular neutrosophic random variable

Denote $\mathcal{L}^p(\Omega, \mathscr{A}, \mathbb{P}; \mathscr{K}(\mathbb{R}))$ by the set of all $\mathscr{A}-$ measurable and L^p- integrably bounded set-valued random variables. Here, we will give some basic concepts of interval neutrosophic random processes:

Definition 4.2. An interval neutrosophic random variable $u : \Omega \to \mathscr{U}$ is said to be L^p- integrably bounded if $[\mathtt{T}_u(x)]$, $[\mathtt{I}_u(x)]$, $[\mathtt{F}_u(x)] \in \mathcal{L}^p(\Omega, \mathscr{A}, \mathbb{P}; \mathscr{K}(\mathbb{R}))$ for each $p \geq 1$. In addition, denote by $\mathcal{L}^p(\Omega, \mathscr{A}, \mathbb{P}; \mathscr{U})$ the set of all L^p- integrably bounded interval neutrosophic random variables, in which $u, v \in \mathcal{L}^p(\Omega, \mathscr{A}, \mathbb{P}; \mathscr{U})$ are considered to be identical if $\mathbb{P}(\{\rho_\infty(u, v) = 0\}) = 1$.

Next, for each interval $J = [0, T] \subset \mathbb{R}$, we consider a complete probability space $(\Omega, \mathscr{A}, \mathbb{P})$ and equip it with a filtration $\{\mathscr{A}_t\}_{t\in J}$ satisfying following hypotheses

(i) $\{\mathscr{A}_t\}_{t\in J}$ is an increasing and right continuous family of sub $\sigma-$ algebras of $\mathscr{A}$;

(ii) $\mathscr{A}_0$ contains all $\mathbb{P}-$ null sets.

Then, the concept of interval neutrosophic stochastic process is given as follows:

Definition 4.3. (a) Mapping $u : J \times \Omega \to \mathscr{U}$ is called an interval neutrosophic stochastic process if

(i) For any fixed $\omega \in \Omega$, $u(\cdot, \omega)$ is a neutrosophic valued function;

(ii) For any fixed $t \in J$, $u(t, \cdot)$ is an interval neutrosophic random variable.

(b) An interval neutrosophic stochastic process $u : J \times \Omega \to \mathscr{U}$ is said to be $\rho_\infty-$ continuous if almost all trajectory $u(\cdot, \omega) : J \to \mathscr{U}$ are continuous functions with respect to ρ_∞.

Definition 4.4. An interval neutrosophic stochastic process $u : J \times \Omega \to \mathscr{U}$ is said to be:

(i) $\{\mathscr{A}_t\}_{t\in J}-$ adapted if for all $t \in J$, all set-valued functions $\left[\mathtt{T}_{u(t)}(x)\right]$, $\left[\mathtt{I}_{u(t)}(x)\right]$, $\left[\mathtt{F}_{u(t)}(x)\right]$ defined on Ω is $\mathscr{A}_t-$ measurable,

(ii) measurable if $[\mathtt{T}_u(x)]$, $[\mathtt{I}_u(x)]$, $[\mathtt{F}_u(x)] : J\times\Omega \to \mathscr{K}(\mathbb{R})$ are $\mathscr{B}(J)\otimes\mathscr{A}-$ measurable multi-functions, in which $\mathscr{B}(J)$ denotes for Borel $\sigma-$ algebra of subsets of J.

(iii) non-anticipating or $\mathcal{N}-$ measurable if it is $\{\mathscr{A}_t\}_{t\in J}-$ adapted and measurable.

For each $p \geq 1$, we denote by $L^p(J\times\Omega,\mathcal{N};\mathbb{R})$ the set of all non-anticipating real-valued stochastic processes $h : J\times\Omega \to \mathbb{R}$ such that

$$\mathbb{E}\left(\int_0^T |h(\tau)|^p d\tau\right) < \infty.$$

Definition 4.5. An interval neutrosophic stochastic process $u : J\times\Omega \to \mathscr{U}$ is said to be L^p- integrably bounded if there exists $h \in L^p(J\times\Omega,\mathcal{N};\mathbb{R})$ such that

$$\rho_\infty\left(u(t,\omega),\{0\}\right) \leq h(t,\omega),$$

for a.e. $(t,\omega) \in J\times\Omega$. Moreover, we denote

$$\mathcal{L}^p\left(J\times\Omega,\mathcal{N};\mathscr{U}\right) = \left\{u : J\times\Omega \to \mathscr{U} : u \text{ non-anticipating and } L^p \text{ -- integrably bounded}\right\}.$$

Remark 4.2. For more simplicity, we denote

- $\eta(\omega) \overset{\mathbb{P}.1}{=} \mu(\omega)$ stands for $\mathbb{P}\left(\{\omega\in\Omega \,|\, \eta(\omega)=\mu(\omega)\}\right) = 1$ for all $\eta,\mu \in \mathscr{M}(\Omega,\mathscr{A},\mathscr{U})$,
- $u(t,\omega) \overset{J\mathbb{P}.1}{=} v(t,\omega)$ stands for $\mathbb{P}\left(\{\omega\in\Omega \,|\, u(t,\omega)=v(t,\omega) \text{ for all } t\in J\}\right) = 1$ for all interval neutrosophic stochastic processes u, v,

and similar notations are also used to define inequalities.

In the following, we introduce the concept of interval neutrosophic random variable, which plays an important role in defining the concept of stochastic solution to the Cauchy problem for interval neutrosophic stochastic differential equations

Definition 4.6. Let $z \in \mathcal{L}^p\left(J\times\Omega,\mathcal{N};\mathscr{U}\right)$ be arbitrary with $p \geq 1$. Then, the interval neutrosophic stochastic Lebesgue integral of z is known as an interval neutrosophic random variable

$$\omega \mapsto \int_J z(\tau,\omega)d\tau \in \mathscr{U}.$$

Then, the notation $\int_0^t z(\tau,\omega)d\tau$ can be understood as $\int_J \mathbb{I}_{[0,t]}(\tau)z(\tau,\omega)d\tau$.

Some properties of interval neutrosophic random variables can be found in the following

Proposition 4.1. *For each $p \geq 1$, if z_1, $z_2 \in \mathcal{L}^p\left(J\times\Omega,\mathcal{N};\mathscr{U}\right)$ then*

(i) $J\times\Omega \ni (t,\omega) \mapsto \int_0^t z_1(\tau,\omega)d\tau \in \mathscr{U}$ *belongs to* $\mathcal{L}^p\left(J\times\Omega,\mathcal{N};\mathscr{U}\right)$,

(ii) *The interval neutrosophic stochastic process* $(t,\omega) \mapsto \int_0^t z_1(\tau,\omega)d\tau$ *is* $\rho_\infty-$ *continuous,*

(iii) $\sup\limits_{s\in[0,t]} \rho_\infty^p\left(\int_0^s z_1(\tau,\omega)d\tau, \int_0^s z_2(\tau,\omega)d\tau\right) \overset{J\mathbb{P}.1}{\leq} t^{p-1}\int_0^t \rho_\infty^p\left(z_1(\tau,\omega),z_2(\tau,\omega)\right)d\tau,$

(iv) *For each $t \in J$, the following estimation holds*

$$\mathbb{E}\sup_{s\in[0,t]} \rho_\infty^p\left(\int_0^s z_1(\tau,\omega)d\tau, \int_0^s z_2(\tau,\omega)d\tau\right) \overset{J\mathbb{P}.1}{\leq} t^{p-1}\mathbb{E}\int_0^t \rho_\infty^p\left(z_1(\tau,\omega),z_2(\tau,\omega)\right)d\tau.$$

Consider the embedding mapping

$$\begin{aligned} \{\cdot\} : \mathbb{R} &\to \mathscr{U} \\ z &\mapsto \{z\} = \langle [1,1],[0,0],[0,0] \rangle / z \end{aligned} \tag{3}$$

that maps each real number z into an interval neutrosophic number $\{z\}$. Here, note that if mapping $z : \Omega \to \mathbb{R}$ is a real-valued random variable defined on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ then mapping $\{z\} : \Omega \to \mathscr{U}$ is an interval neutrosophic random variable.

Remark 4.3. If $z : \Omega \to \mathbb{R}$ is a real-valued random variable which is $\{\mathscr{A}_t\}_{t\in J}-$ adapted (measurable, respectively) then $\{z\} : \Omega \to \mathscr{U}$ is an $\{\mathscr{A}_t\}_{t\in J}-$ adapted (measurable, respectively) neutrosophic valued random variable.

Next, consider a complete probability space $(\Omega, \mathscr{A}, \mathbb{P})$ with a filtration $\{\mathscr{A}_t\}_{t\in J}$ satisfying hypotheses:

(i) $\{\mathscr{A}_t\}_{t\in J}$ is an increasing and right continuous family of sub $\sigma-$ algebras of $\mathscr{A}$;

(ii) $\mathscr{A}_0$ contains all $\mathbb{P}-$ null sets.

Now, we will recall some basic concepts of interval neutrosophic stochastic Itô integral. First of all, we denote by $\mathbf{B} = \{\mathbf{B}(t)\}_{t\in J}$ an one-dimensional $\{\mathscr{A}_t\}_{t\in J}-$ Brownian motion. Then, for each $z \in \mathcal{L}^2(J \times \Omega, \mathcal{N}; \mathbb{R})$, the classical stochastic Itô integral is defined by

$$\int_0^T z(\tau,\omega) d\mathbf{B}(\tau).$$

Definition 4.7. An interval neutrosophic stochastic Itô integral of z is an interval neutrosophic random variable

$$\Omega \ni \omega \mapsto \left\{ \int_0^T z(\tau,\omega) d\mathbf{B}(\tau)(\omega) \right\} \in \mathscr{U}.$$

Here, for each $t \in J$, the interval neutrosophic stochastic Itô integral $\left\{ \int_0^t z(\tau,\omega) d\mathbf{B}(\tau)(\omega) \right\}$ is known as

$$\left\{ \int_0^t z(\tau,\omega) d\mathbf{B}(\tau)(\omega) \right\} := \left\{ \int_0^T \mathbb{I}_{[0,t]}(\tau) z(\tau,\omega) d\mathbf{B}(\tau)(\omega) \right\}.$$

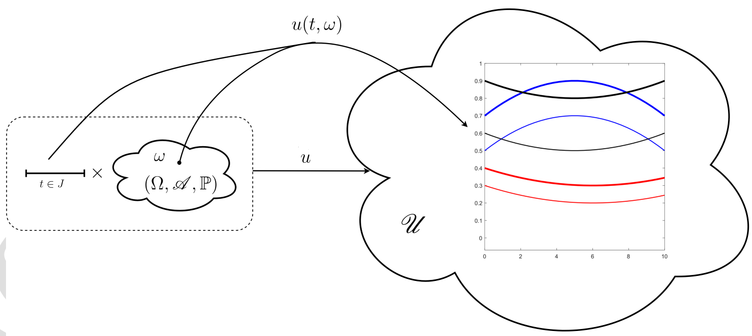

Figure 5: An illustration of interval neutrosophic stochastic process

The following propositions play key roles in the proof of the existence and uniqueness of stochastic solution in the next section

Proposition 4.2. *For each* $z \in \mathcal{L}^2(J \times \Omega, \mathcal{N}; \mathbb{R})$*, the mapping*

$$(t, \omega) \mapsto \left\{ \int_0^t z(\tau, \omega) d\mathbf{B}(\tau)(\omega) \right\}$$

belongs to $\mathcal{L}^2(J \times \Omega, \mathcal{N}; \mathscr{U})$.

Proposition 4.3. *The neutrosophic stochastic process* $\left\{ \int_0^t z(\tau, \omega) d\mathbf{B}(\tau) \right\}_{t \in J}$ *is continuous with respect to metric* ρ_∞ *for each* $z \in \mathcal{L}^2(J \times \Omega, \mathcal{N}; \mathbb{R})$.

