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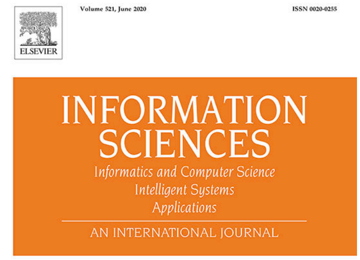

Dynamic three-way neighborhood decision model for multi-dimensional variation of incomplete hybrid data

Qianqian Huang, Yanyong Huang, Tianrui Li, Xin Yang

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Highlights

- To handle the uncertainty decision problems in incomplete hybrid data, a generalized three-way neighborhood decision model is proposed by distributing the interval-valued loss function to each object and averaging the interval-valued loss functions of all objects in the data-driven neighborhood class.
- A matrix-based approach for representing three-way regions in the generalized three-way neighborhood decision model is presented by introducing the matrix forms of related concepts and the matrix operators.
- An efficient framework for dynamically updating the three-way regions is provided when objects and attributes increase simultaneously.
- An incremental algorithm based on matrix is designed for maintaining the three-way regions.
- Experimental results demonstrate that the proposed incremental algorithm has an advantage in improving the computational performance.

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Dynamic three-way neighborhood decision model for multi-dimensional variation of incomplete hybrid data

Qianqian Huang[a,b], Yanyong Huang[c], Tianrui Li[a,b,*], Xin Yang[d]

[a]School of Computing and Artificial Intelligence, Southwest Jiaotong University, Chengdu 611756, China
[b]National Engineering Laboratory of Integrated Transportation Big Data Application Technology, Southwest Jiaotong University, Chengdu 611756, China
[c]School of Statistics, Southwestern University of Finance and Economics, Chengdu 611130, China
[d]School of Economic Information Engineering, Southwestern University of Finance and Economics, Chengdu 611130, China

Abstract

The theory of three-way decisions, as a powerful methodology of granular computing, has been widely used in making decision under uncertainty environments. Decision tasks in incomplete hybrid data including heterogeneous and missing features are of abundance in realistic situations. To deal with these tasks, some work based on three-way decisions has been investigated. However, the losses used for evaluating objects are precise real numbers, which makes these decision models have some limitations in applications when there exist missing values in incomplete hybrid data. Thus, this paper constructs a generalized three-way neighborhood decision model by assigning the interval-valued loss function to each object and further adopting an average strategy to integrate the interval-valued loss functions of objects in each data-driven neighborhood class. Moreover, considering that the objects and attributes of incomplete hybrid data will simultaneously change over time, this paper also provides an efficient framework to dynamically maintain three-way regions of the proposed model. An approach based on matrix to compute the three-way regions is first presented by introducing the matrix operations and the matrix forms of related concepts. Then, with the simultaneous variation of objects and attributes, the matrix-based incremental mechanism and algorithm are proposed for updating the three-way regions, respectively. Experimental results on nine datasets indicate that the proposed incremental algorithm can effectively improve the computational performance for evolving data in comparison with the static algorithm.

Keywords: Three-way decisions, Incomplete hybrid data, Matrix approach, Incremental learning.

1. Introduction

As an important cognitive approach to deal with the issues of uncertain classification, the theory of three-way decisions has become an emerging and significant application direction of rough set theory and granular computing since it was introduced by Yao [40–43]. The kernel idea of three-way decisions is to divide a domain into three pairwise disjoint positive, negative and boundary regions based on a pair of thresholds, and then make three different decisions for these three regions, which are acceptance, rejection and deferment, respectively. Over the past decades, three-way decisions have been extensively employed to address the complex and uncertain problems in various realms, such as image recognition [26], cluster analysis [44], medical diagnosis [37] and recommendation system [47].

To further improve and popularize the theory of three-way decisions, there are a large number of researchers committed to investigating it in depth from the perspectives of theory and application. For instance, Liang et al. [16–18] constructed three novel three-way decision models by generalizing the loss functions in decision-theoretic rough sets with interval numbers, intuitionistic fuzzy numbers and dual hesitant fuzzy elements, respectively. Meanwhile,

*Corresponding author.
Email addresses: huangqnqn@my.swjtu.edu.cn (Qianqian Huang), huangyy@swufe.edu.cn (Yanyong Huang), trli@swjtu.edu.cn (Tianrui Li), yangxin@swufe.edu.cn (Xin Yang)

Zhang et al. [46] presented a new three-way decision model associated with the intuitionistic fuzzy numbers when both cost parameters and attribute values are intuitionistic fuzzy numbers. Li and Huang [15] introduced the fuzzy T cos-equivalence relation with the aid of Gaussian kernel method, and advanced a novel three-way decision model in fuzzy condition decision information systems where the condition attributes are all fuzzy. Hu [3, 4] detailedly discussed three-way decision spaces and three-way decisions on account of the partially ordered sets, and further established three-way decisions based on semi-three-way decision spaces by introducing the concept of semi-decision evaluation functions. Jiao et al. [11] introduced two three-way decision models with single-valued neutrosophic sets on the basis of the cosine similarity measure and the Euclidean distance. Li et al. [13] constructed a framework of neighborhood based decision-theoretic rough set model to handle numerical data associated with noisy values, and then gave a three-way decisions based neighborhood classifier to solve the classification problems. Yang et al. [33] respectively proposed two dominance three-way decision models for interval-valued decision systems with the categorical and interval-valued decision attributes on account of a θ-overall dominance relation. Liu and Liang [20] established an ordering three-way decision model based on an integrated decision system consisting of the ordered information and the losses. Zhao and Hu [49] developed an extended three-way decision model in multiset-valued information systems through expanding the loss function into multiset values. Li et al. [14] systematically investigated a series of three-way decision models on two universes.

Unfortunately, these proposed three-way decision models can not be employed for resolving the uncertainty decision-making problems in incomplete data with missing values. To address this issue, Liu et al. [21] redefined a novel similarity relation, and then provided a new three-way decision model with incomplete information by the aid of a hybrid information system based on the combination of the incomplete information table and the loss functions. On this basis, Xu and Wang [32] presented a new aggregation method of the interval loss functions to construct a new three-way decision model. Luo et al. [23] built a conceptual framework for modeling similarity under incomplete information and discussed three-way decisions in incomplete information systems. Yang et al. [36] deeply studied the three-way decision rules by introducing the intuitionistic fuzzy sets to fuzzy incomplete information systems. Furthermore, Huang et al. [7] presented a three-way neighborhood decision model on the basis of the data-driven neighborhood relation in Incomplete Hybrid Information Systems (IHIS) containing the missing categorical and numerical attributes values. However, in practical applications, the losses employed for estimating objects may be imprecise and different when utilizing three-way decisions to deal with the uncertainty decision issues in incomplete hybrid data. Thus, the three-way neighborhood decision model has certain limitations in processing IHIS with missing values considering that the losses are the same precise real numbers. To overcome this drawback, it is necessary to introduce the formats of uncertainty to handle the issues with imprecision. Interval number, as a common format for characterizing incomplete and complex information, exists widely in real world. The advantage of interval number is that it can cater for the cognitions of decision makers and collect all possible variables within two endpoints without too much prior knowledge and assumptions, which is beneficial for proper expression of uncertain information. Motivated by the superiority of interval number, we employ interval number to depict the loss function and assign the interval-valued loss function to each object in IHIS. Further, we introduce a Composite Incomplete Hybrid Information Systems (CIHIS) by combining IHIS and the interval-valued loss functions of objects together. Then, by averaging the interval-valued loss functions of all objects in the data-driven neighborhood class, we propose a generalized three-way neighborhood decision model, which can be regarded as the generalized form of the model in [7].

Along with the popularization and development of information technology, data derived from the practical applications generally emerge the dynamicity, which can be described as the insertion and removal of objects, the addition and deletion of attributes and the modification of attribute values. When analyzing and processing the uncertainty classification problems under dynamic environments, the traditional batch-learning methods are time-consumption or even ineffective since they need to relearn the whole models with regard to the newly updated data. To this end, by simulating human cognitive mechanism, the incremental learning technologies are devised to deal with the continuously changing data. It can effectively improve the computational performance and facilitate knowledge maintenance through using the previously acquired results. Based on this superiority, much attention has been attracted to incorporate the incremental learning technologies into rough set theory, and a tremendous amount of outstanding achievements have been presented when objects, attributes or attribute values in information systems vary individually [5, 6, 19, 22, 24, 25, 45]. However, under dynamic environments, data may be no longer confined to the single-dimensional variation but to the multi-dimensional variations, namely, objects, attributes or attribute values may evolve over time simultaneously [1, 9, 10, 29–31, 34]. Generally, the phenomenon of simultaneous variation

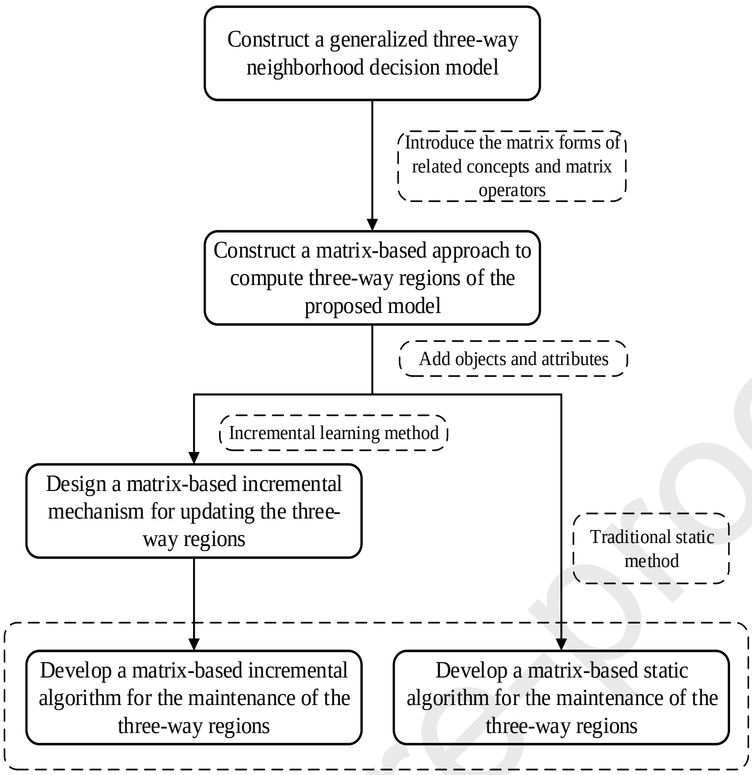

Figure 1: The overall structure diagram of the whole work.

of objects and attributes is frequent in real-life applications. For instance, in the diagnosis of unknown infections diseases, when patients continue to increase over time, new disease symptoms that have not appeared before may arise; In product quality analysis, when more product samples are tested, some new performance indicators may need to be considered. With respect to the simultaneous variation of objects and attributes, Chen et al. [1] provided a dynamic framework for maintaining approximations in decision-theoretic rough sets; Huang et al. [9] presented the incremental principles for the update of the rough fuzzy approximations; Wang et al. [29] introduced the incremental algorithms for the maintenance of the approximations in dominance-based rough sets. But the above-mentioned methods could not be directly applied to process the real-time classification problems in dynamic incomplete hybrid data with multi-dimensional variations. Hence, the another objective of this paper is to establish an efficient framework for dynamically maintaining three-way regions in the generalized three-way neighborhood decision model when objects and attributes are added into CIHIS simultaneously.

As an extremely powerful mathematical tool, matrix method is simple and intuitive for the description and computation of knowledge, which has been successfully applied to rough set theory [2, 8, 12, 35]. In accordance with the advantage of matrix method, we firstly propose a matrix-based approach for the fast calculation of three-way regions in the generalized three-way neighborhood decision model by constructing the matrix forms of related concepts and the matrix operations. Then, considering that objects and attributes in CIHIS simultaneously evolve over time under dynamic environments, we present a matrix-based incremental mechanism for facilitating the continuous generation of three-way regions. Subsequently, we design a matrix-based incremental algorithm for maintaining the three-way regions according to the proposed updating principles. Finally, we conduct the extensive comparative experiments to demonstrate the superiority of the incremental updating approach. To better illustrate the associations between the above-mentioned work, an overall structure diagram about this paper is shown in Figure 1.

The main contributions about this work are summarized as follows. (1) A generalized three-way decision model is proposed to deal with the uncertainty classification problems in IHIS. (2) The matrix-based framework of computing the three-way regions in the proposed model is constructed. (3) The matrix-based incremental principles for updating the three-way regions are discussed in detail when objects and attributes vary over time simultaneously. (4) The comparative experiments are carried out to verify the effectiveness and the efficiency of the incremental method. The rest of this paper is organized as follows. Section 2 reviews the fundamental concepts of the relative knowledge. Section 3 presents a generalized three-way neighborhood decision model. Section 4 introduces a novel matrix approach to represent three-way regions of the proposed model. Section 5 constructs an efficient approach based on matrix for incrementally maintaining the three-way regions with the simultaneous addition of objects and attributes. Section 6 presents the matrix-based static and incremental algorithms for updating the three-way regions when objects and attributes increase with time. Section 7 carries out a series of evaluation experiments to validate the efficiency of the incremental algorithm. Section 8 summarizes the work and discusses the future research topics.

2. Preliminaries

For the convenience of discussion, this section briefly reviews some basic concepts and results about three-way decisions [40], decision-theoretic rough sets [38, 39] and the data-driven neighborhood relation [8], respectively.

2.1. Three-way decisions in decision-theoretic rough sets

In rough set theory, an information system is often represented as a 4-tuple $S = (U, AT, V, f)$, where U denotes a non-empty finite set of objects; AT denotes a non-empty finite set of attributes consisting of condition attributes A and decision attributes D, i.e., $AT = A \cup D$ $(A \cap D = \emptyset)$; $V = \bigcup_{a \in AT} V_a$ and V_a is a non-empty finite set of attribute values of a; $f : U \times AT \to V$ is a mapping function such that $f(u, a) \in V_a$ for each $u \in U$ and $a \in AT$. In classical rough set model, the indiscernibility relation is formally expressed as

$$E_Q = \{(u, v) \in U^2 \mid \forall a \in Q, f(u, a) = f(v, a)\}, \ \forall Q \subseteq A. \tag{1}$$

Apparently, the indiscernibility relation satisfies reflexivity, symmetry and transitivity, which is also called as the equivalence relation. Then, with regard to any concept $X \subseteq U$, the lower and upper approximations of X in terms of E_Q are denoted respectively as

$$\begin{aligned} &\underline{E_Q}(X) = \{u \in U | [u]_{E_Q} \subseteq X\}, \\ &\overline{E_Q}(X) = \{u \in U | [u]_{E_Q} \cap X \neq \emptyset\}, \end{aligned} \tag{2}$$

where $[u]_{E_Q}$ is the equivalence class of u under E_Q, i.e., $[u]_{E_Q} = \{v \in U | (u, v) \in E_Q\}$.

From this pair of approximations, it can easily be found that the classification must be completely correct or certain, which makes the classical rough sets difficult to achieve effective knowledge from the practical applications [39, 50]. To overcome this limitation, a pair of decision thresholds α and β are introduced into the probabilistic rough sets. Then, the lower and upper approximations in the probabilistic rough sets can be represented respectively as

$$\begin{aligned} &\underline{E_Q^{(\alpha,\beta)}}(X) = \{u \in U | Pr(X | [u]_{E_Q}) \geq \alpha\}, \\ &\overline{E_Q^{(\alpha,\beta)}}(X) = \{u \in U | Pr(X | [u]_{E_Q}) > \beta\}, \end{aligned} \tag{3}$$

where $Pr(X | [u]_{E_Q})$ denotes the conditional probability of u in X, i.e., $Pr(X | [u]_{E_Q}) = \frac{|X \cap [u]_{E_Q}|}{|[u]_{E_Q}|}$, and $0 \leq \beta < \alpha \leq 1$. To systematically calculate the parameters α and β, decision-theoretic rough sets are constructed based on Bayesian decision procedure [38].

As is known to all, decision-theoretic rough sets are made up of two states and three actions. The state set is given by $\Omega = \{X, X'\}$ indicating that an element belongs to X and does't belong to X, respectively. The action set is given by $\mathcal{A} = \{\mathcal{A}_P, \mathcal{A}_B, \mathcal{A}_N\}$, where $\mathcal{A}_P$, $\mathcal{A}_B$ and $\mathcal{A}_N$ denote three actions in classifying u, that is, deciding $u \in POS(X)$, deciding $u \in BND(X)$ and deciding $u \in NEG(X)$, which are corresponding to the acceptance decision, the non-commitment decision and the rejection decision, respectively. In addition, the loss function in regard of the risk or cost of different actions in different states is described by a 2×3 matrix, shown in Table 1, where λ_{PP}, λ_{BP} and λ_{NP} respectively denote the costs of taking actions $\mathcal{A}_P$, $\mathcal{A}_B$ and $\mathcal{A}_N$ when the object u is in X, and λ_{PN}, λ_{BN} and λ_{NN} respectively denote the costs of taking same actions when the object u is not in X.

Table 1: The loss function

	$\mathcal{A}_P$	$\mathcal{A}_B$	$\mathcal{A}_N$
$X(P)$	λ_{PP}	λ_{BP}	λ_{NP}
$X'(N)$	λ_{PN}	λ_{BN}	λ_{NN}

According to the given loss function matrix, the expected losses $\mathcal{L}(\mathcal{A}_\star|[u]_{E_Q})$ ($\star = P, B, N$) associated with the actions $\mathcal{A}_P$, $\mathcal{A}_B$ and $\mathcal{A}_N$ can be expressed respectively as

$$\begin{aligned}
&\mathcal{L}(\mathcal{A}_P|[u]_{E_Q}) = \lambda_{PP}Pr(X|[u]_{E_Q}) + \lambda_{PN}Pr(X'|[u]_{E_Q});\\
&\mathcal{L}(\mathcal{A}_B|[u]_{E_Q}) = \lambda_{BP}Pr(X|[u]_{E_Q}) + \lambda_{BN}Pr(X'|[u]_{E_Q});\\
&\mathcal{L}(\mathcal{A}_N|[u]_{E_Q}) = \lambda_{NP}Pr(X|[u]_{E_Q}) + \lambda_{NN}Pr(X'|[u]_{E_Q}).
\end{aligned} \tag{4}$$

By the Bayesian decision procedure, it can easily be deserved the following minimum-cost decision rules.

(P) If $\mathcal{L}(\mathcal{A}_P|[u]_{E_Q}) \leq \mathcal{L}(\mathcal{A}_B|[u]_{E_Q})$ and $\mathcal{L}(\mathcal{A}_P|[u]_{E_Q}) \leq \mathcal{L}(\mathcal{A}_N|[u]_{E_Q})$, decide $u \in POS_Q(X)$;
(B) If $\mathcal{L}(\mathcal{A}_B|[u]_{E_Q}) \leq \mathcal{L}(\mathcal{A}_P|[u]_{E_Q})$ and $\mathcal{L}(\mathcal{A}_B|[u]_{E_Q}) \leq \mathcal{L}(\mathcal{A}_N|[u]_{E_Q})$, decide $u \in BND_Q(X)$;
(N) If $\mathcal{L}(\mathcal{A}_N|[u]_{E_Q}) \leq \mathcal{L}(\mathcal{A}_P|[u]_{E_Q})$ and $\mathcal{L}(\mathcal{A}_N|[u]_{E_Q}) \leq \mathcal{L}(\mathcal{A}_B|[u]_{E_Q})$, decide $u \in NEG_Q(X)$.

In view of the reasonable semantic explanations of the loss functions with $\lambda_{PP} \leq \lambda_{BP} < \lambda_{NP}$ and $\lambda_{NN} \leq \lambda_{BN} < \lambda_{PN}$, the decision rules can also be simplified as

(P') If $Pr(X|[u]_{E_Q}) \geq \alpha$ and $Pr(X|[u]_{E_Q}) \geq \gamma$, decide $u \in POS_Q(X)$;
(B') If $Pr(X|[u]_{E_Q}) \leq \alpha$ and $Pr(X|[u]_{E_Q}) \geq \beta$, decide $u \in BND_Q(X)$;
(N') If $Pr(X|[u]_{E_Q}) \leq \beta$ and $Pr(X|[u]_{E_Q}) \leq \gamma$, decide $u \in NEG_Q(X)$.

where $\alpha = \frac{(\lambda_{PN}-\lambda_{BN})}{(\lambda_{PN}-\lambda_{BN})+(\lambda_{BP}-\lambda_{PP})}$, $\beta = \frac{(\lambda_{BN}-\lambda_{NN})}{(\lambda_{BN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{BP})}$, $\gamma = \frac{(\lambda_{PN}-\lambda_{NN})}{(\lambda_{PN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{PP})}$, and $0 \leq \beta < \gamma < \alpha \leq 1$. Therefore, the minimum-cost decision rules can be re-represented respectively as

(P'') If $Pr(X|[u]_{E_Q}) \geq \alpha$, decide $u \in POS_Q(X)$;
(B'') If $\beta < Pr(X|[u]_{E_Q}) < \alpha$, decide $u \in BND_Q(X)$;
(N'') If $Pr(X|[u]_{E_Q}) \leq \beta$, decide $u \in NEG_Q(X)$.

It is obvious that the minimum-cost decision rules don't contain γ. Then, the positive, boundary and negative regions can be expressed respectively as

$$\begin{aligned}
&POS_Q^{(\alpha,\beta)}(X) = \{u \in U|\ Pr(X|[u]_{E_Q}) \geq \alpha\};\\
&BND_Q^{(\alpha,\beta)}(X) = \{u \in U|\ \beta < Pr(X|[u]_{E_Q}) < \alpha\};\\
&NEG_Q^{(\alpha,\beta)}(X) = \{u \in U|\ Pr(X|[u]_{E_Q}) \leq \beta\}.
\end{aligned} \tag{5}$$

2.2. Data-driven neighborhood relation

In many real applications, the data always show the heterogeneity and incompleteness [27, 28, 48]. Such data can be represented by IHIS. First, we simply introduce the definition of IHIS [8].

Definition 2.1. *[8] Let $S = (U, AT = A \cup D, V, f)$ be an information system. If the condition attribute set A is composed of the categorical attribute set A^C and the numerical attribute set A^N, i.e., $A = A^C \cup A^N$ ($A^C \cap A^N = \emptyset$), and there exist $u \in U$ and $a \in A$ such that $f(u, a) = *$ or $f(u, a) =?$, then S is called as IHIS.*

Table 2: Description of related symbols

Symbol	Meaning
V_a^k	the k-th known categorical attribute value with respect to a;
V_b^l	the l-th known numerical attribute value with respect to b;
$V_a^\dagger$	the set of all known categorical attribute values with respect to a, i.e., $V_a^\dagger = \{V_a^1, \cdots, V_a^{m_1}\}$;
$V_b^\dagger$	the set of all known numerical attribute values with respect to b, i.e., $V_b^\dagger = \{V_b^1, \cdots, V_b^{m_2}\}$;
H_a	the set of objects with known attribute values under a, i.e., $H_a = \{u \in U \| f(u,a) \neq * \wedge f(u,a) \neq ?\}$;
H_b	the set of objects with known attribute values under b, i.e., $H_b = \{u \in U \| f(u,b) \neq * \wedge f(u,b) \neq ?\}$;
s_a^k	the number of objects such that $f(u,a) = V_a^k$ under a, i.e., $s_a^k = \|\{u \in H_a \| f(u,a) = V_a^k\}\|$;
n_b^l	the number of objects such that $\|f(u,b) - V_b^l\| \leq \delta_b$ under b, i.e., $n_b^l = \|\{u \in H_b \| \|f(u,b) - V_b^l\| \leq \delta_b\}\|$;
δ_b	the threshold value with respect to b.

