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WEIGHTED NEUTROSOPHIC SOFT MULTISET AND ITS APPLICATION TO DECISION MAKING

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Abstract: In this paper, we procure the idea of weighted neutrosophic soft multiset (WNSMS) as a generalization of neutrosophic soft multiset (NSMS) and its basic properties are to be showed. Besides, we present a new adjustable approach to WNSMS based on decision-making, for solving decision-making in an indeterminacy situation.

Keywords: Decision-making, soft set, neutrosophic set, neutrosophic soft multiset, weighted neutrosophic soft multiset, weighted neutrosophic soft multiset part.

MSC: 03H10.

1. INTRODUCTION

Molodtsov in 1999 introduced the concept of soft set theory [1] which has been proved as a generic mathematical tool to deal with problems involving uncertainties. Due to the inadequacy of parametrization in the theory of fuzzy sets [2], rough sets [3], vague sets [4], probability theory etc. we become handicapped to use them successfully. Consequently Molodtsov has proved that soft set theory has a potential to use in different fields. Recently, the works on soft set theory is growing very rapidly with all its potentiality and is being used in different fields [5, 6]. Neutrosophic soft sets [7] were introduced and later on other researchers presented some studied taking into account these notions [8]. In the last few years, some applications on decision making have been carried out by using neutrosophic soft sets and multi sets. Riaz et al. [9] introduced some operations on neutrosophic N-soft sets along with their fundamental properties and for multi-attribute decisionmaking (MADM) problems with neutrosophic N-soft sets, they proposed an extended TOPSIS (technique based on order preference by similarity to ideal solution) method. Saqlain et al. [10] presented a new approach to select smart phone, in which environment of decision-making is MCDM. They defined and algorithm in which problem was formulated in the form of neutrosophic soft set and then solved with generalized fuzzy TOPSIS (GFT), thus rankings were compared with other well-known proposed method. Riaz et al. [11] introduced the notion of soft multi-set topology (SMS-topology) defined on a soft multi-set (SMS). Soft multi-set and soft multi-set topology were fundamental tools in computational intelligence, which have a large number of applications in soft computing, fuzzy modeling and decision-making under uncertainty. The idea of power whole multi-subsets of a SMS was defined to explore various rudimentary properties of SMS-topology. Certain properties of SMS-topology like SMS-basis, SMS-subspace, SMS-interior, SMS-closure and boundary of SMS are explored. Furthermore, the multicriteria decision-making (MCDM) algorithms with aggregation operators based on SMS-topology were established by them. For more notions related to the topic, we refer the reader to [12, 13].

It is well known that the world is immersed in some uncertainties that cannot be controlled or determined, however in some situations we have more than one degree of uncertainty. Smarandache [14] defines the n-refined and mentions that the degree of truth, falsity and indeterminacy can have bifurcations (depending on the study). If the literature is reviewed, there are very few studies that show situations where indeterminacy is bifurcated into two or more aspects and applications in decision making. This is why, in this paper we present the thought of weighted neutrosophic soft multiset (WNSMS) and WNSMS-part, besides proposed an adjustable approach to WNSMS based on decision-making, which is another new numerical device for managing instabilities and more possible for some genuine uses of decision making in an uncertainty situation. Some illustrative case is employed to demonstrate the attainability of our methodology in pragmatic applications.

2. PRELIMINARIES

In this section, we remain some well-known notions which will be useful for the development of this paper.

Definition 1 ([1]). *Let $\mathfrak{U}$ be a universe, $\mathfrak{E}$ be a set of parameters and $\mathfrak{P}(\mathfrak{U})$ means the power set of $\mathfrak{U}$ with $\mathfrak{A} \subset \mathfrak{E}$. Then, a couple $(\mathfrak{P}, \mathfrak{U})$ is known as soft set on $\mathfrak{U}$, where $\mathfrak{F} : \mathfrak{A} \to \mathfrak{P}(\mathfrak{U})$ is a mapping.*

Remark 2. *For any* $e \in \mathfrak{A}$, $\mathfrak{F}(e)$ *is referred to as the collection of fuzzy approximate value set of the parameter* e *and it is actually a collection of fuzzy set on* $\mathfrak{U}$.

