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Solution of an EPQ model for imperfect production process under game and neutrosophic fuzzy approach

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ABSTRACT

This article deals with an Economic Production Quantity (EPQ) deteriorating inventory model for non-random uncertain environment. It includes rework process, screening of imperfect items and partial backlogging. The items are partially serviceable, because at the time of production some items are found to be defective which cannot be recoverable or serviceable. At first, we develop a cost minimization problem under several assumptions related to imperfect items and rework process under certain linear constraints. We solve the crisp model (primal nonlinear problem) first, and then we convert this model into equivalent game problem taking the help of the theories related to strong and weak duality theorem. However, this game problem consists of the Lagrangian function that correspond a nonlinear objective function subject to some linear constraints. The main objective of the study is to develop a solution procedure of the problem associated to an imperfect process where all unit cost components might increase or decrease neutrosophically. Thus, according to the experiences gained by the decision maker (DM) we fuzzify all cost components as sub-neutrosophic offset. To defuzzify the model we have utilized the sine cuts of neutrosophic fuzzy numbers followed by a solution procedure developed in solving the matrix game exclusively. To validate the model, a numerical example is studied then we have compared the optimal results among the original problem, the equivalent game problem and the game problem under neutrosophic environment explicitly. Our findings reveal that under negative α-cuts the value of the objective function assumes lower and higher values. Finally, sensitivity analysis, graphical illustrations, conclusions and scope of future works have been discussed.

1. Introduction

In real world scenario, the defectiveness and deterioration of the items are the major concern of any kind of production process which is unavoidable. Some of these items are basically reworkable and the rests are disposed of. For this reason, a situation may come when backlogging of items are partially applied to the production plant for customers' satisfaction. Traditionally, researchers were involved to optimize the items of good quality order quantity, optimum cycle time, optimum screening time, optimum rework time, optimum inventory run time to control imperfect production process. In the literature numerous research articles are available along these directions. Wee et al. [1] developed optimal inventory model for items with imperfect quality and shortage backordering. Cardenas-Barron [2] studied economic production quantity with rework process at a single-stage manufacturing system with planned backorders. Tai [3] developed economic production quantity models for deteriorating/imperfect products and service with rework. Hsu and Hsu [4] developed two backorder EPQ models with imperfect production processes, inspection errors and sales returns. Ruidas et al. [5,6,7] studied on production inventory model for imperfect production system with rework of regular production, shortages and sales return via particle swarm optimization. A single-stage manufacturing system with rework and backorder options was also studied by Kang et al. [8] and Sanjai and Periyasamy [9]. Li et al. [10] developed an EPQ model for deteriorating reworkable items. A manufacturing model of imperfect reworkable items and random breakdown under abort/resume policy has been studied by Chiu et al. [11,12]. Kuzyutin et al. [13] implemented a cooperative multistage multicriteria game problem and its solution procedure. Recently, the production system having synchronous and

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asynchronous flexible rework rates was analysed by Muhammad Al-Salamah [14]. The concept of advertisement cost dependent demand was introduced by Khara et al. [15] and it has been solved by utilizing branch and bound technique which is another kind of extension in imperfect production process.

However, to capture the non-random uncertainty of the real-world phenomenon Zadeh [16] invented the concept of 'Fuzzy sets'. After that several research articles have been developed by the numerous eminent researchers by implementing fuzzy sets in various decision-making problems. Researchers like De [17] studied an economic order quantity (EOQ) model with natural idle time and wrongly measured demand rate using intuitionistic fuzzy set. De and Sana [18,19] discussed backlogging model under fuzzy environment considering promotional effort and selling price sensitive demand. The application of dense fuzzy set into a pollution sensitive production model was developed by Karmakar et al. [20]. The application of embedded fuzzy logic controller for positive and negative pressure control in pneumatic soft robots was wisely introduced by Oguntosin et al. [21]. In addition, De [22]) introduced first time the concept of fuzzy lock sets which is solely based on learning experiences and studied a new defuzzification method after extending the triangular dense fuzzy lock sets into $m \times n$ lock fuzzy matrices. Karmakar et al. [23] applied the fuzzy lock set and its corresponding defuzzification method to analyse a pollution sensitive remanufacturing model with waste items. De and Mahata [24] implemented cloudy fuzzy set and new defuzzification approach in developing EOQ model for imperfect-quality items with allowable proportionate discounts.

Moreover, the concept of game theory was introduced by Karlin [25] through various mathematical methods. Preda [26] extensively analysed convex optimization with nested maxima and consider corresponding matrix game problem. In 1994, he applied matrix game theory in nonlinear programming problem also. Some notable research articles over game theory incorporating linear programming problem under fuzzy environment may be discussed over here. Researchers like Chinchuluun et al. [27] applied game theory in supply chain management problem. Nayak and Pal [28] discussed bi-matrix games with intuitionistic fuzzy goals. Seikh et al. [29] studied matrix games with intuitionistic fuzzy pay-offs. Wu [30] and Metzger and Rieger [31] extensively analysed interval valued dominance cores and non-cooperative games with prospect theory players and dominated strategies respectively.

The concept of neutrosophic fuzzy set is coined by Smarandache [32] in his new book Neutrosophy which is the new branch of Philosophy. In 2005, he also able to discovered that the neutrosophic set is nothing but a generalization of intuitionistic fuzzy set. On the basis of the fundamental concept on neutrosophic set De and Beg [33,34] analysed new defuzzification procedure for triangular dense fuzzy sets and triangular dense fuzzy neutrosophic sets. Through its long journey, the neutrosophic set itself has been classified into several sub neutrosophic sets in various truth values generated from the basic philosophy of science. Smarandache [35] invented neutrosophic overset, neutrosophic underset, and neutrosophic offset to characterize the special class of decision making in different production sectors based on behavioural science and ability to each individual associated in a particular production process. The subject neutrosophic set has also been extended to neutrosophic vague sets with the help of Hashim et al. [36] recently.

From the above study, it is seen that not a single article has been developed yet which includes neutrosophic fuzzy set through solving game theory in imperfect and reworkable production inventory system in which the positive and negative membership degree of neutrosophic fuzzy numbers acts simultaneously for describing the learning experiences of the DM. Therefore, in this study we develop a cost minimization problem of an imperfect production process through the extension of Tai [3]'s model by incorporating all cost components as neutrosophic fuzzy set and we solve the problem utilizing fuzzy game theory in which the concept of lock fuzzy set and a solution algorithm have been employed. We organize the article as follows: Section 1 includes a brief literature review highlighting major research works, Section 2 discusses preliminaries of some basic definitions and theorems which have been used in developing the proposed model, Section 3 includes notations and assumptions of the model, Section 4 defines the formulation of EPQ model followed by four subsections; Section 4.1 gives model formulation over game theory, 4.2 gives solution procedure of the game problem, 4.3 gives neutrosophic fuzzy model and 4.4 gives methodology to solve the neutrosophic model. Section 5 includes numerical illustrations, Section 6 includes sensitivity analysis, Section 7 expresses graphical illustrations and finally Section 8 gives the conclusion.

Indeed, we include a chronological literature review on some major articles for imperfect production process with game theory, crisp and fuzzy environment are included in Table 1 to show the novelty of this article also.

2. Preliminaries

2.1. Single valued Neutrosophic Offset (Smarandache [35])

Definition 1. Let U be a universe of discourse and the neutrosophic set $A \subset U$. Let $T(x), I(x), F(x)$ be the functions that describe the degrees of membership, indeterminate-membership and non-membership respectively, of a generic element $x \in U$, with respect to the set A: $T(x), I(x), F(x): U \rightarrow [\Psi, \Omega]$ where $\Psi < 0 < 1 < \Omega$, and Ψ is called underlimit, while Ω is called overlimit, $T(x), I(x), F(x) \in [\Psi, \Omega]$.

A Single-Valued Neutrosophic Offset A is defined as: $A = \{(x, \langle T(x), I(x), F(x)\rangle), x \in U\}$, such that there exist some elements in A that have at least one neutrosophic component that is > 1, and at least another neutrosophic component that is < 0.

2.2. Fuzzy subset of sub-neutrosophic offset

Definition 2. In Neutrosophic set theory the component triplets $\langle T(x), I(x), F(x)\rangle$ has specific meaning in any kind of decision theory. Now if we wish to draw the subsets of subneutrosophic sets that correspond any one of these components taking one or more at a time then it is called Subsets of Sub-Neutrosophic set. If these subsets satisfy the properties of Neutrosophic offset then we call such subsets as Sub-Neutrosophic offset.

In fuzzy set theory, the membership function and its corresponding α-cuts are always belonging to [0, 1]. But to get a fuzzy sub-Neutrosophic offset we may consider the membership value of the fuzzy set which belongs to $[-1, 1]$. To do this we may take the help of sine-cut of fuzzy membership which itself lies within $[-1, 1]$.

