Advances and Applications in Mathematical Sciences **ISSN 0974-6803**
Volume 22, Issue 6, April 2023, Pages 1045-1057

FUZZY NEUTROSOPHIC BI-IDEALS OF BCK-ALGEBRAS

B. SATYANARAYANA, SHAKE BAJI and U. BINDU MADHAVI

Department of Mathematics
Acharya Nagarjuna University
Nagarjuna Nagar, Guntur-522510
Andhra Pradesh, India
E-mail: drbsn63@yahoo.co.in

Department of Mathematics
Sir C.R. Reddy College of Engineering
Eluru-534007, Andhra Pradesh, India
E-mail: shakebaji6@gmail.com

Department of Applied Mathematics
K. U. Dr. MRAR College of Post-Graduation Studies
Nuzvid, Andhra Pradesh, India
E-mail: bindumadhaviu@gmail.com

Abstract

The aim of this paper is to propose a notion of fuzzy neutrosophic Bi-ideal of BCK-algebras and explore some of their properties.

1. Introduction

L. A. Zadeh [1], a professor of computer science at the University of California, introduced the concept of fuzzy set (FS) in 1965. Fuzzy sets analyzed the degree of membership of members of set and Xi [2] applied this concept to the ideals of BCK/BCI algebras. In 1986 Atanassove [3] generalized a fuzzy set to an intuitionistic fuzzy set (IFS) by including another function called a non-membership function and Jun and Kim [4] introduced intuitionistic fuzzy ideals of BCK-algebras. In 1995 Smarandache

2020 Mathematics Subject Classification: 06F35, 08A72.
Keywords: fuzzy Bi-ideal (FBi-I), Intuitionistic fuzzy Bi-ideal (IFBi-I), fuzzy neutrosophic Bi-ideal (FNBi-I).
Received November 7, 2022; Accepted February 4, 2023

([5], [6]) introduced the neutrosophic set (NS), which discuss the degree of uncertainty. In [7], Arockiarani et al. introduced the concept of fuzzy neutrosophic set (FNS). In [8] E. V. Sindhu add M. H. Jaleela Begum given the idea of intuitionistic fuzzy Bi-ideals of BCK-algebras. This article introduces the fuzzy neutrosophic Bi-ideals of BCK-algebras and its properties.

2. Preliminaries

Definition 2.1[4]**.** Let $\mathcal{K}$ be a non-empty set with a binary operation "$\Diamond$" and a constant "0". Then $(\mathcal{K}, \Diamond, 0)$ is called BCK-algebra if it follows the following axioms for all $p_0, r_0, u_0 \in \mathcal{K}$

(i) $((p_0 \Diamond r_0) \Diamond (p_0 \Diamond u_0)) \Diamond (u_0 \Diamond r_0) = 0$

(ii) $(p_0 \Diamond (p_0 \Diamond r_0)) \Diamond r_0 = 0$

(iii) $p_0 \Diamond p_0 = 0$

(iv) $0 \Diamond p_0 = 0$

(v) $p_0 \Diamond r_0 = 0$ and $r_0 \Diamond p_0 = 0 \Rightarrow p_0 = r_0$.

Definition 2.2[1]**.** We can define a binary operation "$\leq$" on $\mathcal{K}$ by assuming $p_0 \leq r_0$ if and only if $p_0 \Diamond r_0 = 0$.

Definition 2.3[4]**.** A non-empty subset $\mathcal{I}$ of a BCK-algebra $\mathcal{K}$ is called sub algebra of $\mathcal{K}$, if for any $p_0, r_0 \in \mathcal{I} \Rightarrow p_0 \Diamond r_0 \in \mathcal{I}$.

Definition 2.4[4]**.** A non-empty subset $\mathcal{I}$ of a BCK-algebra $\mathcal{K}$ is called an ideal, if it satisfies

(I-1) $0 \in \mathcal{I}$

(I-2) $p_0 \Diamond r_0 \in \mathcal{I}$ and $r_0 \in \mathcal{I} \Rightarrow p_0 \in \mathcal{I}$ for all $p_0, r_0 \in \mathcal{K}$.

Definition 2.5[1]**.** Let $\mathcal{K}$ be a non-empty set. A fuzzy set in a set $\mathcal{K}$ is a mapping $\mathcal{N}_T : \mathcal{K} \to [0, 1]$.

Definition 2.6[1]**.** The complement of a fuzzy set $\mathcal{N}_T$ denoted by $(\mathcal{N}_T)^c$ is also a fuzzy set defined as $(\mathcal{N}_T)^c = 1 - \mathcal{N}_T$, for all $p_0 \in \mathcal{K}$.

Definition 2.7[9]**.** A fuzzy set $\mathcal{N}_T : \mathcal{K} \to [0, 1]$ is called fuzzy sub-algebra of $\mathcal{K}$, if $\mathcal{N}_T(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Definition 2.8[9]**.** A fuzzy set $(\mathcal{K}, \mathcal{N}_T)$ in a BCK-algebra $\mathcal{K}$, is said to be a fuzzy ideal of $\mathcal{K}$ if

(F-1) $\mathcal{N}_T(0) \geq \mathcal{N}_T(p_0)$.