Definition 3 ([14]). *Let* $\mathfrak{U}$ *be a universe of discourse, then the neutrosophic set (NS)* $\mathfrak{A}$ *is an object having the form* $\mathfrak{A} = \{\langle x : \mathfrak{T}_{\mathfrak{A}}(x), \mathfrak{I}_{\mathfrak{A}}(x), \mathfrak{F}_{\mathfrak{A}}(x)\rangle\}$ *where the functions* $\mathfrak{T}, \mathfrak{I}, \mathfrak{F} : \mathfrak{U} \to [0,1]$, *fine respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership (falsity) of the element* $x \in \mathfrak{X}$.

Definition 4 ([8]). *Let* $\{\mathfrak{U}_i : i \in \Omega\}$ *be a collection of universes such that* $\bigcap_{i\in\Omega} \mathfrak{U}_i = \emptyset$ *and let* $\{\mathfrak{E}_{\mathfrak{U}_i} : i \in \Omega\}$ *be a collection of sets of parameters. Let* $\mathfrak{U} = \prod_{i\in\Omega} NS(\mathfrak{U}_i)$ *where* $NS(\mathfrak{U}_i)$ *denotes the set of all NS-subsets of* $\mathfrak{U}_i$ *and* $\mathfrak{E} = \prod_{i\in\Omega} (\mathfrak{E}_{\mathfrak{U}_i})$ *and* $\mathfrak{A} \subseteq \mathfrak{E}$. *A neutrosophic sof multiset (NSMS) over* $\mathfrak{U}$ *is a pair* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A})$ *and will be denoted by* $\mathfrak{N}_{\mathfrak{A}}$ *where* $\mathfrak{N}_{\mathfrak{A}}$ *is a mapping given by* $\mathfrak{N}_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{U}$. *Thus, NSMS* $\mathfrak{N}_{\mathfrak{A}}$ *over* $\mathfrak{U}$ *can be represented by the set of ordered pairs* $\mathfrak{N}_{\mathfrak{A}} = \{(x_1, \mathfrak{N}_{\mathfrak{A}}(x_1)) : x_1 \in \mathfrak{A} \subseteq \mathfrak{U}\}$.

Definition 5 ([8]). *A NSMS* $\mathfrak{N}_{\mathfrak{A}}$ *over* $\mathfrak{U}$ *is said to be null, if all NSMs-parts of* $\mathfrak{N}_{\mathfrak{A}}$ *are equal to* $(0,0,1)$.

Definition 6 ([8]). *A NSMS* $\mathfrak{N}_{\mathfrak{A}}$ *over* $\mathfrak{U}$ *is said to be absolute, if all NSMs-parts of* $\mathfrak{N}_{\mathfrak{A}}$ *are equal to* $(1,0,0)$.

3. WEIGHTED NEUTROSOPHIC SOFT MULTISET

Deli et al. in 2014 [8] presented the idea of neutrosophic soft multiset and showed some important properties. In this section, we make a generalization for mentioned notion and we present the idea of weighted neutrosophic soft multiset (WNSMS) and show some of its principal properties.

Definition 7. *Let* $\{\mathfrak{U}_i : i \in \Omega\}$ *be a collection of universes such that* $\bigcap_{i\in\Omega} \mathfrak{U}_i = \emptyset$ *and let* $\{\mathfrak{E}_{\mathfrak{U}_i} : i \in \Omega\}$ *be a collection of sets of parameters. Let* $\mathfrak{U} = \prod_{i\in\Omega} NS(\mathfrak{U}_i)$ *where* $NS(\mathfrak{U}_i)$ *denotes the set of all NS-subsets of* $\mathfrak{U}_i$ *and* $\mathfrak{E} = \prod_{i\in\Omega} (\mathfrak{E}_{\mathfrak{U}_i})$ *and* $\mathfrak{A} \subseteq \mathfrak{E}$. *A WNSMS is a triple* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ *and will be denoted by* $\mathfrak{N}_{\mathfrak{A}_\omega}$ *where* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A})$ *is a NSMS over* $\mathfrak{U}$ *and* ω *is a weighted for the NSMS* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A})$ *where* $\omega : \mathfrak{A} \to [0,1]$ *is a weighted function indicating the weighted* $w_i = \omega(e_i), e_i \in \mathfrak{E}$.