Definition 3. Let $\tilde{A} = \langle x, \mu(x)\rangle$ is a fuzzy sub-Neutrosophic offset defined in the universal set $x \in X \subseteq \mathbb{R}$ then the sine-cuts of $\mu(x)$ is obtained from $\mu(x) \geq sin(\alpha p)$, (p>0 is the shape parameter of control parameter of the decision maker). In other words, $\alpha \leq \frac{1}{p}\sin^{-1}[\mu(x)] \in [-1, 1]$

Definition 4. Let $\tilde{B} = \langle (x, T_B(x), I_B(x), F_B(x)) : x \in X \subseteq \mathbb{R}\rangle$ be a Neutrosophic set where $T_B(x)$, $I_B(x)$ and $F_B(x)$ refer the true membership function, Indeterminacy membership and falsity membership function respectively. Then the sine-cuts of the corresponding membership functions are given by $T_B(x) \geq sin(\alpha p)$,

Table 1
Literature review on recent major imperfect production process.

Authors	Model	Assumption	Demand	Cost component	Fuzzy/Crisp	Solution procedure
Chiu et al. [11]	EPQ	Rework, random breakdown (Poisson)	Constant	Finite	Crisp	Cost minimization, Convex optimization
Chan and Prakash [37]	EPQ	Reliability, maintenance policy, capital cost	Linguistic fuzzy	Linguistic fuzzy	Triangular Fuzzy Number, Trapezoidal Fuzzy Number	Profit maximization, MCDM, Proximity ratio
Manna et al. [38]	Three-layer supply chain	Two storage facility, Rework, Transportation	Stock dependent fuzzy rough	Nonrandom uncertain	Fuzzy rough set	Profit maximization, Convex optimization
Li et al. [10]	EPQ	Deterioration, system rework, backlog, maintenance, no lost sale	Constant	Finite	Crisp	Cost minimization, Convex optimization
Jauhari et al. [39]	EPQ	Inspection error (Type I, II), warranty, transportation, partial backorder, vender–buyer	Fuzzy stochastic demand	Finite	Fuzzy	Cost minimization, Convex optimization
Khanna et al. [40]	EPQ	Inspection error (Type I, II), sales return, rework, random imperfect item	Constant	Finite	Crisp	Profit maximization, Convex optimization
Nobil et al. [41]	EPQ	Rework (delayed & immediate), Without shortage, Inspection	Constant	Finite	Crisp	Cost minimization, Convex optimization
Taleizadeh et al. [42]	EPQ	Two warranty policy, returns, shortages, maximum budget	Random (Normal distribution)	Random	Crisp	Cost minimization, Metaheuristic, Non-Dominated Sorting Genetic Algorithm
Tayyab et al. [43]	EPQ	Rework, Inspection, n- stage production system	Triangular fuzzy demand	Finite	Fuzzy	Cost minimization, Centre of Gravity, Convex optimization
Kuzyutin et al. [13]	EPQ	Backorder, rework, inspection, Multi criteria multi stage game	Constant	Finite	Crisp	Profit maximization, Unique Pareto Efficient solution
Khedlekar and Tiwari [44]	EPQ	Discount rate, random imperfect quality item, customers' impatient function, partial backorder	Demand is a function of selling price and discount rate	Finite	Crisp	Profit maximization, Convex optimization
This Paper	EPQ	Rework, inspection, learning experience, deterioration, partial backorder	Constant	Neutrosophic fuzzy, lock fuzzy	Neutrosophic fuzzy, lock fuzzy	Primal–dual problem, Lagrangian, Matrix game, Algorithm

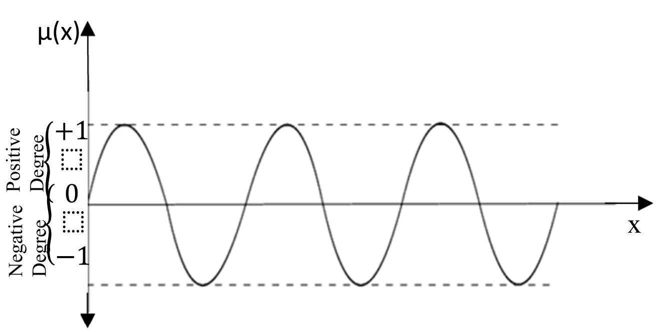

Fig. 1. Neutrosophic fuzzy membership function with sine-cuts.

$I_B(x) \geq sin(\beta q)$ and $F_B(x) \geq sin(\gamma r)$ where $\alpha \in [-1,1]$, $\beta \in [-1,1]$ and $\gamma \in [-1,1]$, $p,q,r > 0$ such that $-3 \leq \alpha+\beta+\gamma \leq +3 \Rightarrow -3 \leq \frac{1}{p}\sin^{-1} T_B(x) + \frac{1}{q}\sin^{-1} I_B(x) + \frac{1}{r}\sin^{-1} F_B(x) \leq +3$. The following diagram shows Fig. 1 the basic nature of sine-cut (degree of subset of sub-neutrosophic offset).

2.3. Concept of game theory

The subject of game theory is strongly associated with two or more than competitors (players) who are competing to gain more profit in one side and to achieve minimum loss from another side. In the literature, several definitions have been found but we put two formal definitions stated below which have been used to develop our proposed model.

Definition 5. A game is described by a set of players and their possibilities to play the game according to some rules, that is, their set of strategies. It is situational (time dependent) where the result for a particular player does not depend only on his own decisions, but also on the behaviour of the other players.

Definition 6. A game G consists of a set of players (leaders/agents) $M = \{1, 2, \ldots, m\}$, an action set denoted by Ω_i (also referred to as a set of strategies S_i) available for each player i and an individual payoff (utility) U_i or cost function F_i for each player $i \in M$. Here, each player individually takes an optimal action which optimizes its own objective function and each player's success in making decisions depends on the decisions of the others. We define a non-co-operative game G as an object specified by (M, S, Ω, F), where $S = S_1 \times S_2 \times \cdots \times S_m$ is known as the strategy space, $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_m$ is the action space, and $F: \Omega \to \mathbb{R}^m$, defined as $F(u) = [F_1(u), F_2(u), \ldots, F_m(u)]^T$, $u \in \Omega$ is the vector of objective functions associated to each of the m players, or agents participating in the game. In some cases, a graph notation might be more appropriate than the set M notation. Conventionally F represents a vector of cost functions to be minimized by the agents.

2.4. Mixed Strategy Game with Linear Constraints (Preda [45])

Let us consider the linearly constrained non-linear programming problem *(P)* together with its Mond–Weir dual problem *(D)*, as follows:

$$(P)\left|\begin{array}{l} \min f(x) \\ \text{subject to}: A(x) \geq b, x \geq 0; \end{array}\right.$$

$$(D)\left|\begin{array}{l} \max g(x,u) = f(x) - u^T(Ax-b) \\ \quad \text{subject to}: \nabla f(x) - A^T u \geq 0 \\ \quad x^T\left[\nabla f(x) - A^T u\right] \leq 0, u \geq 0, \end{array}\right. \tag{1}$$

where $x \in \mathbb{R}^n, b \in \mathbb{R}^m, u \in \mathbb{R}^m, A = (a_{ij})$ is an $m \times n$ real matrix, the symbol ^{T} denotes the transpose, $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable and $\nabla f(x)$ denotes the gradient (column) vector of f at x.

Now we consider the matrix game associated with the following $(n+1) \times (m+1)$ matrix $M_1(x)$ (depending on x), given by

$$M_1(x) = \begin{pmatrix} A^T & \nabla f(x) \\ -b^T & x^T \nabla f(x) \end{pmatrix} \tag{2}$$

Theorem 1. *Let* $P^0 = \begin{pmatrix} x^0 \\ z_1^0 \end{pmatrix}$, $Q^0 = \begin{pmatrix} u^0 \\ z_2^0 \end{pmatrix}$, $\bar{x} = x^0/z_1^0$, $\bar{u} = u^0/z_2^0$, *with* $z_1^0, z_2^0 > 0$. *Let* (P^0, Q^0) *solve the matrix game* $M_1(\bar{x})$ *and* $P^{0T} M_1(\bar{x}) Q^0 = 0$. *Then* $\bar{x}$ *and* $(\bar{x}, \bar{u})$ *are feasible solution to* (P) *and* (D) *respectively with* $f(\bar{x}) = g(\bar{x}, \bar{u})$. *In addition, if there exists weak duality between* (P) *and* (D) *then* $\bar{x}$ *and* $(\bar{x}, \bar{u})$ *are optimal to respective problems.*

Proof. We know that if the value of the game (in random extension) is zero then (P^0, Q^0) is the equilibrium point of the given problem. Then we write: $M_1(\bar{x})Q^0 \leq 0$ and $M_1(\bar{x})^T P^0 \geq 0$; which gives

$$\begin{cases} A^T u^0 - z_1^0 \nabla f(\bar{x}) \leq 0 \\ -b^T u^0 + z_1^0 \bar{x}^T \nabla f(\bar{x}) \leq 0 \\ A x^0 \geq z_2^0 b \\ -x^{0T} \nabla f(\bar{x}) + z_2^0 \bar{x}^T \nabla f(\bar{x}) \geq 0 \end{cases} \tag{3}$$

But we are given $x^0 \geq 0, u^0 \geq 0, z_1^0 > 0, z_2^0 > 0$ and therefore from above we get

$$\begin{cases} A^T \bar{u} - \nabla f(\bar{x}) \leq 0 \\ -b^T \bar{u} + \bar{x}^T \nabla f(\bar{x}) \leq 0 \\ A\bar{x} \geq b \\ -\bar{x}^T \nabla f(\bar{x}) + \bar{x}^T \nabla f(\bar{x}) \geq 0 \end{cases} \tag{4}$$

The above relations reduce to $\bar{x}^T \nabla f(\bar{x}) \leq b^T \bar{u} \leq \bar{x}^T A^T \bar{u} \leq \bar{x}^T \nabla f(\bar{x})$

$$\Rightarrow b^T \bar{u} = \bar{x}^T A^T \bar{u} = \bar{x}^T \nabla f(\bar{x}) \tag{5}$$

Now, $g(\bar{x}, \bar{u}) = f(\bar{x}) - \bar{u}^T(A\bar{x} - b) = f(\bar{x})$. Thus using (3)–(5) we have: $\bar{x}^T\left[\nabla f(\bar{x}) - A^T \bar{u}\right] = 0$. Hence $\bar{x}$, $(\bar{x}, \bar{u})$ are feasible solution for (P) and (D) respectively. When a weak duality exists between (P) and (D) then $\bar{x}$ is optimal for (P) and $(\bar{x}, \bar{u})$ is optimal for (D).