(F-2) $\mathcal{N}_T(p_0) \geq \min\{\mathcal{N}_T(p_0 \lozenge r_0), \mathcal{N}_T(r_0)\}$, for all $p_0, r_0 \in \mathcal{K}$.

Definition 2.9[3]**.** An intuitionistic fuzzy set $\mathcal{N}$ in a non-empty set $\mathcal{K}$ is an object having the form $\mathcal{N} = \{(p_0, \mathcal{N}_T(p_0), \mathcal{N}_F(p_0))/p_0 \in \mathcal{K}\}$ where the functions $\mathcal{N}_T : \mathcal{K} \to [0, 1]$ and $\mathcal{N}_F : \mathcal{K} \to [0, 1]$ denote the grade of membership and non-membership of each element $p_0 \in \mathcal{K}$ to the set $\mathcal{N}$ respectively and $0 \leq \mathcal{N}_T(p_0) + \mathcal{N}_F(p_0) \leq 1$ for all $p_0 \in \mathcal{K}$.

Definition 2.10[3]**.** An IFS $\mathcal{N} = (\mathcal{K}, \mathcal{N}_T, \mathcal{N}_F)$ of $\mathcal{K}$ is IFSA of $\mathcal{K}$ if it follows the conditions

(IFSA-1) $\mathcal{N}_T(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$

(IFSA-2) $\mathcal{N}_F(p_0 \lozenge r_0) \leq \max\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Proposition 2.11. *Every IFSA* $\mathcal{N} = (\mathcal{K}, \mathcal{N}_T, \mathcal{N}_F)$ *of a BCK-algebra* $\mathcal{K}$ *satisfies the inequalities* $\mathcal{N}_T(0) \geq \mathcal{N}_T(p_0)$ *and* $\mathcal{N}_F(0) \geq \mathcal{N}_F(p_0)$ *for all* $p_0 \in \mathcal{K}$.

Definition 2.12[4]**.** An IFS $\mathcal{N} = (\mathcal{K}, \mathcal{N}_T, \mathcal{N}_F)$ of a BCK-algebra $\mathcal{K}$ is IFI of $\mathcal{K}$ if it follows the conditions

(IFI-1) $\mathcal{N}_T(0) \geq \mathcal{N}_T(p_0)$ and $\mathcal{N}_F(0) \geq \mathcal{N}_F(p_0)$

(IFI-2) $\mathcal{N}_T(p_0) \geq \min\{\mathcal{N}_T(p_0 \lozenge r_0), \mathcal{N}_T(r_0)\}$

(IFI-3) $\mathcal{N}_F(p_0) \leq \max\{\mathcal{N}_F(p_0 \lozenge r_0), \mathcal{N}_F(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Definition 2.13[8]**.** A fuzzy subset $(\mathcal{K}, \mathcal{N}_T)$ in a BCK-algebra $\mathcal{K}$ is called a fuzzy Bi-ideal if (FBi-I 1) $\mathcal{N}_T(0) \geq \mathcal{N}_T(p_0)$

(FBi-I 2) $\mathcal{N}_T(p_0) \geq \min\{\mathcal{N}_T(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_T(u_0)\}$ for all $p_0, r_0, u_0 \in \mathcal{K}$.

Definition 2.14[8]. An IFS $\mathcal{N} = (\mathcal{K}, \mathcal{N}_T, \mathcal{N}_F)$ of a BCK-algebra $\mathcal{K}$ is called intuitionistic fuzzy Bi-ideal of $\mathcal{K}$ if

(IFBi-I 1) $\mathcal{N}_T(0) \geq \mathcal{N}_T(p_0)$ and $\mathcal{N}_F(0) \geq \mathcal{N}_F(p_0)$

(IFBi-I 2) $\mathcal{N}_T(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_T(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_T(u_0)\}$

(IFBi-I 3) $\mathcal{N}_F(p_0 \lozenge r_0) \leq \max\{\mathcal{N}_F(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F(u_0)\}$ for all $p_0, r_0, u_0 \in \mathcal{K}$.

Definition 2.15[7]. A fuzzy neutrosophic set (FNS) in a non-empty set $\mathcal{K}$ is a structure of the form $\mathcal{N} = \{\langle p_0, \mathcal{N}_T(p_0), \mathcal{N}_I(p_0), \mathcal{N}_F(p_0)\rangle / p_0 \in \mathcal{K}\}$

Where $\mathcal{N}_T : \mathcal{K} \to [0, 1]$, $\mathcal{N}_I : \mathcal{K} \to [0, 1]$ and $\mathcal{N}_F : \mathcal{K} \to [0, 1]$ represents grade of belongingness, grade of indeterminacy and grade of non-belongingness of each element $p_0 \in \mathcal{K}$ to the set $\mathcal{N}$ respectively and $0 \leq \mathcal{N}_T(p_0) + \mathcal{N}_I(p_0) + \mathcal{N}_F(p_0) \leq 3$.