To illustrate this, let's consider the following example:

Example 8. *Let's consider three universe* $\mathfrak{U}_1 = \{a_1, a_2, a_3\}$, $\mathfrak{U}_2 = \{b_1, b_2, b_3\}$ *and* $\mathfrak{U}_3 = \{c_1, c_2, c_3, c_4\}$ *which are the arrangements of university, food and transportation, respectively. Assume that Mr. Y has a financial plan to pay the university* *semester, food during the semester and transportation during the semester. Consider a NSMS* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A})$ *which depicts university, food and transportation that Mr. Y is considering. Let* $\{\mathfrak{E}_{\mathfrak{U}_1}, \mathfrak{E}_{\mathfrak{U}_2}, \mathfrak{E}_{\mathfrak{U}_3}\}$ *be a collection of sets of choice parameters identified with the above universes, where*

$$\mathfrak{E}_{\mathfrak{U}_1} = \{e_{\mathfrak{U}_1}, 1 = Loan, e_{\mathfrak{U}_1}, 2 = Scholarship, e_{\mathfrak{U}_1}, 3 = Cashpayment\}$$
$$\mathfrak{E}_{\mathfrak{U}_2} = \{e_{\mathfrak{U}_2}, 1 = Restaurant, e_{\mathfrak{U}_2}, 2 = Cooking, e_{\mathfrak{U}_2}, 3 = Junkfood\}$$
$$\mathfrak{E}_{\mathfrak{U}_3} = \{e_{\mathfrak{U}_3}, 1 = Bus, e_{\mathfrak{U}_3}, 2 = Taxi, e_{\mathfrak{U}_3}, 3 = Walking, e_{\mathfrak{U}_3}, 4 = Bike\}.$$

Let $\mathfrak{U} = \prod_{i=1}^{3} NS(\mathfrak{U}_i)$, $\mathfrak{E} = \prod_{i=1}^{3} \mathfrak{E}_i$ and $\mathfrak{A} \subseteq \mathfrak{E}$, such that

$$\mathfrak{A} = \{x_1 = (e_{\mathfrak{U}_1}, 1, e_{\mathfrak{U}_2}, 2, e_{\mathfrak{U}_3}, 4), x_2 = (e_{\mathfrak{U}_1}, 2, e_{\mathfrak{U}_2}, 1, e_{\mathfrak{U}_3}, 3),$$
$$x_3 = (e_{\mathfrak{U}_1}, 3, e_{\mathfrak{U}_2}, 2, e_{\mathfrak{U}_3}, 1), x_4 = (e_{\mathfrak{U}_1}, 3, e_{\mathfrak{U}_2}, 3, e_{\mathfrak{U}_3}, 2)\}$$

Assume Mr. XY needs to pick objects from the sets of given objects regarding the sets of decision parameters. Let the resultant NSMS in Table 1.

Table 1: The NSMS $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A})$

$\mathfrak{U}_i$		x_1	x_2	x_3	x_4
$\mathfrak{U}_1$					
	a_1	(0.1,0.3.0.9)	(0.5,0.6,0.5)	(0.3,0.1,0.7)	(0.8,0.3,0.2)
	a_2	(0.8,0.7,0.2)	(0.4,0.6,0.6)	(0.3,0.9,0.7)	(0.1,0.7,0.9)
	a_3	(0.25,0.5,0.75)	(0.4,0.8,0.6)	(0.6,0.2,0.4)	(0.1,0.05,0.9)
$\mathfrak{U}_2$					
	b_1	(0.1,1,0.9)	(0,0.5,1)	(0.3,0.5,0.7)	(0.2,0.7,0.8)
	b_2	(0.9,0.7,0.1)	(0.2,0.6,0.8)	(0.2,.0.9,0.8)	(0.34,0.7,0.64)
	b_3	(0.1,.0.9,0.2)	(0.2,0.1,0.8)	(0.25,0.1,0.75)	(0.11,0.2,0.89)
$\mathfrak{U}_3$					
	c_1	(0.9,0.1,0.1)	(0.2,0.8,0.8)	(1,0.2,0)	(0.23,0.5.,0.77)
	c_2	(0.1,0.8.0.9)	(0.5,0.25,0.5)	(0.3,0.17,0.7)	(0.8,0.37,0.2)
	c_3	(0.1,0,0.9)	(0,0.6,1)	(0.3,0.1,0.7)	(0.2,0.24,0.8)
	c_4	(0.1,1,0.2)	(0.2,0,0.8)	(0.25,0.6,0.75)	(0.11,0.24,0.89)