With respect to IHIS, Zhao and Qin proposed a neighborhood-tolerance relation by integrating the neighborhood relation and the tolerance relation when all unknown values are "do not care" condition [48]. Similarly, Huang et al. presented a neighborhood-characteristic relation to handle IHIS with "do not care" and "lost value" unknown values [8]. Obviously, the characteristics of data distribution are not well considered during the construction of the binary relations. Therefore, Huang et al. also constructed a data-driven neighborhood relation based on two kinds of pseudo-metric functions only with the reflexivity [8], which can be considered as a generalized form of other neighborhood relations that deal with IHIS. For the convenience of understanding, the detailed explanations of some symbols in the pseudo-metric functions are summarized in Table 2.

Definition 2.2. *[8] Let S be an IHIS and $Q = Q^C \cup Q^N \subseteq A$. Suppose that $\rho^C : U^2 \times Q^C \to \mathbb{R}$ and $\rho^N : U^2 \times Q^N \to \mathbb{R}$ denote two pseudo-metric functions with regard to Q^C and Q^N, respectively. Then, for any $a \in Q^C$ and $b \in Q^N$,*

$$\rho^C(u,v,a) = \begin{cases} 0, & u = v \vee f(u,a) = ?; \\ 0, & (f(u,a), f(v,a) \in V_a^\dagger) \wedge f(u,a) = f(v,a); \\ 0, & s_a^k/|H_a| \geq \kappa_1 \wedge ((f(u,a) = V_a^k \wedge f(v,a) = *) \vee (f(u,a) = * \wedge f(v,a) = V_a^k)); \\ 0, & (u \neq v) \wedge (\sum_{k=1}^{m_1} (s_a^k/|H_a|)^2 \geq \kappa_1 \wedge (f(u,a) = * \wedge f(v,a) = *)); \\ 0, & f(u,a) = * \wedge f(v,a) = ?; \\ \infty, & otherwise, \end{cases} \tag{6}$$

and

$$\rho^N(u,v,b) = \begin{cases} 0, & u = v \vee f(u,b) = ?; \\ |f(u,b) - f(v,b)|, & f(u,b), f(v,b) \in V_b^\dagger; \\ 0, & n_b^l/|H_b| \geq \kappa_2 \wedge ((f(u,b) = V_b^l \wedge f(v,b) = *) \vee (f(u,b) = * \wedge f(v,b) = V_b^l)); \\ 0, & (u \neq v) \wedge (\sum_{l=1}^{m_2} (n_b^l/|H_b|)^2/|H_b| \geq \kappa_2 \wedge (f(u,b) = * \wedge f(v,b) = *)); \\ 0, & f(u,b) = * \wedge f(v,b) = ?; \\ \infty, & otherwise, \end{cases} \tag{7}$$

where κ_1 and κ_2 denote two given threshold values, respectively.

Definition 2.3. *[8] Let S be an IHIS, and $Q = Q^C \cup Q^N \subseteq A$. The data-driven neighborhood relation with respect to Q can be defined as*

$$\mathcal{N}_Q^{(\delta,\kappa_1,\kappa_2)} = \{(u,v) \in U^2 | \forall q \in Q, (q \in Q^C \to \rho^C(u,v,q) = 0) \wedge (q \in Q^N \to \rho^N(u,v,q) \leq \delta_q)\}, \tag{8}$$

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where $\delta = \{\delta_q | q \in Q^N\}$.

For simplicity, we employ the symbol $\mathcal{N}_Q^{\delta}$ to replace $\mathcal{N}_Q^{(\delta,\kappa_1,\kappa_2)}$ in the following. Then, the data-driven neighborhood class of u can be expressed as $\mathcal{N}_Q^{\delta}(u) = \{v \in U | (u, v) \in \mathcal{N}_Q^{\delta}\}$. Furthermore, by Eqs. (6) and (7), the two pseudo-distance functions ρ^C and ρ^N satisfy regularity, but do not necessarily satisfy symmetry or triangle inequality. Hence, the data-driven neighborhood relation is only reflexive.

3. The generalized three-way neighborhood decision model in IHIS

In this section, we focus on exploring an effective three-way decision model to process the uncertainty decision problems in IHIS. First, a CIHIS is introduced by integrating IHIS and the interval-valued loss functions of all objects. Then, by averaging the interval-valued loss functions of all objects in the data-driven neighborhood class, a generalized three-way neighborhood decision model is presented.

Definition 3.1. *Let* $\mathbb{S} = (S, \bar{\lambda}, g)$ *be a CIHIS, where* $S = (U, AT = A \cup D, V, f)$ *is an IHIS;* $\bar{\lambda}$ *is the set concluding the interval-valued loss functions of all objects, i.e.,* $\bar{\lambda} = \bigcup_{u \in U}(\bar{\lambda}_{PP}(u), \bar{\lambda}_{BP}(u), \bar{\lambda}_{NP}(u), \bar{\lambda}_{NN}(u), \bar{\lambda}_{BN}(u), \bar{\lambda}_{PN}(u))$; $g : U \to \bar{\lambda}$ *denotes a mapping function such that* $g(u) = (\bar{\lambda}_{PP}(u), \bar{\lambda}_{BP}(u), \bar{\lambda}_{NP}(u), \bar{\lambda}_{NN}(u), \bar{\lambda}_{BN}(u), \bar{\lambda}_{PN}(u))$ *for any* $u \in U$.

It is noteworthy that $\bar{\lambda}_{\star\diamond}(u) = [\lambda^-_{\star\diamond}(u), \lambda^+_{\star\diamond}(u)]$ ($\star = P, B, N; \diamond = P, N$), where $\lambda^-_{\star\diamond}(u)$ and $\lambda^+_{\star\diamond}(u)$ denote the lower and upper bounds of $\bar{\lambda}_{\star\diamond}(u)$, respectively. Assume that $\bar{\lambda}_1 = [\lambda_1^-, \lambda_1^+]$ and $\bar{\lambda}_2 = [\lambda_2^-, \lambda_2^+]$ respectively represent two interval numbers, then we have:

(i) $\bar{\lambda}_1 + \bar{\lambda}_2 = [\lambda_1^- + \lambda_2^-, \lambda_1^+ + \lambda_2^+]$;
(ii) $\bar{\lambda}_1 - \bar{\lambda}_2 = [\lambda_1^- - \lambda_2^+, \lambda_1^+ - \lambda_2^-]$;
(iii) $\bar{\lambda}_1 \times \bar{\lambda}_2 = [\min(\lambda_1^-\lambda_2^-, \lambda_1^-\lambda_2^+, \lambda_1^+\lambda_2^-, \lambda_1^+\lambda_2^+), \max(\lambda_1^-\lambda_2^-, \lambda_1^-\lambda_2^+, \lambda_1^+\lambda_2^-, \lambda_1^+\lambda_2^+)]$;
(iv) $\bar{\lambda}_1 / \bar{\lambda}_2 = [\lambda_1^-, \lambda_1^+] \times [1/\lambda_2^-, 1/\lambda_2^+]$, where $0 \notin [\lambda_2^-, \lambda_2^+]$;
(v) $k\bar{\lambda}_1 = [k\lambda_1^-, k\lambda_1^+]$, where $k \in \mathbf{R}$ and $k \geq 0$.

Furthermore, taking the reasonable semantic explanations of the loss functions into account, we can get the following conditions:

$$\begin{aligned} &\lambda^-_{PP}(u) \leq \lambda^-_{BP}(u) < \lambda^-_{NP}(u),\ \lambda^+_{PP}(u) \leq \lambda^+_{BP}(u) < \lambda^+_{NP}(u); \\ &\lambda^-_{NN}(u) \leq \lambda^-_{BN}(u) < \lambda^-_{PN}(u),\ \lambda^+_{NN}(u) \leq \lambda^+_{BN}(u) < \lambda^+_{PN}(u). \end{aligned} \tag{9}$$

Given a CIHIS $\mathbb{S}$ in which $Q \subseteq A$. Considering that the objects in a data-driven neighborhood class $\mathcal{N}_Q^{\delta}(u)$ may have different interval-valued loss functions, it is significant to investigate available approaches to aggregate the interval-valued loss functions of all objects in $\mathcal{N}_Q^{\delta}(u)$. Liu and Liang [21] introduced two strategies to integrate the interval-valued loss functions of all objects belonging to an indiscernibility class, which are the optimistic aggregation strategy and the pessimistic aggregation strategy, respectively. The optimistic and pessimistic aggregation strategies respectively employ the union and intersection operations to integrate these losses to generate the aggregated loss functions. Obviously, both two strategies have their own drawbacks, that is, the length of the aggregated loss function is too broad and too narrow, respectively. To tackle these issues, we utilize the average value of the interval-valued loss functions to represent the aggregated loss function, which can use the the characteristics of the interval-valued loss functions of objects in $\mathcal{N}_Q^{\delta}(u)$ and make the length of the aggregated loss function neither too broad nor too narrow. Then, the expected losses of taking actions $\mathcal{A}_P$, $\mathcal{A}_B$ and $\mathcal{A}_N$ for object u can be represented respectively as

$$\begin{aligned} \mathcal{L}(\mathcal{A}_P | \mathcal{N}_Q^{\delta}(u)) &= \frac{\sum_{v \in \mathcal{N}_Q^{\delta}(u)} \bar{\lambda}_{PP}(v)}{|\mathcal{N}_Q^{\delta}(u)|} Pr(X | \mathcal{N}_Q^{\delta}(u)) + \frac{\sum_{v \in \mathcal{N}_Q^{\delta}(u)} \bar{\lambda}_{PN}(v)}{|\mathcal{N}_Q^{\delta}(u)|} Pr(X' | \mathcal{N}_Q^{\delta}(u)) \\ &= \frac{\sum_{v \in \mathcal{N}_Q^{\delta}(u)} [\lambda^-_{PP}(v), \lambda^+_{PP}(v)]}{|\mathcal{N}_Q^{\delta}(u)|} Pr(X | \mathcal{N}_Q^{\delta}(u)) + \frac{\sum_{v \in \mathcal{N}_Q^{\delta}(u)} [\lambda^-_{PN}(v), \lambda^+_{PN}(v)]}{|\mathcal{N}_Q^{\delta}(u)|} Pr(X' | \mathcal{N}_Q^{\delta}(u)); \end{aligned} \tag{10}$$

$$\begin{aligned}\mathcal{L}(\mathcal{A}_B|\mathcal{N}_Q^\delta(u)) &= \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}\bar{\lambda}_{BP}(v)}{|\mathcal{N}_Q^\delta(u)|}Pr(X|\mathcal{N}_Q^\delta(u)) + \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}\bar{\lambda}_{BN}(v)}{|\mathcal{N}_Q^\delta(u)|}Pr(X'|\mathcal{N}_Q^\delta(u)) \\ &= \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}[\lambda_{BP}^-(v),\lambda_{BP}^+(v)]}{|\mathcal{N}_Q^\delta(u)|}Pr(X|\mathcal{N}_Q^\delta(u)) + \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}[\lambda_{BN}^-(v),\lambda_{BN}^+(v)]}{|\mathcal{N}_Q^\delta(u)|}Pr(X'|\mathcal{N}_Q^\delta(u));\end{aligned} \tag{11}$$

$$\begin{aligned}\mathcal{L}(\mathcal{A}_N|\mathcal{N}_Q^\delta(u)) &= \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}\bar{\lambda}_{NP}(v)}{|\mathcal{N}_Q^\delta(u)|}Pr(X|\mathcal{N}_Q^\delta(u)) + \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}\bar{\lambda}_{NN}(v)}{|\mathcal{N}_Q^\delta(u)|}Pr(X'|\mathcal{N}_Q^\delta(u)) \\ &= \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}[\lambda_{NP}^-(v),\lambda_{NP}^+(v)]}{|\mathcal{N}_Q^\delta(u)|}Pr(X|\mathcal{N}_Q^\delta(u)) + \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}[\lambda_{NN}^-(v),\lambda_{NN}^+(v)]}{|\mathcal{N}_Q^\delta(u)|}Pr(X'|\mathcal{N}_Q^\delta(u)).\end{aligned} \tag{12}$$

Based on the Bayesian decision procedure, we need to compare the expected losses $\mathcal{L}(\mathcal{A}_P|\mathcal{N}_Q^\delta(u))$, $\mathcal{L}(\mathcal{A}_B|\mathcal{N}_Q^\delta(u))$ and $\mathcal{L}(\mathcal{A}_N|\mathcal{N}_Q^\delta(u))$ to acquire the minimum-cost decision rules. However, since the aggregated loss function of $\mathcal{N}_Q^\delta(u)$ is also an interval number, it is hard to compare these expected losses intuitively. To tackle this matter, it is necessary to explore an effective method to rank the interval-valued losses. The θ ranking method [17], as a typical certain method for ranking interval numbers, utilizes the lower and upper bounds of interval numbers to transform interval numbers into real numbers. Considering that the θ ranking method has advantages in easy implementation and simplifying computation when ranking interval numbers, we employ it to compare the interval-valued expected losses.

Definition 3.2. *[17] Let $\bar{\lambda} = [\lambda^-, \lambda^+]$ be an interval number and $\theta \in [0, 1]$. The transformed formula of $\bar{\lambda}$ can be defined as*

$$h^\theta(\bar{\lambda}) = (1-\theta)\lambda^- + \theta\lambda^+, \tag{13}$$

where $h^\theta(\bar{\lambda})$ and θ denote a transformed outcome and the risk attitude of decision maker, respectively.

From Definition 3.2, it can easily be found that $h^\theta(\bar{\lambda}) \in \mathbb{R}$. Then, combined with Condition (9), we have

$$\begin{aligned}&h^\theta(\bar{\lambda}_{PP}(u)) \le h^\theta(\bar{\lambda}_{BP}(u)) < h^\theta(\bar{\lambda}_{NP}(u));\\ &h^\theta(\bar{\lambda}_{NN}(u)) \le h^\theta(\bar{\lambda}_{BN}(u)) < h^\theta(\bar{\lambda}_{PN}(u)).\end{aligned} \tag{14}$$

By applying the θ ranking method, the Bayesian decision procedure can deduce the following three minimum-risk decision rules:

$(P1)$ If $h^\theta(\mathcal{L}(\mathcal{A}_P|\mathcal{N}_Q^\delta(u))) \le h^\theta(\mathcal{L}(\mathcal{A}_B|\mathcal{N}_Q^\delta(u)))$ and $h^\theta(\mathcal{L}(\mathcal{A}_P|\mathcal{N}_Q^\delta(u))) \le h^\theta(\mathcal{L}(\mathcal{A}_N|\mathcal{N}_Q^\delta(u)))$, decide $u \in POS_Q(X)$;

$(B1)$ If $h^\theta(\mathcal{L}(\mathcal{A}_B|\mathcal{N}_Q^\delta(u))) \le h^\theta(\mathcal{L}(\mathcal{A}_P|\mathcal{N}_Q^\delta(u)))$ and $h^\theta(\mathcal{L}(\mathcal{A}_B|\mathcal{N}_Q^\delta(u))) \le h^\theta(\mathcal{L}(\mathcal{A}_N|\mathcal{N}_Q^\delta(u)))$, decide $u \in BND_Q(X)$;

$(N1)$ If $h^\theta(\mathcal{L}(\mathcal{A}_N|\mathcal{N}_Q^\delta(u))) \le h^\theta(\mathcal{L}(\mathcal{A}_P|\mathcal{N}_Q^\delta(u)))$ and $h^\theta(\mathcal{L}(\mathcal{A}_N|\mathcal{N}_Q^\delta(u))) \le h^\theta(\mathcal{L}(\mathcal{A}_B|\mathcal{N}_Q^\delta(u)))$, decide $u \in NEG_Q(X)$.

where

$$\begin{aligned}h^\theta(\mathcal{L}(\mathcal{A}_P|\mathcal{N}_Q^\delta(u)) &= \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}h^\theta(\bar{\lambda}_{PP}(v))}{|\mathcal{N}_Q^\delta(u)|}Pr(X|\mathcal{N}_Q^\delta(u)) + \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}h^\theta(\bar{\lambda}_{PN}(v))}{|\mathcal{N}_Q^\delta(u)|}Pr(X'|\mathcal{N}_Q^\delta(u));\\ h^\theta(\mathcal{L}(\mathcal{A}_B|\mathcal{N}_Q^\delta(u)) &= \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}h^\theta(\bar{\lambda}_{BP}(v))}{|\mathcal{N}_Q^\delta(u)|}Pr(X|\mathcal{N}_Q^\delta(u)) + \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}h^\theta(\bar{\lambda}_{BN}(v))}{|\mathcal{N}_Q^\delta(u)|}Pr(X'|\mathcal{N}_Q^\delta(u));\\ h^\theta(\mathcal{L}(\mathcal{A}_N|\mathcal{N}_Q^\delta(u)) &= \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}h^\theta(\bar{\lambda}_{NP}(v))}{|\mathcal{N}_Q^\delta(u)|}Pr(X|\mathcal{N}_Q^\delta(u)) + \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}h^\theta(\bar{\lambda}_{NN}(v))}{|\mathcal{N}_Q^\delta(u)|}Pr(X'|\mathcal{N}_Q^\delta(u)).\end{aligned} \tag{15}$$

Considering that $Pr(X|\mathcal{N}_Q^\delta(u)) + Pr(X'|\mathcal{N}_Q^\delta(u)) = 1$, the decision rules (P1)-(N1) can intuitively be represented as follows.

$(P1')$ If $Pr(X|\mathcal{N}_Q^\delta(u)) \geq \bar{\alpha}^u$ and $Pr(X|\mathcal{N}_Q^\delta(u)) \geq \bar{\gamma}^u$, decide $u \in POS_Q(X)$;

$(B1')$ If $Pr(X|\mathcal{N}_Q^\delta(u)) \leq \bar{\alpha}^u$ and $Pr(X|\mathcal{N}_Q^\delta(u)) \geq \bar{\beta}^u$, decide $u \in BND_Q(X)$;

$(N1')$ If $Pr(X|\mathcal{N}_Q^\delta(u)) \leq \bar{\beta}^u$ and $Pr(X|\mathcal{N}_Q^\delta(u)) \leq \bar{\gamma}^u$, decide $u \in NEG_Q(X)$.

where

$$\begin{aligned}
\bar{\alpha}^u &= \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}(h^\theta(\bar{\lambda}_{PN}(v)) - h^\theta(\bar{\lambda}_{BN}(v)))}{\sum_{v\in\mathcal{N}_Q^\delta(u)}(h^\theta(\bar{\lambda}_{PN}(v)) - h^\theta(\bar{\lambda}_{BN}(v)) + h^\theta(\bar{\lambda}_{BP}(v)) - h^\theta(\bar{\lambda}_{PP}(v)))};\\
\bar{\beta}^u &= \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}(h^\theta(\bar{\lambda}_{BN}(v)) - h^\theta(\bar{\lambda}_{NN}(v)))}{\sum_{v\in\mathcal{N}_Q^\delta(u)}(h^\theta(\bar{\lambda}_{BN}(v)) - h^\theta(\bar{\lambda}_{NN}(v)) + h^\theta(\bar{\lambda}_{NP}(v)) - h^\theta(\bar{\lambda}_{BP}(v)))};\\
\bar{\gamma}^u &= \frac{\sum_{v\in\mathcal{N}_Q^\delta(u)}(h^\theta(\bar{\lambda}_{PN}(v)) - h^\theta(\bar{\lambda}_{NN}(v)))}{\sum_{v\in\mathcal{N}_Q^\delta(u)}(h^\theta(\bar{\lambda}_{PN}(v)) - h^\theta(\bar{\lambda}_{NN}(v)) + h^\theta(\bar{\lambda}_{NP}(v)) - h^\theta(\bar{\lambda}_{PP}(v)))}.
\end{aligned} \tag{16}$$

Apparently, it follows from Condition (14) that $0 < \bar{\alpha}^u \leq 1$, $0 \leq \bar{\beta}^u < 1$ and $0 < \bar{\gamma}^u < 1$ for any $u \in U$. Besides, to make rule (B1′) meaningful, it is reasonable to assume that $\bar{\beta}^u \leq \bar{\alpha}^u$. By a series of uncomplicated calculations, it is easy to yield that $0 \leq \bar{\beta}^u < \bar{\gamma}^u < \bar{\alpha}^u \leq 1$. Then, the decision rules (P1′)-(N1′) are simplified respectively as follows.

$(P1'')$ If $Pr(X|\mathcal{N}_Q^\delta(u)) \geq \bar{\alpha}^u$, decide $u \in POS_Q(X)$;

$(B1'')$ If $\bar{\beta}^u < Pr(X|\mathcal{N}_Q^\delta(u)) < \bar{\alpha}^u$, decide $u \in BND_Q(X)$;

$(N1'')$ If $Pr(X|\mathcal{N}_Q^\delta(u)) \leq \bar{\beta}^u$, decide $u \in NEG_Q(X)$.

It can obviously be found that the minimum-cost decision rules are only related to parameters $\bar{\alpha}$ and $\bar{\beta}$, where $\bar{\alpha} = \{\bar{\alpha}^u|\forall u \in U\}$ and $\bar{\beta} = \{\bar{\beta}^u|\forall u \in U\}$. On this basis, the positive, boundary and negative regions in the generalized three-way neighborhood decision model can be expressed respectively as follows.

$$\begin{aligned}
&POS_Q^{(\bar{\alpha},\bar{\beta})}(X) = \{u|Pr(X|\mathcal{N}_Q^\delta(u)) \geq \bar{\alpha}^u, \forall u \in U\};\\
&BND_Q^{(\bar{\alpha},\bar{\beta})}(X) = \{u|\bar{\beta}^u < Pr(X|\mathcal{N}_Q^\delta(u)) < \bar{\alpha}^u, \forall u \in U\};\\
&NEG_Q^{(\bar{\alpha},\bar{\beta})}(X) = \{u|Pr(X|\mathcal{N}_Q^\delta(u)) \leq \bar{\beta}^u, \forall u \in U\}.
\end{aligned} \tag{17}$$

Similarly, with respect to the multi-category case, we also construct the positive, boundary and negative regions in the generalized three-way neighborhood decision model.

Definition 3.3. *Given a CIHIS* $\mathbb{S}$ *and* $Q \subseteq A$. *If* $U/D = \{D_1, D_2, \cdots, D_m\}$, *the three-way regions of* D *with regard to* N_Q^δ *are defined by*

$$\begin{aligned}
&POS_Q^{(\bar{\alpha},\bar{\beta})}(D) = \bigcup_{j=1}^{m} POS_Q^{(\bar{\alpha},\bar{\beta})}(D_j) = \bigcup_{j=1}^{m}\{u|Pr(D_j|\mathcal{N}_Q^\delta(u)) \geq \bar{\alpha}^u, \forall u \in U\};\\
&BND_Q^{(\bar{\alpha},\bar{\beta})}(D) = \bigcup_{j=1}^{m} BND_Q^{(\bar{\alpha},\bar{\beta})}(D_j) = \bigcup_{j=1}^{m}\{u|\bar{\beta}^u < Pr(D_j|\mathcal{N}_Q^\delta(u)) < \bar{\alpha}^u, \forall u \in U\};\\
&NEG_Q^{(\bar{\alpha},\bar{\beta})}(D) = U - (POS_Q^{(\bar{\alpha},\bar{\beta})}(D) \cup BND_Q^{(\bar{\alpha},\bar{\beta})}(D)) = \bigcap_{j=1}^{m}\{u|Pr(D_j|\mathcal{N}_Q^\delta(u)) \leq \bar{\beta}^u, \forall u \in U\}.
\end{aligned} \tag{18}$$

To visually view the process of the above discussions, an example about the neonatal disease screening is employed to explain the generalized three-way neighborhood decision model. As is known to all, the neonatal disease screening is the population census of some serious congenital and genetic diseases at the birth of newborns, which includes the neonatal hearing screening, the neonatal fundus screening and the genetic metabolic diseases screening, etc. Its purpose is early diagnosis and treatment to avoid irreversible physical and intellectual development disorders.