We shall use the symbol $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ for the FNS $\mathcal{N} = \{\langle p_0, \mathcal{N}_T(p_0), \mathcal{N}_I(p_0), \mathcal{N}_F(p_0)\rangle / p_0 \in \mathcal{K}\}$.

Definition 2.16[10]. A FNS $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ in $\mathcal{K}$ is fuzzy neutrosophic sub-algebra (FNSA) of $\mathcal{K}$ if it follows the conditions

(FNSA-1) $\mathcal{N}_T(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$

(FNSA-2) $\mathcal{N}_I(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\}$

(FNSA-3) $\mathcal{N}_F(p_0 \lozenge r_0) \leq \max\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Definition 2.17. A FNS $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ in $\mathcal{K}$ is fuzzy neutrosophic ideal (FNI) of $\mathcal{K}$ if it follows the conditions

(FNI-1) $\mathcal{N}_T(0) \geq \mathcal{N}_T(p_0)$, $\mathcal{N}_I(0) \geq \mathcal{N}_I(p_0)$ and $\mathcal{N}_F(0) \leq \mathcal{N}_F(p_0)$

(FNI-2) $\mathcal{N}_T(p_0) \geq \min\{\mathcal{N}_T(p_0 \lozenge r_0), \mathcal{N}_T(r_0)\}$

(FNI -3) $\mathcal{N}_I(p_0) \geq \min\{\mathcal{N}_I(p_0 \lozenge r_0), \mathcal{N}_I(r_0)\}$

(FNI-4) $\mathcal{N}_F(p_0) \leq \max\{\mathcal{N}_F(p_0 \lozenge r_0), \mathcal{N}_F(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

3. Fuzzy Neutrosophic Bi-Ideals of BCK-algebras

Definition 3.1. A FNS $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is called a fuzzy neutrosophic Bi-ideal (FNBi-I) of $\mathcal{K}$ if it satisfies

(FNBi-I 1) $\mathcal{N}_T(0) \geq \mathcal{N}_T(p_0)$, $\mathcal{N}_I(0) \geq \mathcal{N}_I(p_0)$ and $\mathcal{N}_F(0) \leq \mathcal{N}_F(p_0)$

(FNBi-I 2) $\mathcal{N}_T(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_T(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_T(u_0)\}$

(FNBi-I 3) $\mathcal{N}_I(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_I(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_I(u_0)\}$

(FNBi-I 4) $\mathcal{N}_F(p_0 \lozenge r_0) \leq \max\{\mathcal{N}_F(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F(u_0)\}$ for all p_0, r_0, $u_0 \in \mathcal{K}$.

Example 3.2. Consider a BCK-algebra $\mathcal{K} = \{0, 1, 2, 3\}$ with the Cayley table as shown in following Figure 3.1

$\lozenge$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Figure 3.1.

Define a FNS $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ in $\mathcal{K}$ as follows in Figure 3.2

$\mathcal{K}$	$\mathcal{N}_T$	$\mathcal{N}_I$	$\mathcal{N}_F$
0	0.7	0.6	0.2
1	0.7	0.5	0.3
2	0.3	0.1	0.5
3	0.4	0.2	0.7

Figure 3.2.

Then by routine calculation $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNBi-I of $\mathcal{K}$.

Lemma 3.3. *Let a FNS* $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ *in* $\mathcal{K}$ *be a FNBi-I of* $\mathcal{K}$. *If the inequality* $p_0 \lozenge r_0 \le u_0$ *holds in* $\mathcal{K}$ *then*

(i) $\mathcal{N}_T(p_0 \lozenge r_0) \ge \min\{\mathcal{N}_T(p_0), \mathcal{N}_T(u_0)\}$

(ii) $\mathcal{N}_I(p_0 \lozenge r_0) \ge \min\{\mathcal{N}_I(p_0), \mathcal{N}_I(u_0)\}$

(iii) $\mathcal{N}_F(p_0 \lozenge r_0) \le \max\{\mathcal{N}_F(p_0), \mathcal{N}_F(u_0)\}$ *for all* $p_0, r_0, u_0 \in \mathcal{K}$.