Consider that Mr. Y has forced the accompanying weights for the parameters in $\mathfrak{A}$ for the parameters

$$x_1 = (Loan, Cooking, Bike), \omega_1 = 0.3;$$
$$x_2 = (Scholarship, Restaurant, Walking), \omega_2 = 0.5;$$
$$x_3 = (Bus, Cooking, Bus), \omega_3 = 0.7;$$
$$x_4 = (Cashpayment, Junkfood, Taxi), \omega_4 = 0.4.$$

Then, we have a weighted ω on NSMS $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A})$ where $\omega : \mathfrak{A} \to [0, 1]$ and the NSMS $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A})$ is changed into a WNSMS $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ as can be seen in Table 2.

Table 2: The WNSMS $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$

$\mathfrak{U}_i$		$x_1 0.3$	$x_2 0.5$	$x_3 0.7$	$x_4 0.4$
$\mathfrak{U}_1$					
	a_1	(0.1,0.3.0.9)	(0.5,0.6,0.5)	(0.3,0.1,0.7)	(0.8,0.3,0.2)
	a_2	(0.8,0.7,0.2)	(0.4,0.6,0.6)	(0.3,0.9,0.7)	(0.1,0.7,0.9)
	a_3	(0.25,0.5,0.75)	(0.4,0.8,0.6)	(0.6,0.2,0.4)	(0.1,0.05,0.9)
$\mathfrak{U}_2$					
	b_1	(0.1,1,0.9)	(0,0.5,1)	(0.3,0.5,0.7)	(0.2,0.7,0.8)
	b_2	(0.9,0.7,0.1)	(0.2,0.6,0.8)	(0.2,.0.9,0.8)	(0.34,0.7,0.64)
	b_3	(0.1,.0.9,0.2)	(0.2,0.1,0.8)	(0.25,0.1,0.75)	(0.11,0.2,0.89)
$\mathfrak{U}_3$					
	c_1	(0.9,0.1,0.1)	(0.2,0.8,0.8)	(1,0.2,0)	(0.23,0.5.,0.77)
	c_2	(0.1,0.8.0.9)	(0.5,0.25,0.5)	(0.3,0.17,0.7)	(0.8,0.37,0.2)
	c_3	(0.1,0,0.9)	(0,0.6,1)	(0.3,0.1,0.7)	(0.2,0.24,0.8)
	c_4	(0.1,1,0.2)	(0.2,0,0.8)	(0.25,0.6,0.75)	(0.11,0.24,0.89)

Definition 9. *Let $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ be WNSMS over $\mathfrak{U}$. Then, the pair $(e_{\mathfrak{U}_i,j}, \mathfrak{N}_{\mathfrak{A}}(e_{\mathfrak{U}_i,j}), \omega)$ is said to be $\mathfrak{U}_i$-WNSMS-part of $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ over $\mathfrak{U}$, where $(e_{\mathfrak{U}_i,j}, \mathfrak{N}_{\mathfrak{A}}(e_{\mathfrak{U}_i,j}))$ is a $\mathfrak{U}_i$-NSMS-part of $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A})$.*

Example 10. *Let $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ be a WNSMS in the Table 2, the $\mathfrak{U}_1$-WNSMS-part, $\mathfrak{U}_2$-WNSMS-part and $\mathfrak{U}_3$-WNSMS-part are represented in Table 3, Table 4 and Table 5, respectively.*

Table 3: $\mathfrak{U}_1$-WNSMS-part of $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$

$\mathfrak{U}_1$	$x_1 0.3$	$x_2 0.5$	$x_3 0.7$	$x_4 0.4$
a_1	(0.1,0.3.0.9)	(0.5,0.6,0.5)	(0.3,0.1,0.7)	(0.8,0.3,0.2)
a_2	(0.8,0.7,0.2)	(0.4,0.6,0.6)	(0.3,0.9,0.7)	(0.1,0.7,0.9)
a_3	(0.25,0.5,0.75)	(0.4,0.8,0.6)	(0.6,0.2,0.4)	(0.1,0.05,0.9)