Proposition 4.4. *For each* $z_1, z_2 \in \mathcal{L}^2(J \times \Omega, \mathcal{N}; \mathbb{R})$*, the estimation*

$$\mathbb{E} \sup_{s \in [0,t]} \rho_\infty^2 \left(\left\{ \int_0^s z_1(\tau, \omega) d\mathbf{B}(\tau) \right\}, \left\{ \int_0^s z_2(\tau, \omega) d\mathbf{B}(\tau) \right\} \right) \leq 8\mathbb{E} \int_0^t \rho_\infty^2 \left(\{z_1(\tau, \omega)\}, \{z_2(\tau, \omega)\} \right) d\tau.$$

holds for every $t \in J$.

5. Interval neutrosophic stochastic differential equations

Consider a complete probability space $(\Omega, \mathscr{A}, \mathbb{P})$, where the filtration $\{\mathscr{A}_t\}_{t \in J}$ satisfies the usual hypotheses:

(i) $\{\mathscr{A}_t\}_{t \in J}$ is an increasing and right continuous family of sub $\sigma-$ algebras of $\mathscr{A}$;

(ii) $\mathscr{A}_0$ contains all $\mathbb{P}-$ null sets.

5.1. State the problem

This section is devoted to investigating the unique solvability for a class of interval neutrosophic stochastic differential equations that can be symbolized in the following form

$$z'(t)dt \overset{J\mathbb{P}.1}{=} f(t, z(t))dt + \{\Phi(t, z(t))d\mathbf{B}(t)\}, \qquad t \in J = [0, T], \tag{4}$$

subject to the initial condition

$$z(0) \overset{J\mathbb{P}.1}{=} z_0, \tag{5}$$

in which $\{\mathbf{B}(t)\}_{t \in J}$ is the one dimensional $\{\mathscr{A}_t\}-$ Brownian motion defined on $(\Omega, \mathscr{A}, \{\mathscr{A}_t\}_{t \in J}, \mathbb{P})$, z_0 is an interval neutrosophic random variable and $f : J \times \Omega \times \mathscr{U} \to \mathscr{U}$ and $\Phi : J \times \Omega \times \mathscr{U} \to \mathbb{R}$ are given functions that will be specified later. Firstly, we define the concept of stochastic solution to Cauchy problem (4) - (5):

Definition 5.1. An interval neutrosophic stochastic process $z : J \times \Omega \to \mathscr{U}$ is said to be a stochastic solution of the Cauchy problem (4) - (5) if it satisfies following conditions:

(i) $z \in \mathcal{L}^2(J \times \Omega, \mathcal{N}; \mathscr{U})$;

(ii) z is continuous w.r.t ρ_∞;

(iii) For each $t \in J$, $z(t, \omega) \overset{J\mathbb{P}.1}{=} z_0 + \int_0^t f(\tau, \omega, z(\tau, \omega))d\tau + \left\{ \int_0^t \Phi(\tau, \omega, z(\tau, \omega))d\mathbf{B}(\tau) \right\}$.

Definition 5.2. An interval neutrosophic stochastic process $z : J \times \Omega \to \mathscr{U}$ is said to be a unique stochastic solution of the Cauchy problem (4) - (5) if and only if

(i) z is a stochastic solution of Cauchy problem (4) - (5);

(ii) If y is another stochastic solution of Cauchy problem (4) - (5) then $z(t, \omega) \overset{J\mathbb{P}.1}{=} y(t, \omega)$.

Next, the existence and uniqueness of Cauchy problem (4) - (5) is obtained under following hypotheses:

(A1) The mapping $f : J \times \Omega \times \mathscr{U} \to \mathscr{U}$ is $\mathcal{N} \otimes \mathscr{B}_{\rho_\infty} | \mathscr{B}_{\rho_\infty} -$ measurable and $\Phi : J \times \Omega \times \mathscr{U} \to \mathbb{R}$ is $\mathcal{N} \otimes \mathscr{B}_{\rho_\infty} | \mathscr{B}(\mathbb{R}) -$ measurable;

(A2) There exists $L > 0$ such that the inequality

$$\max\left\{\rho_\infty^2\left(f(t,\omega,u), f(t,\omega,v)\right); |\Phi(t,\omega,u) - \Phi(t,\omega,v)|^2\right\} \le L\rho_\infty^2(u,v);$$

holds $\mathbb{P}-$ a.e. for every $t \in J = [0,T]$ and $u,\, v \in \mathscr{U}$;

(A3) There exists $C > 0$ such that the inequality

$$\max\left\{\rho_\infty^2\left(f(t,\omega,\{0\}), \{0\}\right); |\Phi(t,\omega,\{0\})|^2\right\} \le \kappa,$$

holds $\mathbb{P}-$ a.e. for every $t \in J = [0,T]$.

The main technique used to prove the existence and uniqueness of Cauchy problem (4) - (5) is the Picard successive approximation method. For this aim, we define a Picard type sequence $\{z_n\} : J \times \Omega \to \mathscr{U}$ as follows:

$$\begin{cases} z_0(t) & \overset{J\mathbb{P}.1}{=} z_0, \\ z_n(t) & \overset{J\mathbb{P}.1}{=} z_0 + \displaystyle\int_0^t f(\tau,\omega,z_{n-1}(\tau,\omega))d\tau + \left\{\int_0^t \Phi(\tau,\omega,z_{n-1}(\tau,\omega))d\mathbf{B}(\tau)\right\}. \end{cases} \tag{6}$$

It should be noted that the sequence $\{z_n\}$ is well-defined $\rho_\infty-$ continuous interval neutrosophic stochastic processes from $\mathcal{L}^2(J\times\Omega,\mathcal{N};\mathscr{U})$. Thus, it follows that each element of the Picard approximation $\{z_n\}$ defined by (6) is an interval neutrosophic stochastic process which is non-anticipating and L^2- integrably bounded. Next, we will show that the sequence $\{z_n\}$ satisfies following property:

Lemma 5.1. *For each $z_0 \in \mathcal{L}^2(\Omega,\mathscr{A},\mathbb{P};\mathscr{U})$, if all hypotheses (**A1**), (**A2**) and (**A3**) are fulfilled then the sequence $\{z_n\}$ satisfies the following estimation*

$$\mathbb{E}\sup_{t\in J}\rho_\infty^2(z_n(t),\{0\}) \le \left(\lambda_1 + \lambda_2 T\mathbb{E}\rho_\infty^2(z_0,\{0\})\right)e^{\lambda_2 T} \quad \text{for each } n \in \mathbb{N},$$

where $\lambda_1 = 3\left[\mathbb{E}\rho_\infty^2(z_0,\{0\}) + 2\kappa T^2 + 16\kappa T\right]$ *and* $\lambda_2 = 6L(T+8)$.

Proof. See Appendix. □

5.2. Main results

Theorem 5.1. *Assume that $z_0 \in \mathcal{L}^2(\Omega,\mathscr{A},\mathbb{P};\mathscr{U})$ and all hypotheses (**A1**), (**A2**) and (**A3**) are fulfilled. Then, Cauchy problem (4) - (5) has a unique stochastic solution.*

Proof. The proof is divided into two following steps:
(Existence). For each $n \in \mathbb{N}$ and $t \in J$, we denote $e_n(t) = \mathbb{E}\sup_{\tau\in[0,t]}\rho_\infty^2(z_n(\tau), z_{n-1}(\tau))$. Then, by using mathematical induction principle, we will estimate the value of $e_n(t)$ for each $t \in J$. Indeed, if $n = 1$ then we have

$$\begin{aligned} e_1(t) =& \mathbb{E}\sup_{\tau\in[0,t]}\rho_\infty^2\left(\int_0^\tau f(s,z_0)ds + \left\{\int_0^\tau \Phi(s,z_0)d\mathbf{B}(s)\right\}, \{0\}\right) \\ \le& 2\mathbb{E}\sup_{\tau\in[0,t]}\left[\rho_\infty^2\left(\int_0^\tau f(s,z_0(s))ds, \{0\}\right) + \rho_\infty^2\left(\left\{\int_0^\tau \Phi(s,z_0(s))d\mathbf{B}(s)\right\}, \{0\}\right)\right] \\ \le& 4\mathbb{E}\sup_{\tau\in[0,t]}\left[\rho_\infty^2\left(\int_0^\tau f(s,z_0(s))ds, \int_0^\tau f(s,\{0\})ds\right) + \rho_\infty^2\left(\int_0^\tau f(s,\{0\})ds, \{0\}\right) + \right. \\ & \left.\rho_\infty^2\left(\left\{\int_0^\tau \Phi(s,z_0(s))d\mathbf{B}(s)\right\}, \left\{\int_0^\tau \Phi(s,\{0\})d\mathbf{B}(s)\right\}\right) + \rho_\infty^2\left(\left\{\int_0^\tau \Phi(s,\{0\})d\mathbf{B}(s)\right\}, \{0\}\right)\right]. \end{aligned}$$

According to Proposition 4.1 and Proposition 4.4, we infer that

$$e_1(t) \leq 4\left[t\mathbb{E}\int_0^t \left(\rho_\infty^2\left(f(s,z_0), f(s,\{0\})\right) + \rho_\infty^2\left(f(s,\{0\}),\{0\}\right)\right) ds \right.$$
$$\left. + 8\mathbb{E}\int_0^t \left(\rho_\infty^2\left(\{\Phi(s,x_0)\},\{\Phi(s,\{0\})\}\right) + \rho_\infty^2\left(\{f(s,\{0\})\},\{0\}\right)\right) ds\right].$$

By (**A2**) and (**A3**), it implies that $e_1(t) \leq M_1 t$, where $M_1 = 4\left[LT\mathbb{E}\rho_\infty^2(z_0,\{0\}) + \kappa(T+8) + 8L\mathbb{E}\rho_\infty^2(z_0,\{0\})\right]$. By doing similar arguments, we also obtain

$$\begin{aligned} e_{n+1}(t) &\leq 4L(t+8)\mathbb{E}\int_0^t \rho_\infty^2\left(z_n(s), z_{n-1}(s)\right) ds \\ &\leq 4L(T+8)\int_0^t \mathbb{E}\sup_{\tau\in[0,s]} \rho_\infty^2\left(z_n(\tau), z_{n-1}(\tau)\right) ds \\ &= 4L(T+8)\int_0^t e_n(s) ds. \end{aligned}$$

Therefore, by mathematical induction principle, we obtain

$$e_n(t) \leq \frac{M_1}{2L(T+8)}\frac{[2L(T+8)t]^n}{n!} \quad \text{for all } n \in \mathbb{N}. \tag{7}$$

As a consequence, by Chebyshev inequality and the inequality (7), we directly get

$$\mathbb{P}\left(\sup_{\tau\in[0,T]} \rho_\infty^2(z_n(\tau), z_{n-1}(\tau)) > \frac{1}{4^n}\right) \leq 4^n e_n(T) \leq \frac{M_1}{2L(T+8)}\frac{[8L(T+8)T]^n}{n!}.$$

Note that the series $\sum_{n=1}^{\infty} \frac{[8L(T+8)T]^n}{n!}$ converges. Thus, by applying Borel-Cantelli lemma, we have

$$\mathbb{P}\left(\sup_{\tau\in[0,T]} \rho_\infty^2(z_n(\tau), z_{n-1}(\tau)) > \frac{1}{2^n} \text{ infinitely often}\right) = 0,$$

which means that for a.e $\omega \in \Omega$, there exists $N_0 = N_0(\omega) \in \mathbb{N}$ such that

$$\sup_{\tau\in[0,T]} \rho_\infty^2(z_n(\tau), z_{n-1}(\tau)) \leq \frac{1}{2^n} \quad \text{for all } n \geq N_0.$$

Let us consider the set $\Omega_c \in \mathscr{A}$ satisfying $\mathbb{P}(\Omega_c) = 1$. Then, we can conclude that for each $\omega \in \Omega_c$, the sequence $\{z_n(\cdot,\omega)\}$ uniformly converges to a function $\tilde{z}(\cdot,\omega) : J \to \mathscr{U}$. Moreover, it is easy to see that the function $\overline{z}$ is $\rho_\infty-$ continuous. Next, we define a mapping $z : J \times \Omega \to \mathscr{U}$ as follows:

$$z(\cdot,\omega) = \begin{cases} \tilde{z}(\cdot,\omega) & \text{if } \omega \in \Omega_c, \\ \text{freely chosen function} & \text{if } \omega \in \Omega \setminus \Omega_c \end{cases}$$