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Table 3: A CIHIS about the neonatal disease screening

U	a_1	a_2	a_3	a_4	d	$\bar{\lambda}_{PP}$	$\bar{\lambda}_{BP}$	$\bar{\lambda}_{NP}$	$\bar{\lambda}_{NN}$	$\bar{\lambda}_{BN}$	$\bar{\lambda}_{PN}$
u_1	F	F	3.2	0.2	1	[0,1ℏ]	[1ℏ,2ℏ]	[4ℏ,6ℏ]	[1ℏ,1.5ℏ]	[2ℏ,4ℏ]	[5ℏ,8ℏ]
u_2	*	P	8.3	0.7	1	[0.5ℏ,1.5ℏ]	[2ℏ,4ℏ]	[5ℏ,7ℏ]	[1ℏ,2ℏ]	[2.5ℏ,4ℏ]	[4.5ℏ,6.5ℏ]
u_3	*	?	2.9	0.3	0	[1.5ℏ,3ℏ]	[3ℏ,5ℏ]	[6ℏ,8ℏ]	[1.5ℏ,2.5ℏ]	[3ℏ,4ℏ]	[4.5ℏ,7ℏ]
u_4	P	P	2.7	0.4	1	[0,1.5ℏ]	[1.5ℏ,3ℏ]	[3.5ℏ,5ℏ]	[0ℏ,2ℏ]	[2.5ℏ,4.5ℏ]	[5ℏ,7.5ℏ]
u_5	P	P	1.9	?	0	[1ℏ,2.5ℏ]	[3.5ℏ,5ℏ]	[5.5ℏ,7.5ℏ]	[0.5ℏ,2.5ℏ]	[2.5ℏ,4.5ℏ]	[5ℏ,7ℏ]
u_6	F	F	3.1	*	0	[1ℏ,2ℏ]	[2.5ℏ,4ℏ]	[4.5ℏ,6.5ℏ]	[1ℏ,3ℏ]	[3.5ℏ,5.5ℏ]	[6ℏ,8ℏ]

Note: ℏ is an unit cost determined by the individual administration.

Table 4: The results of the thresholds $\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$

U	$h^{0.5}(\bar{\lambda}_{PP})$	$h^{0.5}(\bar{\lambda}_{BP})$	$h^{0.5}(\bar{\lambda}_{NP})$	$h^{0.5}(\bar{\lambda}_{NN})$	$h^{0.5}(\bar{\lambda}_{BN})$	$h^{0.5}(\bar{\lambda}_{PN})$	$\bar{\alpha}$	$\bar{\beta}$	$\bar{\gamma}$
u_1	0.50	1.50	5.00	1.25	3.00	6.50	0.6857	0.4250	0.5467
u_2	1.00	3.00	6.00	1.50	3.25	5.50	0.5294	0.3684	0.4444
u_3	2.25	4.00	7.00	2.00	3.50	5.75	0.6471	0.4342	0.5347
u_4	0.75	2.25	4.25	1.00	3.50	6.25	0.6471	0.5556	0.6000
u_5	1.75	4.25	6.50	1.50	3.50	6.00	0.5000	0.4706	0.4865
u_6	1.50	3.25	5.50	2.00	4.50	7.00	0.6857	0.4250	0.5467

Example 3.1. *Table 3 shows a CIHIS about the neonatal disease screening $\mathbb{S} = (S, \bar{\lambda}, g)$, where $S = (U, AT = A \cup D, V, f)$. In Table 3, $U = \{u_i | 1 \leq i \leq 6\}$ denotes 6 different newborns; $A = A^C \cup A^N = \{a_1, a_2\} \cup \{a_3, a_4\} = \{a_1, a_2, a_3, a_4\}$ denotes the 4 different disease screening programs, where a_1, a_2, a_3 and a_4 represent the hearing screening, the fundus screening, Congenital Hypothyroidism (CH) and Phenylketonuria (PKU), respectively; $D = \{d\}$ denotes the decision attribute set. With respect to attributes a_1 and a_2, the symbols "P" and "F" indicate that the newborn passed and failed the primary hearing (fundus) screening, respectively. With respect to attributes a_3 and a_4, the values represent the concentrations of Thyroid Stimulating Hormone (in mIU/L) and Phenylalanine (in mg/dL) in neonatal plantar bloods, respectively. With respect to the decision attribute D, the values* 0 *and* 1 *represent "normal" and "abnormal", respectively. In addition, "*" and "?" represent two kinds of unknown values in the screening process, respectively.*

Let $\kappa_1 = \kappa_2 = 0.5$ and $\delta = \{\delta_{a_3}, \delta_{a_4}\} = \{0.30, 0.20\}$. According to Definition 2.3, we have: $\mathcal{N}_A^\delta(u_1) = \{u_1, u_6\}$, $\mathcal{N}_A^\delta(u_2) = \{u_2\}$, $\mathcal{N}_A^\delta(u_3) = \{u_1, u_3, u_4, u_6\}$, $\mathcal{N}_A^\delta(u_4) = \{u_4\}$, $\mathcal{N}_A^\delta(u_5) = \{u_5\}$, $\mathcal{N}_A^\delta(u_6) = \{u_1, u_6\}$.

For the decision attribute set $D = \{d\}$, we have $U/D = \{D_1, D_2\}$, where $D_1 = \{u_1, u_2, u_4\}$ and $D_2 = \{u_3, u_5, u_6\}$. Then, the conditional probabilities of all objects can be computed as: $Pr(D_1|\mathcal{N}_A^\delta(u_1)) = 0.5$, $Pr(D_1|\mathcal{N}_A^\delta(u_2)) = 1$, $Pr(D_1|\mathcal{N}_A^\delta(u_3)) = 0.5$, $Pr(D_1|\mathcal{N}_A^\delta(u_4)) = 1$, $Pr(D_1|\mathcal{N}_A^\delta(u_5)) = 0$, $Pr(D_1|\mathcal{N}_A^\delta(u_6)) = 0.5$; $Pr(D_2|\mathcal{N}_A^\delta(u_1)) = 0.5$, $Pr(D_2|\mathcal{N}_A^\delta(u_2)) = 0$, $Pr(D_2|\mathcal{N}_A^\delta(u_3)) = 0.5$, $Pr(D_2|\mathcal{N}_A^\delta(u_4)) = 0$, $Pr(D_2|\mathcal{N}_A^\delta(u_5)) = 1$, $Pr(D_2|\mathcal{N}_A^\delta(u_6)) = 0.5$.

Moreover, suppose that $\theta = 0.5$, the interval-valued loss functions can be transformed into real numbers by Definition 3.2. On this basis, the thresholds $\bar{\alpha}^u$, $\bar{\beta}^u$ and $\bar{\gamma}^u$ with regard to object u can be obtained according to Eq. (16). The detailed results are listed in Table 4.

Then, by Eq. (17), the three-way regions of D_1 and D_2 are calculated as: $POS_A^{(\bar{\alpha},\bar{\beta})}(D_1) = \{u_2, u_4\}$, $BND_A^{(\bar{\alpha},\bar{\beta})}(D_1) = \{u_1, u_3, u_6\}$, $NEG_A^{(\bar{\alpha},\bar{\beta})}(D_1) = \{u_5\}$; $POS_A^{(\bar{\alpha},\bar{\beta})}(D_2) = \{u_5\}$, $BND_A^{(\bar{\alpha},\bar{\beta})}(D_2) = \{u_1, u_3, u_6\}$, $NEG_A^{(\bar{\alpha},\bar{\beta})}(D_2) = \{u_2, u_4\}$.

Finally, it follows from Definition 3.3 that $POS_A^{(\bar{\alpha},\bar{\beta})}(D) = \{u_2, u_4, u_5\}$, $BND_A^{(\bar{\alpha},\bar{\beta})}(D) = \{u_1, u_3, u_6\}$, $NEG_A^{(\bar{\alpha},\bar{\beta})}(D) = \emptyset$.

4. Matrix approach to the generalized three-way neighborhood decision model

This section focuses on studying the generalized three-way neighborhood decision model from the view of matrix. Firstly, the matrix forms of related concepts in the generalized three-way neighborhood decision model are introduced.

Then, with the help of several effective matrix operators, the matrix-based representation of the three-way regions in the generalized three-way neighborhood decision model is presented.

Definition 4.1. *Let $\mathbb{S}$ be a CIHIS, where $U = \{u_1, u_2, \cdots, u_n\}$ and $X \subseteq U$. The characteristic vector of X can be described by an $n \times 1$ column vector, defined as*

$$G(X) = [g_1, g_2, \cdots, g_n]^T, \text{ where } g_i = \begin{cases} 1, & u_i \in X, 1 \le i \le n; \\ 0, & u_i \notin X, 1 \le i \le n, \end{cases} \tag{19}$$

and "T" is the transpose operation.

Definition 4.2. *Let $\mathbb{S}$ be a CIHIS, and $U/D = \{D_1, D_2, \cdots, D_m\}$. The decision matrix is defined as*

$$\mathbb{D} = [d_{ij}]_{n \times m}, \text{ where } d_{ij} = \begin{cases} 1, & u_i \in D_j, 1 \le i \le n \wedge 1 \le j \le m; \\ 0, & u_i \notin D_j, 1 \le i \le n \wedge 1 \le j \le m. \end{cases} \tag{20}$$

From Definition 4.2, it is apparent that $\mathbb{D} = [G(D_1), G(D_2), \cdots, G(D_m)]$. In addition, because the binary relation under the decision attribute set D is an equivalence relation, the summation of the elements in every row in $\mathbb{D}$ is identically equal to 1, i.e., $\sum_{j=1}^{m} d_{ij} = 1$.

As is known to all, the traditional relation matrix is employed to describe the binary relations between objects. which plays a crucial role in matrix-based method of computing the related knowledge in rough set theory [5, 10]. Unfortunately, the traditional relation matrix is unfeasible to update the three-way regions in our proposed model under dynamic data environments. Therefore, we defined a new relation matrix in our previous work [7], which can be viewed as the generalized form of the tradition relation matrix. For the discussion, we call it as the double-value relation matrix.

Definition 4.3. *Let $\mathbb{S}$ be a CIHIS, where $U = \{u_1, u_2, \cdots, u_n\}$ and $a \in A$. The double-value relation matrix with respect to a is denoted as $\mathbb{M}^a = [m_{ij}^a]_{n \times n}$, where*

(1) when $(u_i \neq u_j) \wedge [(f(u_i, a) = * \wedge f(u_j, a) \neq ?) \vee (f(u_i, a) \neq ? \wedge f(u_j, a) = *)]$, then

$$m_{ij}^a = \begin{cases} (1, 1), & u_j \in \mathcal{N}_a^{\delta}(u_i); \\ (0, 1), & otherwise. \end{cases} \tag{21}$$

(2) when $(u_i = u_j) \vee (f(u_i, a) = ?) \vee (f(u_j, a) = ?) \vee (f(u_i, a), f(u_j, a) \in V_a^{\dagger})$, then

$$m_{ij}^a = \begin{cases} (1, 0), & u_j \in \mathcal{N}_a^{\delta}(u_i); \\ (0, 0), & otherwise. \end{cases} \tag{22}$$

Considering that the elements in the double-value relation matrix are a pair of values consisting of 0 and 1, i.e., $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$, the multiplication operation rule between a pair of values is different from that between single values. Let $m_{ij}^a = (x^a, y^a)$ and $m_{ij}^b = (x^b, y^b)$ be two elements in the double-value relation matrices $\mathbb{M}^a$ and $\mathbb{M}^b$, respectively. The multiplication operation "$\times$" between m_{ij}^a and m_{ij}^b can be denoted as follows.

$$\text{If } x^a = x^b = 1, \text{ then } m_{ij}^a \times m_{ij}^b = (x^a \wedge x^b, y^a \vee y^b). \tag{23}$$

$$\text{If } x^a = x^b = 0, \text{ then } m_{ij}^a \times m_{ij}^b = (x^a \wedge x^b, y^a \wedge y^b). \tag{24}$$

$$\text{If } x^a = 1 \text{ and } x^b = 0, \text{ then } m_{ij}^a \times m_{ij}^b = \begin{cases} (x^a \wedge x^b, y^a \vee y^b), & y^b = 1; \\ (x^a \wedge x^b, y^a \wedge y^b), & y^b = 0. \end{cases} \tag{25}$$

$$\text{If } x^a = 0 \text{ and } x^b = 1, \text{ then } m_{ij}^a \times m_{ij}^b = \begin{cases} (x^a \wedge x^b, y^a \vee y^b), & y^a = 1; \\ (x^a \wedge x^b, y^a \wedge y^b), & y^a = 0. \end{cases} \tag{26}$$

where "$\vee$" and "$\wedge$" are the disjunction and conjunction operations, respectively. On this basis, it can easily be calculated the double-value relation matrix with regard to an attribute set.

Definition 4.4. *Let $\mathbb{S}$ be a CIHIS and $a,\ b \in Q$. Let $\mathbb{M}^a = [m^a_{ij}]_{n\times l}$ and $\mathbb{M}^b = [m^b_{ij}]_{n\times l}$ be the double-value relation matrices. The Hadamard product "$\otimes$" of $\mathbb{M}^a$ and $\mathbb{M}^b$ is defined as $\mathbb{M}^a \otimes \mathbb{M}^b = [m^{a\cup b}_{ij}]_{n\times l}$, where $m^{a\cup b}_{ij} = m^a_{ij} \times m^b_{ij}$.*

Lemma 4.1. *Given a CIHIS $\mathbb{S}$, $Q = \{a_1, a_2, \cdots, a_{|Q|}\} \subseteq A$. Let $\mathbb{M}^{a_k} = [m^{a_k}_{ij}]_{n\times n}$ and $\mathbb{M}^Q = [m^Q_{ij}]_{n\times n}$ be the double-value relation matrices with respect to a_k $(1 \le k \le |Q|)$ and Q, respectively. Then,*

$$\mathbb{M}^Q = \mathbb{M}^{a_1} \otimes \mathbb{M}^{a_2} \cdots \otimes \mathbb{M}^{a_{|Q|}}. \tag{27}$$

For simplicity of presentation, we use symbols r^Q_{ij} and s^Q_{ij} to describe a pair of values of m^Q_{ij} in $\mathbb{M}^Q$, respectively, i.e., $m^Q_{ij} = (r^Q_{ij}, s^Q_{ij})$. Based on Definition 4.3 and Lemma 4.1, several simple results can be easily observed: (1) the diagonal elements in $\mathbb{M}^Q$ are identically equal to $(1, 0)$, i.e., $m^Q_{ii} = (1, 0)$; (2) the double-value relation matrix $\mathbb{M}^Q$ is not necessarily symmetric, i.e., $m^Q_{ij} \neq m^Q_{ji}$ $(i \neq j)$; (3) the sum of the first values of all elements in i-th row of $\mathbb{M}^Q$ is the cardinality of the data-driven neighborhood class containing u_i, i.e., $|\mathcal{N}^\delta_Q(u_i)| = \sum_{j=1}^n r^Q_{ij}$. Furthermore, the related induced matrices presented later are merely related to the value r^Q_{ij} in $\mathbb{M}^Q$. Therefore, in the following, a new function is introduced.

Definition 4.5. *Let $\mathbb{M}^Q = [m^Q_{ij}]_{n\times n}$ be a double-value relation matrix. The relation valued function on $\mathbb{M}^Q$ can be defined as*

$$F(\mathbb{M}^Q) = [F(m^Q_{ij})]_{n\times n},\ \text{where } F(m^Q_{ij}) = r^Q_{ij}. \tag{28}$$

Definition 4.6. *Let $\mathbb{S}$ be a CIHIS, $Q \subseteq A$. $\mathbb{M}^Q = [m^Q_{ij}]_{n\times n}$ denotes the double-value relation matrix under $\mathcal{N}^\delta_Q$. The induced diagonal matrix of $\mathbb{M}_Q$ is defined as*

$$\Gamma^Q = \mathrm{diag}[1/\gamma^Q_1, 1/\gamma^Q_2, \cdots, 1/\gamma^Q_n],\ \text{where } \gamma^Q_i = \sum_{j=1}^n F(m^Q_{ij}). \tag{29}$$

Definition 4.7. *Let $X = [x_{ij}]_{n\times l}$ and $Y = [y_{ij}]_{l\times p}$ denote two matrices. The inner product "$\bullet$" of matrices X and Y can be defined as $X \bullet Y = [z_{ij}]_{n\times p}$, where $z_{ij} = \sum_{k=1}^l x_{ik}y_{kj}$.*

Definition 4.8. *Let $\mathbb{S}$ be a CIHIS, $Q \subseteq A$. $\mathbb{D} = [d_{ik}]_{n\times m}$, $\mathbb{M}^Q = [m^Q_{ij}]_{n\times n}$ and $\Gamma^Q = \mathrm{diag}[1/\gamma^Q_1, 1/\gamma^Q_2, \cdots, 1/\gamma^Q_n]$ are the decision matrix, the double-value relation matrix and the induced diagonal matrix, respectively. The condition matrix in regard to Q is defined as*

$$\Phi^Q = \Gamma^Q \bullet (F(\mathbb{M}^Q) \bullet \mathbb{D}) = \Gamma^Q \bullet \mathbb{X}^Q = [\phi^Q_{ij}]_{n\times m}, \tag{30}$$

where $\phi^Q_{ij} = (\sum_{k=1}^n F(m^Q_{ik})d_{kj})/\gamma^Q_i$ and $\mathbb{X}^Q = F(\mathbb{M}^Q) \bullet \mathbb{D}$, named as the intersection matrix.

Apparently, Definition 4.8 provides an effective matrix approach to calculate the conditional probability, and the element ϕ^Q_{ij} in Φ^Q is equal to $Pr(D_j|\mathcal{N}^\delta_Q(u_i))$, i.e., $\phi^Q_{ij} = |\mathcal{N}^\delta_Q(u_i) \cap D_j|/|\mathcal{N}^\delta_Q(u_i)|$. In what follows, we concentrate on exploring a matrix-based method for computing the threshold parameters of all objects, i.e., $\bar{\alpha}$ and $\bar{\beta}$.

Definition 4.9. *Let $\mathbb{S}$ be a CIHIS, $U = \{u_1, u_2, \cdots, u_n\}$. The loss matrix is denoted by an $n \times 6$ matrix, defined as*

$$\mathbb{L}^\theta = [\lambda_{ij}]_{n\times 6},\ \text{where } \lambda_{ij} = \begin{cases} h^\theta(\bar{\lambda}_{PP}(u_i)), & j = 1; \\ h^\theta(\bar{\lambda}_{BP}(u_i)), & j = 2; \\ h^\theta(\bar{\lambda}_{NP}(u_i)), & j = 3; \\ h^\theta(\bar{\lambda}_{NN}(u_i)), & j = 4; \\ h^\theta(\bar{\lambda}_{BN}(u_i)), & j = 5; \\ h^\theta(\bar{\lambda}_{PN}(u_i)), & j = 6, \end{cases} \tag{31}$$

where h^θ denotes a transformed function, introduced by Definition 3.2.

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Definition 4.10. *Let $X = [x_{ij}]_{n\times l}$ and $Y = [y_{ij}]_{n\times l}$ denote two matrices. The dot divide "$\oslash$" of matrices X and Y can be defined as $X \oslash Y = [w_{ij}]_{n\times l}$, where $w_{ij} = \frac{x_{ij}}{y_{ij}}$.*

Definition 4.11. *Let $\mathbb{S}$ be a CIHIS, $Q \subseteq A$. $\mathbb{M}^Q$ and $\mathbb{L}^\theta$ are the double-value relation matrix with regard to $\mathcal{N}_Q^\delta$ and the loss matrix, respectively. $\mathbb{T}^\alpha = [\iota_1, \iota_2, \cdots, \iota_n]^T$ and $\mathbb{T}^\beta = [o_1, o_2, \cdots, o_n]^T$ are called the α-threshold and β-threshold vectors, defined respectively as follows.*

$$\begin{aligned}
&\mathbb{T}^\alpha = (F(\mathbb{M}^Q) \bullet (\mathbb{L}^\theta(\cdot,6) - \mathbb{L}^\theta(\cdot,5))) \oslash (F(\mathbb{M}^Q) \bullet (\mathbb{L}^\theta(\cdot,2) - \mathbb{L}^\theta(\cdot,1) + \mathbb{L}^\theta(\cdot,6) - \mathbb{L}^\theta(\cdot,5)));\\
&\mathbb{T}^\beta = (F(\mathbb{M}^Q) \bullet (\mathbb{L}^\theta(\cdot,5) - \mathbb{L}^\theta(\cdot,4))) \oslash (F(\mathbb{M}^Q) \bullet (\mathbb{L}^\theta(\cdot,3) - \mathbb{L}^\theta(\cdot,2) + \mathbb{L}^\theta(\cdot,5) - \mathbb{L}^\theta(\cdot,4))),
\end{aligned} \tag{32}$$

where $\mathbb{L}^\theta(\cdot, i)$ $(1 \le i \le 6)$ denotes the i-th column in $\mathbb{L}^\theta$.