Proof. Let $p_0, r_0, u_0 \in \mathcal{K}$ be such that $p_0 \lozenge r_0 \le u_0$

(i) Now $\mathcal{N}_T(p_0 \lozenge r_0) \ge \min\{\mathcal{N}_T(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_T(u_0)\}$

$= \min\{\mathcal{N}_T(0), \mathcal{N}_T(u_0)\}$

$\ge \min\{\min\{\mathcal{N}_T(p_0), \mathcal{N}_T(u_0)\}, \mathcal{N}_T(u_0)\}$

$= \min\{\mathcal{N}_T(p_0), \mathcal{N}_T(u_0)\}$

Therefore, $\mathcal{N}_T(p_0 \lozenge r_0) \ge \min\{\mathcal{N}_T(p_0), \mathcal{N}_T(u_0)\}$

(ii) $\mathcal{N}_I(p_0 \lozenge r_0) \ge \min\{\mathcal{N}_I(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_I(u_0)\}$

$= \min\{\mathcal{N}_I(0), \mathcal{N}_I(u_0)\}$

$\ge \min\{\min\{\mathcal{N}_I(p_0), \mathcal{N}_I(u_0)\}, \mathcal{N}_I(u_0)\}$

$= \min\{\mathcal{N}_I(p_0), \mathcal{N}_I(u_0)\}$

Therefore, $\mathcal{N}_I(p_0 \lozenge r_0) \ge \min\{\mathcal{N}_I(p_0), \mathcal{N}_I(u_0)\}$

(iii) $\mathcal{N}_F(p_0 \lozenge r_0) \le \max\{\mathcal{N}_F(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F(u_0)\}$

$= \max\{\mathcal{N}_F(0), \mathcal{N}_F(u_0)\}$

$\ge \max\{\max\{\mathcal{N}_F(p_0), \mathcal{N}_F(u_0)\}, \mathcal{N}_F(u_0)\}$

$= \max\{\mathcal{N}_F(p_0), \mathcal{N}_F(u_0)\}$

Therefore, $\mathcal{N}_F(p_0 \lozenge r_0) \le \max\{\mathcal{N}_F(p_0), \mathcal{N}_F(u_0)\}$

Hence $\mathcal{N}_T(p_0 \lozenge r_0) \ge \min\{\mathcal{N}_T(p_0), \mathcal{N}_T(u_0)\}$

$\mathcal{N}_I(p_0 \lozenge r_0) \ge \min\{\mathcal{N}_I(p_0), \mathcal{N}_I(u_0)\}$

$\mathcal{N}_F(p_0 \lozenge r_0) \le \max\{\mathcal{N}_F(p_0), \mathcal{N}_F(u_0)\}$ for all $p_0, r_0, u_0 \in \mathcal{K}$.

Lemma 3.4. *Let a FNS* $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ *be a FNBi-I of* $\mathcal{K}$. *If the inequality* $p_0 \lozenge r_0 \leq u_0$ *holds in* $\mathcal{K}$ *then* $\mathcal{N}_T(p_0 \lozenge r_0) \geq \mathcal{N}_T(u_0)$, $\mathcal{N}_I(p_0 \lozenge r_0) \geq \mathcal{N}_I(u_0)$ *and* $\mathcal{N}_F(p_0 \lozenge r_0) \leq \mathcal{N}_F(u_0)$ *that is* $\mathcal{N}_T, \mathcal{N}_I$ *are order-reversing and* $\mathcal{N}_I$ *is order-preserving.*

Proof. Let $p_0, r_0, u_0 \in \mathcal{K}$ be such that $p_0 \lozenge r_0 \leq u_0$. Then $p_0 \lozenge r_0 \lozenge u_0 = 0$ and so

(i) $\mathcal{N}_T(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_T(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_T(u_0)\}$

$$= \min\{\mathcal{N}_T(0), \mathcal{N}_T(u_0)\}$$

$$= \mathcal{N}_T(u_0)$$

Therefore, $\mathcal{N}_T(p_0 \lozenge r_0) \geq \mathcal{N}_T(u_0)$

(ii) $\mathcal{N}_I(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_I(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_I(u_0)\}$

$$= \min\{\mathcal{N}_I(0), \mathcal{N}_I(u_0)\}$$

$$= \mathcal{N}_I(u_0)$$

Therefore, $\mathcal{N}_I(p_0 \lozenge r_0) \geq \mathcal{N}_I(u_0)$

(iii) $\mathcal{N}_F(p_0 \lozenge r_0) \leq \max\{\mathcal{N}_F(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F(u_0)\}$

$$= \max\{\mathcal{N}_F(0), \mathcal{N}_F(u_0)\}$$

$$= \mathcal{N}_F(u_0)$$

Therefore, $\mathcal{N}_F(p_0 \lozenge r_0) \leq \mathcal{N}_I(u_0)$

Hence $\mathcal{N}_T, \mathcal{N}_I$ are order-reserving and $\mathcal{N}_F$ is order-preserving.

Theorem 3.5. *If a FNS* $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ *be a FNBi-I of* $\mathcal{K}$ *then for any* $p_0 \lozenge r_0, a_1, a_2, a_3, \ldots a_n \in \mathcal{K}, (\ldots((p_0 \lozenge r_0 \lozenge a_1) \lozenge a_2) \lozenge \ldots) \lozenge a_n = 0$ *implies* $\mathcal{N}_T(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_T(a_1), \mathcal{N}_T(a_2) \ldots \mathcal{N}_T(a_n)\}$

$\mathcal{N}_I(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_I(a_1), \mathcal{N}_I(a_2) \ldots \mathcal{N}_I(a_n)\}$ *and*

$\mathcal{N}_F(p_0 \lozenge r_0) \leq \max\{\mathcal{N}_F(a_1), \mathcal{N}_F(a_2) \ldots \mathcal{N}_F(a_n)\}$.