Firstly, we can see that for every $t \in J$, $d_H\left([z_n(t,\omega)]_{(\alpha,\beta,\gamma)}, [z(t,\omega)]_{(\alpha,\beta,\gamma)}\right) \to 0$ as $n \to \infty$, which follows that $[z(t,\cdot)]_{(\alpha,\beta,\gamma)} : \Omega \to \mathscr{U}$ is an $\mathscr{A}_t-$ measurable multi-function. Thus, the process z is continuous and $\{\mathscr{A}\}_{t\in J}-$ adapted and is also non-anticipating. On the other hand, since $z_n(t) \in \mathcal{L}^2(\Omega, \mathscr{A}, \mathbb{P}; \mathscr{U})$, we deduce that

$$\mathbb{E}\int_0^T \rho_\infty^2(z(t),\{0\}) \leq T\mathbb{E}\sup_{t\in J} \rho_\infty^2(z(t),\{0\}) < \infty,$$

which follows that $z \in \mathcal{L}^2(J \times \Omega, \mathcal{N}; \mathscr{U})$. Finally, we will show that the process z is a stochastic solution of Cauchy problem (4) - (5). Indeed, one can observe that

$$\begin{aligned}
&\mathbb{E}\rho_\infty^2\left(z(t), z_0 + \int_0^t f(s,z(s))ds + \left\{\int_0^t \Phi(s,z(s))d\mathbf{B}(s)\right\}\right) \\
\leq 2&\left[\mathbb{E}\rho_\infty^2\left(\int_0^t f(s,z(s))ds + \left\{\int_0^t \Phi(s,z(s))d\mathbf{B}(s)\right\}, \int_0^t f(s,z_n(s))ds + \left\{\int_0^t \Phi(s,z_n(s))d\mathbf{B}(s)\right\}\right)\right. \\
&\left. + \mathbb{E}\rho_\infty^2(z(t), z_n(t))\right].
\end{aligned}$$

Due to $\mathbb{E}\rho_\infty^2(z(t), z_n(t)) \to 0$ as $n \to \infty$, it suffices to show that the first term of right-hand side also converges to 0. Indeed, by Proposition 4.1, Proposition 4.4 and the hypothesis (**A2**), we immediately get

$$\begin{aligned}
&\mathbb{E}\rho_\infty^2\left(\int_0^t f(s,z(s))ds + \left\{\int_0^t \Phi(s,z(s))d\mathbf{B}(s)\right\}, \int_0^t f(s,z_n(s))ds + \left\{\int_0^t \Phi(s,z_n(s))d\mathbf{B}(s)\right\}\right) \\
\leq &2\left[\mathbb{E}\rho_\infty^2\left(\int_0^t f(s,z(s))ds, \int_0^t f(s,z_n(s))ds\right) + \mathbb{E}\rho_\infty^2\left(\left\{\int_0^t \Phi(s,z(s))d\mathbf{B}(s)\right\}, \left\{\int_0^t \Phi(s,z_n(s))d\mathbf{B}(s)\right\}\right)\right] \\
\leq &2L(T+8)\int_0^T \mathbb{E}\rho_\infty^2(z_{n-1}(s), z(s))ds \to 0 \quad \text{ as } n \to \infty.
\end{aligned}$$

Thus, we receive $\mathbb{E}\rho_\infty^2\left(z(t), z_0 + \int_0^t f(s,z(s))ds + \left\{\int_0^t \Phi(s,z(s))d\mathbf{B}(s)\right\}\right) = 0$, which yields

$$\rho_\infty^2\left(z(t), z_0 + \int_0^t f(s,z(s))ds + \left\{\int_0^t \Phi(s,z(s))d\mathbf{B}(s)\right\}\right) \overset{\mathbb{P}.1}{=} 0,$$

or equivalently, $\rho_\infty^2\left(z(t), z_0 + \int_0^t f(s,z(s))ds + \left\{\int_0^t \Phi(s,z(s))d\mathbf{B}(s)\right\}\right) \overset{J\mathbb{P}.1}{=} 0$ due to the $\rho_\infty-$ continuity of the proposed stochastic process. Hence, the function z is a stochastic solution of the Cauchy problem (4) - (5).

(Uniqueness). In order to show the unique existence of stochastic solution z, we assume by contrary that there exist two distinct stochastic processes $z, \overline{z} : J \times \Omega \to \mathscr{U}$ which are stochastic solutions of Cauchy problem (4) - (5). For each $t \in J$, we denote $e(t) := \mathbb{E} \sup_{\tau \in [0,t]} \rho_\infty^2(z(\tau), \overline{z}(\tau))$. Then, we have

$$e(t) = 3L(t+8)\mathbb{E}\int_0^t \rho_\infty^2(z(s), \overline{z}(s))ds \leq 3L(t+8)\mathbb{E}\int_0^t e(s)ds.$$

By Gronwall inequality, it implies that $e(t) = 0$ for each $t \in J$, or equivalently, $\sup_{t\in J} \rho_\infty(z(t), \overline{z}(t)) \overset{\mathbb{P}.1}{=} 0$. Hence, we can conclude that $z(t) = \overline{z}(t)$ for all $t \in J$, that is, $z \equiv \overline{z}$. The proof is completed. □

Remark 5.1. Numerical methods are widely known due to their effective capacities in solving various mathematical, physical and engineering problems even when they are unable to obtain analytical solutions by using analysis methods and transformations. Numerical methods give the approximate values of solutions of considered problems at a certain point. In this paper, to solve the Cauchy problem for the neutrosophic stochastic differential equation, we will conduct an numerical algorithm based on Euler-Maruyama (EM) method which was discussed in [15].

Consider the following Cauchy problem to interval neutrosophic differential equation

$$\begin{cases} X'(t)dt \overset{J\mathbb{P}.1}{=} f(t, X(t))dt + \{\Phi(t, X(t))d\mathbf{B}(t)\}, & t \in J = [0,T], \\ X(0) \overset{J\mathbb{P}.1}{=} X_0. \end{cases} \tag{8}$$

Theorem 5.2. *If the hypotheses (**A1**), (**A2**) and (**A3**) are fulfilled then the difference equation generated by Euler − Maruyama method*

$$X_{j+1} = X_j + f(\tau_j, X_j)\,\Delta t + \{\Phi(\tau_j, X_j)\Delta W_j\}, \quad j = 0, 2, \dots, N-1,$$

approximately solves the Cauchy problem (8) and the global error at any $t \in [0, T]$ *is* $\mathcal{O}\left((\Delta t)^2\right)$.

Proof. Firstly, we divide the interval $[0, T]$ into N equal sub-intervals whose lengths are $\Delta t = \frac{T}{N}$ for a big enough natural number N. For each $j = 1, 2, \dots, N$, we denote $\tau_j = j\Delta t$. Then, our proof is divided into following steps: **Step 1.** In this step, we will compute the sequence $\{X_n\} \approx \{X(\tau_n)\}$ to approximate the stochastic solution of the Cauchy problem (8). First of all, we will simulate a Brownian motion by using discretized Brownian motion, in which $W(t)$ is sampled at the discrete t values. To this end, we consider

$$\Delta W_j = W_{j+1} - W_j, \quad j = 0, 2, \dots, N-1,$$

where $\Delta W_j \sim \sqrt{\Delta t}\,\mathcal{N}(0, 1)$. In addition, we use the function `randn` to generate a random number drawn from the $\mathcal{N}(0, 1)$ distribution. Next, let $X(\tau_j)$ be abbreviated as X_j. Then, at $t = t_j$ and $t = t_{j+1}$, we obtain

$$\begin{aligned} X(\tau_{j+1}) &= X_0 + \int_0^{\tau_{j+1}} f(s, X(s))ds + \left\{\int_0^{\tau_{j+1}} \Phi(s, X(s))dW(s)\right\}, \\ X(\tau_j) &= X_0 + \int_0^{\tau_j} f(s, X(s))ds + \left\{\int_0^{\tau_j} \Phi(s, X(s))dW(s)\right\}. \end{aligned}$$

Then, by subtracting the first equation to the second one, it yields

$$X(\tau_{j+1}) - X(\tau_j) = \int_{\tau_j}^{\tau_{j+1}} f(s, X(s))ds + \left\{\int_{\tau_j}^{\tau_{j+1}} \Phi(s, X(s))dW(s)\right\}.$$

For the first integral, by using the conventional deterministic quadrature rule, we have

$$\int_{\tau_j}^{\tau_{j+1}} f(s, X(s))ds \approx f(\tau_j, X(\tau_j))\,(\tau_{j+1} - \tau_j) = f(\tau_j, X_j)\,\Delta t.$$

The second integral can be estimated by using Itô's formula

$$\int_{\tau_j}^{\tau_{j+1}} \Phi(s, X(s))dW(s) \approx \Phi(\tau_j, X(\tau_j))\,[W(\tau_{j+1}) - W(\tau_j)] = \Phi(\tau_j, X_j)\,[W_{j+1} - W_j] = \Phi(\tau_j, X_j)\Delta W_j.$$

Therefore, we obtain

$$X_{j+1} = X_j + f(\tau_j, X_j)\,\Delta t + \{\Phi(\tau_j, X_j)\Delta W_j\}, \quad j = 0, 2, \dots, N-1, \tag{9}$$

is the difference equation that approximately computes the stochastic solution of the Cauchy problem (8).

Step 2. *(Error estimation)* Our aim is to estimate the error $\rho_\infty(X_n, X(\tau_n))$.
Firstly, denote $\varepsilon_j = X_j - X(\tau_j)$ and $\Delta\varepsilon_j = \varepsilon_{j+1} - \varepsilon_j$. Then, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \Delta\varepsilon_n &= \varepsilon_{n+1} - \varepsilon_n = (X_{n+1} - X(\tau_{n+1})) - (X_n - X(\tau_n)) \\ &= (X_{n+1} - X_n) - (X(\tau_{n+1}) - X(\tau_n)) \\ &= [f(\tau_n, X_n)\,\Delta t + \{\Phi(\tau_n, X_n)\Delta W_n\}] - \left[\int_{\tau_n}^{\tau_{n+1}} f(s, X(s))ds + \left\{\int_{\tau_n}^{\tau_{n+1}} \Phi(s, X(s))dW(s)\right\}\right] \\ &= \left[f(\tau_n, X_n)\,\Delta t - \int_{\tau_n}^{\tau_{n+1}} f(s, X(s))ds\right] + \left[\{\Phi(\tau_n, X_n)\Delta W_n\} - \left\{\int_{\tau_n}^{\tau_{n+1}} \Phi(s, X(s))dW(s)\right\}\right], \end{aligned}$$

which follows that

$$\begin{aligned}\rho_\infty(\Delta\varepsilon_n,\{0\}) &\le \rho_\infty\left(f(\tau_n,X_n)\Delta t,\int_{\tau_n}^{\tau_{n+1}} f(s,X(s))ds\right)\\&\quad+\rho_\infty\left(\{\Phi(\tau_n,X_n)\Delta W_n\},\left\{\int_{\tau_n}^{\tau_{n+1}}\Phi(s,X(s))dW(s)\right\}\right)\\&=E_1+E_2,\end{aligned}$$

where

$$\begin{aligned}E_1&\le\rho_\infty\left(f(\tau_n,X_n)\Delta t,f(\tau_n,X(\tau_n))\Delta t\right)+\rho_\infty\left(f(\tau_n,X(\tau_n))\Delta t,\int_{\tau_n}^{\tau_{n+1}}f(s,X(s))ds\right)\\&\le\rho_\infty\left(f(\tau_n,X_n),f(\tau_n,X(\tau_n))\right)\Delta t+\int_{\tau_n}^{\tau_{n+1}}\rho_\infty\left(f(\tau_n,X(\tau_n)),f(s,X(s))\right)ds\\&\le\sqrt{L}\rho_\infty\left(X_n,X(\tau_n)\right)\Delta t+\int_{\tau_n}^{\tau_{n+1}}\sqrt{L}\rho_\infty\left(X(\tau_n),X(s)\right)ds\\&=\sqrt{L}\rho_\infty(\varepsilon_n,\{0\})\Delta t+\frac{M\sqrt{L}\,(\Delta t)^2}{2},\end{aligned}$$

and

$$\begin{aligned}E_2&=\rho_\infty\left(\{\Phi(\tau_n,X_n)\Delta W_n\},\left\{\int_{\tau_n}^{\tau_{n+1}}\Phi(s,X(s))dW(s)\right\}\right)\\&\le\sqrt{L}\rho_\infty\left(X_n,X(\tau_n)\right)\Delta W_n\sim\sqrt{L}\rho_\infty(\varepsilon_n,\{0\})B_n\sqrt{\Delta t},\qquad\text{for } B_n\sim\mathcal{N}(0,1).\end{aligned}$$