Table 4: $\mathfrak{U}_2$-WNSMS-part of $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$

$\mathfrak{U}_2$	$x_1 0.3$	$x_2 0.5$	$x_3 0.7$	$x_4 0.4$
b_1	(0.1,1,0.9)	(0,0.5,1)	(0.3,0.5,0.7)	(0.2,0.7,0.8)
b_2	(0.9,0.7,0.1)	(0.2,0.6,0.8)	(0.2,.0.9,0.8)	(0.34,0.7,0.64)
b_3	(0.1,.0.9,0.2)	(0.2,0.1,0.8)	(0.25,0.1,0.75)	(0.11,0.2,0.89)

Table 5: $\mathfrak{U}_3$-WNSMS-part of $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$

$\mathfrak{U}_2$	$x_1 0.3$	$x_2 0.5$	$x_3 0.7$	$x_4 0.4$
c_1	(0.9,0.1,0.1)	(0.2,0.8,0.8)	(1,0.2,0)	(0.23,0.5.,0.77)
c_2	(0.1,0.8.0.9)	(0.5,0.25,0.5)	(0.3,0.17,0.7)	(0.8,0.37,0.2)
c_3	(0.1,0,0.9)	(0,0.6,1)	(0.3,0.1,0.7)	(0.2,0.24,0.8)
c_4	(0.1,1,0.2)	(0.2,0,0.8)	(0.25,0.6,0.75)	(0.11,0.24,0.89)

Next, we study and show some basic properties on WNSMSs.

Definition 11. *A WNSMS $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ over $\mathfrak{U}$ is said to be a null WNSMS, denoted by $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)_{\emptyset}$, if $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A})$ is a null NSMS and for all $e \in \mathfrak{A}$, $\omega(e) = 0$.*

Example 12. *Let's consider the values presented in Table 2, then the* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)_{\emptyset}$ *is given by Table 6.*

Table 6: The null WNSMS $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)_{\emptyset}$

$\mathfrak{U}_i$		$x_1 0$	$x_2 0$	$x_3 0$	$x_4 0$
$\mathfrak{U}_1$					
	a_1	(0,0.1)	(0,0,1)	(0,0,01)	(0,0,1)
	a_2	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,1)
	a_3	(0,0,1)	(0,0,01)	(0,0,1)	(0,0,1)
$\mathfrak{U}_2$					
	b_1	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,1)
	b_2	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,1)
	b_3	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,1)
$\mathfrak{U}_3$					
	c_1	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,1)
	c_2	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,1)
	c_3	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,1)
	c_4	(0,0,1)	(0,0,1)	(0,0,1)	(0,0,1)

Definition 13. *A WNSMS* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ *over* $\mathfrak{U}$ *is said to be absolute WNSMS, denoted by* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)_{\mathfrak{U}}$*, if* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A})$ *is an absolute NSMS and for all* $e \in \mathfrak{A}$, $\omega(e) = 1$.

Example 14. *Let's consider the values presented in Table 2, then the* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)_{\mathfrak{U}}$ *is given by Table 7.*

Table 7: The absolute WNSMS $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)_{\mathfrak{U}}$

$\mathfrak{U}_i$		$x_1 1$	$x_2 1$	$x_3 1$	$x_4 1$
$\mathfrak{U}_1$					
	a_1	(1,0,0)	(1,0,0)	(1,0,0)	(1,0,0)
	a_2	(1,0,0)	(1,0,0)	(1,0,0)	(1,0,0)
	a_3	(1,0,0)	(1,0,0)	(1,0,0)	(1,0,0)
$\mathfrak{U}_2$					
	b_1	(1,0,0)	(1,0,0)	(1,0,0)	(1,0,0)
	b_2	(1,0,0)	(1,0,0)	(1,0,0)	(1,0,0)
	b_3	(1,0,0)	(1,0,0)	(1,0,0)	(1,0,0)
$\mathfrak{U}_3$					
	c_1	(1,0,0)	(1,0,0)	(1,0,0)	(1,0,0)
	c_2	(1,0,0)	(1,0,0)	(1,0,0)	(1,0,0)
	c_3	(1,0,0)	(1,0,0)	(1,0,0)	(1,0,0)
	c_4	(1,0,0)	(1,0,0)	(1,0,0)	(1,0,0)