Proof. Using induction on n and Lemma 3.3, Lemma 3.4, the proof is straightforward.

Theorem 3.6. *Every FNBi-I of $\mathcal{K}$ is FNSA of $\mathcal{K}$.*

Proof. Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a FNBi-I of $\mathcal{K}$.

Since $p_0 \lozenge r_0 \leq p_0$ it follows from Lemma 3.4 $\mathcal{N}_T(p_0 \lozenge r_0) \geq \mathcal{N}_T(p_0)$, $\mathcal{N}_I(p_0 \lozenge r_0) \geq \mathcal{N}_I(p_0)$ and $\mathcal{N}_F(p_0 \lozenge r_0) \leq \mathcal{N}_F(p_0)$

Now $\mathcal{N}_T(p_0 \lozenge r_0) \geq \mathcal{N}_T(p_0) \geq \min\{\mathcal{N}_T(p_0 \lozenge r_0), \mathcal{N}_T(r_0)\}$

$\geq \min\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$

Therefore, $\mathcal{N}_T(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_T(p_0), \mathcal{N}_T(r_0)\}$

$\mathcal{N}_I(p_0 \lozenge r_0) \geq \mathcal{N}_I(p_0) \geq \min\{\mathcal{N}_I(p_0 \lozenge r_0), \mathcal{N}_I(r_0)\}$

$\geq \min\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\}$

Therefore, $\mathcal{N}_I(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_I(p_0), \mathcal{N}_I(r_0)\}$

$\mathcal{N}_F(p_0 \lozenge r_0) \leq \mathcal{N}_F(p_0) \leq \max\{\mathcal{N}_F(p_0 \lozenge r_0), \mathcal{N}_F(r_0)\}$

$\leq \max\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\}$

Therefore, $\mathcal{N}_F(p_0 \lozenge r_0) \leq \max\{\mathcal{N}_F(p_0), \mathcal{N}_F(r_0)\}$.

Thus $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNSA of $\mathcal{K}$.

Lemma 3.7. *A FNS $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNBi-I of $\mathcal{K}$ iff the fuzzy sets $\mathcal{N}_T, \mathcal{N}_I$ and $\mathcal{N}_F^c$ are fuzzy Bi-ideals of $\mathcal{K}$.*

Proof. Let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a FNBi-I of $\mathcal{K}$ then for every $p_0, r_0, u_0 \in \mathcal{K}$ we have $\mathcal{N}_T(0) \geq \mathcal{N}_T(p_0), \mathcal{N}_I(0) \geq \mathcal{N}_I(p_0)$ and $\mathcal{N}_F(0) \leq \mathcal{N}_F(p_0)$

$\mathcal{N}_T(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_T(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_T(u_0)\}$,

$\mathcal{N}_I(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_I(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_I(u_0)\}$ and

$\mathcal{N}_F(p_0 \lozenge r_0) \leq \max\{\mathcal{N}_F(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F(u_0)\}$, Clearly $\mathcal{N}_T$, $\mathcal{N}_I$ are fuzzy Bi-ideals of $\mathcal{K}$.

$\mathcal{N}_F^c(0) = 1 - \mathcal{N}_F(0) \geq 1 - \mathcal{N}_F(p_0) = \mathcal{N}_F^c(p_0)$. Therefore $N_F^c(0) \geq N_F^c(p_0)$.

$\mathcal{N}_F^c(p_0 \lozenge r_0) = 1 - \mathcal{N}_F(p_0 \lozenge r_0) \geq 1 - \max\{\mathcal{N}_F(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F(u_0)\}$

$= \min\{1 - \mathcal{N}_F(p_0 \lozenge r_0 \lozenge u_0), 1 - \mathcal{N}_F(u_0)\}$

$= \min\{\mathcal{N}_F^c(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F^c(u_0)\}$

Therefore, $\mathcal{N}_F^c(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_F^c(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F^c(u_0)\}$.

Hence $\mathcal{N}_F^c$ fuzzy Bi-ideals of $\mathcal{K}$.

Conversely assume that $\mathcal{N}_T$, $\mathcal{N}_I$ and $\mathcal{N}_F^c$ are fuzzy Bi-ideals of $\mathcal{K}$.

For every $p_0, r_0, u_0 \in \mathcal{K}$ we have $\mathcal{N}_T(0) \geq \mathcal{N}_T(p_0)$, $\mathcal{N}_I(0) \geq \mathcal{N}_I(p_0)$ and $\mathcal{N}_F^c(0) \geq \mathcal{N}_F^c(p_0)$.

Now $1 - \mathcal{N}_F(0) = \mathcal{N}_F^c(0) \geq \mathcal{N}_F^c(p_0) = 1 - \mathcal{N}_F(p_0) \Rightarrow \mathcal{N}_F(0) \leq \mathcal{N}_F(p_0)$.