Therefore, we directly obtain

$$\begin{aligned}\rho_\infty(\Delta\varepsilon_n,\{0\})&\le\sqrt{L}\rho_\infty(\varepsilon_n,\{0\})\Delta t+\frac{M\sqrt{L}\,(\Delta t)^2}{2}+\sqrt{L}\rho_\infty(\varepsilon_n,\{0\})B_n\sqrt{\Delta t}\\&\le\sqrt{L}\rho_\infty(\varepsilon_n,\{0\})\left(\Delta t+B_n\sqrt{\Delta t}\right)+\frac{M\sqrt{L}\,(\Delta t)^2}{2},\end{aligned}$$

or equivalently, $\rho_\infty(\varepsilon_{n+1},\{0\})\le\rho_\infty(\varepsilon_n,\{0\})\left(1+\sqrt{L}\Delta t+\sqrt{L}B_n\sqrt{\Delta t}\right)+\frac{M\sqrt{L}\,(\Delta t)^2}{2}$. Here, for simplicity in representation, we denote $C_0=1+\sqrt{L}\Delta t+B_n\sqrt{L\Delta t}$ and $C_1=\frac{M\sqrt{L}(\Delta t)^2}{2}$. Then, we have

$$\begin{aligned}\rho_\infty(\varepsilon_{n+1},\{0\})&\le C_0\rho_\infty(\varepsilon_n,\{0\})+C_1\\&\le C_0\left(C_0\rho_\infty(\varepsilon_{n-1},\{0\})+C_1\right)+C_1\\&=C_0^2\rho_\infty(\varepsilon_{n-1},\{0\})+C_1(1+C_0)\\&\le C_0^2\left(C_0\rho_\infty(\varepsilon_{n-2},\{0\})+C_1\right)+C_1(1+C_0)\\&=C_0^3\rho_\infty(\varepsilon_{n-2},\{0\})+C_1(1+C_0+C_0^2)\\&\cdots\cdots\end{aligned}$$

By mathematical induction principle, we receive

$$\begin{aligned}\rho_\infty(\varepsilon_{n+1},\{0\})&\le C_0^n\rho_\infty(\varepsilon_0,\{0\})+C_1\left(1+C_0+C_0^2+\ldots+C_0^{n-1}\right)\\&=C_0^n\rho_\infty(\varepsilon_0,\{0\})+C_1\frac{C_0^n-1}{C_0-1}.\end{aligned}$$

In addition, since the fact that $\varepsilon_0=X_0-X(\tau_0)=\{0\}$, it implies that

$$\begin{aligned}\rho_\infty(\varepsilon_{n+1},\{0\})&\le C_1\frac{C_0^n-1}{C_0-1}=\frac{M\sqrt{L}\,(\Delta t)^2}{2}\frac{\left(1+\sqrt{L}\Delta t+B_n\sqrt{L\Delta t}\right)^n-1}{\sqrt{L}\Delta t+B_n\sqrt{L\Delta t}}\\&\le\frac{M\sqrt{L}\,(\Delta t)^2}{2}\left(e^{n\sqrt{L}\Delta t+n\sigma B_n\sqrt{L\Delta t}}-1\right)\\&=\frac{M\sqrt{L}\,(\Delta t)^2}{2}\left(e^{\sqrt{L}\tau_n+B_n\sqrt{nL\tau_n}}-1\right)\to0\quad\text{as }\Delta t\to0.\end{aligned}\tag{10}$$

Therefore, we can conclude that $X_n \to X(\tau_n)$ as $\Delta \to 0$, which means that the EM method converges and the global error is $\mathcal{O}\left((\Delta t)^2\right)$. Moreover, the error estimation of the EM method is defined in (10). $\square$

Remark 5.2. If the function $\Phi(t, X(t)) = 0$ for all $t \in [0, T]$ then the interval neutrosophic stochastic Cauchy problem (8) becomes a Cauchy problem to interval neutrosophic differential systems. Then, the scheme presented in Theorem 5.2 is similar to the familiar Euler scheme in numerical analysis. Thus, a modification of the Euler scheme can be applied to solve the numerical solution of the problem.

Remark 5.3. In the following, we will summarize a procedure to numerically solve Cauchy problem (8) to the interval neutrosophic stochastic differential systems:

Step 1 Discrete the time domain $[0, T]$ into N sub-intervals $[t_i, t_{i+1}]$ with partition's length $\Delta t = \frac{T}{N}$;

Step 2 Construct the difference formula (9) corresponding to the considered problem and evaluate the finite terms of difference sequence $\{X_n\}$. Here, note that the random parts are the Brownian motions which are approximated by using the norm distribution;

Step 3 Using mathematical programs to plot the values of sequence $\{X_n\}$ that are approximate to the solution of the considered problem. Here, note that the random part is generated by using Matlab function `rand()` and the graph of the initial condition X_0 is constructed by converting into parametric form (2) and plotting the level sets of each component.

6. Applications

Now, we will apply theoretical results to investigate some real-world models that can be described by interval neutrosophic stochastic differential equations:

Example 6.1. In this example, we investigate the Malthusian model that describes the population growth of a microbes species $N(t)$

$$\begin{cases} dN(t) \overset{J\mathbb{P}.1}{=} a(t)N(t), \\ N(0) \overset{J\mathbb{P}.1}{=} N_0, \end{cases} \qquad t \in J = [0, T], \tag{11}$$

in which $N(t)$ represents for the size of population at time t and $a(t)$ represents for the relative rate of growth at time t, that is not exactly known, but subject to some random environmental effects $a(t) = r(t) + \sigma W(t)$, where the white noise $W(t)$ has the strength $\sigma \in \mathbb{R}$.

It is a given fact that it is possible to observe under the microscope, large number of the microbes that always move continuously and change their positions quickly. Moreover, due to the imprecision of measurement equipment, imperfect human judgments or opinions on parameters, it is too difficult to estimate the initial number of individuals precisely or determine exactly the precise values of the initial density N_0 and hence, every initial datum of the model always contain itself a certain degree of uncertainty. In reality, it is often accepted to communicate the initial datum as linguistic expressions like "small", "large", "huge", which not only contains some neutral information but also has a considerable degree of falsity. Furthermore, the initial data may come from unreliable sources, so the data contains a certain degree of indeterminacy and falsity that need to be taken into account in our model, see Chapter 2 in [32] for more explanation of neutrosophic dynamical systems. The neutrosophic sets are suitable objects which can express both the true, false and indeterminate degree of data in itself. So, it is regarded as the most suitable concept to express the initial density N_0. In this example, it is assumed that "The initial data is approximately 7000", which means that it may be less than or more than a little bit. Here, we propose to use the fuzzy sets of triangular form $(6500, 7000, 7500)$ to present this statement, i.e., the value of initial data can vary from 6500 to 7500 where 7000 is the value getting the highest truth degree. However, this presentation still has a certain degree of unreliability and it is obvious that there are some doubts for this choice because the error in estimation seems to be quite big. Hence, the neutrality and falsity must be all taken into account in the initial data $N_0 = (6500, 7000, 7500)$. In particular, the truth, indeterminacy and falsity membership functions of the number N_0 can be chosen as follows:

$$\mathtt{T}_A^l(x) = \begin{cases} \frac{x}{1000} - \frac{13}{2} & x \in [6500, 7000], \\ \frac{1}{2} & x = 7000, \\ \frac{15}{2} - \frac{x}{1000} & x \in [7000, 7500], \\ 0 & \text{otherwise}, \end{cases} \qquad \mathtt{T}_A^u(x) = 0,$$

$$\mathtt{I}_A^l(x) = \begin{cases} -\frac{x}{2500} + \frac{18}{5} & x \in [6500, 7000], \\ \frac{4}{5} & x = 7000, \\ \frac{x}{2500} + 2 & x \in [7000, 7500], \\ 1 & \text{otherwise}, \end{cases} \qquad \mathtt{I}_A^u(x) = 1,$$

$$\mathtt{F}_A^l(x) = \begin{cases} -\frac{x}{5000} + \frac{23}{10} & x \in [6500, 7000], \\ \frac{9}{10} & x = 7000, \\ \frac{x}{5000} + \frac{1}{2} & x \in [7000, 7500], \\ 1 & \text{otherwise}, \end{cases} \qquad \mathtt{F}_A^u(x) = 1.$$

According to the result introduced in [34], the $(\alpha, \beta, \gamma)-$ cuts of the number N_0 can be represented as follows:

$$[N_0]_{(\alpha,\beta,\gamma)} = \begin{pmatrix} [6500 + 1000\alpha, 7500 - 1000\alpha], \\ [9000 - 2500\beta, 5000 + 2500\beta], \\ [11500 - 5000\gamma, 2500 + 5000\gamma] \end{pmatrix},$$

for each $\alpha \in [0, 0.5]$, $\beta \in [0.8, 1.0]$ and $\gamma \in [0.9, 1.0]$. In this example, let us assume that the function $r(t) = r$ is a constant. Then, the differential equation of the considered problem is equivalent to

$$dN(t) \overset{J\mathbb{P}.1}{=} rN(t)dt + \sigma N(t)d\mathbf{B}(t), \tag{12}$$

where $\mathbf{B}(t)$ is an one dimensional $\{\mathscr{A}_t\}-$ Brownian motion defined on $(\Omega, \mathscr{A}, \{\mathscr{A}_t\}_{t\in J}, \mathbb{P})$ and $\sigma = 0.3$ represents for the strength of noise. For each $t \in J$, denote

$$f(t, N(t)) = rN(t) \qquad \Phi(t, N(t)) = \sigma N(t)$$

It can be seen that the mapping f and Φ satisfy all assumptions (**A1**) - (**A3**). Thus, Theorem 5.1 guarantees the unique existence of a stochastic solution of the problem (12).

Numerical Solution: Now, based on the numerical method presented in Theorem 5.2, we will introduce the algorithm of EM method to solve the ecosystem (12). Note that if $\sigma = 0$, the difference equations (9) becomes the Euler approximation to the logistic model (11).

Algorithm 1: Euler-Maruyama method to solve the ecosystem (12)

Input: The interval $[0, 10]$, the value of parameters r, σ, n, 'num', the partition δ and initial condition N_0.

Output: The numerical solution & stochastic solution of the ecosystem (11)

Data: Set the variable and initial state vectors

```
        Y = zeros(num, n);
        Ye = zeros(1, n);
        meantime = zeros(n, 10);
        extinction = 0;
/* Numerical stochastic solution of the ecosystem (11)                 */
for i = 1, 2, ..., num do
    dW = sqrt(delta)*randn(1, n);    // random number is generated by function randn
    Yt = N0;                         // set initial condition
    for j = 1, 2, ..., n do
        Yt = Yt + r*delta*Yt + sig*Yt*dW(j);
        Y(i, j) = Yt;
Set bool = Y(i, :) > 0;
if sum(bool)==n then
    plot([0 : delta : T],[N0,Y(i, :)],'g')
else
    meantime(:, extinction + 1) = 1 − bool;
    extinction = extinction + 1;
    time = find(1 − bool, 1);
    plot([0 : delta : time ∗ delta],[N0,Y(i, 1 : time)],'r');
                                     // plot the numerical stochastic solution
/* Numerical solution of the ecosystem (11)                            */
for i = 1, 2, ..., n do
    Yte = Yte + r*Yte*delta;
    Ye(j) = Yte;
plot([0 : delta : T], [X0, Ye])      // plot the numerical solution
```

As a consequence, by applying the above algorithm, the numerical solution and numerical stochastic solution of the ecosystem (12) are obtained. In addition, graphical representations of numerical solutions are shown in Figure 6 and Figure 7 by using the Matlab program.