From Definition 4.11, it can easily be calculated a pair of thresholds $\bar{\alpha}$ and $\bar{\beta}$ from the perspective of matrix, and the elements ι_i and o_i in $\mathbb{T}^\alpha$ and $\mathbb{T}^\beta$ are equal to $\bar{\alpha}^{u_i}$ and $\bar{\beta}^{u_i}$ in $\bar{\alpha}$ and $\bar{\beta}$, respectively. Moreover, for simplicity of presentation, suppose that $\mathbb{H}^{(1,2)} = F(\mathbb{M}^Q) \bullet (\mathbb{L}^\theta(\cdot,2) - \mathbb{L}^\theta(\cdot,1))$, $\mathbb{H}^{(2,3)} = F(\mathbb{M}^Q) \bullet (\mathbb{L}^\theta(\cdot,3) - \mathbb{L}^\theta(\cdot,2))$, $\mathbb{H}^{(4,5)} = F(\mathbb{M}^Q) \bullet (\mathbb{L}^\theta(\cdot,5) - \mathbb{L}^\theta(\cdot,4))$ and $\mathbb{H}^{(5,6)} = F(\mathbb{M}^Q) \bullet (\mathbb{L}^\theta(\cdot,6) - \mathbb{L}^\theta(\cdot,5))$. Then, Eq. (32) can be represented as follows.

$$\begin{aligned}
&\mathbb{T}^\alpha = \mathbb{H}^{(5,6)} \oslash (\mathbb{H}^{(1,2)} + \mathbb{H}^{(5,6)});\\
&\mathbb{T}^\beta = \mathbb{H}^{(4,5)} \oslash (\mathbb{H}^{(2,3)} + \mathbb{H}^{(4,5)}).
\end{aligned} \tag{33}$$

Definition 4.12. *Let $X = [x_i]_{n\times 1}$ and $Y = [y_i]_{n\times 1}$ be two column vectors, $B = [b_{ij}]_{n\times m}$ be a matrix. The operations "$\trianglerighteq$" and "$\triangleright$" can be defined respectively as follows.*

(1) $X \trianglerighteq B \trianglerighteq Y = [\upsilon^1_{ij}]_{n\times m}$, *where* $\upsilon^1_{ij} = \begin{cases} 1, & x_i \le b_{ij} \le y_i;\\ 0, & otherwise.\end{cases}$

(2) $X \triangleright B \triangleright Y = [\upsilon^2_{ij}]_{n\times m}$, *where* $\upsilon^2_{ij} = \begin{cases} 1, & x_i < b_{ij} < y_i;\\ 0, & otherwise.\end{cases}$

Definition 4.13. *Let $\mathbb{S}$ be a CIHIS, $Q \subseteq A$. $\Phi^Q = [\phi^Q_{ij}]_{n\times m}$ be the condition matrix with regard of Q, $\mathbb{T}^\alpha = [\iota_i]_{n\times 1}$ and $\mathbb{T}^\beta = [o_i]_{n\times 1}$ be the α-threshold and β-threshold vectors, respectively. Three cut matrices of Φ^Q can be defined as follows.*

$$\begin{aligned}
&\check{\Psi}^Q = \mathbb{T}^\alpha \trianglerighteq \Phi^Q \trianglerighteq \boldsymbol{1} = [\check{\psi}^Q_{ij}]_{n\times m};\\
&\bar{\Psi}^Q = \mathbb{T}^\beta \triangleright \Phi^Q \triangleright \mathbb{T}^\alpha = [\bar{\psi}^Q_{ij}]_{n\times m};\\
&\hat{\Psi}^Q = \boldsymbol{0} \trianglerighteq \Phi^Q \trianglerighteq \mathbb{T}^\beta = [\hat{\psi}^Q_{ij}]_{n\times m}.
\end{aligned} \tag{34}$$

*where **1** denotes the column vector that all elements are 1, and **0** denotes the column vector that all elements are 0.*

Theorem 4.1. *Given a CIHIS $\mathbb{S}$, $Q \subseteq A$. Let $\check{\Psi}^Q = [\check{\psi}^Q_{ij}]_{n\times m}$, $\bar{\Psi}^Q = [\bar{\psi}^Q_{ij}]_{n\times m}$ and $\hat{\Psi}^Q = [\hat{\psi}^Q_{ij}]_{n\times m}$ be three cut matrices in regard of Q. Then, the characteristic vectors of three-way regions with respect to D, i.e., $G(POS_Q^{(\bar{\alpha},\bar{\beta})})$, $G(BND_Q^{(\bar{\alpha},\bar{\beta})})$ and $G(NEG_Q^{(\bar{\alpha},\bar{\beta})})$ can be calculated respectively as follows.*

$$\begin{aligned}
&G(POS_Q^{(\bar{\alpha},\bar{\beta})}(D)) = [\mu_i]_{n\times 1},\ where\ \mu_i = \begin{cases} 1, & \bigvee_{j=1}^m \check{\psi}^Q_{ij} = 1;\\ 0, & otherwise.\end{cases}\\
&G(BND_Q^{(\bar{\alpha},\bar{\beta})}(D)) = [\nu_i]_{n\times 1},\ where\ \nu_i = \begin{cases} 1, & \bigvee_{j=1}^m \bar{\psi}^Q_{ij} = 1;\\ 0, & otherwise.\end{cases}\\
&G(NEG_Q^{(\bar{\alpha},\bar{\beta})}(D)) = [\omega_i]_{n\times 1},\ where\ \omega_i = \begin{cases} 1, & \bigwedge_{j=1}^m \hat{\psi}^Q_{ij} = 1;\\ 0, & otherwise.\end{cases}
\end{aligned} \tag{35}$$

Proof. If $\bigvee_{j=1}^{m}\check{\psi}_{ij}^{Q}=1$, there exists $j\in[1,m]$ such that $\check{\psi}_{ij}^{Q}=1$. It follows from Definitions 4.12 and 4.13 that $\iota_i\leq\phi_{ij}^{Q}\leq 1$, namely, $Pr(D_j|\mathcal{N}_Q^{\delta}(u_i))\geq\bar{\alpha}^{u_i}$. Then, according to Definition 3.3, we have $u_i\in POS_Q^{(\bar{\alpha},\bar{\beta})}(D)$, namely, $\mu_i=1$. If $\bigvee_{j=1}^{m}\check{\psi}_{ij}^{Q}=0$, then $\check{\psi}_{ij}^{Q}=0$ for any $j\in[1,m]$. From Definitions 4.12 and 4.13, we have $\phi_{ij}^{Q}<\iota_i$, namely, $Pr(D_j|\mathcal{N}_Q^{\delta}(u_i))<\bar{\alpha}^{u_i}$. Then, by Definition 3.3, we have $u_i\notin POS_Q^{(\bar{\alpha},\bar{\beta})}(D)$, namely, $\mu_i=0$. The remainder of this theorem follows in a similar manner. □

Example 4.1. *(Continued from Example 3.1) First, we compute the decision matrix. By Definition 4.2, we have*

$$\mathbb{D}=[G(D_1),G(D_2)]=\begin{bmatrix}1&1&0&1&0&0\\0&0&1&0&1&1\end{bmatrix}^T.$$

Next, we compute the double-value relation matrix under A. According to Definition 4.3, the double-value relation matrices with regard to a_1, a_2, a_3 and a_4 can be computed respectively as

$$\mathbb{M}^{a_1}=\begin{bmatrix}(1,0)&(1,1)&(1,1)&(0,0)&(0,0)&(1,0)\\(1,1)&(1,0)&(1,1)&(1,1)&(1,1)&(1,1)\\(1,1)&(1,1)&(1,0)&(1,1)&(1,1)&(1,1)\\(0,0)&(1,1)&(1,1)&(1,0)&(1,0)&(0,0)\\(0,0)&(1,1)&(1,1)&(1,0)&(1,0)&(0,0)\\(1,0)&(1,1)&(1,1)&(0,0)&(0,0)&(1,0)\end{bmatrix},\ \mathbb{M}^{a_2}=\begin{bmatrix}(1,0)&(0,0)&(0,0)&(0,0)&(0,0)&(1,0)\\(0,0)&(1,0)&(0,0)&(1,0)&(1,0)&(0,0)\\(1,0)&(1,0)&(1,0)&(1,0)&(1,0)&(1,0)\\(0,0)&(1,0)&(0,0)&(1,0)&(1,0)&(0,0)\\(0,0)&(1,0)&(0,0)&(1,0)&(1,0)&(0,0)\\(1,0)&(0,0)&(0,0)&(0,0)&(0,0)&(1,0)\end{bmatrix},$$

$$\mathbb{M}^{a_3}=\begin{bmatrix}(1,0)&(0,0)&(1,0)&(0,0)&(0,0)&(1,0)\\(0,0)&(1,0)&(0,0)&(0,0)&(0,0)&(0,0)\\(1,0)&(0,0)&(1,0)&(1,0)&(0,0)&(1,0)\\(0,0)&(0,0)&(1,0)&(1,0)&(0,0)&(0,0)\\(0,0)&(0,0)&(0,0)&(0,0)&(1,0)&(0,0)\\(1,0)&(0,0)&(0,0)&(0,0)&(0,0)&(1,0)\end{bmatrix},\ \mathbb{M}^{a_4}=\begin{bmatrix}(1,0)&(0,0)&(1,0)&(1,0)&(0,0)&(1,1)\\(0,0)&(1,0)&(0,0)&(0,0)&(0,0)&(0,1)\\(1,0)&(0,0)&(1,0)&(1,0)&(0,0)&(1,1)\\(1,0)&(0,0)&(1,0)&(1,0)&(0,0)&(1,1)\\(1,0)&(1,0)&(1,0)&(1,0)&(1,0)&(1,0)\\(1,1)&(0,1)&(1,1)&(1,1)&(1,0)&(1,0)\end{bmatrix}.$$

Based on Lemma 4.1, it is easy to obtain

$$\mathbb{M}^{A}=\mathbb{M}^{a_1}\otimes\mathbb{M}^{a_2}\otimes\mathbb{M}^{a_3}\otimes\mathbb{M}^{a_4}=\begin{bmatrix}(1,0)&(0,0)&(0,0)&(0,0)&(0,0)&(1,1)\\(0,0)&(1,0)&(0,0)&(0,0)&(0,0)&(0,0)\\(1,1)&(0,0)&(1,0)&(1,1)&(0,0)&(1,1)\\(0,0)&(0,0)&(0,0)&(1,0)&(0,0)&(0,0)\\(0,0)&(0,0)&(0,0)&(0,0)&(1,0)&(0,0)\\(1,1)&(0,0)&(0,0)&(0,0)&(0,0)&(1,0)\end{bmatrix}.$$

Then, by Definition 4.6, the induced diagonal matrix of $\mathbb{M}^A$ can be computed as $\Gamma^A=\mathrm{diag}[\frac{1}{2},1,\frac{1}{4},1,1,\frac{1}{2}]$. Hence, from Definition 4.8, the condition matrix with respect to A can be calculated as

$$\Phi^A=\Gamma^A\bullet(F(\mathbb{M}^A)\bullet\mathbb{D})=\mathrm{diag}[\frac{1}{2},1,\frac{1}{4},1,1,\frac{1}{2}]\bullet\left(\begin{bmatrix}1&0&0&0&0&1\\0&1&0&0&0&0\\1&0&1&1&0&1\\0&0&0&1&0&0\\0&0&0&0&1&0\\1&0&0&0&0&1\end{bmatrix}\bullet\begin{bmatrix}1&0\\1&0\\0&1\\1&0\\0&1\\0&1\end{bmatrix}\right)=\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\1&0\\\frac{1}{2}&\frac{1}{2}\\1&0\\0&1\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}.$$

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In the following, it follows from Definition 4.9 that the loss matrix can be computed as

$$\mathbb{L}^{0.5} = \begin{bmatrix} 0.50 & 1.50 & 5.00 & 1.25 & 3.00 & 6.50 \\ 1.00 & 3.00 & 6.00 & 1.50 & 3.25 & 5.50 \\ 2.25 & 4.00 & 7.00 & 2.00 & 3.50 & 5.75 \\ 0.75 & 2.25 & 4.25 & 1.00 & 3.50 & 6.25 \\ 1.75 & 4.25 & 6.50 & 1.50 & 3.50 & 6.00 \\ 1.50 & 3.25 & 5.50 & 2.00 & 4.50 & 7.00 \end{bmatrix}.$$

According to Definition 4.11, we have

$$\mathbb{T}^{\alpha} = (F(\mathbb{M}^{A}) \bullet (\mathbb{L}^{0.5}_{\cdot 6} - \mathbb{L}^{0.5}_{\cdot 5})) \oslash (F(\mathbb{M}^{A}) \bullet (\mathbb{L}^{0.5}_{\cdot 2} - \mathbb{L}^{0.5}_{\cdot 1} + \mathbb{L}^{0.5}_{\cdot 6} - \mathbb{L}^{0.5}_{\cdot 5}))$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 3.50 \\ 2.25 \\ 2.25 \\ 2.75 \\ 2.50 \\ 2.50 \end{bmatrix}\right) \oslash \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 4.50 \\ 4.25 \\ 4.00 \\ 4.25 \\ 5.00 \\ 4.25 \end{bmatrix}\right) = \begin{bmatrix} 6.00 \\ 2.25 \\ 11.00 \\ 2.75 \\ 2.50 \\ 6.00 \end{bmatrix} \oslash \begin{bmatrix} 8.75 \\ 4.25 \\ 17.00 \\ 4.25 \\ 5.00 \\ 8.75 \end{bmatrix} = \begin{bmatrix} 0.6857 \\ 0.5294 \\ 0.6471 \\ 0.6471 \\ 0.5000 \\ 0.6857 \end{bmatrix},$$

$$\mathbb{T}^{\beta} = (F(\mathbb{M}^{A}) \bullet (\mathbb{L}^{0.5}_{\cdot 5} - \mathbb{L}^{0.5}_{\cdot 4})) \oslash (F(\mathbb{M}^{A}) \bullet (\mathbb{L}^{0.5}_{\cdot 3} - \mathbb{L}^{0.5}_{\cdot 2} + \mathbb{L}^{0.5}_{\cdot 5} - \mathbb{L}^{0.5}_{\cdot 4}))$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 1.75 \\ 1.75 \\ 1.50 \\ 2.50 \\ 2.00 \\ 2.50 \end{bmatrix}\right) \oslash \left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \bullet \begin{bmatrix} 5.25 \\ 4.75 \\ 4.50 \\ 4.50 \\ 4.25 \\ 4.75 \end{bmatrix}\right) = \begin{bmatrix} 4.25 \\ 1.75 \\ 8.25 \\ 2.50 \\ 2.00 \\ 4.25 \end{bmatrix} \oslash \begin{bmatrix} 10.00 \\ 4.75 \\ 19.00 \\ 4.50 \\ 4.25 \\ 10.00 \end{bmatrix} = \begin{bmatrix} 0.4250 \\ 0.3684 \\ 0.4342 \\ 0.5556 \\ 0.4706 \\ 0.4250 \end{bmatrix}.$$

Then, it follows from Definitions 4.12 and 4.13 that

$$\check{\Psi}^{Q} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \bar{\Psi}^{Q} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad \hat{\Psi}^{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Combined with Theorem 4.1, we can obtain the characteristic vectors of three-way regions with respect to D.

$$G(POS_{A}^{(\bar{\alpha},\bar{\beta})}(D)) = [0, 1, 0, 1, 1, 0]^{T};$$

$$G(BND_{A}^{(\bar{\alpha},\bar{\beta})}(D)) = [1, 0, 1, 0, 0, 1]^{T};$$

$$G(NEG_{A}^{(\bar{\alpha},\bar{\beta})}(D)) = [0, 0, 0, 0, 0, 0]^{T}.$$

5. Matrix-based incremental mechanism for updating three-way regions in dynamic CIHIS

In this section, an efficient incremental framework for maintaining three-way regions of the generalized three-way neighborhood decision model based on matrix is constructed when objects and attributes in CIHIS increase with

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time simultaneously. As we have stated in the previous section, the three-way regions can be derived from three cut matrices through the construction of the decision matrix, the double-value relation matrix and the loss matrix. It is evident that the loss matrix can be intuitively built in view of the loss functions. Consequently, the important step for the maintenance of the three-way regions is to update the decision matrix and the double-value relation matrix. In what follows, we mainly focus on investigating the incremental strategies for the update of the decision matrix and the double-value relation matrix under the simultaneous variation of objects and attributes.

For simplicity of presentation, let $\mathbb{S}^t = (S^t, \bar{\lambda}^t, g^t)$ be a CIHIS at time t, where $S^t = (U^t, A^t \cup D, V^t, f^t)$ and $U^t/D = \{D_1^t, D_2^t, \cdots, D_m^t\}$. When new object set ΔU and attribute set ΔA are added to $\mathbb{S}^t$ at time $t+1$, $\mathbb{S}^t$ is updated as $\mathbb{S}^{t+1} = (S^{t+1}, \bar{\lambda}^{t+1}, g^{t+1})$, where $S^{t+1} = (U^{t+1}, A^{t+1} \cup D, V^{t+1}, f^{t+1})$, $U^{t+1} = U^t \cup \Delta U$, $A^{t+1} = A^t \cup \Delta A$ and $U^{t+1}/D = \{D_1^{t+1}, D_2^{t+1}, \cdots, D_m^{t+1}, D_{m+1}^{t+1}, \cdots, D_{m+m'}^{t+1}\}$. Suppose that $|U^t| = n$, $|\Delta U| = n'$, $|A^t| = l$, $|\Delta A| = l'$, then $|U^{t+1}| = n + n'$ and $|A^{t+1}| = l + l'$. In addition, let $[\mathcal{N}_{A^t}^{\delta}]^{U^{t+1}}$ and $[\mathcal{N}_{A^{t+1}}^{\delta}]^{U^{t+1}}$ be the data-driven neighborhood relations with regard to A^t and A^{t+1} on U^{t+1}, respectively.

Obviously, the variation of the condition attributes can not give rise to the change of the decision matrix. Thus, we merely need to discuss the updating principle of the decision matrix with the addition of new objects.

Theorem 5.1. *Let $\mathbb{S}^t$ be a CIHIS at time t, and $\mathbb{D}^t = [d_{ij}^t]_{n\times m}$ be the decision matrix at time t. When new object set ΔU and attribute set ΔA are added to $\mathbb{S}^t$, suppose that $\mathbb{D}^{t+1} = [d_{ij}^{t+1}]_{(n+n')\times(m+m')}$ denotes the decision matrix at time $t+1$. Then the decision matrix $\mathbb{D}^{t+1}$ can be updated as follows.*

$$d_{ij}^{t+1} = \begin{cases} d_{ij}^t, & 1 \le i \le n \wedge 1 \le j \le m; \\ 1, & u_i \in D_j^{t+1} \wedge n+1 \le i \le n+n'; \\ 0, & (1 \le i \le n \wedge m+1 \le j \le m+m') \vee (u_i \notin D_j^{t+1} \wedge n+1 \le i \le n+n'). \end{cases} \tag{36}$$

Proof. When adding ΔU and ΔA simultaneously, it is straightforward to yield that the decision matrix is updated as $(n+n')\times(m+m')$ matrix, i.e., $\mathbb{D}^{t+1} = [d'_{ij}]_{(n+n')\times(m+m')}$. For the ease of description, the process of proof can be divided into the following three parts.

(1) For any $1 \le i \le n$ and $1 \le j \le m$, if $u_i \in D_j^t$, i.e., $d_{ij}^t = 1$, then $u_i \in D_j^{t+1}$, i.e., $d_{ij}^{t+1} = 1$; otherwise, $u_i \notin D_j^{t+1}$, i.e., $d_{ij}^{t+1} = 0$. It follows that $d_{ij}^{t+1} = d_{ij}^t$ when $1 \le i \le n$ and $1 \le j \le m$.

(2) For any $1 \le i \le n$ and $m+1 \le j \le m+m'$, we have $\sum_{j=m+1}^{m+m'} d_{ij}^{t+1} = \sum_{j=1}^{m+m'} d_{ij}^{t+1} - \sum_{j=1}^{m} d_{ij}^{t+1}$. Considering that $\sum_{j=1}^{m+m'} d_{ij}^{t+1} = 1$ and $\sum_{j=1}^{m} d_{ij}^{t+1} = \sum_{j=1}^{m} d_{ij}^t = 1$, then $\sum_{j=m+1}^{m+m'} d_{ij}^{t+1} = 0$. So, $d_{ij}^{t+1} = 0$ when $1 \le i \le n$ and $m+1 \le j \le m+m'$.

(3) For any $n+1 \le i \le n+n'$, the elements in the updated decision matrix need to be calculated from scratch according to Definition 4.2. Thus, if $u_i \in D_j^{t+1}$, then $d_{ij}^{t+1} = 1$; otherwise, $d_{ij}^{t+1} = 0$. □

Theorem 5.2. *Let $\mathbb{S}^t$ be a CIHIS at time t, and $\mathbb{M}_{U^t}^{A^t} = [m_{ij}^{A^t}]_{n\times n} = [(r_{ij}^{A^t}, s_{ij}^{A^t})]_{n\times n}$ be the double-value relation matrix under A^t on U^t at time t. When new object set ΔU and attribute set ΔA are added to $\mathbb{S}^t$, suppose that $\mathbb{M}_{U^t}^{\Delta A} = [m_{ij}^{\Delta A}]_{n\times n}$ and $\mathbb{M}_{U^{t+1}}^{A^{t+1}} = [m_{ij}^{A^{t+1}}]_{(n+n')\times(n+n')}$ denote the double-value relation matrices in regard to ΔA and A^{t+1} on U^t and U^{t+1} at time $t+1$, respectively. Then the double-value relation matrix $\mathbb{M}_{U^{t+1}}^{A^{t+1}}$ can be updated as follows.*

(1) When $1 \le i \le n$ and $1 \le j \le n$,

- *If $s_{ij}^{A^t} = 0$, then*

$$m_{ij}^{A^{t+1}} = \begin{cases} m_{ij}^{A^t}, & r_{ij}^{A^t} = 0; \\ m_{ij}^{\Delta A}, & otherwise. \end{cases} \tag{37}$$

- *If $s_{ij}^{A^t} = 1$, then*

$$m_{ij}^{A^{t+1}} = \begin{cases} m_{ij}^{\Delta A}, & r_{ij}^{\Delta A} = 0; \\ (1,1), & r_{ij}^{\Delta A} = 1 \wedge u_j \in [\mathcal{N}_{A^t}^{\delta}]^{U^{t+1}}(u_i); \\ (0,1), & r_{ij}^{\Delta A} = 1 \wedge u_j \notin [\mathcal{N}_{A^t}^{\delta}]^{U^{t+1}}(u_i)). \end{cases} \tag{38}$$

(2) When $n+1 \leq i \leq n+n'$ or $n+1 \leq i \leq n+n'$,

$$m_{ij}^{A^{t+1}} = \begin{cases} (1,0), & s_{ij}^{A^{t+1}} = 0 \wedge u_j \in [\mathcal{N}_{A^{t+1}}^{\delta}]^{U^{t+1}}(u_i); \\ (0,0), & s_{ij}^{A^{t+1}} = 0 \wedge u_j \notin [\mathcal{N}_{A^{t+1}}^{\delta}]^{U^{t+1}}(u_i); \\ (1,1), & s_{ij}^{A^{t+1}} = 1 \wedge u_j \in [\mathcal{N}_{A^{t+1}}^{\delta}]^{U^{t+1}}(u_i); \\ (0,1), & s_{ij}^{A^{t+1}} = 1 \wedge u_j \notin [\mathcal{N}_{A^{t+1}}^{\delta}]^{U^{t+1}}(u_i). \end{cases} \tag{39}$$

Proof. When adding ΔU and ΔA simultaneously, it can obviously be obtained that the double-value relation matrix is updated as $(n+n') \times (n+n')$ matrix, i.e., $\mathbb{M}_{U^{t+1}}^{A^{t+1}} = [m_{ij}^{A^{t+1}}]_{(n+n')\times(n+n')}$. For any $1 \leq i, j \leq n$, assume that $\bar{\mathbb{M}}_{U^t}^{A^t} = [\bar{m}_{ij}^{A^t}]_{n\times n}$ denotes the updated double-value relation matrix with respect to A^t on U^t after adding objects, then we have $m_{ij}^{A^{t+1}} = \bar{m}_{ij}^{A^t} \times m_{ij}^{\Delta A}$ according to Lemma 4.1. (i) When $s_{ij}^{A^t} = 0$, it can easily be observed that the element $\bar{m}_{ij}^{A^t}$ in $\bar{\mathbb{M}}_{U^t}^{A^t}$ keeps unchanged with the addition of objects, i.e., $\bar{m}_{ij}^{A^t} = m_{ij}^{A^t}$. Consequently, if $r_{ij}^{A^t} = 0$, we have $m_{ij}^{A^{t+1}} = m_{ij}^{A^t}$; otherwise, we have $m_{ij}^{A^{t+1}} = m_{ij}^{\Delta A}$. (ii) When $s_{ij}^{A^t} = 1$, it is easy to find that the element $\bar{m}_{ij}^{A^t}$ in $\bar{\mathbb{M}}_{U^t}^{A^t}$ may vary with the increase of objects, i.e., if $u_j \in [\mathcal{N}_{A^t}^{\delta}]^{U^{t+1}}(u_i)$, then $\bar{m}_{ij}^{A^t} = (1,1)$; otherwise, $\bar{m}_{ij}^{A^t} = (0,1)$. Accordingly, when $r_{ij}^{\Delta A} = 0$, i.e., $m_{ij}^{\Delta A} = (0,0)$ or $m_{ij}^{\Delta A} = (0,1)$, it can easily be yielded that $m_{ij}^{A^{t+1}} = m_{ij}^{\Delta A}$; when $r_{ij}^{\Delta A} = 1$, i.e., $m_{ij}^{\Delta A} = (1,0)$ or $m_{ij}^{\Delta A} = (1,1)$, if $\bar{m}_{ij}^{A^t} = (1,1)$, then $m_{ij}^{A^{t+1}} = (1,1)$; if $\bar{m}_{ij}^{A^t} = (0,1)$, then $m_{ij}^{A^{t+1}} = (0,1)$. Furthermore, for any $n+1 \leq i \leq n+n'$ or $n+1 \leq j \leq n+n'$, we need to compute the element $m_{ij}^{A^{t+1}}$ in $\mathbb{M}_{U^{t+1}}^{A^{t+1}}$ from scratch based on Definition 4.3 and Lemma 4.1 due to the lack of priori knowledge, namely, when $s_{ij}^{A^{t+1}} = 0$, if $u_j \in [\mathcal{N}_{A^{t+1}}^{\delta}]^{U^{t+1}}(u_i)$, then $m_{ij}^{A^{t+1}} = (1,0)$; otherwise, $m_{ij}^{A^{t+1}} = (0,0)$; when $s_{ij}^{A^{t+1}} = 1$, if $u_j \in [\mathcal{N}_{A^{t+1}}^{\delta}]^{U^{t+1}}(u_i)$, then $m_{ij}^{A^{t+1}} = (1,1)$; otherwise, $m_{ij}^{A^{t+1}} = (0,1)$. □

Based on the above results, we can easily obtain the following strategies for dynamically updating the induced diagonal matrix and the condition matrix.