Theorem 2. *Let* $\bar{x}$ *and* $(\bar{x}, \bar{u})$ *be the feasible solutions to* (P) *and* (D) *respectively, such that* $\bar{u}^T(A\bar{x} - b) = 0$. *We define* $z_1^0 = 1/(1+\sum_{i=1}^n \bar{x}_i)$, $z_2^0 = 1/\left(1+\sum_{j=1}^m \bar{u}_j\right)$, $P^0 = \begin{pmatrix} \bar{x} z_1^0 \\ z_1^0 \end{pmatrix}$ *and* $Q^0 = \begin{pmatrix} \bar{u} z_2^0 \\ z_2^0 \end{pmatrix}$. *Then* (P^0, Q^0) *solves the matrix game* $M_1(\bar{x})$ *and the value of this game is zero.*

Proof. Taking the equilibrium point (P^0, Q^0) over matrix game $M_1(\bar{x})$ we have

$$\begin{aligned} &P^{0T} M_1(\bar{x}) Q^0 \\ &= \left(z_1^0 \bar{x}^T A^T - z_1^0 b^T, -z_1^0 \bar{x}^T \nabla f(\bar{x}) + z_1^0 \bar{x}^T \nabla f(\bar{x})\right) \begin{pmatrix} \bar{u} z_2^0 \\ z_2^0 \end{pmatrix} \\ &= z_1^0 z_2^0 \left[\bar{x}^T A^T \bar{u} - b^T \bar{u} - \bar{x}^T \nabla f(\bar{x}) + \bar{x}^T \nabla f(\bar{x})\right] \\ &= z_1^0 z_2^0 \bar{u}^T (A\bar{x} - b) = 0. \end{aligned}$$

Since, $\bar{x}$ and $(\bar{x}, \bar{u})$ are feasible solutions to (P) and (D) respectively, and $\bar{u}^T(A\bar{x} - b) = 0$, we obtain

$$\bar{x}^T A^T \bar{u} = \bar{x}^T \nabla f(\bar{x}) = b^T \bar{u} \tag{6}$$

Now utilizing weak duality theorem and the condition (6) we have

$M_1(\bar{x}) Q^0 = z_2^0 \begin{pmatrix} A^T \bar{u} - \nabla f(\bar{x}) \\ -b^T \bar{u} + \bar{x}^T \nabla f(\bar{x}) \end{pmatrix} \leq 0$ and $P^{0T} M_1(\bar{x}) = z_1^0 \left(\bar{x}^T A^T - b, -\bar{x}^T \nabla f(\bar{x}) + \bar{x}^T \nabla f(\bar{x})\right) \geq 0.$

Thus (P^0, Q^0) solves the matrix game $M_1(\bar{x})$ and the value of the game is zero.

3. Assumption and notations

The following notations and assumptions are used to develop the model.

Notations

p Production rate per unit time
p_r Rate of rework process per unit time
α' Percentage of good quality items produced
α_r Percentage of imperfect quality items recovered
λ Demand rate per unit time
β Percentage of customers who accept backlogging
θ Percentage of items deteriorated per unit time
γ Percentage of deteriorated items screened out from the inventory
I_b Unfilled order backlogged
I_s Inventory level of serviceable items
I_m Maximum inventory level of serviceable items
I_c Maximum inventory level of imperfect quality items
K Setup cost per cycle ($)
C Deterioration cost per unit time per unit item ($)
C_d Penalty cost of selling deteriorated items to customers per unit item ($)
C_p Cost of unrecoverable perfect quality items per unit time ($)
C_s Shortage cost per unit item per unit time ($)
C_u Unsatisfied demands penalty cost per unit time ($)
h_s Holding cost of serviceable items per unit item per unit time ($)
h_r Holding cost of imperfect quality items per unit item per unit time ($)
T_1 Recover time of backlogged items (year)
T_2 Screening of serviceable item (year)
T_3 Duration of recovering serviceable items (year)
T_4 Normal inventory time after the production stops (year)
T_5 Duration of backlogging time (year)
T Inventory cycle time (year)

Assumption

i. The imperfect production system involves single period and single item.
ii. Rework is processed instantly and all defective items are recovered to good quality items.
iii. Only good and serviceable items are deteriorating with constant rate θ.
iv. Shortages are partially backlogged and the rests are treated as unsatisfied demand.
v. Backlogged demands are meet up at the beginning of each cycle.
vi. Deteriorated items and unrecoverable imperfect quality items are disposed of.
vii. For fuzzy model, all cost parameters are assumed to be neutrosophic fuzzy number.

4. Formulation of EPQ model

We consider the above assumptions and notations for developing an imperfect production process studied by Tai [3], the schematic diagram of the production flow in given in Fig. 2 and subsequently the average inventory cost function of the proposed model is discussed as follows

$$\begin{aligned} TC = &[Holding\ cost\ for\ (serviceable\ items + imperfect\ items)\\ &+ Deterioration\ cost + Shortage\ cost + Penalty\ cost\\ &+ Unsatisfied\ cost + Unrecoverable\ cost + Setup\ cost]/\\ &Total\ cycle\ time \end{aligned}$$

$$\begin{aligned} TC = &\frac{1}{T}\left[\eta_1 T_4^2 + \eta_2 T_3^2 + K + \eta_3 T_2^2 + 2\eta_3 T_2 T_3 + \eta_4 T_3^2 + \eta_5 T_4^2\right.\\ &\left. + \eta_6 T_3 + \eta_7 (T - T_2 - T_3 - T_4)^2 + \eta_8 (T - T_2 - T_3 - T_4)\right] \end{aligned}$$

where

$$\begin{cases} \eta_1 = \frac{\lambda\theta[\gamma C + (1-\gamma)C_d]}{2}, \quad \eta_2 = \frac{h_r\left[p_r^2 + (1-\alpha')p.p_r\right]}{2(1-\alpha')p}, \quad \eta_3 = \frac{h_s(\alpha' p - \lambda)}{2}\\ \eta_4 = \frac{h_s(\alpha_r p_r - \lambda)}{2}, \quad \eta_5 = \frac{h_s\lambda}{2}, \quad \eta_6 = C_p(1-\alpha_r)p_r\\ \eta_7 = \frac{C_s\beta\lambda(\alpha' p - \lambda)}{2(\alpha' p - \beta'\lambda)}, \quad \eta_8 = \frac{C_u\beta'\lambda(\alpha' p - \lambda)}{2(\alpha' p - \beta'\lambda)} \end{cases} \tag{7}$$

Therefore, our given problem can be developed as follows:

$$\begin{cases} \min TC(T, T_4) = f(T, T_4)\\ = \psi_1 T - \psi_2 T_4 + \psi_3 \frac{T_4^2}{T} - \psi_5 \frac{T_4}{T} + \frac{K}{T} + \psi_4\\ \text{Subject to, } \begin{bmatrix} a_{11} & -a_{12}\\ -a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} T\\ T_4 \end{bmatrix} = \begin{bmatrix} T_1\\ T_2 \end{bmatrix}\\ q = (a_{11} - a_{21}) pT - (a_{12} - a_{22}) pT_4 \end{cases} \tag{8}$$

where

$$\begin{cases} \psi_1 = \eta_2 a_{31}^2 + \eta_3 a_{21}^2 - 2\eta_3 a_{21} a_{31} + \eta_4 a_{31}^2 + \eta_7 (1 + a_{21} - a_{31})^2\\ \psi_2 = -2(\eta_2 a_{31} a_{32} + \eta_3 a_{21} a_{22} + \eta_3 a_{21} a_{32} + \eta_3 a_{22} a_{31}\\ \quad + \eta_4 a_{31} a_{32} - \eta_7 (1 + a_{21} - a_{31})(1 + a_{22} + a_{32}))\\ \psi_3 = \eta_1 + \eta_2 a_{32}^2 + \eta_3 a_{22}^2 + 2\eta_3 a_{22} a_{32}\\ \quad + \eta_4 a_{32}^2 + \eta_5 + \eta_7 (1 + a_{22} + a_{32})^2\\ \psi_4 = \eta_6 a_{31} + \eta_8 (1 + a_{21} - a_{31})\\ \psi_5 = -\eta_6 a_{32} + \eta_8 (1 + a_{22} + a_{32}) \end{cases} \tag{9}$$

and the other relations are given below:

$$\begin{cases} T_3 = a_{31} T_4 + a_{32} T\\ T_5 = \frac{(\alpha' p - \lambda)}{\beta\lambda} T_1\\ \omega = \beta\lambda + \frac{(\alpha' p - \beta'\lambda) p_r}{(1-\alpha')p}\\ \begin{bmatrix} a_{11} & -a_{12}\\ -a_{21} & a_{22}\\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} \frac{\beta\lambda(1 - a_{21} - a_{31})}{(\alpha' p - \beta'\lambda)} & \frac{-\beta\lambda(1 + a_{22} + a_{32})}{(\alpha' p - \beta'\lambda)}\\ -\frac{\beta\lambda - a_{31}\omega}{(\alpha' p - \lambda)} & \frac{a_{32}\omega + \beta\lambda}{(\alpha' p - \lambda)}\\ \frac{\beta\lambda}{\omega + \alpha_r p_r - \lambda} & \frac{\beta'\lambda}{\omega + \alpha_r p_r - \lambda} \end{bmatrix} \end{cases} \tag{10}$$