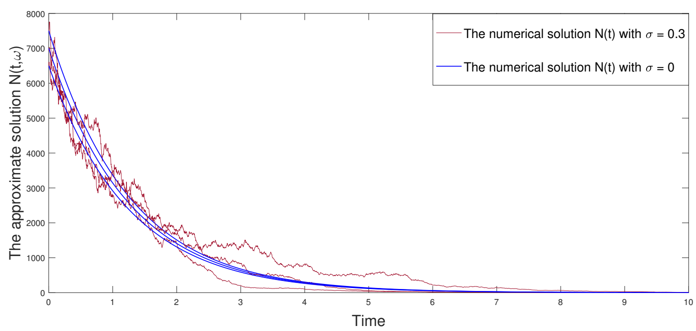

Figure 6: The truth membership function of the numerical solution of the problem (12) with $r = -0.8$, where the blue smooth curve presents the numerical solution of the ecosystem (11) (no random noise) and the brown non-smooth curve is the numerical stochastic solution of the stochastic differential equation (12).

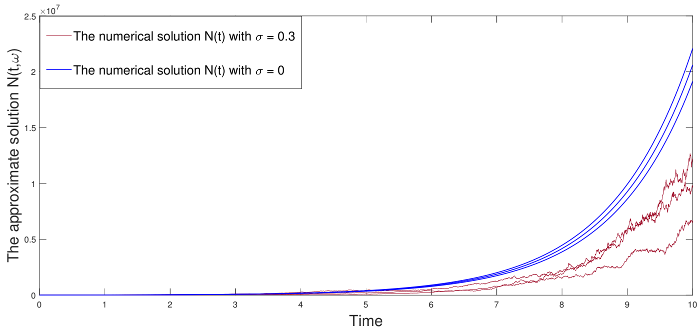

Figure 7: The truth membership function of the numerical solution of the problem (12) with $r = 0.8$, where the blue smooth curve presents the numerical solution of the ecosystem (11) (no random noise) and the brown non-smooth curve is the numerical stochastic solution of the stochastic differential equation (12).

Analytical Solution: Firstly, the interval neutrosophic stochastic differential equation (12) is equivalent to

$$\frac{dN(t)}{N(t)} \overset{J\mathbb{P}.1}{=} rdt + \sigma d\mathbf{B}(t).$$

Due to the fact that $\mathbf{B}(0) = 0$, the above equality becomes

$$\int_0^t \frac{dN(\tau)}{N(\tau)} \overset{J\mathbb{P}.1}{=} rt + \sigma\mathbf{B}(t). \tag{13}$$

In order to evaluate the above integral, we will apply the Itô formula for a function $\Psi(t) = \ln t$ and obtain

$$d(\ln N(t)) \overset{J\mathbb{P}.1}{=} \frac{d(N(t))}{N(t)} - \frac{1}{2N^2(t)}\left[d(N(t))\right]^2 \overset{J\mathbb{P}.1}{=} \frac{d(N(t))}{N(t)} - \frac{\sigma^2 N^2(t)}{2N^2(t)}dt \overset{J\mathbb{P}.1}{=} \frac{d(N(t))}{N(t)} - \frac{\sigma^2}{2}dt.$$

Hence, it follows that $\frac{d(N(t))}{N(t)} \overset{J\mathbb{P}.1}{=} d(\ln N(t)) + \frac{\sigma^2}{2}dt$. Thus, from (13), we can conclude that

$$\ln\left(\frac{N(t)}{N(0)}\right) \overset{J\mathbb{P}.1}{=} \left(r - \frac{\sigma^2}{2}\right)t + \sigma\mathbf{B}(t),$$

that implies

$$N(t) \overset{J\mathbb{P}.1}{=} N_0 e^{(r-\frac{\sigma^2}{2})t+\sigma\mathbf{B}(t)}, \qquad t \geq 0. \tag{14}$$

Finally, the graphical representations of the analytic stochastic solution $N(t,\omega)$ compared with the numerical stochastic solution obtained by proposed method are shown in Figure 8 and Figure 9 by using Matlab programs.

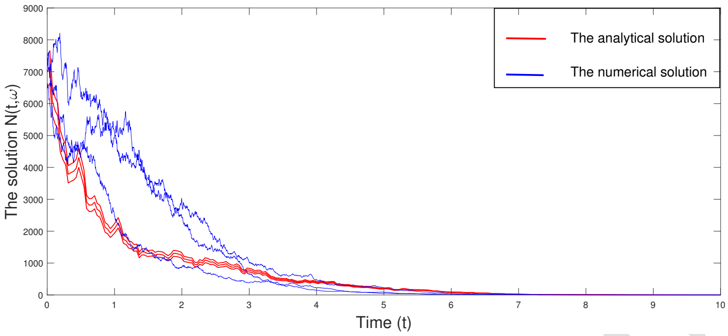

Figure 8: The truth membership function of the stochastic solution $N(t)$ of the problem (12) with $r = -0.8$ and $\sigma = 0.3$, where the red curve represents for the analytical stochastic solution given in the formula (14) and the blue curve is the numerical solution of the problem (12) obtained by Euler-Maruyama method

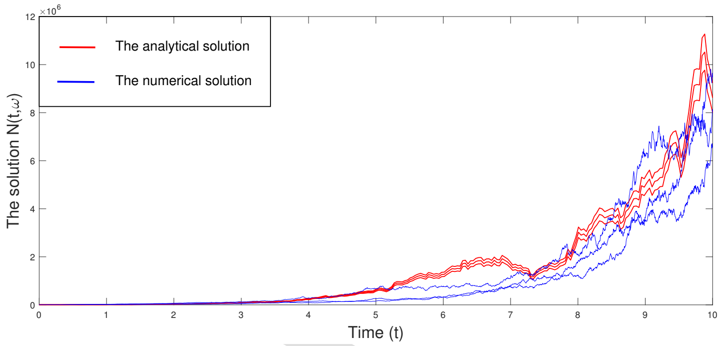

Figure 9: The truth membership function of the stochastic solution $N(t)$ of the problem (12) with $r = 0.8$ and $\sigma = 0.3$, where the red curve represents for the analytical stochastic solution given in the formula (14) and the blue curve is the numerical solution of the problem (12) obtained by Euler-Maruyama method

Example 6.2. In this example, we will apply the theoretical foundations on interval neutrosophic differential equations and interval neutrosophic stochastic processes to model and investigate Lotka - Volterra predator-prey model. According to the discussion of Lotka in 1920, the interaction between predators Y_2 and its preys Y_1 satisfies following reactions:

$$\begin{aligned} &\textbf{(Reaction 1)} \quad && X + Y_1 \xrightarrow{c_1} 2Y_1, \\ &\textbf{(Reaction 2)} \quad && Y_1 + Y_2 \xrightarrow{c_2} 2Y_2, \\ &\textbf{(Reaction 3)} \quad && Y_3 \xrightarrow{c_3} Z. \end{aligned}$$

The above reactions imply some remarkable dynamical properties of Lotka - Volterra predator-prey model

Reaction 1: Under the full feed conditions, the prey population Y_1 increases with rate constant c_1;

Reaction 2: The population of the predators Y_2 increases with a rate constant c_2 as they eat preys Y_1;

Reaction 3: The predators Y_2 die in natural environment with rate constant c_3.

Here, the amount of species X in Reaction 1 is assumed constant. Several years later, Volterra studied the use of following reaction-rate differential equations

$$\begin{cases} \dfrac{dY_1(t)}{dt} &= c_1 X Y_1 - c_2 Y_1 Y_2, \\ \dfrac{dY_2(t)}{dt} &= c_2 Y_1 Y_2 - c_3 Y_2. \end{cases} \tag{15}$$

On the other hand, since the fact that natural environments are stochastic and deterministic models fail to describe basic phenomenon of natural systems in the changing environment which may cause random variations in the predator-prey growth rate and death rate, stochastic models are considered for an accurate approximation of such dynamics of interactions. It is well-known that the noise plays a vital role in the structure of biological systems. Here, the Gaussian white noise is considered as a useful concept to model rapidly fluctuating phenomena. Moreover, the initial data and parameters of the population state always contain both truth and falsity degrees that is the reason why it can not be given in a certain form. Hence, we can conclude that the modeling of initial data as an interval neutrosophic random variable is natural and necessary. Motivated by aforesaid, a realistic model of population dynamic is investigated under the combination of two interesting concepts: neutrosophic and random. Then, the differential equation (15) becomes the following neutrosophic stochastic differential equation

$$\begin{cases} dY_1(t) & \overset{J\mathbb{P}.1}{=} (c_1XY_1 - c_2Y_1Y_2)\,dt + \sigma Y_1(t)d\mathbf{B}(t), \\ dY_2(t) & \overset{J\mathbb{P}.1}{=} (c_2Y_1Y_2 - c_3Y_2)\,dt + \sigma Y_2(t)d\mathbf{B}(t), \end{cases} \tag{16}$$

for each $t \in J = [0, T]$, subject to the initial conditions

$$\begin{cases} Y_1(0) \overset{J\mathbb{P}.1}{=} Y_1^0 \\ Y_2(0) \overset{J\mathbb{P}.1}{=} Y_2^0. \end{cases} \tag{17}$$

Here, the values of parameters are

$$c_1X = 10, \qquad c_2 = 0.01 \qquad c_3 = 10 \qquad \sigma = 0.1.$$

In addition, the initial numbers of predators (coyotes) and preys (rabbits) are given in following table

Table 1: The initial data of the Lotka-Volterra stochastic predator-prey model

Membership function	The initial prey Y_1^0	The initial predator Y_2^0
Truth m.f	$\mathtt{T}^l_{Y_1^0}(x) = \begin{cases} \frac{x}{250} - \frac{36}{5} & x \in [1800, 2000], \\ \frac{4}{5} & x = 2000, \\ \frac{44}{5} - \frac{x}{250} & x \in [2000, 2200], \\ 0 & \text{otherwise}, \end{cases}$	$\mathtt{T}^l_{Y_2^0}(x) = \begin{cases} \frac{x}{125} - \frac{56}{5} & x \in [1400, 1500], \\ \frac{4}{5} & x = 1500, \\ \frac{64}{5} - \frac{x}{125} & x \in [1500, 1600], \\ 0 & \text{otherwise}, \end{cases}$
	$\mathtt{T}^u_{Y_1^0}(x) = 0$	$\mathtt{T}^u_{Y_2^0}(x) = 0$
Indeterminacy m.f	$\mathtt{I}^l_{Y_1^0}(x) = \begin{cases} -\frac{x}{2000} + \frac{19}{10} & x \in [1800, 2000], \\ \frac{9}{10} & x = 2000, \\ \frac{x}{2000} - \frac{1}{10} & x \in [2000, 2200], \\ 1 & \text{otherwise}, \end{cases}$	$\mathtt{I}^l_{Y_2^0}(x) = \begin{cases} -\frac{x}{1000} + \frac{12}{5} & x \in [1400, 1500], \\ \frac{9}{10} & x = 1500, \\ \frac{x}{1000} - \frac{3}{5} & x \in [1500, 1600], \\ 1 & \text{otherwise}, \end{cases}$
	$\mathtt{I}^u_{Y_1^0}(x) = 1$	$\mathtt{I}^u_{Y_2^0}(x) = 1$
Falsity m.f	$\mathtt{F}^l_{Y_1^0}(x) = \begin{cases} -\frac{x}{1000} + \frac{14}{5} & x \in [1800, 2000], \\ \frac{4}{5} & x = 2000, \\ \frac{x}{1000} - \frac{6}{5} & x \in [2000, 2200], \\ 1 & \text{otherwise}, \end{cases}$	$\mathtt{F}^l_{Y_2^0}(x) = \begin{cases} -\frac{x}{500} + \frac{19}{5} & x \in [1400, 1500], \\ \frac{4}{5} & x = 1500, \\ \frac{x}{500} - \frac{11}{5} & x \in [1500, 1600], \\ 1 & \text{otherwise}, \end{cases}$
	$\mathtt{F}^u_{Y_1^0}(x) = 1$	$\mathtt{F}^u_{Y_2^0}(x) = 1$

According to Definition 2.7, the parametric forms of the initial data Y_0^1 and Y_0^2 are

$$\left[Y_1^0\right]_{(\alpha,\beta,\gamma)} = \begin{pmatrix} [1800 + 250\alpha, 2200 - 250\alpha], \\ [3800 - 2000\beta, 200 + 2000\beta], \\ [2800 - 1000\gamma, 1200 + 1000\gamma] \end{pmatrix},$$

$$\left[Y_2^0\right]_{(\alpha,\beta,\gamma)} = \begin{pmatrix} [1400 + 125\alpha, 1600 - 125\alpha], \\ [2400 - 1000\beta, 900 + 1000\beta], \\ [1900 - 500\gamma, 1100 + 500\gamma] \end{pmatrix},$$

for each $\alpha \in [0, 0.8]$, $\beta \in [0.9, 1.0]$ and $\gamma \in [0.8, 1.0]$.