Corollary 5.1. *Let $\mathbb{S}^t$ be a CIHIS at time t, $\mathbb{M}_{U^t}^{A^t} = [m_{ij}^{A^t}]_{n\times n}$ and $\Gamma_{U^t}^{A^t} = \mathrm{diag}[1/\gamma_1^{A^t}, 1/\gamma_2^{A^t}, \cdots, 1/\gamma_n^{A^t}]$ be the double-value relation matrix and the induced diagonal matrix of $\mathbb{M}_{U^t}^{A^t}$ at time t, respectively. When new object set ΔU and attribute set ΔA are added to $\mathbb{S}^t$, let $\mathbb{M}_{U^{t+1}}^{A^{t+1}} = [m_{ij}^{A^{t+1}}]_{(n+n')\times(n+n')}$ and $\Gamma_{U^{t+1}}^{A^{t+1}} = \mathrm{diag}[1/\gamma_1^{A^{t+1}}, 1/\gamma_2^{A^{t+1}}, \cdots, 1/\gamma_n^{A^{t+1}}, \cdots, 1/\gamma_{n+n'}^{A^{t+1}}]$ denote the double-value relation matrix and the induced diagonal matrix of $\mathbb{M}_{U^{t+1}}^{A^{t+1}}$ at time $t+1$, respectively. Then the induced diagonal matrix $\Gamma_{U^{t+1}}^{A^{t+1}}$ can be updated as follows.*

(1) If $1 \leq i \leq n$, then

$$\gamma_i^{A^{t+1}} = \gamma_i^{A^t} + \sum_{j=1}^{n} (F(m_{ij}^{A^{t+1}}) - F(m_{ij}^{A^t}))(F(m_{ij}^{A^{t+1}}) \oplus F(m_{ij}^{A^t})) + \sum_{j=n+1}^{n+n'} F(m_{ij}^{A^{t+1}}); \tag{40}$$

(2) If $n+1 \leq i \leq n+n'$, then

$$\gamma_i^{A^{t+1}} = \sum_{j=1}^{n+n'} F(m_{ij}^{A^{t+1}}), \tag{41}$$

where "⊕" indicates the logical exclusion OR operation.

Proof. According to Theorem 5.2, the results can be easily proved. □

Corollary 5.2. *Let $\mathbb{S}^t$ be a CIHIS at time t, $\mathbb{D}^t = [d_{ij}^t]_{n\times m}$, $\mathbb{M}_{U^t}^{A^t} = [m_{ij}^{A^t}]_{n\times n}$ and $\mathbb{X}_{U^t}^{A^t} = [\chi_{ij}^{A^t}]_{n\times m}$ be the decision matrix, the double-value relation matrix and the intersection matrix at time t, respectively. When new object set ΔU and attribute set ΔA are added to $\mathbb{S}^t$, suppose that $\mathbb{D}^{t+1} = [d_{ij}^{t+1}]_{(n+n')\times(m+m')}$ and $\mathbb{M}_{U^{t+1}}^{A^{t+1}} = [m_{ij}^{A^{t+1}}]_{(n+n')\times(n+n')}$ denote the decision matrix and the double-value relation matrix at time $t+1$, respectively. Then $\mathbb{X}_{U^{t+1}}^{A^{t+1}} = [\chi_{ij}^{A^{t+1}}]_{(n+n')\times(m+m')}$ can be updated as follows.*

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(1) If $1 \le i \le n$ *and* $1 \le j \le m$*, then*

$$\chi_{ij}^{A^{t+1}} = \chi_{ij}^{A^t} + \sum_{k=1}^{n}(F(m_{ik}^{A^{t+1}}) - F(m_{ik}^{A^t}))(F(m_{ik}^{A^{t+1}}) \oplus F(m_{ik}^{A^t}))d_{kj}^t + \sum_{k=n+1}^{n+n'} F(m_{ik}^{A^{t+1}})d_{kj}^{t+1}. \tag{42}$$

(2) If $n+1 \le i \le n+n'$ *and* $1 \le j \le m$*, then*

$$\chi_{ij}^{A^{t+1}} = \sum_{k=1}^{n} F(m_{ik}^{A^{t+1}})d_{kj}^t + \sum_{k=n+1}^{n+n'} F(m_{ik}^{A^{t+1}})d_{kj}^{t+1}. \tag{43}$$

(3) If $m+1 \le j \le m+m'$*, then*

$$\chi_{ij}^{A^{t+1}} = \sum_{k=n+1}^{n+n'} F(m_{ik}^{A^{t+1}})d_{kj}^{t+1}. \tag{44}$$

Proof. It is apparent from Theorems 5.1 and 5.2 that the conclusions hold. □

Obviously, when new objects and attributes are added to CIHIS simultaneously, by Corollaries 5.1 and 5.2, the condition matrix can be updated as $\Phi_{U^{t+1}}^{A^{t+1}} = [\phi_{ij}^{A^{t+1}}]_{(n+n')\times(m+m')}$, where $\phi_{ij}^{A^{t+1}} = \chi_{ij}^{A^{t+1}}/\gamma_i^{A^{t+1}}$. Moreover, it can intuitively be found that the thresholds $\bar{\alpha}$ and $\bar{\beta}$ may change with the variation of objects and attributes based on the introduction in Section 3. Therefore, in what follows, we focus on discussing the principles for updating the thresholds $\bar{\alpha}$ and $\bar{\beta}$.

Theorem 5.3. *Let* $\mathbb{S}^t$ *be a CIHIS at time t, and* $\mathbb{L}_t^\theta = [\lambda_{ij}^t]_{n\times 6}$ *be the loss matrix at time t. When new object set* ΔU *and attribute set* ΔA *are added to* $\mathbb{S}^t$*, suppose that* $\mathbb{L}_{t+1}^\theta = [\lambda_{ij}^{t+1}]_{(n+n')\times 6}$ *denotes the loss matrix at time* $t+1$*. Then the loss matrix* $\mathbb{L}_{t+1}^\theta$ *can be updated as follows.*

$$\lambda_{ij}^{t+1} = \begin{cases} \lambda_{ij}^t, & 1 \le i \le n; \\ h^\theta(\bar{\lambda}_{PP}(u_i)), & n+1 \le i \le n+n' \wedge j=1; \\ h^\theta(\bar{\lambda}_{BP}(u_i)), & n+1 \le i \le n+n' \wedge j=2; \\ h^\theta(\bar{\lambda}_{NP}(u_i)), & n+1 \le i \le n+n' \wedge j=3; \\ h^\theta(\bar{\lambda}_{NN}(u_i)), & n+1 \le i \le n+n' \wedge j=4; \\ h^\theta(\bar{\lambda}_{BN}(u_i)), & n+1 \le i \le n+n' \wedge j=5; \\ h^\theta(\bar{\lambda}_{PN}(u_i)), & n+1 \le i \le n+n' \wedge j=6. \end{cases} \tag{45}$$

Proof. From Definition 4.9, it is obvious to obtain this result. □

Corollary 5.3. *Let* $\mathbb{S}^t$ *be a CIHIS at time t,* $\mathbb{M}_{U^t}^{A^t} = [m_{ij}^{A^t}]_{n\times n}$ *and* $\mathbb{L}_t^\theta = [\lambda_{ij}^t]_{n\times 6}$ *be the double-value relation matrix and the loss matrix at time t, respectively, and* $\mathbb{H}_t^{(1,2)} = F(\mathbb{M}_{U^t}^{A^t}) \bullet (\mathbb{L}_t^\theta(\cdot,2) - \mathbb{L}_t^\theta(\cdot,1)) = [\zeta_i^t]_{n\times 1}$*,* $\mathbb{H}_t^{(2,3)} = F(\mathbb{M}_{U^t}^{A^t}) \bullet (\mathbb{L}_t^\theta(\cdot,3) - \mathbb{L}_t^\theta(\cdot,2)) = [\eta_i^t]_{n\times 1}$*,* $\mathbb{H}_t^{(4,5)} = F(\mathbb{M}_{U^t}^{A^t}) \bullet (\mathbb{L}_t^\theta(\cdot,5) - \mathbb{L}_t^\theta(\cdot,4)) = [\vartheta_i^t]_{n\times 1}$*,* $\mathbb{H}_t^{(5,6)} = F(\mathbb{M}_{U^t}^{A^t}) \bullet (\mathbb{L}_t^\theta(\cdot,6) - \mathbb{L}_t^\theta(\cdot,5)) = [\xi_i^t]_{n\times 1}$*. When new object set* ΔU *and attribute set* ΔA *are added to* $\mathbb{S}^t$*, suppose that* $\mathbb{M}_{U^{t+1}}^{A^{t+1}} = [m_{ij}^{A^{t+1}}]_{(n+n')\times(n+n')}$ *and* $\mathbb{L}_{t+1}^\theta = [\lambda_{ij}^{t+1}]_{(n+n')\times 6}$ *denote the double-value relation matrix and the loss matrix at time* $t+1$*, respectively. Then, we have*

$$\mathbb{H}_{t+1}^{(1,2)} = F(\mathbb{M}_{U^{t+1}}^{A^{t+1}}) \bullet (\mathbb{L}_{t+1}^\theta(\cdot,2) - \mathbb{L}_{t+1}^\theta(\cdot,1)) = [\zeta_i^{t+1}]_{(n+n')\times 1};$$
$$\mathbb{H}_{t+1}^{(2,3)} = F(\mathbb{M}_{U^{t+1}}^{A^{t+1}}) \bullet (\mathbb{L}_{t+1}^\theta(\cdot,3) - \mathbb{L}_{t+1}^\theta(\cdot,2)) = [\eta_i^{t+1}]_{(n+n')\times 1};$$
$$\mathbb{H}_{t+1}^{(4,5)} = F(\mathbb{M}_{U^{t+1}}^{A^{t+1}}) \bullet (\mathbb{L}_{t+1}^\theta(\cdot,5) - \mathbb{L}_{t+1}^\theta(\cdot,4)) = [\vartheta_i^{t+1}]_{(n+n')\times 1};$$
$$\mathbb{H}_{t+1}^{(5,6)} = F(\mathbb{M}_{U^{t+1}}^{A^{t+1}}) \bullet (\mathbb{L}_{t+1}^\theta(\cdot,6) - \mathbb{L}_{t+1}^\theta(\cdot,5)) = [\xi_i^{t+1}]_{(n+n')\times 1},$$

where

(1) if $1 \leq i \leq n$*, then*

$$
\begin{aligned}
\zeta_i^{t+1} &= \zeta_i^t + \sum_{j=1}^{n}(F(m_{ij}^{A^{t+1}}) - F(m_{ij}^{A^t}))(F(m_{ij}^{A^{t+1}}) \oplus F(m_{ij}^{A^t}))(\lambda_{j2}^t - \lambda_{j1}^t) + \sum_{j=n+1}^{n+n'} F(m_{ij}^{A^{t+1}})(\lambda_{j2}^{t+1} - \lambda_{j1}^{t+1});\\
\eta_i^{t+1} &= \eta_i^t + \sum_{j=1}^{n}(F(m_{ij}^{A^{t+1}}) - F(m_{ij}^{A^t}))(F(m_{ij}^{A^{t+1}}) \oplus F(m_{ij}^{A^t}))(\lambda_{j3}^t - \lambda_{j2}^t) + \sum_{j=n+1}^{n+n'} F(m_{ij}^{A^{t+1}})(\lambda_{j3}^{t+1} - \lambda_{j2}^{t+1});\\
\vartheta_i^{t+1} &= \vartheta_i^t + \sum_{j=1}^{n}(F(m_{ij}^{A^{t+1}}) - F(m_{ij}^{A^t}))(F(m_{ij}^{A^{t+1}}) \oplus F(m_{ij}^{A^t}))(\lambda_{j5}^t - \lambda_{j4}^t) + \sum_{j=n+1}^{n+n'} F(m_{ij}^{A^{t+1}})(\lambda_{j5}^{t+1} - \lambda_{j4}^{t+1});\\
\xi_i^{t+1} &= \xi_i^t + \sum_{j=1}^{n}(F(m_{ij}^{A^{t+1}}) - F(m_{ij}^{A^t}))(F(m_{ij}^{A^{t+1}}) \oplus F(m_{ij}^{A^t}))(\lambda_{j6}^t - \lambda_{j5}^t) + \sum_{j=n+1}^{n+n'} F(m_{ij}^{A^{t+1}})(\lambda_{j6}^{t+1} - \lambda_{j5}^{t+1}).
\end{aligned} \tag{46}
$$

(2) if $n + 1 \leq i \leq n + n'$*, then*

$$
\begin{aligned}
\zeta_i^{A^{t+1}} &= \sum_{j=1}^{n} F(m_{ij}^{A^{t+1}})(\lambda_{j2}^t - \lambda_{j1}^t) + \sum_{j=n+1}^{n+n'} F(m_{ij}^{A^{t+1}})(\lambda_{j2}^{t+1} - \lambda_{j1}^{t+1});\\
\eta_i^{A^{t+1}} &= \sum_{j=1}^{n} F(m_{ij}^{A^{t+1}})(\lambda_{j3}^t - \lambda_{j2}^t) + \sum_{j=n+1}^{n+n'} F(m_{ij}^{A^{t+1}})(\lambda_{j3}^{t+1} - \lambda_{j2}^{t+1});\\
\vartheta_i^{A^{t+1}} &= \sum_{j=1}^{n} F(m_{ij}^{A^{t+1}})(\lambda_{j5}^t - \lambda_{j4}^t) + \sum_{j=n+1}^{n+n'} F(m_{ij}^{A^{t+1}})(\lambda_{j5}^{t+1} - \lambda_{j4}^{t+1});\\
\xi_i^{A^{t+1}} &= \sum_{j=1}^{n} F(m_{ij}^{A^{t+1}})(\lambda_{j6}^t - \lambda_{j5}^t) + \sum_{j=n+1}^{n+n'} F(m_{ij}^{A^{t+1}})(\lambda_{j6}^{t+1} - \lambda_{j5}^{t+1}).
\end{aligned} \tag{47}
$$

Proof. According to Definition 4.11 and Theorem 5.2, the results can be easily proved. □

Based on the above analysis, when objects and attributes in CIHIS increase with time simultaneously, it follows from Definition 4.11 that two threshold vectors can be updated as $\mathbb{T}_{t+1}^{\alpha} = [\iota_i^{t+1}]_{(n+n')\times 1}$ and $\mathbb{T}_{t+1}^{\beta} = [o_i^{t+1}]_{(n+n')\times 1}$, where $\iota_i^{t+1} = \xi_i^{t+1}/(\xi_i^{t+1}+\zeta_i^{t+1})$ and $o_i^{t+1} = \vartheta_i^{t+1}/(\vartheta_i^{t+1}+\eta_i^{t+1})$. Then, the updated three-way regions in the generalized three-way neighborhood decision model can be yielded directly in accordance with Definition 4.13 and Theorem 4.1.

In the following, an example is introduced to illustrate the proposed incremental mechanism of updating three-way regions in the generalized three-way neighborhood decision model.

Example 5.1. *Let $\mathbb{S}^t = (S^t, \bar{\lambda}^t, g^t)$ be a CIHIS about the neonatal disease screening at time t, which is shown in Table 3. From Table 3, due to the limitation of medical conditions at time t, hospitals in some areas only provide free screening of hearing, fundus, CH and PKU for newborns. With the continuous development of medical technology, hospitals have added two disease screening projects with high incidence, i.e., Congenital Heart Disease (CHD) and Glucose-6-Phosphate Dehydrogenase Deficiency (G6PDD). Therefore, at time $t+1$, for new newborns, there are six disease screening programs. For simplicity of expression, let $\Delta U = \{u_7, u_8, u_9, u_{10}\}$ be the newly added newborns, i.e., the added object set, and $\Delta A = \{a_5, a_6\}$ be the newly added disease screening programs, i.e., the added attribute set, where a_5 and a_6 represent CHD and G6PDD, respectively. With respect to the added attribute a_5, the symbols "+" and "-" indicate that the screening results of CHD are positive and negative, respectively. With respect to the added attribute a_6, the attribute values denote the activities of Glucose-6-Phosphate Dehydrogenase (in U/g Hb) in neonatal plantar bloods. Then, as the new object set ΔU and the new attribute set ΔA are appended to $\mathbb{S}^t$ at time $t+1$, $\mathbb{S}^t$ is updated as $\mathbb{S}^{t+1}$, which is shown in Table 5.*

First, based on the previously acquired decision matrix from Example 4.1 and Theorem 5.1, the decision matrix at

Table 5: A CIHIS about the neonatal disease screening at time $t+1$

U	a_1	a_2	a_3	a_4	a_5	a_6	d	$\bar{\lambda}_{PP}$	$\bar{\lambda}_{BP}$	$\bar{\lambda}_{NP}$	$\bar{\lambda}_{NN}$	$\bar{\lambda}_{BN}$	$\bar{\lambda}_{PN}$
u_1	F	F	3.2	0.2	+	2.2	1	[0,1ℏ]	[1ℏ,2ℏ]	[4ℏ,6ℏ]	[1ℏ,1.5ℏ]	[2ℏ,4ℏ]	[5ℏ,8ℏ]
u_2	*	P	8.3	0.7	+	1.9	1	[0.5ℏ,1.5ℏ]	[2ℏ,4ℏ]	[5ℏ,7ℏ]	[1ℏ,2ℏ]	[2.5ℏ,4ℏ]	[4.5ℏ,6.5ℏ]
u_3	*	?	2.9	0.3	-	*	0	[1.5ℏ,3ℏ]	[3ℏ,5ℏ]	[6ℏ,8ℏ]	[1.5ℏ,2.5ℏ]	[3ℏ,4ℏ]	[4.5ℏ,7ℏ]
u_4	P	P	2.7	0.4	*	2.0	1	[0,1.5ℏ]	[1.5ℏ,3ℏ]	[3.5ℏ,5ℏ]	[0ℏ,2ℏ]	[2.5ℏ,4.5ℏ]	[5ℏ,7.5ℏ]
u_5	P	P	1.9	?	-	5.1	0	[1ℏ,2.5ℏ]	[3.5ℏ,5ℏ]	[5.5ℏ,7.5ℏ]	[0.5ℏ,2.5ℏ]	[2.5ℏ,4.5ℏ]	[5ℏ,7ℏ]
u_6	F	F	3.1	*	-	4.9	0	[1ℏ,2ℏ]	[2.5ℏ,4ℏ]	[4.5ℏ,6.5ℏ]	[1ℏ,3ℏ]	[3.5ℏ,5.5ℏ]	[6ℏ,8ℏ]
u_7	F	P	8.0	0.6	+	2.1	1	[0.5ℏ,2.5ℏ]	[3ℏ,4.5ℏ]	[5.5ℏ,7ℏ]	[1ℏ,2ℏ]	[2.5ℏ,5ℏ]	[5.5ℏ,7.5ℏ]
u_8	F	F	*	0.9	-	2.3	1	[2ℏ,3.5ℏ]	[4.5ℏ,6.5ℏ]	[7ℏ,9ℏ]	[2ℏ,3ℏ]	[4ℏ,6.5ℏ]	[6.5ℏ,8.5ℏ]
u_9	P	P	4.6	0.5	+	6.2	0	[0.5ℏ,2ℏ]	[2ℏ,4ℏ]	[5ℏ,7.5ℏ]	[1.5ℏ,3ℏ]	[3.5ℏ,5.5ℏ]	[6ℏ,9ℏ]
u_{10}	P	P	1.5	0.3	-	7.7	0	[0ℏ,1.5ℏ]	[2ℏ,3.5ℏ]	[4.5ℏ,5ℏ]	[1ℏ,2.5ℏ]	[2.5ℏ,5ℏ]	[6.5ℏ,8ℏ]

time $t+1$ can be updated as follows.

$$\mathbb{D}^{t+1} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}^T.$$

In the light of the double-value relation matrix from Example 4.1 and Theorem 5.2, we just need to compute $m_{16}^{\Delta A}$, $m_{31}^{\Delta A}$, $m_{34}^{\Delta A}$, $m_{36}^{\Delta A}$ and $m_{61}^{\Delta A}$ in $\mathbb{M}_{U^t}^{\Delta A}$. Let $\delta_{a_6} = 0.3$. It follows from Definition 4.3 that $m_{16}^{\Delta A} = (0,0)$, $m_{31}^{\Delta A} = (0,0)$, $m_{34}^{\Delta A} = (1,1)$, $m_{36}^{\Delta A} = (0,1)$ and $m_{61}^{\Delta A} = (0,0)$. Then, according to Theorem 5.2, the double-value relation matrix at time $t+1$ can be updated as follows.

$$\mathbb{M}_{U^{t+1}}^{A^{t+1}} = \begin{bmatrix} (1,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (1,0) & (0,0) & (0,0) & (0,0) & (0,0) & (1,1) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (1,0) & (1,1) & (0,0) & (0,1) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (1,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (1,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (1,0) & (0,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (1,1) & (0,0) & (0,0) & (0,0) & (0,0) & (1,0) & (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (1,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (1,0) & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (0,0) & (1,0) \end{bmatrix}.$$

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According to Corollary 5.1, the elements in the induced diagonal matrix at time $t+1$ *can be updated as follows.*

$$\begin{cases} \gamma_1^{A^{t+1}} = \gamma_1^{A^t} - F(m_{16}^{A^t}) + \sum_{j=7}^{10} F(m_{1j}^{A^{t+1}}) = 2 - 1 + 0 = 1; \\ \gamma_2^{A^{t+1}} = \gamma_2^{A^t} + \sum_{j=7}^{10} F(m_{2j}^{A^{t+1}}) = 1 + 1 = 2; \\ \gamma_3^{A^{t+1}} = \gamma_3^{A^t} - F(m_{31}^{A^t}) - F(m_{36}^{A^t}) + \sum_{j=7}^{10} F(m_{3j}^{A^{t+1}}) = 4 - 2 + 0 = 2; \\ \gamma_4^{A^{t+1}} = \gamma_4^{A^t} + \sum_{j=7}^{10} F(m_{4j}^{A^{t+1}}) = 1 + 0 = 1; \\ \gamma_5^{A^{t+1}} = \gamma_5^{A^t} + \sum_{j=7}^{10} F(m_{5j}^{A^{t+1}}) = 1 + 0 = 1; \\ \gamma_6^{A^{t+1}} = \gamma_6^{A^t} - F(m_{61}^{A^t}) + \sum_{j=7}^{10} F(m_{6j}^{A^{t+1}}) = 2 - 1 + 0 = 1; \\ \gamma_7^{A^{t+1}} = \sum_{j=1}^{10} F(m_{7j}^{A^{t+1}}) = 2; \\ \gamma_8^{A^{t+1}} = \sum_{j=1}^{10} F(m_{8j}^{A^{t+1}}) = 1; \\ \gamma_9^{A^{t+1}} = \sum_{j=1}^{10} F(m_{9j}^{A^{t+1}}) = 1; \\ \gamma_{10}^{A^{t+1}} = \sum_{j=1}^{10} F(m_{10j}^{A^{t+1}}) = 1. \end{cases}$$

Hence, we have $\Gamma_{U^{t+1}}^{A^{t+1}} = \mathrm{diag}[1, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \frac{1}{2}, 1, 1, 1]$.