4.1. Formulation of EPQ model under game theory

Let us consider the objective function, to be minimized as

$$f(T, T_4) = \psi_1 T - \psi_2 T_4 + \psi_3 \frac{T_4^2}{T} - \psi_5 \frac{T_4}{T} + \frac{K}{T} + \psi_4$$

Subject to the constraint $\begin{bmatrix} a_{11} & -a_{12}\\ -a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} T\\ T_4 \end{bmatrix} = \begin{bmatrix} T_1\\ T_2 \end{bmatrix}$

Rewriting the fundamental decision variables (T, T_4) in terms of (x, y) and replacing the coefficient matrix by A, the decision variable by X and the requirement time vector by B; then the given problem reduces to

$$\begin{cases} \min f(x, y) = \psi_1 x - \psi_2 y + \psi_3 \frac{y^2}{x} - \psi_5 \frac{y}{x} + \frac{K}{x} + \psi_4\\ \text{Subject to, } AX = B \end{cases} \tag{11}$$

where

$$A = \begin{bmatrix} a_{11} & -a_{12}\\ -a_{21} & a_{22} \end{bmatrix}, X = \begin{bmatrix} x\\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} T_1\\ T_2 \end{bmatrix} \tag{12}$$

Utilizing the Section 2.4 the equivalent game problem can be defined as follows

$$(P)\left| \begin{array}{l} \min f(X)\\ \text{subject to: } AX \geq B, X \geq 0 \end{array}\right. \tag{13}$$

$$(D)\left| \begin{array}{l} \max g(X, U) = f(X) - U^T(AX - B)\\ \quad \text{subject to: } \nabla f(X) - A^T U \geq 0\\ \quad X^T\left[\nabla f(X) - A^T U\right] \leq 0, U \geq 0 \end{array}\right. \tag{14}$$

where $X, B, U \in \mathbb{R}^2, A = (a_{ij})_{2\times 2}$ real matrix, U denotes the Lagrangian multiplier vector defined by $U = [\zeta_1, \zeta_2]^T$, the symbol ^{T} denotes the usual transpose operator.

4.2. Solution procedure of the game problem

To solve (13) and (14) we shall proceed as follows:

Step 1: Take the gradient vector of f at X , defined by $\nabla f = \begin{bmatrix} \psi_1 - \psi_3 \frac{y^2}{x^2} + \psi_5 \frac{y}{x^2} - \frac{K}{x^2}\\ -\psi_2 + 2\psi_3 \frac{y}{x} - \frac{\psi_5}{x} \end{bmatrix}$

Step 2: Construct the matrix game utilizing the relations (11)–(12) as

$$M(X) = \begin{bmatrix} a_{11} & -a_{21} & \psi_1 - \psi_3 \frac{y^2}{x^2} + \psi_5 \frac{y}{x^2} - \frac{K}{x^2}\\ -a_{12} & a_{22} & -\psi_2 + 2\psi_3 \frac{y}{x} - \frac{\psi_5}{x}\\ -T_1 & -T_2 & \psi_1 x - \psi_2 y + \psi_3 \frac{y^2}{x} - \frac{K}{x} \end{bmatrix}$$

Step 3 Let (P^*, Q^*) solve the matrix game $M(\overline{X})$, such that $P^{*T} M(\overline{X}) Q^* = 0$ then calculate $\overline{X}$ and $(\overline{X}, \overline{U})$ such that they are the feasible solutions of (P) and (D) respectively with $f(\overline{X}) = g(\overline{X}, \overline{U})$ where $P^* = (X^*, z_1^*)^T = (x^*, y^*, z_1^*)^T, Q^* = (U^*, z_2^*)^T = (\zeta_1^*, \zeta_2^*, z_2^*)^T, \overline{X} = \left(\frac{x^*}{z_1^*}, \frac{y^*}{z_1^*}\right)^T$ and $\overline{U} = \left(\frac{\zeta_1^*}{z_2^*}, \frac{\zeta_2^*}{z_2^*}\right)^T$. Utilizing the

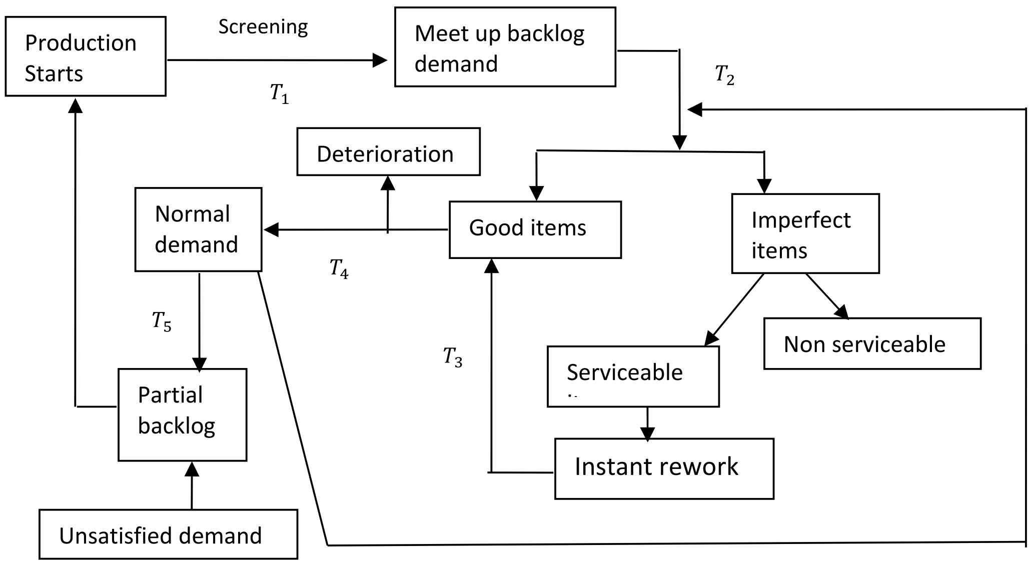

Fig. 2. Imperfect production flow over time.

condition $P^{*T}M(\overline{X})Q^* = 0$ the nonlinear functional satisfies the equation

$$\begin{aligned}&\zeta_1\left\{a_{11}x + a_{12}y - z_1\left(a_{11}x - a_{12}y\right)\right\}\\ &+ \zeta_2\left\{a_{21}x + a_{22}y - z_1\left(-a_{21}x + a_{22}y\right)\right\}\\ &+ 2z_2\left\{\psi_1 x - \psi_2 y + \psi_3\frac{y^2}{x} - \frac{Kz_1^2}{x}\right\} = 0\end{aligned}$$

Step 4: Find X^* and $f(X*)$ satisfying the matrix inequality $M(\overline{X})Q^* \leq 0$ that is

$$\begin{bmatrix} a_{11}\zeta_1 - a_{21}\zeta_2 + \psi_1 z_2 - \psi_3\frac{y^2z_2}{x^2} + \psi_5\frac{yz_1z_2}{x^2} - \frac{Kz_1^2z_2}{x} \\ -a_{12}\zeta_1 + a_{22}\zeta_2 - \psi_2 z_2 + 2\psi_3\frac{yz_2}{x} - \psi_5\frac{z_1z_2}{x} \\ -a_{11}\zeta_1 x + a_{12}\zeta_1 y + a_{21}\zeta_2 x - a_{22}\zeta_2 y \\ +\psi_1\frac{xz_2}{z_1} - \psi_2\frac{yz_2}{z_1} + \psi_3\frac{y^2z_2}{xz_1} - \frac{Kz_1z_2}{x} \end{bmatrix} \leq 0$$

and $M(\overline{X})^T P^* \geq 0$ that is $\begin{bmatrix} a_{11}x - a_{12}y - z_1a_{11}x + z_1a_{12}y \\ -a_{21}x + a_{22}y + z_1a_{21}x - z_1a_{22}y \\ 2\psi_1 x - 2\psi_2 y + 2\psi_3\frac{y^2}{x} - \frac{Kz_1^2}{x} \end{bmatrix} \geq 0.$

4.3. Formulation of fuzzy neutrosophic EPQ model

Let all the cost components associated with the imperfect production process behave as fuzzy sub-Neutrosophic offset by means of interval valued lock fuzzy number. The basic characteristic of the lock fuzzy number is, it refers the special class of α-cuts namely sine-cuts as developed in Section 2.2. If κ be the learning parameter over the cycle time T, then we may assume the degree of learning achieved by $\alpha = \sin(\kappa T)$. Let the fuzzy intervals are of the form $[x_{i1}, x_{i1} + \delta_i]$ if $\delta_i > 0$ and it is $[x_{i1} + \delta_i, x_{i1}]$ if $\delta_i < 0$ for $i = 0, 1, 2, \ldots.$, with interval length (tolerance parameter) δ_i. Now we may define the membership function of the cost parameter given in (15)

$$\mu_{\tilde{x}_i}(x) = \begin{cases} 1 & for \quad x \leq x_0 \\ \frac{x - x_{i1}}{\delta_i} & for \quad x_{i1} \leq x \leq x_{i1} + \delta_i \\ 0 & for \quad x \geq x_{i1} + \delta_i \end{cases} \tag{15}$$

along with its graphical representation shown in Fig. 3.