For illustration, the SSA stochastic solver Matlab program is used to simulate the uncertain behavior the stochastic Lotka-Volterra predator-prey model (16) with initial condition (17). In particular, Figure 10 is plotted to show the populations of predators (coyotes) Y_2 and preys (rabbits) Y_1 against time over the interval $[0, 10]$, where the leading and lagging curves represent for the Y_1 (prey) population and the Y_2 (predator) population, respectively. The second figure (Figure 11) presents the numbers of predators Y_2 and preys Y_1 versus time $t \in [0, 30]$.

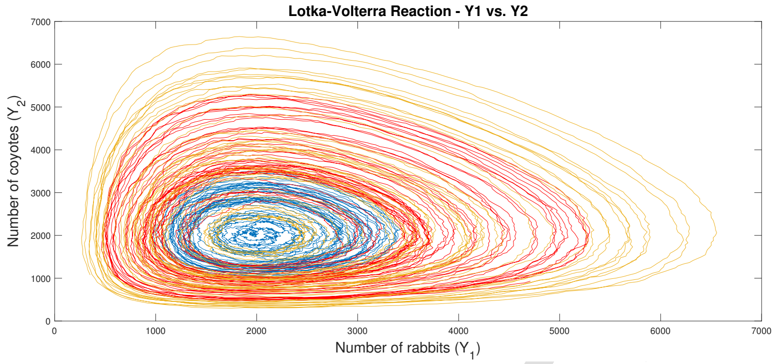

Figure 10: The phrase portrait of the number of rabbits Y_1 to the number of coyotes Y_2

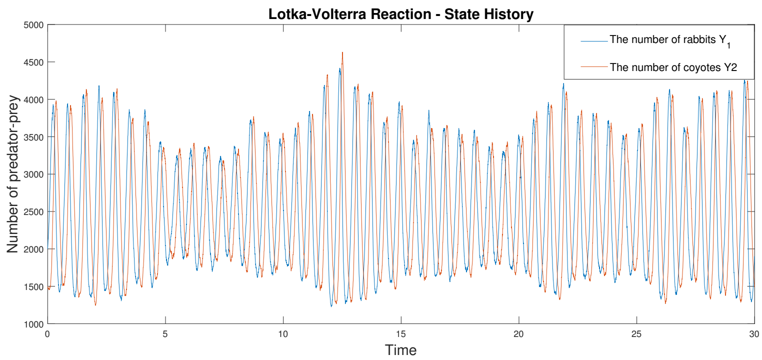

Figure 11: The $(0.8, 0.9, 0.8)-$ cut of number of rabbits and coyotes over the time $[0, 30]$

Example 6.3. This example is devoted to discussing a simple model for the spread of SARS epidemic disease in a given community, that is based on the two-compartment "S−I" (or Susceptible-Infectious) epidemic model. Here, the proposed SARS model is mathematically modeled in the following logistic equation form

$$\frac{dS(t)}{dt} = \lambda S(t) \left(\mathtt{N} - S(t)\right), \tag{18}$$

in which

- The quantity $S(t)$ represents for the number of infected and susceptible individuals at time t (days);
- The coefficient λ is proportional constant;
- The quantity $\mathtt{N}$ represents for the total number of individuals.

In the real-world scenario, SARS epidemic model provides a predictive result only to some extent, it is natural to consider that there is a need to take into account the stochastic environment. Thus, a stochastic component can be added into this model to give a new form of differential equation with stochastic noise. Now, our aim is to investigate the following stochastic differential equation

$$dS(t) = \lambda S(t)\left(\mathtt{N} - S(t)\right)dt + \mu S(t)dW(t) \tag{19}$$

subject to the initial condition

$$S(0) = S_0, \tag{20}$$

where λ, μ are real constant and $W(t)$ represents a Brownian motion.

In addition, we can see that in reality, it is natural to consider the initial datum S_0 as an uncertain quantity. Indeed, because of a lack of knowledge or incomplete statistical information, it cannot be measured exactly. Moreover, the measured data also contains a certain degree of indeterminacy and falsity. Hence, we represent the initial datum S_0 as the following interval neutrosophic number whose truth, indeterminacy and falsity membership functions are given in the following table.

Membership function	**The left membership function**	**The right membership function**
Truth m.f	$\mathtt{T}^l_{S_0}(x) = \begin{cases} \frac{x}{25} - 4 & x \in [100, 120], \\ \frac{4}{5} & x = 120, \\ \frac{28}{5} - \frac{x}{25} & x \in [120, 140], \\ 0 & \text{otherwise}, \end{cases}$	$\mathtt{T}^u_{S_0}(x) = \begin{cases} \frac{x}{40} - \frac{5}{2} & x \in [100, 120], \\ \frac{1}{2} & x = 120, \\ \frac{7}{2} - \frac{x}{40} & x \in [120, 140], \\ 0 & \text{otherwise}, \end{cases}$
Indeterminacy m.f	$\mathtt{I}^l_{S_0}(x) = \begin{cases} -\frac{x}{50} + 3 & x \in [100, 110], \\ \frac{4}{5} & x = 110, \\ \frac{x}{100} - \frac{3}{10} & x \in [110, 130], \\ 1 & \text{otherwise}, \end{cases}$	$\mathtt{I}^u_{S_0}(x) = 1$
Falsity m.f	$\mathtt{F}^l_{S_0}(x) = \begin{cases} -\frac{3x}{100} + \frac{43}{10} & x \in [110, 120], \\ \frac{7}{10} & x = 120, \\ \frac{3x}{200} + \frac{11}{10} & x \in [120, 140], \\ 1 & \text{otherwise}, \end{cases}$	$\mathtt{F}^u_{S_0}(x) = 1$

The respective numerical solution and numerical stochastic solution of the SARS epidemic disease model (19) with the initial condition (20) are numerically solved by employing the Euler-Maruyama (EM) method discussed in previous example. The graphical representations of the numerical solutions are shown in Figure 12.

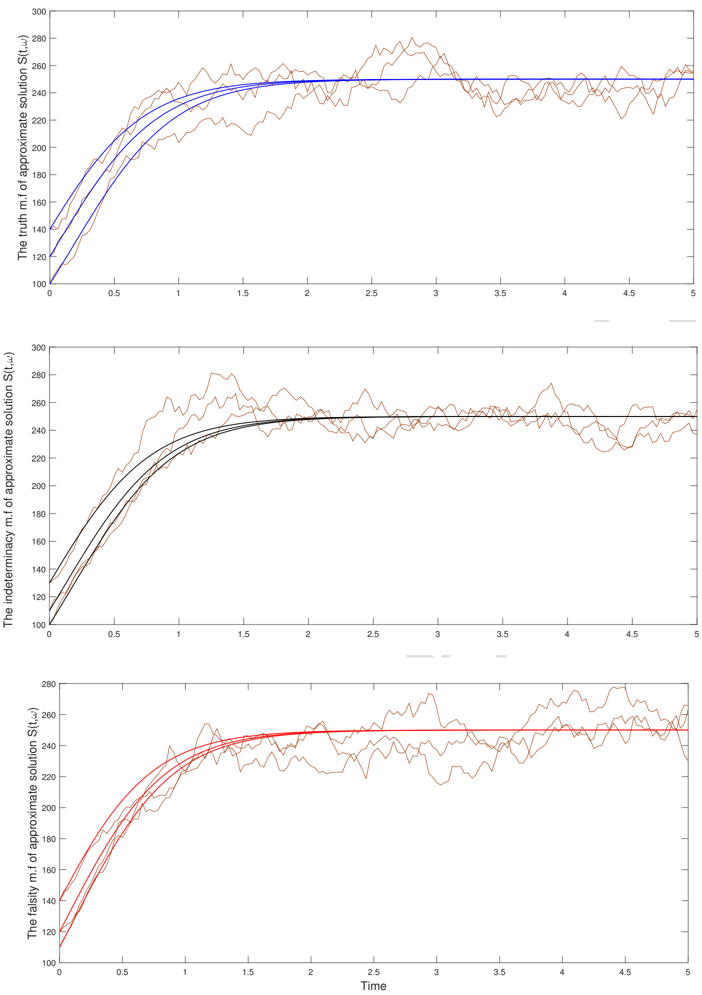

Figure 12: The membership functions of the numerical solution of the SARS epidemic model over the interval $[0, 5]$, where the smooth curves represent for the numerical solution of the SARS epidemic model (18) - (20) (no noise) and the non-smooth curves represents for the numerical stochastic solution of the stochastic SARS epidemic model (19) - (20)

Remark 6.1. The obtained theoretical results of this work are the foundation for the investigations of neutrosophic dynamical systems and neutrosophic stochastic dynamical systems. Indeed, it proves the applicability in better modeling of various real-world phenomena and processes in many fields of science due to the generalization of the proposed model. Additionally, it also shows the capability to develop and study some related problems

- The construction of a solution formula for the initial value problem to interval neutrosophic stochastic differential systems is one of the important contributions of this work. Motivated by this result, we can develop it to study some more complex problems such as Cauchy problems with non-local initial data

$$\begin{cases} z'(t)dt & \overset{J\mathbb{P}.1}{=} f(t, z(t))dt + \{\Phi(t, z(t))d\mathbf{B}(t)\}, \qquad t \in [0, T], \\ z(0) & \overset{J\mathbb{P}.1}{=} g(z(t)), \end{cases}$$

  or some qualitative problems of solutions such as the data dependence, stability or sensitivity, etc.

- The control problems for the neutrosophic stochastic differential systems are still new and updated problems that open up many attractive problems in the theory of neutrosophic sets and systems. Here, we consider a controlled neutrosophic stochastic differential systems of the form

$$z'(t)dt \overset{J\mathbb{P}.1}{=} f(t, z(t))dt + g(u(t))dt + \{\Phi(t, z(t))d\mathbf{B}(t)\}. \tag{21}$$

  By applying the method of constant variation, we can construct the solution formula with respect to the control input u and then, we can use some familiar techniques in stochastic analysis and functional analysis to investigate the controllability or observability of the controlled system (21). In addition, if we consider an optimal control problem that receives the neutrosophic stochastic differential system (21) as the dynamical constraint [12, 35, 36], we obtain a neutrosophic stochastic optimal control problem. Which may be a promising research in both theoretical and practical domains.

7. Conclusions

The paper is concerned with the calculus on the space of interval neutrosophic numbers, the interval neutrosophic stochastic processes and their applications in studying the Cauchy problem to interval neutrosophic stochastic differential equations. This type of stochastic differential equations consists of two types of uncertainties that appear in various real-world problems, i.e. ambiguity driven by interval neutrosophic mappings and randomness caused by stochastic noises. Our approach is a generalized combination of many abstract theories including stochasticity, the theory of interval neutrosophic sets and the theory of stochastic differential equations. In addition, it is well-known that one of the common limitations of the introduction of uncertain quantities in dynamical systems is that the additional information that is used for modeling must come from the expert knowledge of the researchers and certainly they only reflect the personal viewpoints of researchers about the processes. Hence, the consequence is that the considered systems are often more complicated. Moreover, there is also a difficulty in dealing with the input and output information due to the fact that the more input information we require, the harder it is to elicit such information in output. Despite these difficulties, this paper defines some fundamental arithmetic properties of the space of interval neutrosophic numbers and the calculus of interval neutrosophic mappings in order to early establish the basic foundation of the study of analysis and stochastic analysis for neutrosophic valued functions, which are the new fields of research.