By Corollary 5.2, the elements in the intersection matrix at time $t+1$ *can be updated as follows.*

$$\begin{cases} \chi_{11}^{A^{t+1}} = \chi_{11}^{A^t} - F(m_{16}^{A^t})d_{61}^t + \sum_{j=7}^{10} F(m_{1j}^{A^{t+1}})d_{j1}^{t+1} = 1 - 0 + 0 = 1; \\ \chi_{21}^{A^{t+1}} = \chi_{21}^{A^t} + \sum_{j=7}^{10} F(m_{2j}^{A^{t+1}})d_{j1}^{t+1} = 1 + 1 = 2; \\ \chi_{31}^{A^{t+1}} = \chi_{31}^{A^t} - F(m_{31}^{A^t})d_{11}^t - F(m_{36}^{A^t})d_{61}^t + \sum_{j=7}^{10} F(m_{3j}^{A^{t+1}})d_{j1}^{t+1} = 2 - 1 - 0 + 0 = 1; \\ \chi_{41}^{A^{t+1}} = \chi_{41}^{A^t} + \sum_{j=7}^{10} F(m_{4j}^{A^{t+1}})d_{j1}^{t+1} = 1 + 0 = 1; \\ \chi_{51}^{A^{t+1}} = \chi_{51}^{A^t} + \sum_{j=7}^{10} F(m_{5j}^{A^{t+1}})d_{j1}^{t+1} = 0 + 0 = 0; \\ \chi_{61}^{A^{t+1}} = \chi_{61}^{A^t} - F(m_{61}^{A^t})d_{11}^t + \sum_{j=7}^{10} F(m_{6j}^{A^{t+1}})d_{j1}^{t+1} = 1 - 1 + 0 = 0; \\ \chi_{71}^{A^{t+1}} = \sum_{j=1}^{6} F(m_{7j}^{A^{t+1}})d_{j1}^t + \sum_{j=7}^{10} F(m_{7j}^{A^{t+1}})d_{j1}^{t+1} = 1 + 1 = 2; \\ \chi_{81}^{A^{t+1}} = \sum_{j=1}^{6} F(m_{8j}^{A^{t+1}})d_{j1}^t + \sum_{j=7}^{10} F(m_{8j}^{A^{t+1}})d_{j1}^{t+1} = 0 + 1 = 1; \\ \chi_{91}^{A^{t+1}} = \sum_{j=1}^{6} F(m_{9j}^{A^{t+1}})d_{j1}^t + \sum_{j=7}^{10} F(m_{9j}^{A^{t+1}})d_{j1}^{t+1} = 0 + 0 = 0; \\ \chi_{10,1}^{A^{t+1}} = \sum_{j=1}^{6} F(m_{10j}^{A^{t+1}})d_{j1}^t + \sum_{j=7}^{10} F(m_{10j}^{A^{t+1}})d_{j1}^{t+1} = 0 + 0 = 0. \end{cases}$$

$$\begin{cases} \chi_{12}^{A^{t+1}} = \chi_{12}^{A^t} - F(m_{16}^{A^t})d_{62}^t + \sum_{j=7}^{10} F(m_{1j}^{A^{t+1}})d_{j2}^{t+1} = 1 - 1 + 0 = 0; \\ \chi_{22}^{A^{t+1}} = \chi_{22}^{A^t} + \sum_{j=7}^{10} F(m_{2j}^{A^{t+1}})d_{j2}^{t+1} = 1 + 1 = 2; \\ \chi_{32}^{A^{t+1}} = \chi_{32}^{A^t} - F(m_{31}^{A^t})d_{12}^t - F(m_{36}^{A^t})d_{62}^t + \sum_{j=7}^{10} F(m_{3j}^{A^{t+1}})d_{j2}^{t+1} = 2 - 0 - 1 + 0 = 1; \\ \chi_{42}^{A^{t+1}} = \chi_{42}^{A^t} + \sum_{j=7}^{10} F(m_{4j}^{A^{t+1}})d_{j2}^{t+1} = 0 + 0 = 0; \\ \chi_{51}^{A^{t+1}} = \chi_{52}^{A^t} + \sum_{j=7}^{10} F(m_{5j}^{A^{t+1}})d_{j2}^{t+1} = 1 + 0 = 0; \\ \chi_{62}^{A^{t+1}} = \chi_{62}^{A^t} - F(m_{61}^{A^t})d_{12}^t + \sum_{j=7}^{10} F(m_{6j}^{A^{t+1}})d_{j2}^{t+1} = 1 - 0 + 0 = 1; \\ \chi_{72}^{A^{t+1}} = \sum_{j=1}^{6} F(m_{7j}^{A^{t+1}})d_{j2}^t + \sum_{j=7}^{10} F(m_{7j}^{A^{t+1}})d_{j2}^{t+1} = 0 + 0 = 0; \\ \chi_{82}^{A^{t+1}} = \sum_{j=1}^{6} F(m_{8j}^{A^{t+1}})d_{j2}^t + \sum_{j=7}^{10} F(m_{8j}^{A^{t+1}})d_{j2}^{t+1} = 0 + 0 = 0; \\ \chi_{92}^{A^{t+1}} = \sum_{j=1}^{6} F(m_{9j}^{A^{t+1}})d_{j2}^t + \sum_{j=7}^{10} F(m_{9j}^{A^{t+1}})d_{j2}^{t+1} = 0 + 1 = 1; \\ \chi_{10,2}^{A^{t+1}} = \sum_{j=1}^{6} F(m_{10j}^{A^{t+1}})d_{j2}^t + \sum_{j=7}^{10} F(m_{10j}^{A^{t+1}})d_{j2}^{t+1} = 0 + 1 = 1. \end{cases}$$

Then, we have

$$\mathbb{X}_{U^{t+1}}^{A^{t+1}} = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}^T.$$

Based on this, the condition matrix at time $t+1$ can be updated as follows.

$$\Phi^{A^{t+1}}_{U^{t+1}} = \begin{bmatrix} 1 & 1 & \frac{1}{2} & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}^T .$$

In the following, according to Theorem 5.3, the loss matrix at time $t+1$ can be updated as follows.

$$\mathbb{L}^{0.5}_{t+1} = \begin{bmatrix} 0.50 & 1.50 & 5.00 & 1.25 & 3.00 & 6.50 \\ 1.00 & 3.00 & 6.00 & 1.50 & 3.25 & 5.50 \\ 2.25 & 4.00 & 7.00 & 2.00 & 3.50 & 5.75 \\ 0.75 & 2.25 & 4.25 & 1.00 & 3.50 & 6.25 \\ 1.75 & 4.25 & 6.50 & 1.50 & 3.50 & 6.00 \\ 1.50 & 3.25 & 5.50 & 2.00 & 4.50 & 7.00 \\ 1.50 & 3.75 & 6.25 & 1.50 & 3.75 & 6.50 \\ 2.75 & 5.50 & 8.00 & 2.50 & 5.25 & 7.50 \\ 1.25 & 3.00 & 6.25 & 2.25 & 4.50 & 7.50 \\ 0.75 & 2.75 & 4.75 & 1.75 & 3.75 & 7.25 \end{bmatrix}.$$

According to Corollary 5.3, the elements in $\mathbb{H}^{(1,2)}_{t+1}$ can be updated as follows.

$$\begin{cases} \zeta_1^{A^{t+1}} = \zeta_1^{A^t} - F(m_{16}^{A^t})(\lambda_{62}^t - \lambda_{61}^t) + \sum_{j=7}^{10} F(m_{1j}^{A^{t+1}})(\lambda_{j2}^{t+1} - \lambda_{j1}^{t+1}) = 2.75 - 1.75 + 0 = 1.00; \\ \zeta_2^{A^{t+1}} = \zeta_2^{A^t} + \sum_{j=7}^{10} F(m_{2j}^{A^{t+1}})(\lambda_{j2}^{t+1} - \lambda_{j1}^{t+1}) = 2 + 2.5 = 4.50; \\ \zeta_3^{A^{t+1}} = \zeta_3^{A^t} - F(m_{31}^{A^t})(\lambda_{12}^t - \lambda_{11}^t) - F(m_{36}^{A^t})(\lambda_{62}^t - \lambda_{61}^t) + \sum_{j=7}^{10} F(m_{3j}^{A^{t+1}})(\lambda_{j2}^{t+1} - \lambda_{j1}^{t+1}) = 6 - 1 - 1.75 + 0 = 3.25; \\ \zeta_4^{A^{t+1}} = \zeta_4^{A^t} + \sum_{j=7}^{10} F(m_{4j}^{A^{t+1}})(\lambda_{j2}^{t+1} - \lambda_{j1}^{t+1}) = 1.5 + 0 = 1.50; \\ \zeta_5^{A^{t+1}} = \zeta_5^{A^t} + \sum_{j=7}^{10} F(m_{5j}^{A^{t+1}})(\lambda_{j2}^{t+1} - \lambda_{j1}^{t+1}) = 2.5 + 0 = 2.50; \\ \zeta_6^{A^{t+1}} = \zeta_6^{A^t} - F(m_{61}^{A^t})(\lambda_{12}^t - \lambda_{11}^t) + \sum_{j=7}^{10} F(m_{6j}^{A^{t+1}})(\lambda_{j2}^{t+1} - \lambda_{j1}^{t+1}) = 2.75 - 1 = 1.75; \\ \zeta_7^{A^{t+1}} = \sum_{j=1}^{6} F(m_{7j}^{A^{t+1}})(\lambda_{j2}^t - \lambda_{j1}^t) + \sum_{j=7}^{10} F(m_{7j}^{A^{t+1}})(\lambda_{j2}^{t+1} - \lambda_{j1}^{t+1}) = 2 + 2.5 = 4.50; \\ \zeta_8^{A^{t+1}} = \sum_{j=1}^{6} F(m_{8j}^{A^{t+1}})(\lambda_{j2}^t - \lambda_{j1}^t) + \sum_{j=7}^{10} F(m_{8j}^{A^{t+1}})(\lambda_{j2}^{t+1} - \lambda_{j1}^{t+1}) = 0 + 2.75 = 2.75; \\ \zeta_9^{A^{t+1}} = \sum_{j=1}^{6} F(m_{9j}^{A^{t+1}})(\lambda_{j2}^t - \lambda_{j1}^t) + \sum_{j=7}^{10} F(m_{9j}^{A^{t+1}})(\lambda_{j2}^{t+1} - \lambda_{j1}^{t+1}) = 0 + 1.75 = 1.75; \\ \zeta_{10}^{A^{t+1}} = \sum_{j=1}^{6} F(m_{10j}^{A^{t+1}})(\lambda_{j2}^t - \lambda_{j1}^t) + \sum_{j=7}^{10} F(m_{10j}^{A^{t+1}})(\lambda_{j2}^{t+1} - \lambda_{j1}^{t+1}) = 0 + 2 = 2.00. \end{cases}$$

So we have

$$\mathbb{H}^{(1,2)}_{t+1} = [1.00, 4.50, 3.25, 1.50, 2.50, 1.75, 4.50, 2.75, 1.75, 2.00]^T.$$

By analogy, $\mathbb{H}^{(2,3)}_{t+1}$, $\mathbb{H}^{(4,5)}_{t+1}$ and $\mathbb{H}^{(5,6)}_{t+1}$ can be updated respectively as follows.

$$\mathbb{H}^{(2,3)}_{t+1} = [3.50, 5, 50, 5.00, 2.00, 2.25, 2.25, 5.50, 2.50, 3.25, 2.00]^T;$$
$$\mathbb{H}^{(4,5)}_{t+1} = [1.75, 4.25, 4.00, 2.50, 2.00, 2.50, 2.75, 2.25, 1.75, 2.00]^T;$$
$$\mathbb{H}^{(5,6)}_{t+1} = [3.50, 5.00, 5.00, 2.75, 2.50, 2.50, 5.00, 2.25, 3.00, 3.50]^T.$$

Then, by Definition 4.11, the threshold vectors $\mathbb{T}^{\alpha}_{t+1}$ and $\mathbb{T}^{\beta}_{t+1}$ are updated respectively as follows.

$$\mathbb{T}^{\alpha}_{t+1} = [0.7778, 0.5263, 0.6061, 0.6471, 0.500, 0.5882, 0.5263, 0.4500, 0.6316, 0.6363]^T;$$
$$\mathbb{T}^{\beta}_{t+1} = [0.3333, 0.4359, 0.4444, 0.5556, 0.4706, 0.5263, 0.3333, 0.4737, 0.3500, 0.5000]^T.$$

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According to Definition 4.13, it is easy to obtain the three cut matrices at time $t+1$.

$$\check{\Psi}_{U^{t+1}}^{A^{t+1}} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}^T;$$

$$\bar{\Psi}_{U^{t+1}}^{A^{t+1}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T;$$

$$\hat{\Psi}_{U^{t+1}}^{A^{t+1}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}^T.$$

Based on Theorem 4.1, the characteristic vectors of the three-way regions at time $t+1$ can be computed as follows.

$$POS_{A^{t+1}}^{(\bar{\alpha},\bar{\beta})}(D) = [1,1,0,1,1,1,1,1,1,1]^T;$$
$$BND_{A^{t+1}}^{(\bar{\alpha},\bar{\beta})}(D) = [0,0,1,0,0,0,0,0,0,0]^T;$$
$$NEG_{A^{t+1}}^{(\bar{\alpha},\bar{\beta})}(D) = [0,0,0,0,0,0,0,0,0,0]^T.$$

Therefore, $\{u_1, u_2, u_4, u_5, u_6, u_7, u_8, u_9, u_10\} \in POS_{A^{t+1}}^{(\bar{\alpha},\bar{\beta})}(D)$ and $\{u_3\} \in BND_{A^{t+1}}^{(\bar{\alpha},\bar{\beta})}(D)$.

6. Matrix-based static and incremental algorithms for maintaining three-way regions when adding objects and attributes in CIHIS

Through the detailed analysis stated before, the three-way regions in the generalized three-way neighborhood decision model are always evolving over time under dynamic environments. To highlight the superiority of the updating strategies more intuitively, this section focuses on developing and analyzing the static and incremental algorithms for maintaining the three-way regions based on the matrix when objects and attributes in CIHIS increase simultaneously.

6.1. The static algorithm for updating the three-way regions

Algorithm 1, abbreviated as MSACTR, is a matrix-based static algorithm for the update of the three-way regions with the simultaneous variation of objects and attributes in CIHIS. Step 2 is to calculate the partition determined by the decision attribute set D, whose time complexity is $O((n+n')^2)$. Based on this partition, step 3 aims at constructing the decision matrix $\mathbb{D}^{t+1}$ at time $t+1$ by Definition 4.2, and its time complexity is $O((m+m')(n+n'))$. Steps 4-11 are the computation of the double-value relation matrix $\mathbb{M}_{U^{t+1}}^{A^{t+1}}$ at time $t+1$, in which Steps 4-10 are building the double-value relation matrix with respect to each attribute $a \in A^{t+1}$ according to Definition 4.3 and Step 11 is computing the double-value relation matrix in terms of the attribute set A^{t+1} according to Lemma 4.1. Without doubt, the total time complexity is $O((l+l')(n+n')^2)$. Step 12 is the calculation of the induced diagonal matrix $\Gamma_{U^{t+1}}^{A^{t+1}}$ at time $t+1$ by Definition 4.6, whose time complexity is $O((n+n')^2)$. Then, on the basis of the three matrices generated from the above steps, the condition matrix $\Phi_{U^{t+1}}^{A^{t+1}}$ at time $t+1$ can be easily computed by Definition 4.8 in Steps 13-18, and the time complexity is $O((m+m')(n+n')^2)$. Step 19 aims at constructing the loss matrix L_{t+1}^{θ} at time $t+1$ according to Definition 4.9, whose time complexity is $O((n+n'))$. Step 20-26 and Step 27 are to compute two threshold vectors and three cut matrices at time $t+1$ in the light of Definitions 4.11 and 4.13, respectively, and the corresponding time complexities are $O((n+n')^2)$ and $O((m+m')(n+n'))$, respectively. Then, based on the three cut matrices, it can directly be obtained the characteristic vectors of the three-way regions according to Theorem 4.1 in Step 28, whose time complexity is $O((m+m')(n+n'))$. Therefore, the total time complexity of Algorithm MSACTR is $O((l+l'+m+m')(n+n')^2)$.

Algorithm 1: Matrix-based static algorithm for computing three-way regions when adding objects and attributes simultaneously (MSACTR)

Input:

(1) A CIHIS $\mathbb{S}^t = (S^t, \bar{\lambda}^t, g^t)$ at time t, where $S^t = (U^t, A^t \cup D, V^t, f^t)$, $U^t = \{u_1, u_2, \cdots, u_n\}$, $A^t = \{a_1, a_2, \cdots, a_l\}$;

(2) The four parameters δ, κ_1, κ_2 and θ;

(3) The newly added object set $\Delta U = \{u_{n+1}, u_{n+2}, \cdots, u_{n+n'}\}$ and attribute set $\Delta A = \{a_{l+1}, a_{l+2}, \cdots, a_{l+l'}\}$.

Output: The updated three-way regions: $POS^{(\bar{\alpha},\bar{\beta})}_{A^{t+1}}(D)$, $BND^{(\bar{\alpha},\bar{\beta})}_{A^{t+1}}(D)$ and $NEG^{(\bar{\alpha},\bar{\beta})}_{A^{t+1}}(D)$.

```
begin
    Compute the partition determined by D: U^{t+1}/D = {D_1^{t+1}, D_2^{t+1}, ···, D_{m+m'}^{t+1}};
    Construct the decision matrix 𝔻^{t+1} = [d_{ij}^{t+1}]_{(n+n')×(m+m')} according to Definition 4.2;
    for 1 ≤ i ≤ n + n' do  // Construct the double-value relation matrix 𝕄^a_{U^{t+1}} = [m^a_{ij}]_{(n+n')×(n+n')} by Definition 4.3;
        for 1 ≤ j ≤ n + n' do
            for a ∈ A^{t+1} do
                Compute m^a_{ij};
            end
        end
    end
    Compute the double-value relation matrix 𝕄^{A^{t+1}}_{U^{t+1}} = [m^{A^{t+1}}_{ij}]_{(n+n')×(n+n')} according to Lemma 4.1;
    Compute the induced diagonal matrix Γ^{A^{t+1}}_{U^{t+1}} = diag[1/γ_1^{A^{t+1}}, 1/γ_2^{A^{t+1}}, ···, 1/γ_{n+n'}^{A^{t+1}}] according to Definition 4.6;
    for 1 ≤ i ≤ n + n' do          // Compute the condition matrix Φ^{A^{t+1}}_{U^{t+1}} = [φ^{A^{t+1}}_{ij}]_{(n+n')×(m+m')} by Definition 4.8;
        for 1 ≤ j ≤ m + m' do
            χ^{A^{t+1}}_{ij} = Σ_{k=1}^{n+n'} F(m^{A^{t+1}}_{ik}) d^{t+1}_{kj};     // Compute 𝕏^{A^{t+1}}_{U^{t+1}} = F(𝕄^{A^{t+1}}_{U^{t+1}}) • 𝔻^{t+1} = [χ^{A^{t+1}}_{ij}]_{(n+n')×(m+m')};
        end
        φ^{A^{t+1}}_{ij} = χ^{A^{t+1}}_{ij} / γ_i^{A^{t+1}};
    end
    Construct the loss matrix L^θ_{t+1} = [λ^{t+1}_{ij}]_{(n+n')×6} according to Definition 4.9;
    for 1 ≤ i ≤ n + n' do   // Compute two threshold vectors 𝕋^α_{t+1} = [ι_i^{t+1}]_{(n+n')×1} and 𝕋^β_{t+1} = [o_i^{t+1}]_{(n+n')×1} by Definition 4.11;
        ζ_i^{t+1} = Σ_{j=1}^{n+n'} F(m^{A^{t+1}}_{ij})(λ^{t+1}_{j2} − λ^{t+1}_{j1});
        η_i^{t+1} = Σ_{j=1}^{n+n'} F(m^{A^{t+1}}_{ij})(λ^{t+1}_{j3} − λ^{t+1}_{j2});
        ϑ_i^{t+1} = Σ_{j=1}^{n+n'} F(m^{A^{t+1}}_{ij})(λ^{t+1}_{j5} − λ^{t+1}_{j4});
        ξ_i^{t+1} = Σ_{j=1}^{n+n'} F(m^{A^{t+1}}_{ij})(λ^{t+1}_{j6} − λ^{t+1}_{j5});
        ι_i^{t+1} = ξ_i^{t+1} / (ζ_i^{t+1} + ξ_i^{t+1}), o_i^{t+1} = ϑ_i^{t+1} / (η_i^{t+1} + ϑ_i^{t+1});
    end
    Compute three cut matrices of Ψ̌^{A^{t+1}}_{U^{t+1}}, Ψ̄^{A^{t+1}}_{U^{t+1}} and Ψ̂^{A^{t+1}}_{U^{t+1}} according to Definition 4.13;
    Compute the characteristic vectors of three-way regions G(POS^{(ᾱ,β̄)}_{A^{t+1}}(D)), G(BND^{(ᾱ,β̄)}_{A^{t+1}}(D)) and G(NEG^{(ᾱ,β̄)}_{A^{t+1}}(D)) according to Theorem 4.1;
    Output the updated three-way regions: POS^{(ᾱ,β̄)}_{A^{t+1}}(D), BND^{(ᾱ,β̄)}_{A^{t+1}}(D) and NEG^{(ᾱ,β̄)}_{A^{t+1}}(D).
end
```

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6.2. The incremental algorithm for updating the three-way regions

From the incremental updating principles discussed in Section 5, it can distinctly be found that the core step of dynamically maintaining three-way regions in the generalized three-way neighborhood decision model is the update of the double-value relation matrix under the simultaneous variation of objects and attributes. To this end, a matrix-based incremental algorithm for updating the double-value relation matrix is devised based on Theorem 5.2, which is summarized in Algorithm 2. Subsequently, an incremental algorithm for the maintenance of the three-way regions based on matrix is developed by cooperation with Algorithm 2 when new objects and attributes are appended to CIHIS simultaneously, which is outlined in Algorithm 3, abbreviated as MIAUTR. Step 2 is to initialize these related matrices with the achieved knowledge at time t. Step 3 is the update of the partition determined by decision attribute set D with the addition of objects, whose time complexity is $O((m+n')n')$. On this basis, Steps 4-9 are the update of the decision matrix $\mathbb{D}^{t+1}$ in accordance with Theorem 5.1, whose time complexity is $O((m+m')n')$. Step 10 is the update of the double-value relation matrix $\mathbb{M}_{U^{t+1}}^{A^{t+1}}$ in line with Algorithm 2, and the time complexity is $O(((n+n')^2-\varepsilon_0)(l+l')-\varepsilon_1 l)$, where ε_0 and ε_1 denote the number of elements "$(0,0)$" and "$(1,0)$" in $\mathbb{M}_{U^t}^{A^t}$, respectively, and $\varepsilon_0+\varepsilon_1 \leq n^2$. Step 11 is the update of the loss matrix L_{t+1}^{θ} by Theorem 5.3, whose time complexity is $O(n')$. Combined with the updated results of the above matrices, the condition matrix $\Phi_{U^{t+1}}^{A^{t+1}}$, the α-threshold vector $\mathbb{T}_{t+1}^{\alpha}$ and the β-threshold vector $\mathbb{T}_{t+1}^{\beta}$ can be easily calculated based on the incremental strategies described in Corollaries 5.1, 5.2 and 5.3 by Steps 12-50, and the time complexity is $O((\eta+nn')m+(n+n')n'(m+m'))$, where η expresses the number of "$r_{ij}^{A^t}$" whose value changes during the update of the extended relation matrix, and $\eta \leq n^2$. In particular, Steps 12-36 and Steps 37-50 aim at updating and computing the original elements and new elements in $\Gamma_{U^{t+1}}^{A^{t+1}}$, $\mathbb{X}_{U^{t+1}}^{A^{t+1}}$, $\mathbb{H}_{t+1}^{(1,2)}$, $\mathbb{H}_{t+1}^{(2,3)}$, $\mathbb{H}_{t+1}^{(4,5)}$ and $\mathbb{H}_{t+1}^{(5,6)}$, respectively. Then, in line with Definition 4.13 and Theorem 4.1, the three cut matrices and the characteristic vectors of three-way regions can be simply computed in Step 51 and Step 52, respectively, and their time complexities are all $O((n+n')(m+m'))$. Consequently, the total time complexity of Algorithm MIAUTR is $O(((n+n')^2-\varepsilon_0)(l+l')-\varepsilon_1 l+(\eta+nn')m+(n+n')n'(m+m'))$, which is less than that of Algorithm MSACTR when objects and attributes increase with time simultaneously.