Let the cost components may vary according to the learning experiences (gain or loss) designed by the decision maker. Thus, we may assign the cost vector as fuzzy sub-Neutrosophic offset and the given problem can be stated as

$$\begin{cases} \widetilde{min}\tilde{f}(T, T_4) \tilde{=} \widetilde{\psi_1}T - \widetilde{\psi_2}T_4 + \widetilde{\psi_3}\frac{T_4^2}{T} - \widetilde{\psi_5}\frac{T_4}{T} + \frac{\tilde{K}}{T} + \widetilde{\psi_4} \\ subject\ to{:} \begin{bmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} T \\ T_4 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \\ q = \left(a_{11} - a_{21}\right)pT - \left(a_{12} - a_{22}\right)pT_4 \end{cases} \tag{16}$$

where

$$\begin{cases} \widetilde{\psi_1} \tilde{=} \widetilde{\eta_2}a_{31}^2 + \widetilde{\eta_3}a_{21}^2 - 2\widetilde{\eta_3}a_{21}a_{31} + \widetilde{\eta_4}a_{31}^2 + \widetilde{\eta_7}\left(1 + a_{21} - a_{31}\right)^2 \\ \widetilde{\psi_2} \tilde{=} -2\left(\widetilde{\eta_2}a_{31}a_{32} + \widetilde{\eta_3}a_{21}a_{22} + \widetilde{\eta_3}a_{21}a_{32} + \widetilde{\eta_3}a_{22}a_{31}\right. \\ \left. + \widetilde{\eta_4}a_{31}a_{32} - \widetilde{\eta_7}\left(1 + a_{21} - a_{31}\right)\left(1 + a_{22} + a_{32}\right)\right) \\ \widetilde{\psi_3} \tilde{=} \widetilde{\eta_1} + \widetilde{\eta_2}a_{32}^2 + \widetilde{\eta_3}a_{22}^2 + 2\widetilde{\eta_3}a_{22}a_{32} + \widetilde{\eta_4}a_{32}^2 \\ + \widetilde{\eta_5} + \widetilde{\eta_7}(1 + a_{22} + a_{32})^2 \\ \widetilde{\psi_4} \tilde{=} \widetilde{\eta_6}a_{31} + \widetilde{\eta_8}\left(1 + a_{21} - a_{31}\right) \\ \widetilde{\psi_5} \tilde{=} -\widetilde{\eta_6}a_{32} + \widetilde{\eta_8}\left(1 + a_{22} + a_{32}\right) \\ \widetilde{\eta_1} \tilde{=} \frac{\lambda\theta\left[\gamma\tilde{C} + (1-\gamma)\widetilde{C_d}\right]}{2}, \quad \widetilde{\eta_2} \tilde{=} \frac{\widetilde{h_r}\left[p_r^2 + (1-\alpha')p.p_r\right]}{2(1-\alpha')p}, \quad \widetilde{\eta_3} \tilde{=} \frac{\widetilde{h_s}(\alpha' p - \lambda)}{2} \\ \widetilde{\eta_4} \tilde{=} \frac{\widetilde{h_s}(\alpha_r p_r - \lambda)}{2}, \quad \widetilde{\eta_5} \tilde{=} \frac{\widetilde{h_s}\lambda}{2}, \quad \widetilde{\eta_6} \tilde{=} \widetilde{C_p}(1 - \alpha_r)p_r \\ \widetilde{\eta_7} \tilde{=} \frac{\widetilde{C_s}\beta\lambda(\alpha' p - \lambda)}{2(\alpha' p - \beta'\lambda)}, \quad \widetilde{\eta_8} \tilde{=} \frac{\widetilde{C_u}\beta'\lambda(\alpha' p - \lambda)}{2(\alpha' p - \beta'\lambda)} \end{cases} \tag{17}$$

4.4. Methodology to solve the neutrosophic fuzzy problem

To defuzzify the proposed neutrosophic fuzzy problem we shall use sine-cut approach as stated in Section 2.2. Now taking the sine-cut of the objective function (16), the equivalent deterministic problem of fuzzy neutrosophic model can be written as

$$\begin{cases} min(-\alpha) \\ \alpha = \sin\left(\kappa T\right) \\ Subject\ to\ f_\alpha(T, T_4, \kappa) \geq \psi_{1\alpha}T - \psi_{2\alpha}T_4 \\ + \psi_{3\alpha}\frac{T_4^2}{T} - \psi_{5\alpha}\frac{T_4}{T} + \frac{K}{T} + \psi_{4\alpha} \end{cases} \tag{18}$$

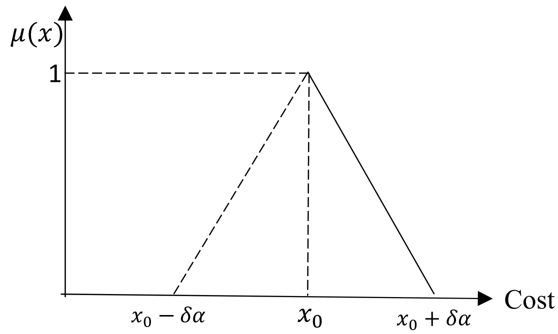

Fig. 3. Nature of membership value of the cost vector.

where

$$\begin{cases} \psi_{1\alpha} = \eta_{2\alpha}a_{31}^2 + \eta_{3\alpha}a_{21}^2 - 2\eta_{3\alpha}a_{21}a_{31} + \eta_{4\alpha}a_{31}^2 \\ \quad +\eta_{7\alpha}\left(1+a_{21}-a_{31}\right)^2 \\ \psi_{2\alpha} = -2\left(\eta_{2\alpha}a_{31}a_{32} + \eta_{3\alpha}a_{21}a_{22} + \eta_{3\alpha}a_{21}a_{32} + \eta_{3\alpha}a_{22}a_{31}\right. \\ \quad \left.+\ \eta_{4\alpha}a_{31}a_{32} - \eta_{7\alpha}\left(1+a_{21}-a_{31}\right)\left(1+a_{22}+a_{32}\right)\right) \\ \psi_{3\alpha} = \eta_{1\alpha} + \eta_{2\alpha}a_{32}^2 + \eta_{3\alpha}a_{22}^2 + 2\eta_{3\alpha}a_{22}a_{32} + \eta_{4\alpha}a_{32}^2 \\ \quad +\eta_{5\alpha} + \eta_{7\alpha}(1+a_{22}+a_{32})^2 \\ \psi_{4\alpha} = \eta_{6\alpha}a_{31} + \eta_{8\alpha}\left(1+a_{21}-a_{31}\right) \\ \psi_{5\alpha} = -\eta_{6\alpha}a_{32} + \eta_{8\alpha}\left(1+a_{22}+a_{32}\right) \\ \eta_{1\alpha} = \frac{\lambda\theta\left[\gamma C_\alpha + (1-\gamma)C_{d_\alpha}\right]}{2}, \quad \eta_{2\alpha} = \frac{h_{r_\alpha}\left[p_r^2 + (1-\alpha')p.p_r\right]}{2(1-\alpha')p}, \\ \eta_{3\alpha} = \frac{h_{s_\alpha}(\alpha' p-\lambda)}{2} \\ \eta_{4\alpha} = \frac{h_{s_\alpha}(\alpha_r p_r-\lambda)}{2}, \quad \eta_{5\alpha} = \frac{h_{s_\alpha}\lambda}{2}, \quad \eta_{6\alpha} = C_{p_\alpha}(1-\alpha_r)p_r \\ \eta_{7\alpha} = \frac{C_{s_\alpha}\beta\lambda(\alpha' p-\lambda)}{2(\alpha' p-\beta'\lambda)}, \quad \eta_{8\alpha} = \frac{C_{u_\alpha}\beta'\lambda(\alpha' p-\lambda)}{2(\alpha' p-\beta'\lambda)} \\ C_\alpha = C_0 + \delta_1\sin(\kappa T), \quad C_{d_\alpha} = C_{d_0} + \delta_2\sin(\kappa T), \\ h_{r_\alpha} = h_{r_0} + \delta_3\sin(\kappa T) \\ h_{s_\alpha} = h_{s_0} + \delta_4\sin(\kappa T), \quad C_{p_\alpha} = C_{p_0} + \delta_5\sin(\kappa T), \\ C_{s_\alpha} = C_{s_0} + \delta_6\sin(\kappa T) \\ C_{u_\alpha} = C_{u_0} + \delta_7\sin(\kappa T), K_\alpha = K_0 + \delta_8\sin(\kappa T) \end{cases} \tag{19}$$

where $\delta_i (i = 1, 2, \ldots, 8)$ represents tolerance level of the cost components of the cost vector $\{C, C_d, h_r, h_s, C_p, C_s, C_u, K\}$ and the elements with 0 suffix indicates their initial values and taking the constraints used in Eqs. (10) and (16).