In the paper [6], the authors studied the relation between advanced fuzzy sets and interval-valued intuitionistic fuzzy sets and claimed that inconsistent intuitionistic fuzzy set, picture fuzzy set and neutrosophic fuzzy set can be represented by an interval-valued intuitionistic fuzzy sets by a trivial normalization. The research of Atanassov and Vassliev has opened up a new perspective on our future studies in the space of advanced fuzzy sets. In this paper, with the foundation of neutrosophic arithmetic operations, one of our contributions is the introduction of calculus of interval neutrosophic-valued functions. Indeed, the concepts of interval neutrosophic derivative and interval neutrosophic stochastic process are firstly presented based on the interval neutrosophic difference, that are the basis for defining the interval neutrosophic stochastic differential equations. Main contributions of this work are about the existence and uniqueness theorem of the stochastic solution (Theorem 5.1) and the convergent numerical method theorem (Theorem 5.2). Additionally, in order to demonstrate the potential applicability of the theoretical results, some biological systems are investigated in terms of interval neutrosophic stochastic differential equations. Our research will be the background for further studies on neutrosophic dynamical systems, neutrosophic differential equations, neutrosophic control system. Moreover, the proposed numerical scheme is constructed for the Cauchy problem of general form, that proves its widely applicable capacity for various classes of real-world models. The current work opens up many potential applications in applied science and engineering that directly employ derivative and integral calculus as the essential tools to study control problems or qualitative properties of the interval neutrosophic stochastic solutions such as stability, attractive property or boundedness or solving numerical of interval neutrosophic dynamical systems by some advance methods [39, 40].

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8. Appendix

Proof of Proposition 2.1. Based on Definitions 2.9 - 2.11, the five first assertions can be easily proved by converting the numbers A, B, C into parametric representations and then, applying arithmetic operations between subsets of $\mathbb{R}$ to get the parametric form of desired results. In the following, we will give the proof of the assertion (vi).

Let A and B be two arbitrary interval neutrosophic numbers such that the difference $C = A \ominus_{neu} B$ exists. Then, according to Definition 2.11, the parametric representation of C is given by

$$[C]_{(\alpha,\beta,\gamma)} = \Big[\Big\{[C^l_-(\alpha), C^l_+(\alpha)]; [C^u_-(\alpha), C^u_+(\alpha)]\Big\}, \Big\{[C^l_-(\beta), C^l_+(\beta)]; [C^u_-(\beta), C^u_+(\beta)]\Big\}, \Big\{[C^l_-(\gamma), C^l_+(\gamma)]; [C^u_-(\gamma), C^u_+(\gamma)]\Big\} \Big],$$

where the level-sets of each components are given by

$$[C^l_-(\mu), C^l_+(\mu)] = \Big[\min\Big\{A^l_-(\mu) - B^l_-(\mu), A^l_+(\mu) - B^l_+(\mu)\Big\}, \max\Big\{A^l_-(\mu) - B^l_-(\mu), A^l_+(\mu) - B^l_+(\mu)\Big\}\Big]$$

$$[C^u_-(\mu), C^u_+(\mu)] = \big[\min\{A^u_-(\mu) - B^u_-(\mu), A^u_+(\mu) - B^u_+(\mu)\}, \max\{A^u_-(\mu) - B^u_-(\mu), A^u_+(\mu) - B^u_+(\mu)\}\big].$$

Here, we use the notation $\mu \in [0,1]$ to denote for the parameters $\alpha, \beta, \gamma \in [0,1]$.

Then, by using the definition of scalar multiplication and the hypothesis $\lambda_1 < 0$, we have

$$\begin{aligned}\lambda_1[C^l_-(\mu), C^l_+(\mu)] &= \Big[\min\Big\{\lambda_1\Big(A^l_-(\mu) - B^l_-(\mu)\Big), \lambda_1\Big(A^l_+(\mu) - B^l_+(\mu)\Big)\Big\}, \\ &\qquad \max\Big\{\lambda_1\Big(A^l_-(\mu) - B^l_-(\mu)\Big), \lambda_1\Big(A^l_+(\mu) - B^l_+(\mu)\Big)\Big\}\Big] \\ &= (-1)\Big[\min\Big\{(-\lambda_1)\Big(A^l_-(\mu) - B^l_-(\mu)\Big), (-\lambda_1)\Big(A^l_+(\mu) - B^l_+(\mu)\Big)\Big\}, \\ &\qquad \max\Big\{(-\lambda_1)\Big(A^l_-(\mu) - B^l_-(\mu)\Big), (-\lambda_1)\Big(A^l_+(\mu) - B^l_+(\mu)\Big)\Big\}\Big].\end{aligned}$$

By doing similar arguments, we also get

$$\begin{aligned}\lambda_1[C^u_-(\mu), C^u_+(\mu)] &= (-1)\Big[\min\Big\{(-\lambda_1)\Big(A^u_-(\mu) - B^u_-(\mu)\Big), (-\lambda_1)\Big(A^u_+(\mu) - B^u_+(\mu)\Big)\Big\}, \\ &\qquad \max\Big\{(-\lambda_1)\Big(A^u_-(\mu) - B^u_-(\mu)\Big), (-\lambda_1)\Big(A^u_+(\mu) - B^u_+(\mu)\Big)\Big\}\Big].\end{aligned}$$

On the other hand, by Definition 2.10 (i), we have

$$\begin{aligned}[(-\lambda_1)A]_{(\alpha,\beta,\gamma)} = \Big[&\Big\{[(-\lambda_1)A^l_-(\alpha), (-\lambda_1)A^l_+(\alpha)]; [(-\lambda_1)A^u_-(\alpha), (-\lambda_1)A^u_+(\alpha)]\Big\}, \\ &\Big\{[(-\lambda_1)A^l_-(\beta), (-\lambda_1)A^l_+(\beta)]; [(-\lambda_1)A^u_-(\beta), (-\lambda_1)A^u_+(\beta)]\Big\}, \\ &\Big\{[(-\lambda_1)A^l_-(\gamma), (-\lambda_1)A^l_+(\gamma)]; [(-\lambda_1)A^u_-(\gamma), (-\lambda_1)A^u_+(\gamma)]\Big\} \Big] \\ [(-\lambda_1)B]_{(\alpha,\beta,\gamma)} = \Big[&\Big\{[(-\lambda_1)B^l_-(\alpha), (-\lambda_1)B^l_+(\alpha)]; [(-\lambda_1)B^u_-(\alpha), (-\lambda_1)B^u_+(\alpha)]\Big\}, \\ &\Big\{[(-\lambda_1)B^l_-(\beta), (-\lambda_1)B^l_+(\beta)]; [(-\lambda_1)B^u_-(\beta), (-\lambda_1)B^u_+(\beta)]\Big\}, \\ &\Big\{[(-\lambda_1)B^l_-(\gamma), (-\lambda_1)B^l_+(\gamma)]; [(-\lambda_1)B^u_-(\gamma), (-\lambda_1)B^u_+(\gamma)]\Big\} \Big].\end{aligned}$$

In addition, since the difference $A \ominus_{neu} B$ exists and $(-\lambda_1) > 0$, the assertion (v) implies that

$$(-\lambda_1)\left[A \ominus_{neu} B\right] = (-\lambda_1)A \ominus_{neu} (-\lambda_1)B,$$

which follows that $[(-\lambda_1)(A \ominus_{neu} B)]_{(\alpha,\beta,\gamma)} = [(-\lambda_1)A \ominus_{neu} (-\lambda_1)B]_{(\alpha,\beta,\gamma)}$.
Denote $E = (-\lambda_1)(A \ominus_{neu} B)$. Then, we immediately obtain

$$[E]_{(\alpha,\beta,\gamma)} = \Big[\Big\{ [E^l_-(\alpha), E^l_+(\alpha)]; [E^u_-(\alpha), E^u_+(\alpha)] \Big\}, \Big\{ [E^l_-(\beta), E^l_+(\beta)]; [E^u_-(\beta), E^u_+(\beta)] \Big\}, \Big\{ [E^l_-(\gamma), E^l_+(\gamma)]; [E^u_-(\gamma), E^u_+(\gamma)] \Big\} \Big],$$

where the level-sets of each components of E are

$$\begin{aligned}
[E^l_-(\mu), E^l_+(\mu)] = \Big[&\min \Big\{ (-\lambda_1)\Big(A^l_-(\mu) - B^l_-(\mu)\Big), (-\lambda_1)\Big(A^l_+(\mu) - B^l_+(\mu)\Big) \Big\}, \\
&\max \Big\{ (-\lambda_1)\Big(A^l_-(\mu) - B^l_-(\mu)\Big), (-\lambda_1)\Big(A^l_+(\mu) - B^l_+(\mu)\Big) \Big\} \Big], \\
[E^u_-(\mu), E^u_+(\mu)] = \Big[&\min \Big\{ (-\lambda_1)\Big(A^u_-(\mu) - B^u_-(\mu)\Big), (-\lambda_1)\Big(A^u_+(\mu) - B^u_+(\mu)\Big) \Big\}, \\
&\max \Big\{ (-\lambda_1)\Big(A^u_-(\mu) - B^u_-(\mu)\Big), (-\lambda_1)\Big(A^u_+(\mu) - B^u_+(\mu)\Big) \Big\} \Big],
\end{aligned}$$

for all $\mu \in [0,1]$. Thus, we directly get that $\lambda_1 (A \ominus_{neu} B) = (-1)\left[(-\lambda_1 A) \ominus_{neu} (-\lambda_1 B)\right]$. □

Proof of Theorem 3.1. Let A, B C be interval neutrosophic numbers whose $(\alpha,\beta,\gamma)-$ cuts are given by

$$\begin{aligned}
[A]_{(\alpha,\beta,\gamma)} &= \left[\left\{ \left[A^l\right]^\alpha; [A^u]^\alpha \right\}, \left\{ \left[A^l\right]^\beta; [A^u]^\beta \right\}, \left\{ \left[A^l\right]^\gamma; [A^u]^\gamma \right\} \right] \\
[B]_{(\alpha,\beta,\gamma)} &= \left[\left\{ \left[B^l\right]^\alpha; [B^u]^\alpha \right\}, \left\{ \left[B^l\right]^\beta; [B^u]^\beta \right\}, \left\{ \left[B^l\right]^\gamma; [B^u]^\gamma \right\} \right], \\
[C]_{(\alpha,\beta,\gamma)} &= \left[\left\{ \left[C^l\right]^\alpha; [C^u]^\alpha \right\}, \left\{ \left[C^l\right]^\beta; [C^u]^\beta \right\}, \left\{ \left[C^l\right]^\gamma; [C^u]^\gamma \right\} \right]
\end{aligned}$$

for each $\alpha,\beta,\gamma \in [0,1]$, respectively. Now, we will show that the $\rho_\infty-$ distance satisfies all conditions of a metric on the space $\mathscr{U}$. Due to the fact that the Hausdorff distances $d_H\left(\left[A^l\right]^{\mu_i}, \left[B^l\right]^{\mu_i}\right)$ and $d_H\left([A^u]^{\mu_i}, [B^u]^{\mu_i}\right)$ are always non-negative, we directly obtain that $\rho_\infty(A,B) \geq 0$. On the other hand, from the equality $\rho_\infty(A,B) = 0$, it implies that

$$d_H\left(\left[A^l\right]^{\mu_i}, \left[B^l\right]^{\mu_i}\right) = 0,$$

for all $\mu_i \in [0,1]$, which means that $\left[A^l\right]^{\mu_i} = \left[B^l\right]^{\mu_i}$ for all $i = 1,2,3$. Hence, according to the assertion (ii) of Definition 2.8, we immediately get that $A = B$.