Definition 15. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *be WNSMSs over* $\mathfrak{U}$*. Then,* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ *is a WNSMS-subset of* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$*, denoted by* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega) \widetilde{\subseteq} (\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *if*

1. $\mathfrak{N}_{\mathfrak{A}} \widetilde{\subseteq} \mathfrak{N}_{\mathfrak{B}}$,
2. *for all* $e \in \mathfrak{A}$, $\omega_1(e) \leq \omega_2(e)$ *and* $\mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}_e}}(u) \leq \mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}_e}}(u)$, $\mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}_e}}(u) \leq \mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}_e}}(u)$ *and* $\mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}_e}}(u) \geq \mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}_e}}(u)$, *for all* $u \in \mathfrak{U}_i$, *for* $i \in \Omega$.

Example 16. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$ *which is given as Table 2 and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *be WNSMSs where* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *is given as Table 8.*

Table 8: The WNSMS $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega)$

$\mathfrak{U}_i$		$x_1 0.6$	$x_2 0.7$	$x_3 0.9$	$x_4 0.6$
$\mathfrak{U}_1$					
	a_1	(0.2,0.3.0.8)	(0.5,0.6,0.5)	(0.3,0.1,0.7)	(0.7,0.3,0.3)
	a_2	(0.3,0.7,0.7)	(0.4,0.6,0.6)	(0.3,0.9,0.7)	(0.3,0.7,0.7)
	a_3	(0.5,0.5,0.5)	(0.4,0.8,0.6)	(0.6,0.2,0.4)	(0.1,0.05,0.9)
$\mathfrak{U}_2$					
	b_1	(0.9,1,0.1)	(0,0.5,1)	(0.3,0.5,0.7)	(0.2,0.7,0.8)
	b_2	(0.3,0.7,0.7)	(0.2,0.6,0.8)	(0.2,.0.9,0.8)	(0.6,0.7,0.4)
	b_3	(0.1,.0.9,0.2)	(0.2,0.1,0.8)	(0.25,0.1,0.75)	(0.1,0.2,0.9)
$\mathfrak{U}_3$					
	c_1	(0,0.1,1)	(0.2,0.8,0.8)	(1,0.2,0)	(0.23,0.5.,0.77)
	c_2	(0,0.8,1)	(0.5,0.25,0.5)	(0.3,0.17,0.7)	(0.8,0.37,0.2)
	c_3	(0.9,0,0.1)	(0,0.6,1)	(0.3,0.1,0.7)	(0.2,0.24,0.8)
	c_4	(0.1,1,0.2)	(0.2,0,0.8)	(0.3,0.6,0.7)	(0.1,0.24,0,9)

Then, $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1) \widetilde{\subseteq} (\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$.

Proposition 17. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ *be WNSMS over* $\mathfrak{U}$. *Then, the following statements holds:*

1. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega) \widetilde{\subseteq} (\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$,
2. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)_{\emptyset} \widetilde{\subseteq} (\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$,
3. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)_{\mathfrak{U}} \widetilde{\supseteq} (\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$.

Proof. The proof follows from Definition 15. □

Definition 18. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *be WNSMSs over* $\mathfrak{U}$. *Then,* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *are WNSMS-equal-set, denoted by* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega) = (\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *if the following conditions holds:*

1. $\mathfrak{N}_{\mathfrak{A}} = \mathfrak{N}_{\mathfrak{B}}$,
2. *for all* $e \in \mathfrak{A}$, $\omega_1(e) = \omega_2(e)$ *and* $\mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}_e}}(u) = \mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}_e}}(u)$, $\mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}_e}}(u) = \mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}_e}}(u)$ *and* $\mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}_e}}(u) = \mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}_e}}(u)$, *for all* $u \in \mathfrak{U}_i$, *for* $i \in \Omega$.