Now we may construct a schematic diagram of the solution procedure (shown in Fig. 4) and compute the numerical results using the solution algorithm developed in Section 4.2.

5. Numerical illustration

For numerical computation, we assume the imperfect production system parameter values $p = 6000, p_r = 4000, \lambda = 1000, \alpha' = 0.9, \alpha_r = 0.7, \gamma = 0.6, \theta = 0.1, \beta = 0.6$ and the costs parameter values $C = \$0.4, C_d = \$100, C_p = \$30, C_s = \$20, C_u = \$6, h_s = \$2.5, h_r = \$1.5$ and $K = \$300$. For Neutrosophic Game Model we also use the neutrosophic learning parameter $\kappa = 3.5$ and obtain the result which is shown in Table 2.

Table 2 reveals the optimum cost of the proposed imperfect rework model under various approaches like crisp, game and neutrosophic fuzzy respectively. For the crisp model the average inventory cost is \$2505.73 with respect to the cycle time 0.41 year and the order quantity is 314.213 units. Here also we see the backlog recovery time requires 0.01 year while the inventory exhaust time reaches to 0.27 year approximately. The game model gives the average inventory cost value to \$2506.41 which is \$0.68 more ($\sim +0.03\%$) with respect to the crisp value. But with the application of learning theory the neutrosophic model giving the cost value \$2438.25 (which is 2.7% less) with respect to the learning parameter $\kappa = 3.5$ with α cut value -0.946 over the cycle time 1.252 years where the inventory run time getting maximum at 0.72 year.

6. Sensitivity analysis

Table 3 shows the optimum solution of the proposed neutrosophic game model while the changes of the learning parameter κ on and from -20% to +20% are performed. At 5% reduction of κ, the average inventory cost gets negligible change. But at -15% and +10% changes the inventory cost becomes slightly change by $+8\%$ and -8% respectively. For the change of -10%, +5% and +20% the objective function gets more than double values (> 100%) with respect to crisp solution. The other cases correspond moderately sensitive. Throughout the whole study it is also observe that the cycle time gets the range [0.32, 4] years with respect to the order quantity range [308, 4231] units. The optimum backlogging recovery time (T_1^*) gets a range [0.005, 0.166] year, the screening time (T_2^*) has the bound [0.046, 0.539] year, the rework time (T_3^*) gets the range [0.008, 0.107] year, normal inventory run time (T_4^*) after production stops gets the range [0.217, 2.565] year and finally the inventory backlogging time duration (T_5^*) assumes the range [0.041, 1.273] year exclusively.

7. Graphical illustration

Fig. 5 shows the comparative study of the total average inventory cost under different methodologies.

It is clear that the cost value of the problem under neutrosophic fuzzy environment assumes minimum with respect to other two cases namely crisp and game problem. The crisp model as well as game model assume values of the objective function more than \$2500 while the neutrosophic fuzzy model assumes value nearly \$2440 which is approximately reduced by -2.7% with respect to the other models.

Fig. 6 indicates the variation and deviation of several time (10^{-2}) components of the model under crisp, game and neutrosophic fuzzy environment. The times of recovery of backlogging items (T_1) due to crisp and game model is very much closer to that of the neutrosophic fuzzy model. If we consider the screening completion time (T_2), the backlogging duration time (T_5) and the normal inventory run time (T_4) then we see that their deviation of the values obtained from neutrosophic fuzzy game problem with respect to the other two models keep an ascending order. However, it is also be noted that the rework time (T_3) for serviceable items gets almost same values for all the cases (models). Although the total cycle time is very much larger (the time gap approximately 10.15 months) to the neutrosophic fuzzy model with respect to the other models. Finally the time bounds for each task completion has the ranges $T_1[0, 10]$, $T_2[5, 15]$, $T_3[0, 5]$, $T_4[25, 72]$, $T_5[5, 35]$ and $T[40, 125]$ respectively.

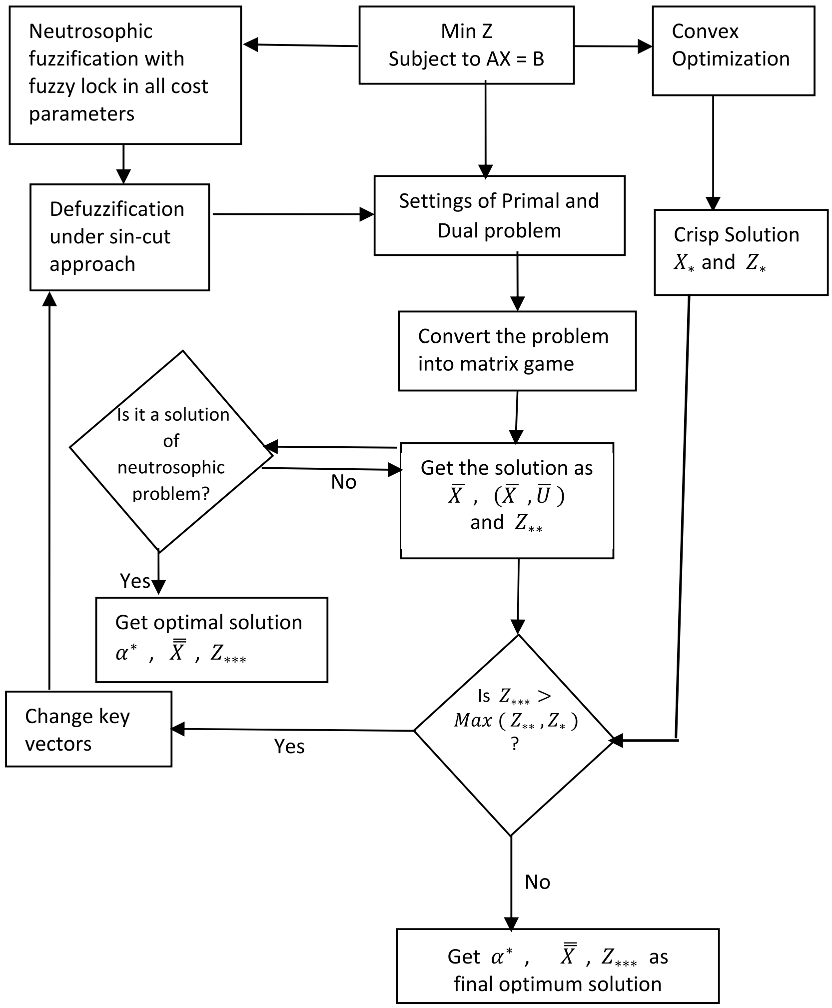

Fig. 4. Schematic overview of solution procedure.

Table 2
Optimal solutions of various models.

Model	T_1^*	T_2^*	T_3^*	T_4^*	T_5^*	T^*	Q^*	α^*	f^*	R_E
Crisp Model	0.0097	0.0576	0.0101	0.2695	0.0708	0.4177	314.213	…	2505.73	…
Game Model	0.0089	0.0572	0.0100	0.2698	0.0705	0.4171	396.811	…	2506.41	+0.03%
Neutrosophic Model	0.0404	0.1512	0.0290	0.7177	0.3115	1.2519	1149.801	−0.946	2438.25	−2.69%

Note: $R_E = \frac{f^*-f_*}{f_*} \times 100\%$ is the relative percentage error where f_* indicates the crisp value.

Table 3
Optimal solution for the change of the learning parameter κ from −20% to +20%.