Since $\mathcal{N}_T, \mathcal{N}_I$ and $\mathcal{N}_F^c$ are fuzzy Bi-ideals of $\mathcal{K}$ we have $\mathcal{N}_T(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_T(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_T(u_0)\}$,

$\mathcal{N}_I(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_I(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_I(u_0)\}$ and

$\mathcal{N}_F^c(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_F^c(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F^c(u_0)\}$.

Now $1 - \mathcal{N}_F(p_0 \lozenge r_0) = \mathcal{N}_F^c(p_0 \lozenge r_0)$

$\geq \min\{\mathcal{N}_F^c(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F^c(u_0)\}$

$= \min\{1 - \mathcal{N}_F(p_0 \lozenge r_0 \lozenge u_0), 1 - \mathcal{N}_F(u_0)\}$

$= 1 - \max\{\mathcal{N}_F(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F(u_0)\}$

$1 - \mathcal{N}_F(p_0 \lozenge r_0) \geq 1 - \max\{\mathcal{N}_F(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F(u_0)\} \Rightarrow \mathcal{N}_F(p_0 \lozenge r_0)$

$\leq \max\{\mathcal{N}_F(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F(u_0)\}$

Hence $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNBi-I of $\mathcal{K}$.

Theorem 3.8. *Let* $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ *is a FNS in* $\mathcal{K}$ *then* $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ *is a FNBi-I of* $\mathcal{K}$ *iff* $\circ\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_T^c)$ *and* $*\mathcal{N} = (\mathcal{N}_F^c, \mathcal{N}_I, \mathcal{N}_F)$ *are FNBi-Is of* $\mathcal{K}$.

Proof. If $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNBi-I of $\mathcal{K}$, then $\mathcal{N}_T, \mathcal{N}_I$ and $\mathcal{N}_F^c$ are fuzzy Bi-ideals of $\mathcal{K}$ from Lemma 3.7 also $\mathcal{N}_T = (\mathcal{N}_T^c)^c$.

Hence $\circ\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_T^c)$ and $*\mathcal{N} = (\mathcal{N}_F^c, \mathcal{N}_I, \mathcal{N}_F)$ are FNBi-Is of $\mathcal{K}$.

Conversely $\circ\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_T^c)$ and $*\mathcal{N} = (\mathcal{N}_F^c, \mathcal{N}_I, \mathcal{N}_F)$ are FNBi-Is of $\mathcal{K}$ then the fuzzy sets $\mathcal{N}_T, \mathcal{N}_I$ and $\mathcal{N}_F^c$ are fuzzy Bi-ideals of $\mathcal{K}$ are fuzzy Bi-ideals of $\mathcal{K}$.

Hence $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNBi-I of $\mathcal{K}$.

Definition 3.9. A mapping $f : \mathcal{K} \to \mathcal{Y}$ of BCK-algebras is called a homomorphism if $f(p_0 \lozenge r_0) = f(p_0) \lozenge f(r_0)$ for all $p_0, r_0 \in \mathcal{K}$.

Note that if $f : \mathcal{K} \to \mathcal{Y}$ is a homomorphism of BCK-algebras then $f(0) = 0$.

Let $f : \mathcal{K} \to \mathcal{Y}$ be a homomorphism of BCK-algebras. For any FNS $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ in $\mathcal{Y}$. We define a new FNS $\mathcal{N}^f = (\mathcal{N}_T^f, \mathcal{N}_I^f, \mathcal{N}_F^f)$ in $\mathcal{K}$ by $\mathcal{N}_T^f(p_0) = \mathcal{N}_T(f(p_0)), \mathcal{N}_I^f(p_0) = \mathcal{N}_I(f(p_0))$ and $\mathcal{N}_F^f(p_0) = \mathcal{N}_F(f(p_0))$ for all $p_0 \in \mathcal{K}$.

Theorem 3.10. *Let* $f : \mathcal{K} \to \mathcal{Y}$ *be a homomorphism of BCK-algebras. If a FNS* $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ *in* $\mathcal{Y}$ *is a FNBi-I of* $\mathcal{Y}$, *then a FNS* $\mathcal{N}^f = (\mathcal{N}_T^f, \mathcal{N}_I^f, \mathcal{N}_F^f)$ *in* $\mathcal{K}$ *is a FNBi-I of* $\mathcal{K}$.