7. Experimental evaluations

To reflect the superiority of the proposed incremental algorithm in updating three-way regions of the generalized three-way neighborhood decision model when objects and attributes increase simultaneously, several groups of comparative experiments are carried out on nine different datasets in this section. In these datastes, six real datasets Molecular Biology (MB), Mushroom (MR), Statlog Image Segmentation (SIS), Ozone Level Detection (OLD), Arrhythmia (AR) and Cylinder Bands (CB) come from the machine learning data repository, University of California at Irvine (UCI) (http://archive.ics.uci.edu/ml/), and three artificial datasets with the missing categorical and numerical attribute values named CIHD1, CIHD2 and CIHD3 are constructed in the light of the method in [7]. The detailed descriptions about the nine datasets are summed up in Table 6. However, considering that these data from real applications can not be directly applied to the comparative experiments, a series of preprocessing steps are adopted to restructure the six real datasets. With respect to the complete datasets without missing values, i.e., MB and SIS, there are 2% of known attribute values transformed to the unknown ones at random, in which half of the unknown attribute values are "the lost" values and the other half are "the do not care" values. With respect to the incomplete datasets with missing values, i.e., MR, OLD, AR and CB, half of the existing "the lost" values are converted to "the do not care" values at random. Subsequently, for the datasets with numerical attributes, we employ a normalization method, that is, $\tilde{f}(u,a)=(f(u,a)-\min(V_a))/(\max(V_a)-\min(V_a))$, to recalculate these numerical attribute values. Moreover, the interval-valued loss functions are constructed on the basis of condition (9), and then appended to each object in the nine datasets. The related algorithms are implemented by MATLAB R2016a, and all comparative experiments are performed on a personal computer with 64-bit Windows 10 operation system and Intel (R) Core(TM) i7-8700 CPU 3.20GHZ, 16.0 GB of memory.

The performance comparisons between static and incremental algorithms for maintaining three-way regions in the generalized three-way neighborhood decision model will be demonstrated from the following two aspects when objects and attributes in CIHIS evolve over time concurrently. One is the comparison of running time between Algorithms MSACTR and MIAUTR with the different sizes of datasets when the increment ratio remains unchanged,

Algorithm 2: Matrix-based incremental algorithm for updating the double-value relation matrix when adding objects and attributes simultaneously

Input:

(1) A CIHIS $\mathbb{S}^t = (S^t, \bar{\lambda}^t, g^t)$ at time t, where $S^t = (U^t, A^t \cup D, V^t, f^t)$, $U^t = \{u_1, u_2, \cdots, u_n\}$ and $A^t = \{a_1, a_2, \cdots, a_l\}$;

(2) The four parameters δ, κ_1, κ_2 and θ;

(3) The result of the double-value relation matrix at time t: $\mathbb{M}_{U^t}^{A^t}$;

(4) The added object set $\Delta U = \{u_{n+1}, u_{n+2}, \cdots, u_{n+n'}\}$ and attribute set $\Delta A = \{a_{l+1}, a_{l+2}, \cdots, a_{l+l'}\}$.

Output: The updated double-value relation matrix $\mathbb{M}_{U^{t+1}}^{A^{t+1}}$.

begin
    $\mathbb{M}_{U^{t+1}}^{A^{t+1}} \leftarrow \mathbb{M}_{U^t}^{A^t}$;
    for $1 \le i \le n$ **do** `// Update the first` n `rows of the double-value relation matrix` $\mathbb{M}_{U^{t+1}}^{A^{t+1}}$ `by Theorem 5.2;`
        for $1 \le j \le n$ **do**
            Compute $m_{ij}^{\Delta A}$;
            if $s_{ij}^{A^t} == 0$ **then**
                if $r_{ij}^{A^t} == 1$ **then** $m_{ij}^{A^{t+1}} = m_{ij}^{\Delta A}$;
            else
                if $r_{ij}^{\Delta A} == 0$ **then**
                    $m_{ij}^{A^{t+1}} = m_{ij}^{\Delta A}$;
                else
                    if $u_j \in [\mathcal{N}_{A^t}^{\delta}]^{U^{t+1}}(u_i)$ **then** $m_{ij}^{A^{t+1}} = (1, 1)$;
                    else $m_{ij}^{A^{t+1}} = (0, 1)$;
                end
            end
        end
        for $n + 1 \le j \le n + n'$ **do**
            Compute $m_{ij}^{A^{t+1}}$;
        end
    end
    for $n + 1 \le i \le n + n'$ **do** `// Update the last` n' `rows of the double-value relation matrix` $\mathbb{M}_{U^{t+1}}^{A^{t+1}}$ `by Theorem 5.2;`
        for $1 \le j \le n + n'$ **do**
            Compute $m_{ij}^{A^{t+1}}$;
        end
    end
    Output the updated double-value relation matrix $\mathbb{M}_{U^{t+1}}^{A^{t+1}}$.
end

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Algorithm 3: Matrix-based incremental algorithm for updating three-way regions when adding objects and attributes simultaneously (MIAUTR)

Input: (1) A CIHIS $\mathbb{S}^t = (S^t, \bar{\lambda}^t, g^t)$ at time t, where $S^t = (U^t, A^t \cup D, V^t, f^t)$, $U^t = \{u_1, u_2, \cdots, u_n\}$, $A^t = \{a_1, a_2, \cdots, a_l\}$, and $U^t/D = \{D^t_1, D^t_2, \cdots, D^t_m\}$; (2) The four parameters δ, κ_1, κ_2 and θ; (3) The related results at time t: $\mathbb{D}^t$, $\mathbb{M}^{A^t}_{U^t}$, $\Gamma^{A^t}_{U^t}$, $\mathbb{X}^{A^t}_{U^t}$, L^θ_t, $\mathbb{H}^{(1,2)}_t$, $\mathbb{H}^{(2,3)}_t$, $\mathbb{H}^{(4,5)}_t$, and $\mathbb{H}^{(5,6)}_t$; (4) The added object set $\Delta U = \{u_{n+1}, u_{n+2}, \cdots, u_{n+n'}\}$ and attribute set $\Delta A = \{a_{l+1}, a_{l+2}, \cdots, a_{l+l'}\}$.

Output: The updated three-way regions: $POS^{(\bar{\alpha},\bar{\beta})}_{A^{t+1}}(D)$, $BND^{(\bar{\alpha},\bar{\beta})}_{A^{t+1}}(D)$ and $NEG^{(\bar{\alpha},\bar{\beta})}_{A^{t+1}}(D)$.

begin
- $\mathbb{D}^{t+1} \leftarrow \mathbb{D}^t$, $\mathbb{M}^{A^{t+1}}_{U^{t+1}} \leftarrow \mathbb{M}^{A^t}_{U^t}$, $\Gamma^{A^{t+1}}_{U^{t+1}} \leftarrow \Gamma^{A^t}_{U^t}$, $\mathbb{X}^{A^{t+1}}_{U^{t+1}} \leftarrow \mathbb{X}^{A^t}_{U^t}$, $L^\theta_{t+1} \leftarrow L^\theta_t$, $\mathbb{H}^{(1,2)}_{t+1} \leftarrow \mathbb{H}^{(1,2)}_t$, $\mathbb{H}^{(2,3)}_{t+1} \leftarrow \mathbb{H}^{(2,3)}_t$, $\mathbb{H}^{(4,5)}_{t+1} \leftarrow \mathbb{H}^{(4,5)}_t$, $\mathbb{H}^{(5,6)}_{t+1} \leftarrow \mathbb{H}^{(5,6)}_t$;
- **Update** the partition induced by D at time $t+1$: $U^{t+1}/D = \{D^{t+1}_1, D^{t+1}_2, \cdots, D^{t+1}_m, \cdots, D^{t+1}_{m+m'}\}$;
- **for** $n+1 \le i \le n+n'$ **do** `// Update the decision matrix` $\mathbb{D}^{t+1} = [d^{t+1}_{ij}]_{(n+n')\times(m+m')}$ `by Theorem 5.1;`
  - **for** $1 \le j \le m+m'$ **do**
    - **if** $u_i \in D^{t+1}_j$ **then** $d^{t+1}_{ij} = 1$;
    - **else** $d^{t+1}_{ij} = 0$;
  - **end**
- **end**
- **Update** the double-value relation matrix $\mathbb{M}^{A^{t+1}}_{U^{t+1}} = [m^{A^{t+1}}_{ij}]_{(n+n')\times(n+n')}$ according to Algorithm 2;
- **Update** the loss matrix $L^\theta_{t+1} = [\lambda^{t+1}_{ij}]_{(n+n')\times 6}$ according to Theorem 5.3;
- **for** $1 \le i \le n$ **do**
  - **for** $1 \le j \le n$ **do**
    - **if** $F(m^{A^t}_{ij}) \oplus F(m^{A^{t+1}}_{ij}) == 1$ **then**
      - **if** $F(m^{A^t}_{ij}) == 0$ *and* $F(m^{A^{t+1}}_{ij}) == 1$ **then**
        - $\gamma^{A^{t+1}}_i = \gamma^{A^{t+1}}_i + 1$;
        - $\zeta^{t+1}_i = \zeta^{t+1}_i + (\lambda^t_{j2} - \lambda^t_{j1})$, $\eta^{t+1}_i = \eta^{t+1}_i + (\lambda^t_{j3} - \lambda^t_{j2})$, $\vartheta^{t+1}_i = \vartheta^{t+1}_i + (\lambda^t_{j5} - \lambda^t_{j4})$, $\xi^{t+1}_i = \xi^{t+1}_i + (\lambda^t_{j6} - \lambda^t_{j5})$;
        - **for** $1 \le k \le m$ **do**
          - $\chi^{A^{t+1}}_{ik} = \chi^{A^{t+1}}_{ik} + d^t_{jk}$;
        - **end**
      - **else**
        - $\gamma^{A^{t+1}}_i = \gamma^{A^{t+1}}_i - 1$;
        - $\zeta^{t+1}_i = \zeta^{t+1}_i - (\lambda^t_{j2} - \lambda^t_{j1})$, $\eta^{t+1}_i = \eta^{t+1}_i - (\lambda^t_{j3} - \lambda^t_{j2})$, $\vartheta^{t+1}_i = \vartheta^{t+1}_i - (\lambda^t_{j5} - \lambda^t_{j4})$, $\xi^{t+1}_i = \xi^{t+1}_i - (\lambda^t_{j6} - \lambda^t_{j5})$;
        - **for** $1 \le k \le m$ **do**
          - $\chi^{A^{t+1}}_{ik} = \chi^{A^{t+1}}_{ik} - d^t_{jk}$;
        - **end**
      - **end**
    - **end**
  - **end**
  - $\gamma^{A^{t+1}}_i = \gamma^{A^{t+1}}_i + \sum_{j=n+1}^{n+n'} F(m^{A^{t+1}}_{ij})$;
  - $\zeta^{t+1}_i = \zeta^{t+1}_i + \sum_{j=n+1}^{n+n'} F(m^{A^{t+1}}_{ij})(\lambda^{t+1}_{j2} - \lambda^{t+1}_{j1})$, $\eta^{t+1}_i = \eta^{t+1}_i + \sum_{j=n+1}^{n+n'} F(m^{A^{t+1}}_{ij})(\lambda^{t+1}_{j3} - \lambda^{t+1}_{j2})$;
  - $\vartheta^{t+1}_i = \vartheta^{t+1}_i + \sum_{j=n+1}^{n+n'} F(m^{A^{t+1}}_{ij})(\lambda^{t+1}_{j5} - \lambda^{t+1}_{j4})$, $\xi^{t+1}_i = \xi^{t+1}_i + \sum_{j=n+1}^{n+n'} F(m^{A^{t+1}}_{ij})(\lambda^{t+1}_{j6} - \lambda^{t+1}_{j5})$;
  - **for** $1 \le k \le m$ **do**
    - $\chi^{A^{t+1}}_{ik} = \chi^{A^{t+1}}_{ik} + \sum_{j=n+1}^{n+n'} F(m^{A^{t+1}}_{ij}) d^{t+1}_{jk}$, $\phi^{A^{t+1}}_{ik} = \chi^{A^{t+1}}_{ik} / \gamma^{A^{t+1}}_i$;
  - **end**
- **end**
- **for** $n+1 \le i \le n+n'$ **do**
  - $\gamma^{A^{t+1}}_i = \sum_{j=1}^{n+n'} F(m^{A^{t+1}}_{ij})$;
  - $\zeta^{t+1}_i = \sum_{j=1}^{n} F(m^{A^{t+1}}_{ij})(\lambda^t_{j2} - \lambda^t_{j1}) + \sum_{j=n+1}^{n+n'} F(m^{A^{t+1}}_{ij})(\lambda^{t+1}_{j2} - \lambda^{t+1}_{j1})$, $\eta^{t+1}_i = \sum_{j=1}^{n} F(m^{A^{t+1}}_{ij})(\lambda^t_{j3} - \lambda^t_{j2}) + \sum_{j=n+1}^{n+n'} F(m^{A^{t+1}}_{ij})(\lambda^{t+1}_{j3} - \lambda^{t+1}_{j2})$;
  - $\vartheta^{t+1}_i = \sum_{j=1}^{n} F(m^{A^{t+1}}_{ij})(\lambda^t_{j5} - \lambda^t_{j4}) + \sum_{j=n+1}^{n+n'} F(m^{A^{t+1}}_{ij})(\lambda^{t+1}_{j5} - \lambda^{t+1}_{j4})$, $\xi^{t+1}_i = \sum_{j=1}^{n} F(m^{A^{t+1}}_{ij})(\lambda^t_{j6} - \lambda^t_{j5}) + \sum_{j=n+1}^{n+n'} F(m^{A^{t+1}}_{ij})(\lambda^{t+1}_{j6} - \lambda^{t+1}_{j5})$;
  - **for** $1 \le k \le m$ **do**
    - $\chi^{A^{t+1}}_{ik} = \sum_{j=1}^{n} F(m^{A^{t+1}}_{ij}) d^t_{jk} + \sum_{j=n+1}^{n+n'} F(m^{A^{t+1}}_{ij}) d^{t+1}_{jk}$, $\phi^{A^{t+1}}_{ik} = \chi^{A^{t+1}}_{ik} / \gamma^{A^{t+1}}_i$;
  - **end**
- **end**
- **for** $1 \le i \le n+n'$ **do**
  - $\iota^{t+1}_i = \xi^{t+1}_i / (\zeta^{t+1}_i + \xi^{t+1}_i)$, $o^{t+1}_i = \vartheta^{t+1}_i / (\eta^{t+1}_i + \vartheta^{t+1}_i)$;
  - **for** $m+1 \le k \le m+m'$ **do**
    - $\chi^{A^{t+1}}_{ik} = \sum_{j=n+1}^{n+n'} F(m^{A^{t+1}}_{ij}) d^{t+1}_{jk}$, $\phi^{A^{t+1}}_{ik} = \chi^{A^{t+1}}_{ik} / \gamma^{A^{t+1}}_i$;
  - **end**
- **end**
- **Compute** three cut matrices of $\check{\Psi}^{A^{t+1}}_{U^{t+1}}$, $\bar{\Psi}^{A^{t+1}}_{U^{t+1}}$ and $\hat{\Psi}^{A^{t+1}}_{U^{t+1}}$ according to Definition 4.13;
- **Compute** the characteristic vectors of three-way regions $G(POS^{(\bar{\alpha},\bar{\beta})}_{A^{t+1}}(D))$, $G(BND^{(\bar{\alpha},\bar{\beta})}_{A^{t+1}}(D))$ and $G(NEG^{(\bar{\alpha},\bar{\beta})}_{A^{t+1}}(D))$ by Theorem 4.1;
- **Output** the updated three-way regions: $POS^{(\bar{\alpha},\bar{\beta})}_{A^{t+1}}(D)$, $BND^{(\bar{\alpha},\bar{\beta})}_{A^{t+1}}(D)$ and $NEG^{(\bar{\alpha},\bar{\beta})}_{A^{t+1}}(D)$.

end

Table 6: The descriptions of datasets

No.	Data Sets	Samples	Attributes			Missing Values	Classes
			Categorical	Numerical	Total		
1	MB	3190	60	0	60	3828	3
2	MR	8124	22	0	22	2480	2
3	SIS	2310	0	18	18	832	7
4	OLD	2536	0	72	72	14938	2
5	AR	452	73	206	279	408	16
6	CB	540	19	20	39	999	2
7	CIHD1	5000	20	20	40	4000	5
8	CIHD2	3000	30	30	60	3600	4
9	CIHD3	2000	50	50	100	4000	3

and the other is the comparison of running time between Algorithms MSACTR and MIAUTR with the different increment ratios when the sizes of original datasets keep constant. Meanwhile, the performance comparisons between the proposed incremental algorithm and other incremental algorithms will also be shown. For convenience, in the comparative experiments, the parameters κ_1, κ_2 and δ to compute the data-driven neighborhood relation are always set as 0.5, 0.5 and 0.3, and the risk attitude of decision maker θ is always set as 0.5. Besides, several experiments will be implemented to study the effect of these parameters on the computational time of Algorithm MIAUTR.

7.1. *Performance comparison between static and incremental algorithms with the increasing sizes of datasets*

This subsection concentrates on comparing the computational performance of the proposed incremental algorithm (Algorithm MIAUTR) with that of the static algorithm (Algorithm MSACTR) when the sizes of datasets increase gradually but the increment ratio of adding objects and attributes keeps unchanged. For each dataset listed in Table 6, we picked out 15%, 30%, 45%, 60% and 75% objects and attributes to make up five original datasets, which are viewed as original dataset 1, original dataset 2, ..., original dataset 5, respectively. In succession, we choose objects and attributes that the size is 20% of each original dataset from the remaining part of each dataset at random to append to the corresponding original dataset. The visual results of comparative experiments on the nine datasets are displayed in Figures 2 and 3, respectively.

Figure 2 exhibits the running time of Algorithms MSACTR and MIAUTR for updating the three-way regions with the increasing sizes of the original datasets when adding objects and attributes simultaneously. With respect to each sub-figure in Figure 2, the x-coordinate and y-coordinate pertain to the five gradually growing original datasets and the computational time of Algorithms MSACTR and MIAUTR, respectively. Based on the trend lines in Figure 2, it can easily be found that the running time of two algorithms raises with the amplification of the scale of the original dataset. In particular, the running time of Algorithm MIAUTR is always lower than that of Algorithm MSACTR, and the differences between the static and incremental algorithms become more and more obvious as the sizes of the original datasets continue to grow. In addition, Figure 3 describes the speedup ratios between Algorithms MSACTR and MIAUTR on nine different datasets when objects and attributes of the same increment ratio are added to the five different original datasets, where the x-coordinate and y-coordinate concern the nine different datasets and the speedup ratio of the two algorithms, respectively. From Figure 3, it is apparent that the speedup ratio with respect to each original dataset of the nine incomplete datasets is always greater than 1, which indicates that the proposed incremental algorithm has a significant advantage in updating the three-way regions compared with the static algorithm when the sizes of the original datasets grow with time.

7.2. *Performance comparison between static and incremental algorithms with the increasing increment ratios*

This subsection concentrates on comparing the computational performance of Algorithm MIAUTR with that of Algorithm MSACTR when the increment ratio of adding objects and attributes grows with time but the sizes of the original datasets keep constant. With regard to each dataset shown in Table 6, the 50% objects and attributes

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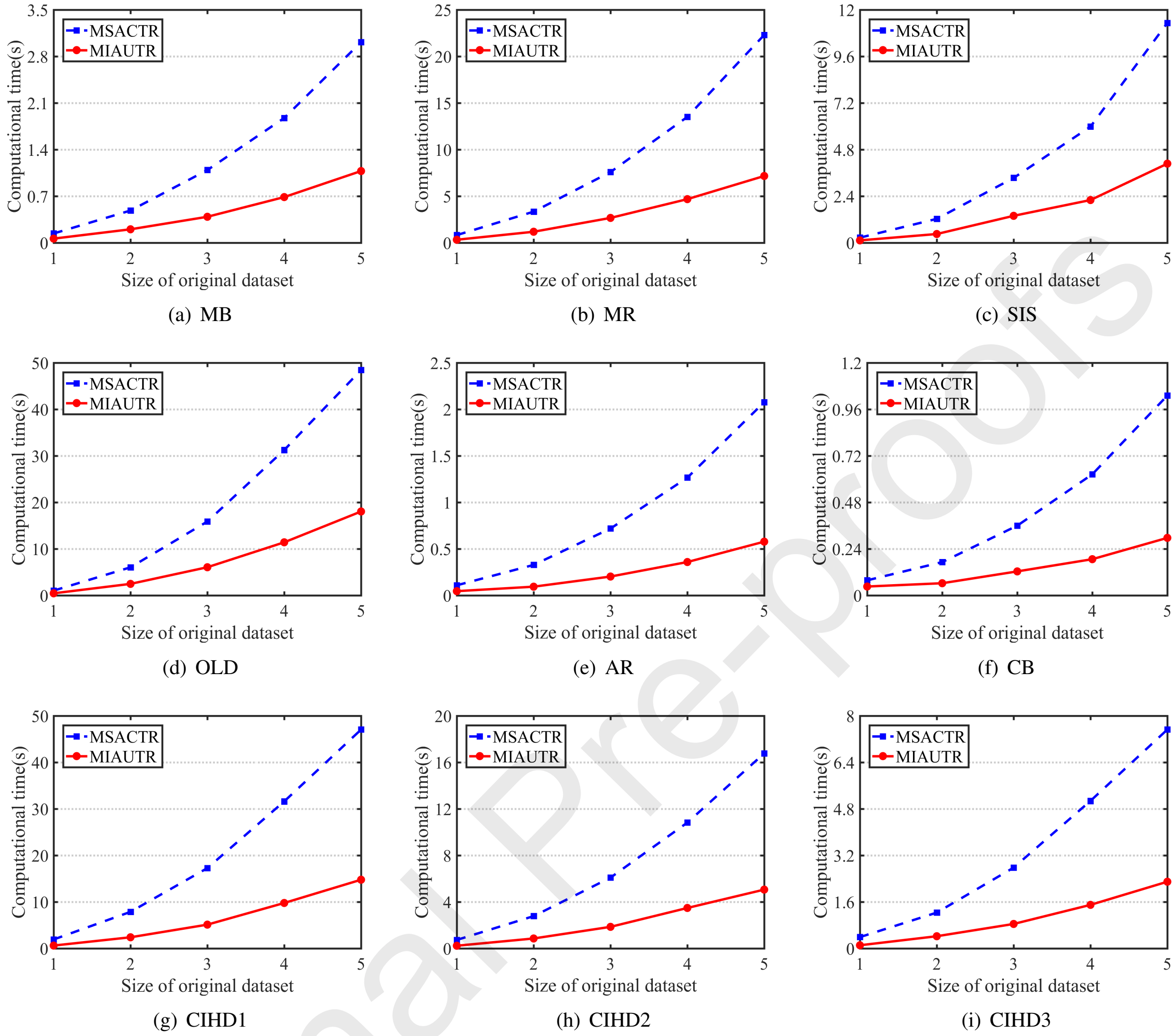

Figure 2: The comparison of computational time between MSACTR and MIAUTR versus different sized datasets.

are randomly taken out to comprise the original dataset. Meanwhile, 20%, 40%, 60%, 80% and 100% objects and attributes are randomly selected from the remaining objects and attributes to form five added datasets. To demonstrate the superiority of the proposed incremental algorithm on updating three-way regions in the generalized three-way neighborhood decision model intuitively, the running time of Algorithms MSACTR and MIAUTR is depicted in Figure 4 when appending objects and attributes of the different increment ratios. In each sub-figure of Figure 4, the x-coordinate corresponds to the increment ratios of adding objects and attributes, and the y-coordinate pertains to the computational time of two algorithms.