Change in κ	T_1^*	T_2^*	T_3^*	T_4^*	T_5^*	T^*	Q^*	α^*	TC^*	% change
−20%	0.005	0.046	0.008	0.217	0.041	0.318	307.999	+0.777	3274.39	30.64
−15%	0.048	0.175	0.034	0.829	0.368	1.456	1334.48	−0.928	2699.67	7.71
−10%	0.12	0.398	0.079	1.892	0.922	3.417	3108.216	−0.973	5093.22	103.21
−5%	0.043	0.158	0.03	0.751	0.329	1.313	1205.176	−0.941	2515.66	0.37
+5%	0.166	0.539	0.107	2.565	1.273	4.658	4231.108	−0.987	6641.26	164.97
+10%	0.037	0.139	0.027	0.66	0.282	1.146	1053.521	−0.955	2306.06	−7.99
+15%	0.035	0.134	0.026	0.634	0.269	1.099	1011.398	−0.959	2249.33	−10.26
+20%	0.145	0.473	0.094	2.25	1.11	4.077	3705.614	−0.988	5902.41	135.49

Fig. 7 expresses the trend studies of the required task completion time duration over the % changes of learning parameter κ. The backlog recovery time (T_1) and rework time (T_3) keep almost same value for any kind of changes of the learning parameter (κ) within ±20%. The rework time (T_3) and the backlog generation time (T_5) have remarkable gap and they have the bounds within $130(10^{-2})$ years throughout. The other time parameters like inventory exhaust time (T_4) and total cycle time (T) are easily separable and interpretable within the real time zone $[20, 460](10^{-2})$ years. Moreover, it is observed that if we wish to change the learning parameter (κ) at −20%, −5%, +10% and +15% then we see that the time curves get 'V' shape for all the task completion independently. But if we wish to change the learning

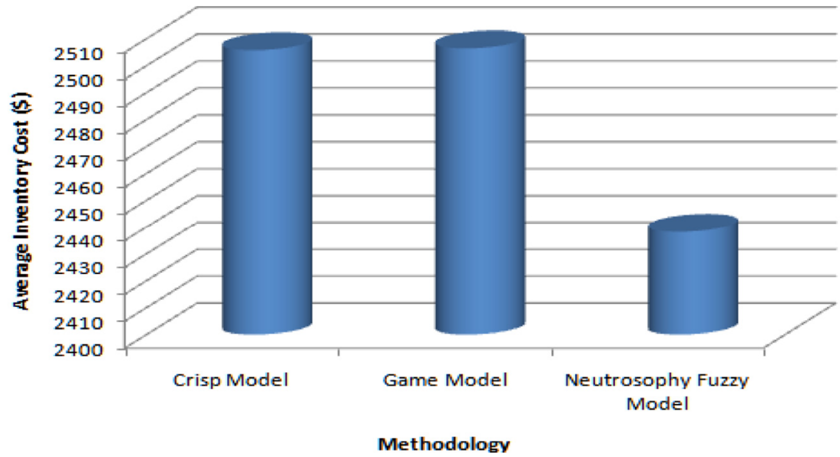

Fig. 5. Inventory cost under several approaches.

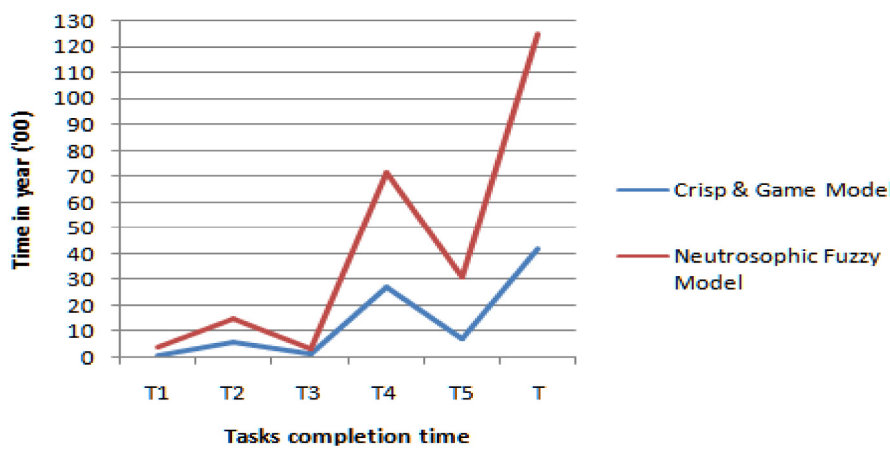

Fig. 6. Task completion time in year under several approaches.

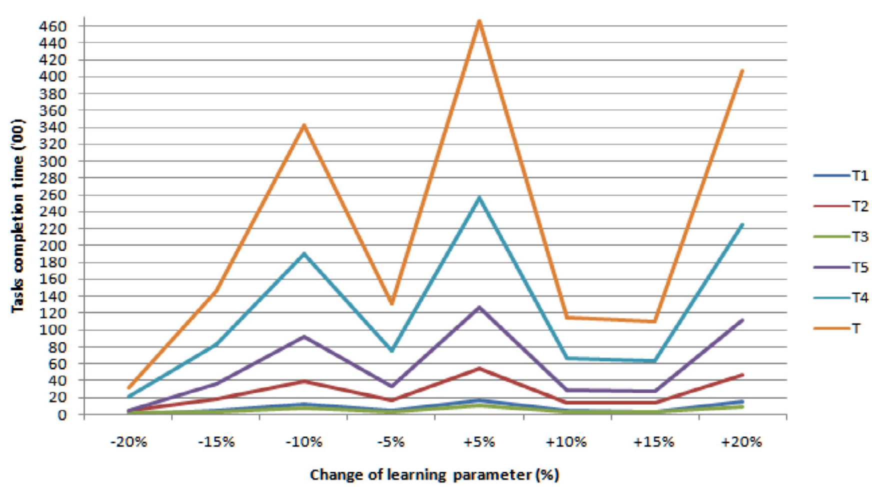

Fig. 7. Tasks completion time under variation of learning parameter (κ)

parameter (κ) at -10%, $+5\%$ and $+20\%$ then the task completion time curves get '$\wedge$' shape independently. Whenever the learning parameter changes from -20% to -15% then the curves generates family of non-intersecting straight lines with different slopes. In addition, when the learning parameter changes from +10% to +15% then the curves becomes almost horizontal non intersecting straight lines having significant distances.

Fig. 8 shows the variation of total average inventory cost with the variation of cycle time. It is observed that the average inventory cost is minimum if it assumes value between [1.099, 1.456] years. Beyond this the total average inventory cost increases. The inventory cost has a paradigm shift up to $6641 approximately within the cycle time range [1.456, 4.7] years alone. Having 'S' shaped cycle time curve, the tail of 'S' indicates the average cost of the inventory $3274 with cycle time 0.318 year that appears

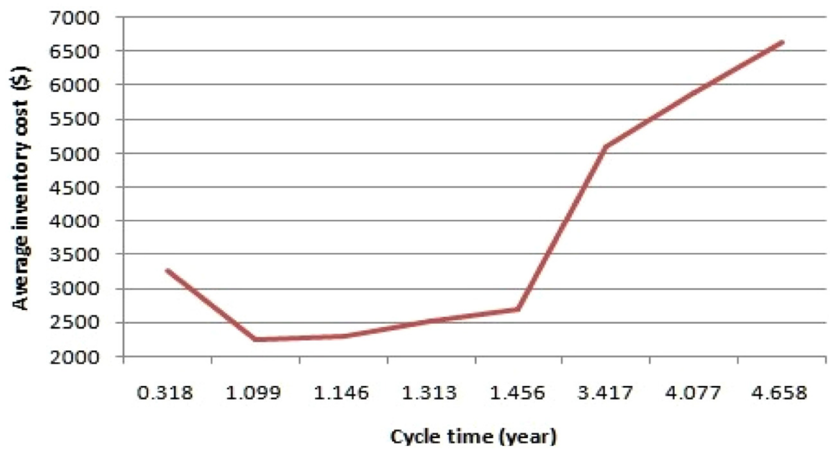

Fig. 8. Variation of average inventory cost with variation of cycle time.

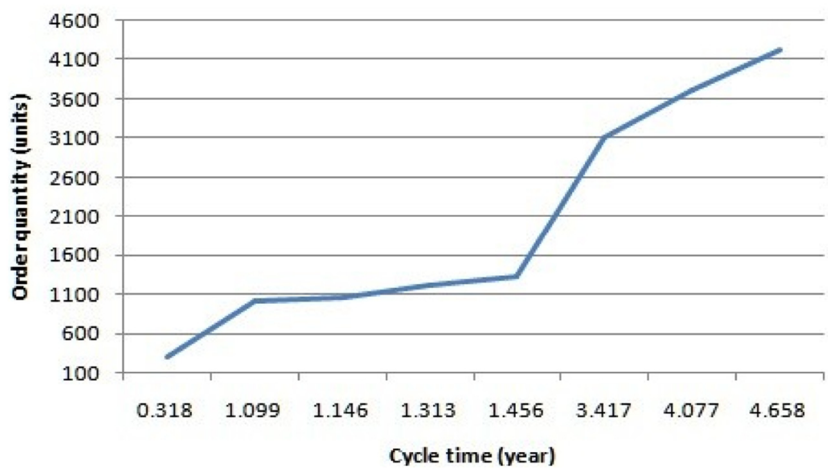

Fig. 9. Variation of order quantity with variation of cycle time.