Proof. i. $\mathcal{N}_T^f(p_0) = \mathcal{N}_T(f(p_0)) \leq \mathcal{N}_T(f(0)) = \mathcal{N}_T^f(0)$

$$\Rightarrow \mathcal{N}_T^f(p_0) \leq \mathcal{N}_T^f(0).$$

ii. $\mathcal{N}_I^f(p_0) = \mathcal{N}_I(f(p_0)) \le \mathcal{N}_I(f(0)) = \mathcal{N}_I^f(0)$

$$\Rightarrow \mathcal{N}_I^f(p_0) \le \mathcal{N}_I^f(0).$$

iii. $\mathcal{N}_F^f(p_0) = \mathcal{N}_F(f(p_0)) \ge \mathcal{N}_F(f(0)) = \mathcal{N}_F^f(0)$

$$\Rightarrow \mathcal{N}_F^f(p_0) \ge \mathcal{N}_F^f(0).$$

Let $p_0, r_0, u_0 \in \mathcal{K}$. Then

$$\min\{\mathcal{N}_T^f(p_0 \diamond r_0 \diamond u_0), \mathcal{N}_T^f(u_0)\} = \min\{\mathcal{N}_T(f(p_0 \diamond r_0 \diamond u_0)), \mathcal{N}_T(f(u_0))\}$$

$$= \min\{\mathcal{N}_T(f(p_0) \diamond f(r_0) \diamond f(u_0)), \mathcal{N}_T(f(u_0))\}$$

$$\le \mathcal{N}_T(f(p_0) \diamond f(r_0)) = \mathcal{N}_T(f(p_0 \diamond r_0)) = \mathcal{N}_T^f(p_0 \diamond r_0)$$

Therefore, $\mathcal{N}_T^f(p_0 \diamond r_0) \ge \min\{\mathcal{N}_T^f(p_0 \diamond r_0 \diamond u_0), \mathcal{N}_T^f(u_0)\}$

$$\min\{\mathcal{N}_I^f(p_0 \diamond r_0 \diamond u_0), \mathcal{N}_I^f(u_0)\} = \min\{\mathcal{N}_I(f(p_0 \diamond r_0 \diamond u_0)), \mathcal{N}_I(f(u_0))\}$$

$$= \min\{\mathcal{N}_I(f(p_0) \diamond f(r_0) \diamond f(u_0)), \mathcal{N}_T(f(u_0))\}$$

$$\le \mathcal{N}_I(f(p_0) \diamond f(r_0)) = \mathcal{N}_I(f(p_0 \diamond r_0)) = \mathcal{N}_I^f(p_0 \diamond r_0)$$

Therefore, $\mathcal{N}_I^f(p_0 \diamond r_0) \ge \min\{\mathcal{N}_I^f(p_0 \diamond r_0 \diamond u_0), \mathcal{N}_I^f(u_0)\}$

$$\max\{\mathcal{N}_F^f(p_0 \diamond r_0 \diamond u_0), \mathcal{N}_F^f(u_0)\} = \max\{\mathcal{N}_F(f(p_0 \diamond r_0 \diamond u_0)), \mathcal{N}_F(f(u_0))\}$$

$$= \max\{\mathcal{N}_F(f(p_0) \diamond f(r_0) \diamond f(u_0)), \mathcal{N}_F(f(u_0))\}$$

$$\ge \mathcal{N}_F(f(p_0) \diamond f(r_0)) = \mathcal{N}_F(f(p_0 \diamond r_0)) = \mathcal{N}_F^f(p_0 \diamond r_0)$$

Therefore, $\mathcal{N}_F^f(p_0 \diamond r_0) \le \max\{\mathcal{N}_F^f(p_0 \diamond r_0 \diamond u_0), \mathcal{N}_F^f(u_0)\}$

Hence $\mathcal{N}_f = (\mathcal{N}_T^f, \mathcal{N}_I^f, \mathcal{N}_F^f)$ is a FNBi-I of $\mathcal{K}$.

Theorem 3.11. *Let $f : \mathcal{K} \to \mathcal{Y}$ be an epimorphism of BCK-algebras and let $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ be a FNS in $\mathcal{Y}$. If $\mathcal{N}_f = (\mathcal{N}_T^f, \mathcal{N}_I^f, \mathcal{N}_F^f)$ is a FNBi-I of $\mathcal{K}$, then $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNBi-I of $\mathcal{Y}$.*

Proof. For any $p_0 \in \mathcal{Y}$ there exists $a \in \mathcal{K}$ such that $f(a) = p_0$ then

$$\mathcal{N}_T(p_0) = \mathcal{N}_T(f(a)) = \mathcal{N}_T^f(a) \leq \mathcal{N}_T^f(0) = \mathcal{N}_T(f(0)) = \mathcal{N}_T(0)$$

$$\Rightarrow \mathcal{N}_T(p_0) \leq \mathcal{N}_T(0).$$

$$\mathcal{N}_I(p_0) = \mathcal{N}_I(f(a)) = \mathcal{N}_I^f(a) \leq \mathcal{N}_I^f(0) = \mathcal{N}_I(f(0)) = \mathcal{N}_I(0)$$

$$\Rightarrow \mathcal{N}_I(p_0) \leq \mathcal{N}_I(0).$$

$$\mathcal{N}_F(p_0) = \mathcal{N}_F(f(a)) = \mathcal{N}_F^f(a) \geq \mathcal{N}_F^f(0) = \mathcal{N}_F(f(0)) = \mathcal{N}_F(0)$$