According to the changing trends in Figure 4, we observe that the computational time of the static and the proposed incremental algorithms always raises when the increment ratio of adding objects and attributes increases consistently, and the incremental algorithm is invariably faster than the static algorithm. Furthermore, the speedup ratios between Algorithms MSACTR and MIAUTR on the nine datasets are provided in Figure 5 when the different amounts of objects and attributes are added into the original dataset. In line with the experimental results in Figure 5, it is apparent to find that the speedup ratios between two algorithms decrease with the increase of increment ratio, and are always greater than 1. Consequently, Algorithm MIAUTR can be regarded as an effective way for the maintenance of three-

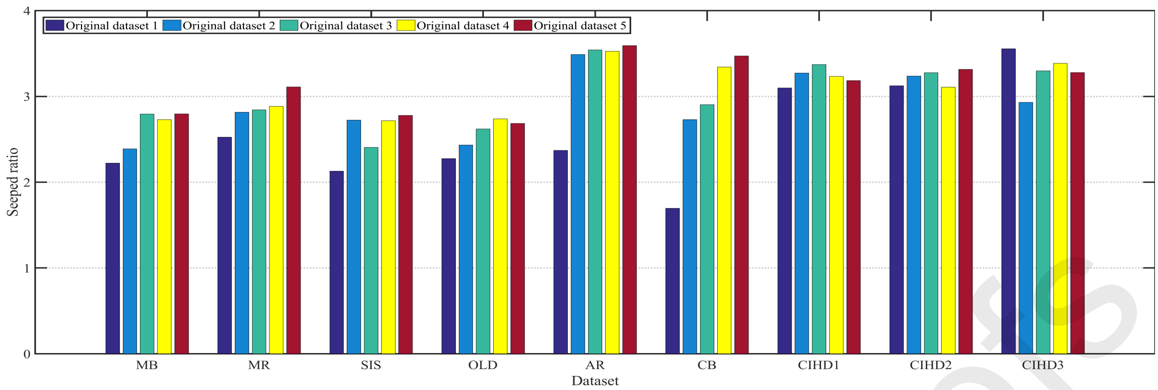

Figure 3: The speedup ratio between MSACTR and MIAUTR versus different sized datasets.

way regions in the generalized three-way neighborhood decision model when objects and attributes are appended to CIHIS simultaneously.

7.3. Performance comparison with other incremental algorithms

To process evolving data with the simultaneous variation of objects and attributes efficiently, a number of incremental approaches have been developed for facilitating knowledge maintenance under dynamic environments. However, these approaches could not be used to incrementally update three-way regions in the generalized three-way neighborhood decision model directly when both objects and attributes in CIHIS continue to increase. Although there are no related incremental algorithms for maintaining the three-way regions with the simultaneous variation of objects and attributes in CIHIS, Huang et al. [7, 8] constructed a matrix-based incremental framework for updating approximations when attributes in IHIS vary with time (referred to as IUA for short) and introduced a dynamic algorithm based on matrix to maintain the three-way regions when objects in IHIS change over time (referred to as IUO for short), respectively. For comparing Algorithm MIAUTR with these two incremental algorithms that applied to handle IHIS with single-dimensional variation, Algorithm IUA and Algorithm IUO are combined together to dynamically update the three-way regions with the simultaneous increase of objects and attributes, which is abbreviated as IUA+IUO in this paper.

Figure 6 shows the running time of Algorithms MIAUTR and IUA+IUO for computing the three-way regions when the increment ratio of adding objects and attributes increases gradually. The x-coordinate and y-coordinate in each sub-figure of Figure 6 pertain to the increment ratios of dataset and the computational time of Algorithms MIAUTR and IUA+IUO, respectively. From Figure 6, it can easily be seen that the running time of the two algorithms rises with the increase of the increment ratio, and the proposed algorithm MIAUTR is better than the combination algorithm IUA+IUO. Furthermore, the speedup ratios between IUA+IUO and MIAUTR on the nine datasets are shown in Figure 7 when adding objects and attributes of different increment ratios. Based on the experimental results in Figure 7, we can easily observe that the speedup ratios between the two incremental algorithms decrease with the increase of increment ratio, and are always greater than 1. Accordingly, when objects and attributes in CIHIS evolve over time simultaneously, our proposed algorithm has superiority in the maintenance of the three-way regions compared with the combination algorithm.

7.4. Experimental evaluation under the variation of parameters

According to the complexity analysis presented in Section 6, it can easily be found that the update of the double-value relation matrix and the corresponding induced matrices has a major influence on the computational performance of the proposed incremental algorithm, which means that the impact of parameter θ on the running time can be ignored compared with parameters κ_1, κ_2 and δ employed for calculating the data-driven neighborhood relation. Therefore, this subsection merely focuses on discussing the effect of parameters κ_1, κ_2 and δ on the time efficiency of the incremental algorithm when objects and attributes are added into CIHIS simultaneously. For simplicity, assume that κ_1 is always equal to κ_2, i.e., $\kappa = \kappa_1 = \kappa_2$, and $\delta_a \in \delta$ is same for each numerical attribute a.

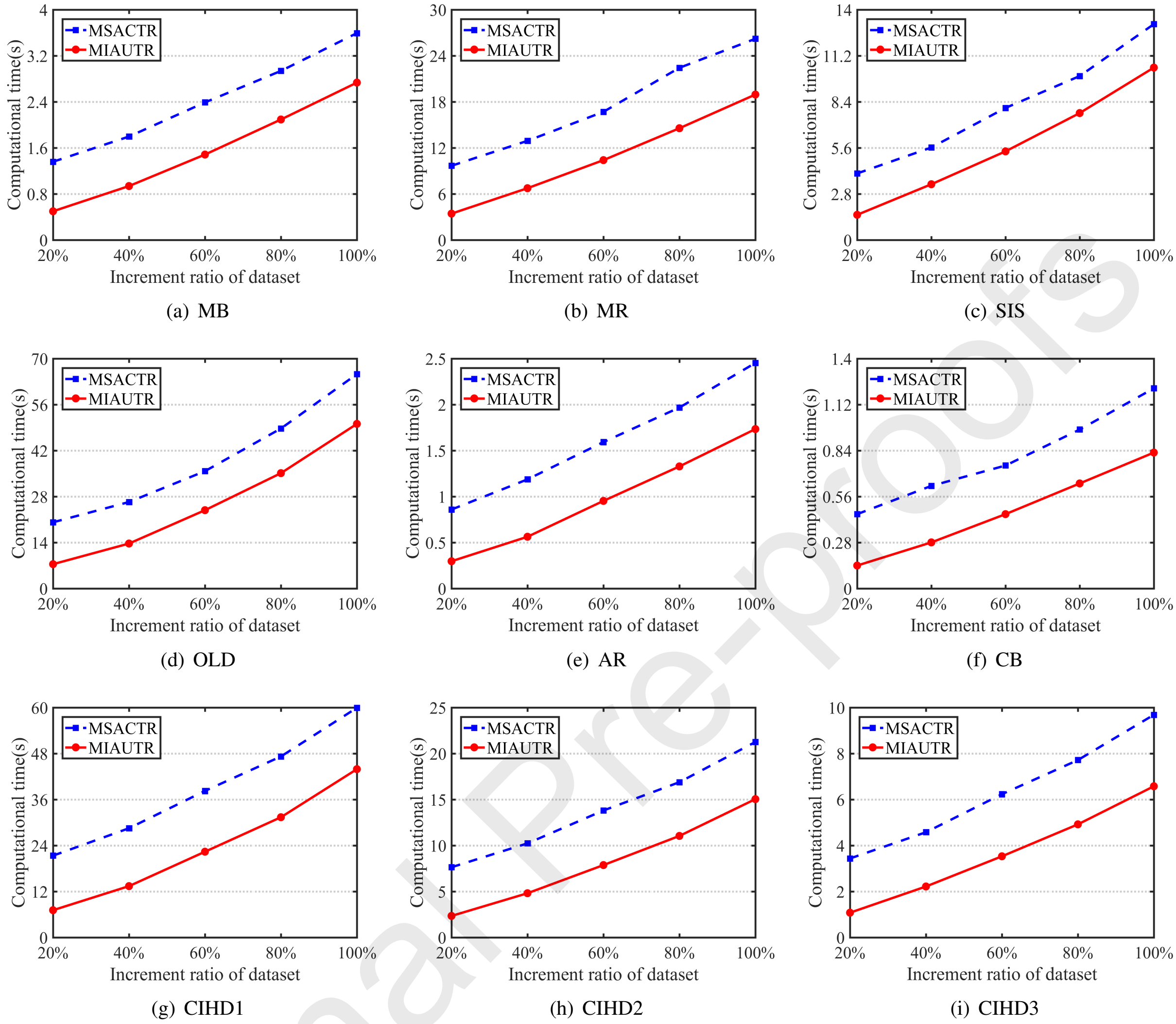

Figure 4: The comparison of computational time between MSACTR and MIAUTR versus different increment ratios.

For each incomplete dataset listed in Table 6, 90% objects and attributes are extracted as the original dataset at random, and the remainder is viewed as the added dataset. The parameters κ and δ increase gradually from 0.1 to 0.9. Figures 8 and 9 depict the detailed variation tendencies with regard to the computational time of the incremental algorithm with the changing of parameters κ and δ, respectively, where the x-coordinate respectively concerns the parameters κ and δ, and the y-coordinate pertains to the computational time of Algorithm MIAUTR.

As demonstrated in Figure 8, we can evidently observe that the running time of Algorithm MIAUTR on nine different incomplete datasets emerges a faint downward trend with the growth of parameter κ, which means that the variation of parameter κ has a slight influence on the computational time of Algorithm MIAUTR when adding objects and attributes simultaneously. Meanwhile, from Figure 9, it can easily be found that the running time of Algorithm MIAUTR on datasets consisting of categorical attributes, i.e., MB and MR, fluctuates a little versus different parameter δ, but that on the rest six datasets rises gradually with the increase of parameter δ. Furthermore, the computational time on datasets SIS, OLD, AR and CB eventually reaches a steady fluctuation. However, the computational time of three artificial datasets CIHD1, CIHD2 and CIHD3 increases slowly when parameter δ rises from 0.1 to 0.5, and then fluctuates greatly when parameter δ rises from 0.5 to 0.9. The main reason for the phenomenon is that the difference

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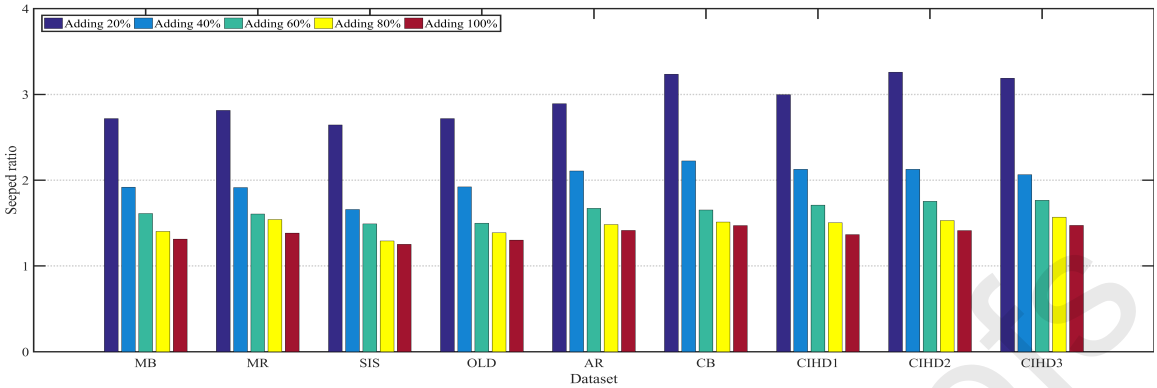

Figure 5: The speedup ratio between MSACTR and MIAUTR versus different increment ratios.

between different attribute values under each numerical attribute in the artificial datasets is large. Then, based on the complexity analysis in Section 5, the time complexity of Algorithm MIAUTR is relevant to the number of $(1,0)$, $(1,1)$ and $(0,1)$ in the double-value relation matrix when the fixed amount of objects and attributes are inserted into CIHIS simultaneously. Obviously, the increase of parameters κ and δ will cause the reduction and addition in the number of $(1,0)$ and $(1,1)$ in the double-value relation matrix, respectively. Therefore, the experimental results are consistent with the complexity analysis of Algorithm MIAUTR.

8. Conclusions

To overcome the limitation of the three-way neighborhood decision model that the loss function employed for estimating all objects in incomplete hybrid data with missing values is the precise real numbers, this paper proposed a generalized three-way neighborhood decision model by distributing the interval-valued loss function to each object and averaging the interval-valued loss functions of all objects in each data-driven neighborhood class. Moreover, considering that the incomplete hybrid data may change with the form of multi-dimensions in realistic scenarios under dynamic environments, a dynamic framework of updating three-way regions in the generalized three-way neighborhood decision model was investigated when adding objects and attributes simultaneously. To this end, an approach based on matrix for computing the three-way regions was proposed by the introduction of the matrix forms of relevant concepts and the matrix operations, which can conduce to resolve the uncertainty classification issues in incomplete hybrid data more intuitive. Then, with the simultaneous increase of objects and attributes, the matrix-based incremental mechanism was established for the maintenance of the three-way regions. On this basis, the matrix-based incremental algorithm of maintaining the three-way regions was developed. Finally, several groups of evaluation experiments were implemented on datasets from UCI as well as artificial datasets to verify the efficiency of the proposed algorithm. The comparative results demonstrate that the incremental approach has an obvious advantage in reducing the computational time of updating the three-way regions in contrast to the static approach when objects and attributes evolve over time simultaneously. In our proposed model, considering that the certain ranking method has advantages in both easy implementation and simplifying computation, we employed the θ ranking method to compare the interval-valued expected losses of taking different actions. However, this method can lead to the loss of fuzzy information in the process of transforming interval numbers into real ones, which may affect the stability of the generalization three-way neighborhood decision model in real applications. In the future work, we will aim to investigate an effective and efficient ranking method to compare the interval numbers, and further improve the framework of the proposed model. Besides, incomplete hybrid data may be composed of the categorical, numerical and interval-valued attributes with the missing values in real-life applications. Therefore, another important task is to seek for an effective model to handle such incomplete hybrid data and then develop a feasible approach to update knowledge under dynamic environments.

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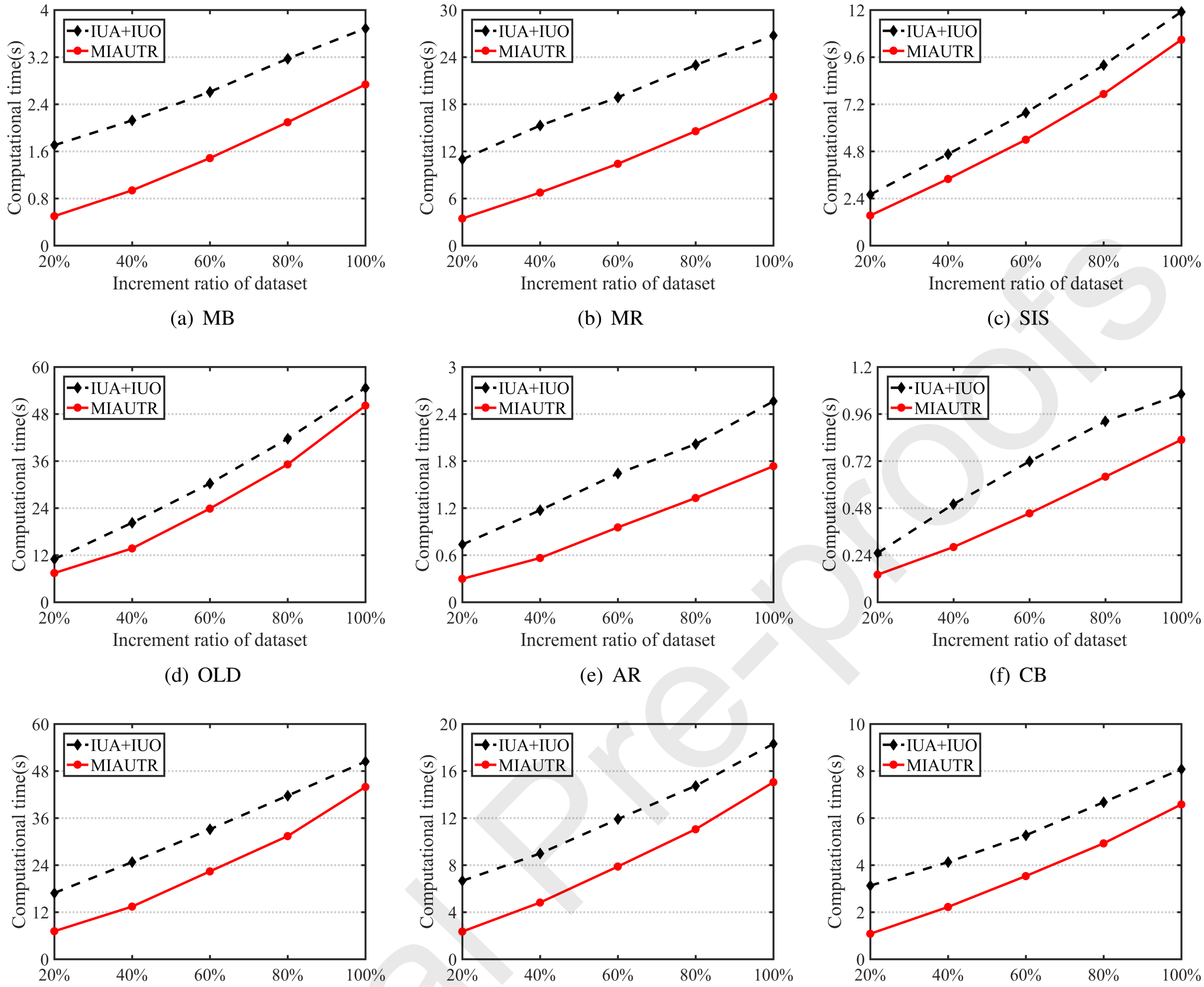

(a) MB (b) MR (c) SIS (d) OLD (e) AR (f) CB (g) CIHD1 (h) CIHD2 (i) CIHD3

Figure 6: The comparison of computational time between MIAUTR and IUA+IUO versus different increment ratios.

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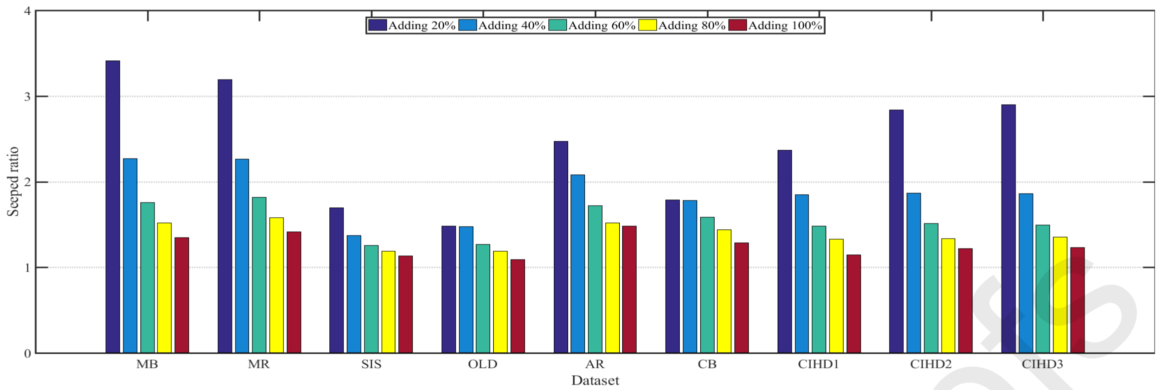

Figure 7: The speedup ratio between IUA+IUO and MIAUTR versus different increment ratios.

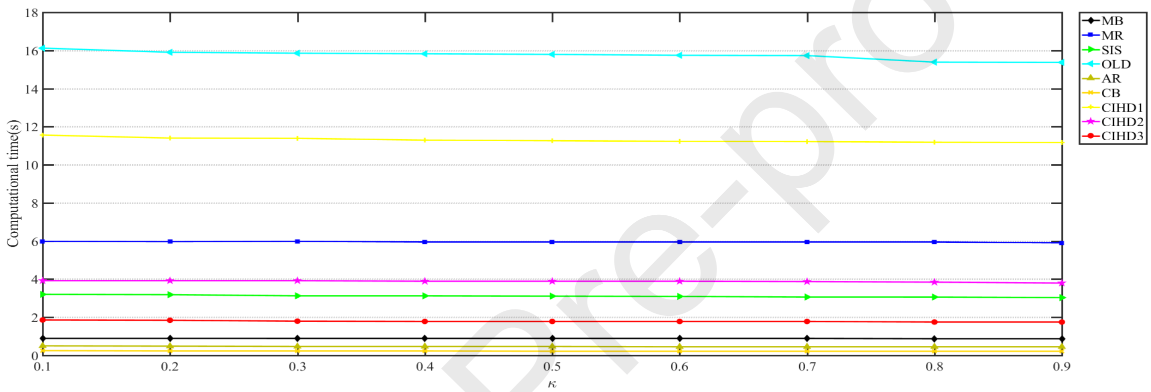

Figure 8: The comparison of Algorithm MIAUTR versus different κ.

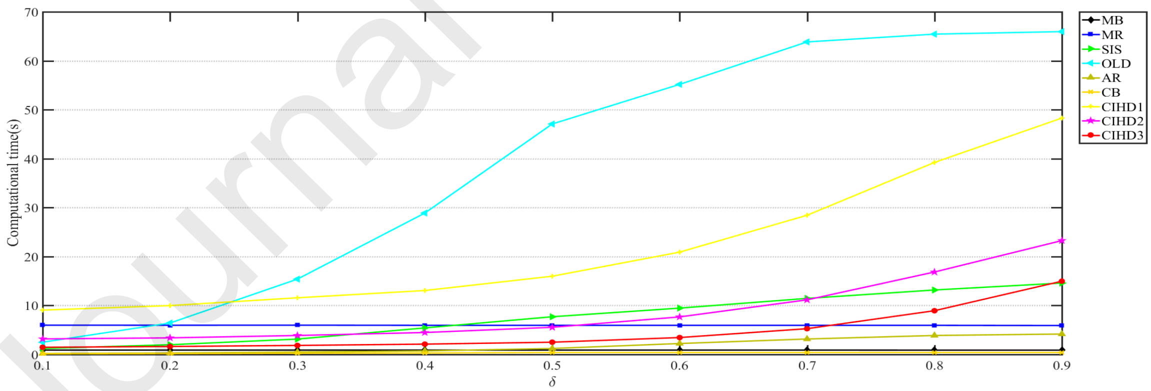

Figure 9: The comparison of Algorithm MIAUTR versus different δ.

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