Proposition 19. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$, $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *and* $(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$ *be WNSMSs over* $\mathfrak{U}$, *then*

1. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1) = (\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2) = (\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$, *then* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1) = (\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$,

2. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\subseteq}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *and* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\subseteq}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$*, then* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1) = (\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$*,*
3. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\subseteq}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)\widetilde{\supseteq}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$*, then* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\subseteq}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$.

Proof. The proof follows from Definitions 15 and 18. □

Definition 20. *Complement of a WNSMS* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ *over* $\mathfrak{U}$ *is denoted by* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)^c$ *and it is represented by* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)^c = (\mathfrak{N}^c_{\mathfrak{A}}, \mathfrak{A}, \omega^c)$ *and it is defined as for all* $e \in \mathfrak{A}$, $\omega^c = 1 - \omega(e)$, $\mathfrak{T}^c_{\mathfrak{N}_{\mathfrak{A}_e}}(u) = \mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}_e}}(u)$, $\mathfrak{I}^c_{\mathfrak{N}_{\mathfrak{A}_e}}(u) = 1 - \mathfrak{I}^c_{\mathfrak{N}_{\mathfrak{A}_e}}(u)$ *and* $\mathfrak{F}^c_{\mathfrak{N}_{\mathfrak{A}_e}}(u) = \mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}_e}}(u)$*, for all* $u \in \mathfrak{U}_i$*, for* $i \in \Omega$.

Example 21. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$ *which is given as Table 2, then* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)^c$ *is given as Table 9.*

Table 9: The WNSMS $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)^c$

$\mathfrak{U}_i$		$x_1 0.3$	$x_2 0.5$	$x_3 0.7$	$x_4 0.4$
$\mathfrak{U}_1$					
	a_1	(0.9,0.7.0.1)	(0.5,0.4,0.5)	(0.7,0.9,0.3)	(0.2,0.7,0.8)
	a_2	(0.2,0.3,0.8)	(0.6,0.4,0.4)	(0.7,0.1,0.3)	(0.9,0.3,0.1)
	a_3	(0.75,0.5,0.25)	(0.6,0.2,0.4)	(0.4,0.8,0.6)	(0.9,0.95,0.1)
$\mathfrak{U}_2$					
	b_1	(0.9,0,0.1)	(1,0.5,0)	(0.7,0.5,0.3)	(0.8,0.3,0.2)
	b_2	(0.1,0.3,0.9)	(0.8,0.4,0.2)	(0.8,.0.1,0.2)	(0.64,0.3,0.34)
	b_3	(0.2,.0.1,0.8)	(0.8,0.9,0.2)	(0.75,0.9,0.25)	(0.89,0.8,0.11)
$\mathfrak{U}_3$					
	c_1	(0.1,0.9,0.9)	(0.8,0.2,0.2)	(0,0.8,1)	(0.77,0.5,0.23)
	c_2	(0.9,0.2.0.1)	(0.5,0.75,0.5)	(0.7,0.73,0.3)	(0.2,0.63,0.8)
	c_3	(0.9,1,0.1)	(1,0.4,0)	(0.7,0.9,0.3)	(0.8,0.76,0.2)
	c_4	(0.2,0,0.8)	(0.8,1,0.2)	(0.75,0.4,0.25)	(0.89,0.76,0.11)

Proposition 22. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ *be a WNSMS over* $\mathfrak{U}$*. Then, the following statements are satisfied:*

1. $((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)^c)^c = (\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$,
2. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)^c_{\emptyset} = (\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)_{\mathfrak{U}}$,
3. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)^c_{\mathfrak{U}} = (\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)_{\emptyset}$.

Proof. The proof follows from Definition 20. □

Definition 23. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *be WNSMSs over* $\mathfrak{U}$*. The union of* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *denoted by* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *and defined as* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2) = (\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$*, where* $\mathfrak{G} = \mathfrak{A} \cup \mathfrak{B}$ *and for all* $e \in \mathfrak{C}$, $\omega_3(e) = \max\{\omega_1(e), \omega_2(e)\}$ *and for all* $u \in \mathfrak{U}_i$, $i \in \Omega$.

$$(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3) = \begin{cases} \max\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}}}\} & if\ e \in \mathfrak{A} \cap \mathfrak{B} \\ \max\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}}}\} & if\ e \in \mathfrak{A} \cap \mathfrak{B} \\ \min\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}}}\} & if\ e \in \mathfrak{A} \cap \mathfrak{B}. \end{cases}$$