$$\Rightarrow \mathcal{N}_F(p_0) \geq \mathcal{N}_F(0).$$

Let $p_0, r_0, u_0 \in \mathcal{Y}$, then $f(a) = p_0, f(b) = r_0$ and $f(c) = u_0$ for some $a, b, c \in \mathcal{K}$. It follows that

$$\mathcal{N}_T(p_0 \lozenge r_0) = \mathcal{N}_T(f(a) \lozenge f(b)) = \mathcal{N}_T(f(a \lozenge b))$$

$$= \mathcal{N}_T^f(a \lozenge b) \geq \min\{\mathcal{N}_T^f(a \lozenge b \lozenge c), \mathcal{N}_T^f(c)\}$$

$$= \min\{\mathcal{N}_T(f(a \lozenge b \lozenge c)), \mathcal{N}_T(f(c))\}$$

$$= \min\{\mathcal{N}_T(f(a) \lozenge f(b) \lozenge f(c)), \mathcal{N}_T(f(c))\}$$

$$= \min\{\mathcal{N}_T(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_T(u_0)\}$$

Therefore, $\mathcal{N}_T(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_T(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_T(u_0)\}$

$$\mathcal{N}_I(p_0 \lozenge r_0) = \mathcal{N}_I(f(a) \lozenge f(b)) = \mathcal{N}_I(f(a \lozenge b))$$

$$= \mathcal{N}_I^f(a \lozenge b) \geq \min\{\mathcal{N}_I^f(a \lozenge b \lozenge c), \mathcal{N}_I^f(c)\}$$

$$= \min\{\mathcal{N}_I(f(a \lozenge b \lozenge c)), \mathcal{N}_I(f(c))\}$$

$$= \min\{\mathcal{N}_I(f(a) \lozenge f(b) \lozenge f(c)), \mathcal{N}_I(f(c))\}$$

$$= \min\{\mathcal{N}_I(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_I(u_0)\}$$

Therefore, $\mathcal{N}_I(p_0 \lozenge r_0) \geq \min\{\mathcal{N}_I(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_I(u_0)\}$

$$\mathcal{N}_F(p_0 \lozenge r_0) = \mathcal{N}_F(f(a) \lozenge f(b)) = \mathcal{N}_F(f(a \lozenge b))$$

$$= \mathcal{N}_F^f(a \lozenge b) \leq \max\{\mathcal{N}_F^f(a \lozenge b \lozenge c), \mathcal{N}_F^f(c)\}$$

$$= \max\{\mathcal{N}_F(f(a \lozenge b \lozenge c)), \mathcal{N}_F(f(c))\}$$

$$= \max\{\mathcal{N}_F(f(a) \lozenge f(b) \lozenge f(c)), \mathcal{N}_F(f(c))\}$$

$$= \max\{\mathcal{N}_F(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F(u_0)\}$$

Therefore, $\mathcal{N}_F(p_0 \lozenge r_0) \leq \max\{\mathcal{N}_F(p_0 \lozenge r_0 \lozenge u_0), \mathcal{N}_F(u_0)\}$

Hence $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ is a FNBi-I of $\mathcal{Y}$.

References

[1] L. A. Zadeh, Fuzzy sets, Inf. Control 8 (1965), 338-353.

[2] O. G. Xi, Fuzzy BCK-algebra, Math. Jpn. 36(5) (1991), 935-942.

[3] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20(1) (1986), 87-96. doi: https://doi.org/10.1016/S0165-0114(86)80034-3

[4] Y. B. Jun and K. Kim, Intuitionistic fuzzy ideals of BCK-algebras, Int. J. Math. Math. Sci. 24(12) (2000), 839-849. doi:10.1155/S0161171200004610

[5] F. Smarandache, A unifying field in logics: neutrosophic logic, neutrosophy, neutrosophic set, neutrosophic probability and statistics, ISBN 1-879585-76-6 American Research Press Rehoboth, 2003.

[6] F. Smarandache, Neutrosophic set-A generalization of the intuitionistic fuzzy set, IEEE Int. Conf. Granul. Comput. (2006), 38-42. doi:10.1109/grc.2006.1635754

[7] I. Arockiarani, I. R. Sumathi and J. M. Jency, Fuzzy neutrosophic soft topological spaces, IJMA 4(10) (2013), 225-238.

[8] E. V. Sindhu and M. H. J. Begum, Intuitionistic Fuzzy Bi-ideals of BCK-Algebras Intuitionistic Fuzzy Bi-ideals of BCK-Algebras, December, 2020.

[9] S. Z. S. Y. B. Jun, S. M. Hong and S. J. Kim, Fuzzy ideals and fuzzy subalgebras of BCK-algebras, J. Fuzzy Math 2 (1999), 411-418.

[10] Shake Baji, U. Madhavi and B. Satyanarayana, Fuzzy neutrosophic implicative and positive implicative ideals of BCK-algebras, Kala Sarovar 25 (2022), 24-31.