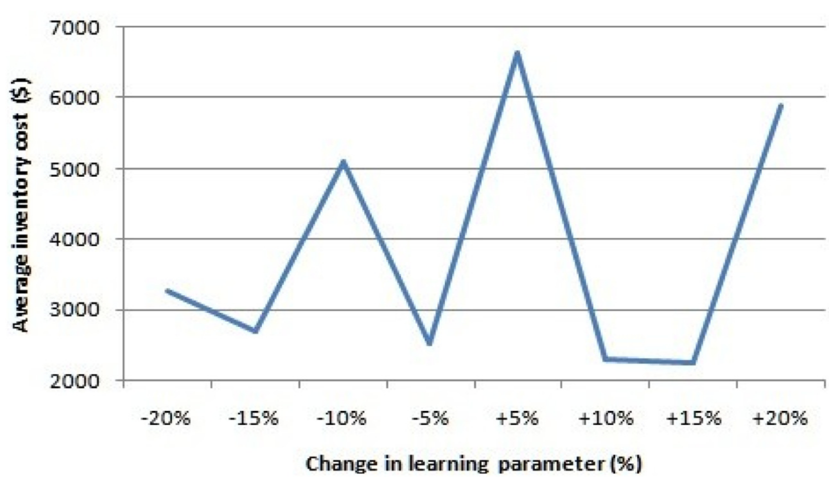

Fig. 10. Variation of average inventory cost with respect to learning parameter *k*.

due to positive α cut ($\alpha = +0.777$) for the neutrosophic fuzzy model.

Fig. 9 shows the variation of order quantity with respect to the variation of cycle time. As the cycle time increases the order quantity is also increases. A sudden jump of order quantity has been viewed at the cycle time duration [1.5, 3.417] years approximately. The curve looks like a continuous monotonic increasing function.

Fig. 10 indicates the variation of average inventory cost with a zigzag path like (saw tooth) over the percentage change of the key vector of the fuzzy lock κ (learning parameter). The average inventory cost assume increasing value for the change

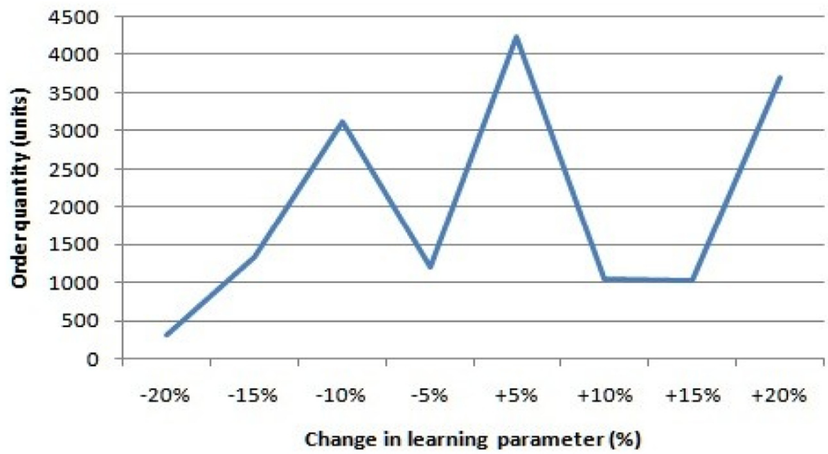

Fig. 11. Variation of order quantity with respect to the learning parameter κ.

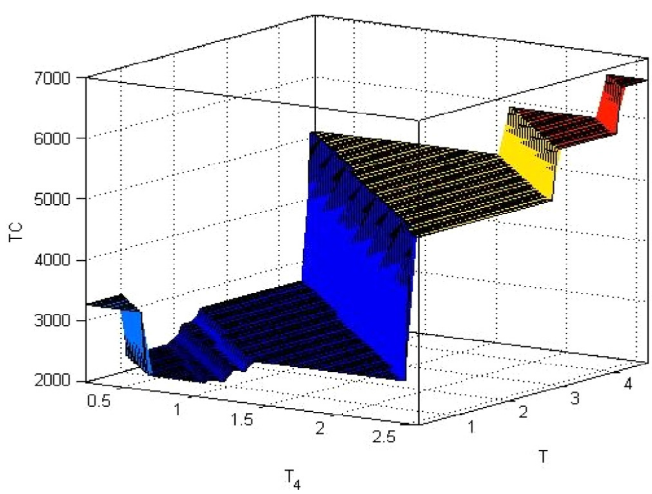

Fig. 12. Variation of average inventory cost with variation of T_4 and T.

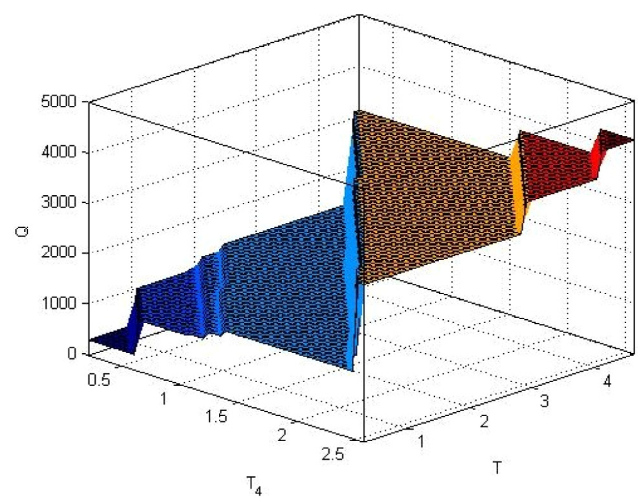

Fig. 13. Variation of order quantity with variation of T_4 and T.

of the lock parameter within the change zone [−15%, −10%], [−5%, +5%], [+15%, +20%] and it is decreasing for the change interval [−20%, −15%], [−10%, −5%], [+5%, +15%] exclusively. Throughout the whole figure the average inventory cost gets a bound $[2100, 6600] approximately which is the maximum range of the objective function of neutrosophic fuzzy model.

Fig. 11 indicates the variation of order quantity with respect to the % change of learning parameter κ. The order quantity has modal(maximum) values at the changes of the learning parameter κ at −10%, 5% and 20% respectively. Also, at the changes in −20%, −5%, 10% and 15% the values of the order quantity get minimum keeping the values within 1205 units. However, the highest peak of the order quantity curve arises at 4231 units whenever the learning parameter κ increases to +5%.

Fig. 12 reveals the variation of total average inventory cost with respect to inventory exhaust time and cycle time simultaneously. As cycle time increase, cost value increases with step size. On the other hand, the cost value decreases with the decrease of inventory exhaust time within time limit (0.5 ∼ 2.5) years but if inventory exhaust time assumes value within 0.5 years the inventory cost refers little more value.

Fig. 13 expresses the variation of order quantity with respect to the variation of inventory exhaust time and variation of cycle time simultaneously. As cycle time increases, order quantity is also increasing with step size. On the other hand, the order quantity decreases with the increase of inventory exhaust time within time limit (0.5 ∼ 2.5) years but if the inventory exhaust time assumes value within 0.5 years the order quantity refers lesser value. The order quantity reaches its maximum value (4231 units) at the cycle time duration 4.7 years approximately.

Fig. 14 expresses the nature of average inventory cost curve under different negative α cuts. We see that at $\alpha = -0.959$ the cost curve reaches minimum point. As α increases the curve slowly goes up following almost a straight line, but the curve gets a sudden jump in the left side of minimum point, reaching a highest peak at $\alpha = -0.987$ and then began to fall down whenever the values of α goes towards −1. Moreover, the curve also indicates the range of average inventory cost assumes values $ [2249, 6641] whenever we are experiencing with negative α cuts throughout.

8. Conclusion

In this study, we have developed an imperfect and reworkable deteriorating production inventory model under fuzzy sub-neutrosophic offset environment. Here, the decision maker has several options to change the real-time managerial strategies so as to reduce (control) several cost components associated with the production inventory problem. Simultaneously, the DM might be able to control the several time parameters as (s)he wishes.

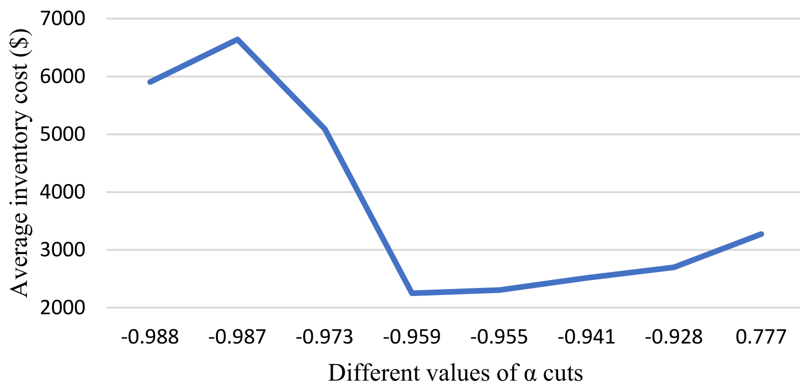

Fig. 14. Variation of inventory cost with the change of α cuts.

The basic merit of this article by using neutrosophic off set is that the DM might be able to get enough time at each and every stages of imperfect production process namely the time duration of backlogs meet up, the screening time, the rework time, the normal inventory exhaust time and the partial backorder time by utilizing lower minimum average inventory cost with respect to other approaches. However, for defuzzification we have utilized α-cuts by means of sine-cuts of neutrosophic fuzzy parameters (several cost components) and then employed game theoretic approach for its solution to primal–dual problem. Our findings reveal that decision making under neutrosophic fuzzy environment is much economical, comfortable and easily applicable even for less qualified decision maker in any kind of management system.

CRediT authorship contribution statement

Sujit Kumar De: Conceptualization, Writing - original draft, Supervision. **Prasun Kumar Nayak:** Methodology, Formal analysis. **Anup Khan:** Validation, Data curation. **Kousik Bhattacharya:** Validation, Data curation. **Florentin Smarandache:** Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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