In addition, the equality $\rho_\infty(A,B) = \rho_\infty(B,A)$ is obvious since the definition of the distance ρ_∞. The rest of proof is to show that

$$\rho_\infty(A,B) \leq \rho_\infty(A,C) + \rho_\infty(B,C).$$

By using the property $d_H(X,Y) \leq d_H(X,Z) + d_H(Z,Y)$ for all subsets $X,Y,Z \subset \mathbb{R}$, the above assertion holds. The proof is completed. □

Proof of Proposition 4.2. Let $z \in \mathcal{L}^2(J \times \Omega, \mathcal{N}; \mathbb{R})$ be arbitrary. It is well-known from the classical Itô integral that the mapping $(t,\omega) \mapsto \int_0^s z(\tau,\omega)d\mathbf{B}(\tau)$ is a non-anticipating real-valued stochastic process. Hence, it implies that

$$(t,\omega) \mapsto \left\{ \int_0^s z(\tau,\omega)d\mathbf{B}(\tau) \right\} = \langle [1,1],[0,0],[0,0] \rangle / \int_0^s z(\tau,\omega)d\mathbf{B}(\tau)$$

is non-anticipating, too. Now, the rest of our proof is to show that z is L^2-integrably bounded. Indeed, for each $s \in [0,T]$, we have

$$\int_J \int_\Omega \rho_\infty^2 \left(\left\{ \int_0^s z(\tau,\omega) d\mathbf{B}(\tau) \right\}, \{0\} \right) P(d\omega)dt$$
$$= \int_J \int_\Omega \sup_{[0,1]} \left\{ d_H \left(\left[\left(\int_0^s z(\tau,\omega) d\mathbf{B}(\tau) \right)^l \right]^\alpha, [\hat{0}]^\alpha \right) + d_H \left(\left[\left(\int_0^s z_1(\tau,\omega) d\mathbf{B}(\tau) \right)^r \right]^\alpha, [\hat{0}]^\alpha \right) \right\}^2 P(d\omega)dt$$
$$= \int_J \int_\Omega \left\{ \left\| \int_0^s z(\tau,\omega) d\mathbf{B}(\tau) \right\| + \left\| \int_0^s z(\tau,\omega) d\mathbf{B}(\tau) \right\| \right\}^2 P(d\omega)dt$$
$$\le 2 \int_J \int_\Omega \left\{ \left\| \int_0^s z(\tau,\omega) d\mathbf{B}(\tau) \right\|^2 + \left\| \int_0^s z(\tau,\omega) d\mathbf{B}(\tau) \right\|^2 \right\} P(d\omega)dt$$
$$\le 2 \int_J \int_\Omega \int_0^s \left\{ \|z(\tau,\omega)\|^2 + \|z(\tau,\omega)\|^2 \right\} d\tau P(d\omega)dt$$
$$\le 4T \int_J \int_\Omega \|z(\tau,\omega)\|^2 P(d\omega)dt.$$

Hence, the proof is completed. □

Proof of Proposition 4.3. Let $z \in \mathcal{L}^2(J \times \Omega, \mathcal{N}; \mathbb{R})$ be fixed. Then, for each $\omega \in \Omega$ and $t, s \in J$ such that $s < t$, we have

$$\rho_\infty \left(\left\{ \int_0^t z(\tau,\omega) d\mathbf{B}(\tau) \right\}, \left\{ \int_0^s z(\tau,\omega) d\mathbf{B}(\tau) \right\} \right) = \rho_\infty \left(\left\{ \int_s^t z(\tau,\omega) d\mathbf{B}(\tau) \right\}, \{0\} \right).$$

By doing similar arguments as in Proposition 4.2, we receive

$$\rho_\infty \left(\left\{ \int_0^t z(\tau,\omega) d\mathbf{B}(\tau) \right\}, \left\{ \int_0^s z(\tau,\omega) d\mathbf{B}(\tau) \right\} \right) \le 2 \int_s^t \|z(\tau,\omega)\| \, ds.$$

Since the process $z(\cdot,\omega)$ is integrably bounded then we have

$$\int_s^t \|z(\tau,\omega)\| \, ds \to 0 \qquad \text{as } s \to t^+,$$

which follows the left-sided continuity of the stochastic process $\left\{ \int_0^t z(\tau,\omega) d\mathbf{B}(\tau) \right\}_{t \in J}$. Similarly, we also obtain the right-sided continuity of this stochastic process. □

Proof of Proposition 4.4. By the definition of embedding mapping (3), it yields

$$\left\{ \int_0^s z_1(\tau,\omega) d\mathbf{B}(\tau) \right\} = \langle [1,1], [0,0], [0,0] \rangle / \int_0^s z_1(\tau,\omega) d\mathbf{B}(\tau)$$
$$\left\{ \int_0^s z_2(\tau,\omega) d\mathbf{B}(\tau) \right\} = \langle [1,1], [0,0], [0,0] \rangle / \int_0^s z_2(\tau,\omega) d\mathbf{B}(\tau).$$

Then, for each $t \in J$ and $z_1, z_2 \in \mathcal{L}^2(J \times \Omega, \mathcal{N}; \mathbb{R})$, we receive

$$
\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} \rho_\infty^2 \left(\left\{ \int_0^s z_1(\tau,\omega) d\mathbf{B}(\tau) \right\}, \left\{ \int_0^s z_2(\tau,\omega) d\mathbf{B}(\tau) \right\} \right) \\
&= \mathbb{E} \sup_{s\in[0,t]} \left\{ \sup_{\alpha\in[0,1]} d_H \left(\left[\left(\int_0^s z_1(\tau,\omega) d\mathbf{B}(\tau) \right)^l \right]^\alpha, \left[\left(\int_0^s z_2(\tau,\omega) d\mathbf{B}(\tau) \right)^l \right]^\alpha \right) \right. \\
&\qquad \left. + \sup_{\alpha\in[0,1]} d_H \left(\left[\left(\int_0^s z_1(\tau,\omega) d\mathbf{B}(\tau) \right)^r \right]^\alpha, \left[\left(\int_0^s z_2(\tau,\omega) d\mathbf{B}(\tau) \right)^r \right]^\alpha \right) \right\}^2 \\
&= \mathbb{E} \sup_{s\in[0,t]} \left\{ \left\| \int_0^s z_1(\tau,\omega) - z_2(\tau,\omega) d\mathbf{B}(\tau) \right\| + \left\| \int_0^s z_1(\tau,\omega) - z_2(\tau,\omega) d\mathbf{B}(\tau) \right\| \right\}^2 \\
&\le \mathbb{E} \sup_{s\in[0,t]} 2 \left\{ \left\| \int_0^s z_1(\tau,\omega) - z_2(\tau,\omega) d\mathbf{B}(\tau) \right\|^2 + \left\| \int_0^s z_1(\tau,\omega) - z_2(\tau,\omega) d\mathbf{B}(\tau) \right\|^2 \right\}.
\end{aligned}
$$

Next, by using Doob inequality and the Itô isometry, we directly obtain

$$
\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} \rho_\infty^2 \left(\left\{ \int_0^s z_1(\tau,\omega) d\mathbf{B}(\tau) \right\}, \left\{ \int_0^s z_2(\tau,\omega) d\mathbf{B}(\tau) \right\} \right) \\
&\le 8\mathbb{E} \left\{ \left\| \int_0^s z_1(\tau,\omega) - z_2(\tau,\omega) d\mathbf{B}(\tau) \right\|^2 + \left\| \int_0^s z_1(\tau,\omega) - z_2(\tau,\omega) d\mathbf{B}(\tau) \right\|^2 \right\} \\
&\le 8\mathbb{E} \int_0^s \left(\| z_1(\tau,\omega) - z_2(\tau,\omega) \|^2 + \| z_1(\tau,\omega) - z_2(\tau,\omega) \|^2 \right) d\tau \\
&\le 8\mathbb{E} \int_0^s \left(\| z_1(\tau,\omega) - z_2(\tau,\omega) \| + \| z_1(\tau,\omega) - z_2(\tau,\omega) \| \right)^2 d\tau \\
&\le 8\mathbb{E} \int_0^s \rho_\infty^2 \left(\{ z_1(\tau,\omega) \}, \{ z_2(\tau,\omega) \} \right) d\tau.
\end{aligned}
$$

□

Proof of Lemma 5.1. Firstly, for each $n \in \mathbb{N}$ and $t \in J$, we denote $X_n(t) = \mathbb{E} \sup_{\tau\in[0,t]} \rho_\infty^2(z_n(\tau), \{0\})$. Then, by the definition of the function $z_n(t)$, we immediately obtain

$$
\begin{aligned}
X_n(t) \le 3 \Bigg[& \mathbb{E}\rho_\infty^2(z_0, \{0\}) + \mathbb{E} \sup_{\tau\in[0,t]} \rho_\infty^2 \left(\int_0^\tau f(s, z_{n-1}(s)) ds, \{0\} \right) \\
& + \mathbb{E} \sup_{\tau\in[0,t]} \rho_\infty^2 \left(\left\{ \int_0^\tau \Phi(s, z_{n-1}(s)) d\mathbf{B}(s) \right\}, \{0\} \right) \Bigg].
\end{aligned}
$$

Then, by using Proposition 4.1, Proposition 4.4 and the triangle inequality, we receive

$$
\begin{aligned}
X_n(t) \le 3 \Bigg[& \mathbb{E}\rho_\infty^2(z_0, \{0\}) + 2t\mathbb{E} \int_0^t \left(\rho_\infty^2 \left(f(s, z_{n-1}(s)), f(s, \{0\}) \right) + \rho_\infty^2 \left(f(s, \{0\}), \{0\} \right) \right) ds \\
& + 16\mathbb{E} \int_0^t \left(\rho_\infty^2 \left(\{ \Phi(s, z_{n-1}(s)) \}, \{ \Phi(s, \{0\}) \} \right) + \rho_\infty^2 \left(\{ \Phi(s, \{0\}) \}, \{0\} \right) \right) ds \Bigg].
\end{aligned}
$$

By using assumptions (**A2**) and (**A3**), one gets

$$
X_n(t) \le 3 \left[\mathbb{E}\rho_\infty^2(z_0, \{0\}) + 2\kappa t^2 + 16\kappa t + 2L(t+8)\mathbb{E} \int_0^t \rho_\infty^2 \left(z_{n-1}(s), \{0\} \right) ds \right] \le \lambda_1 + \lambda_2 \int_0^t X_{n-1}(s) ds.
$$

For each big enough number $N_0 \in \mathbb{N}$, the last inequality implies that

$$\max_{1\le n\le N_0} X_n(t) \le \lambda_1 + \lambda_2 \int_0^t \max_{1\le n\le N_0} X_{n-1}(s)ds.$$

In addition, since $\max_{1\le n\le N_0} X_{n-1}(t) \le X_0(t) + \max_{1\le n\le N_0} X_n(t) = \mathbb{E}\rho_\infty^2(z_0,\{0\})) + \max_{1\le n\le N_0} X_n(t)$, it follows that

$$\max_{1\le n\le N_0} X_n(t) \le \lambda_1 + \lambda_2 T\mathbb{E}\rho_\infty^2(z_0,\{0\}) + \lambda_2 \int_0^t \max_{1\le n\le N_0} X_n(s)ds,$$

for each $k \in \mathbb{N}$ and $t \in J$. Next, by employing Gronwall inequality, we have

$$\max_{1\le n\le N_0} X_n(t) \le \left(\lambda_1 + \lambda_2 T\mathbb{E}\rho_\infty^2(z_0,\{0\})\right) e^{\lambda_2 T}, \quad t \in J.$$

Therefore, the proof is completed. □

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Highlights

- Parametric representation of interval neutrosophic numbers
- Interval neutrosophic stochastic (INS) dynamic systems driven by Brownian motion
- Euler - Maruyama method for numerical INS solution
- INS biological systems such as Lotka-Volterra predator-prey and stochastic SARS model

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Declaration of interests

The authors declare that:

They have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper;

They have no financial interests/personal relationships which may be considered as potential competing interests.

This statement is agreed by all the authors to indicate agreement that the above information is true and correct.