Example 24. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *be WNSMSs which are given as Table 2 and Table 8, respectively. Then, WNSMS* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega)$ *is provided in Table 10.*

Table 10: The WNSMS $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega)$

$\mathfrak{U}_i$		$x_1 0.6$	$x_2 0.7$	$x_3 0.9$	$x_4 0.6$
$\mathfrak{U}_1$					
	a_1	(0.2,0.3.0.8)	(0.5,0.6,0.5)	(0.3,0.1,0.7)	(0.7,0.3,0.3)
	a_2	(0.3,0.7,0.7)	(0.4,0.6,0.6)	(0.3,0.9,0.7)	(0.3,0.7,0.7)
	a_3	(0.5,0.5,0.5)	(0.4,0.8,0.6)	(0.6,0.2,0.4)	(0.1,0.05,0.9)
$\mathfrak{U}_2$					
	b_1	(0.9,1,0.1)	(0,0.5,1)	(0.3,0.5,0.7)	(0.2,0.7,0.8)
	b_2	(0.3,0.7,0.7)	(0.2,0.6,0.8)	(0.2,.0.9,0.8)	(0.6,0.7,0.4)
	b_3	(0.1,.0.9,0.2)	(0.2,0.1,0.8)	(0.25,0.1,0.75)	(0.1,0.2,0.9)
$\mathfrak{U}_3$					
	c_1	(0,0.1,1)	(0.2,0.8,0.8)	(1,0.2,0)	(0.23,0.5.,0.77)
	c_2	(0,0.8,1)	(0.5,0.25,0.5)	(0.3,0.17,0.7)	(0.8,0.37,0.2)
	c_3	(0.9,0,0.1)	(0,0.6,1)	(0.3,0.1,0.7)	(0.2,0.24,0.8)
	c_4	(0.1,1,0.2)	(0.2,0,0.8)	(0.3,0.6,0.7)	(0.1,0.24,0,9)

Proposition 25. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *be WNSMSs over* $\mathfrak{U}$. *Then, the following statements are satisfied:*

1. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1) = (\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$,
2. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)_{\emptyset} = (\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$,
3. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)_{\mathfrak{U}} = (\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$, *[* $(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$ *defined in 23],*
4. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2) = (\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$.

Proof. The proof follows from Definition 23. □

Definition 26. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *be WNSMSs over* $\mathfrak{U}$. *The intersection of* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *denoted by* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *and defined as* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2) = (\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$, *where* $\mathfrak{G} = \mathfrak{A} \cap \mathfrak{B}$ *and for all* $e \in \mathfrak{C}$, $\omega_3(e) = \min\{\omega_1(e), \omega_2(e)\}$ *and for all* $u \in \mathfrak{U}_i$, $i \in \Omega$.

$$(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3) = \begin{cases} \min\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}}}\} & if \ e \in \mathfrak{A} \cap \mathfrak{B} \\ \min\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}}}\} & if \ e \in \mathfrak{A} \cap \mathfrak{B} \\ \max\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}}}\} & if \ e \in \mathfrak{A} \cap \mathfrak{B}. \end{cases}$$

Example 27. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *be WNSMSs which are given as Table 2 and Table 8, respectively. Then, WNSMS* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega)$ *is presented in Table 11.*

Table 11: The WNSMS $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega)$

$\mathfrak{U}_i$		$x_1 0.3$	$x_2 0.5$	$x_3 0.7$	$x_4 0.4$
$\mathfrak{U}_1$					
	a_1	(0.1,0.3.0.9)	(0.5,0.6,0.5)	(0.3,0.1,0.7)	(0.8,0.3,0.2)
	a_2	(0.8,0.7,0.2)	(0.4,0.6,0.6)	(0.3,0.9,0.7)	(0.1,0.7,0.9)
	a_3	(0.25,0.5,0.75)	(0.4,0.8,0.6)	(0.6,0.2,0.4)	(0.1,0.05,0.9)
$\mathfrak{U}_2$					
	b_1	(0.1,1,0.9)	(0,0.5,1)	(0.3,0.5,0.7)	(0.2,0.7,0.8)
	b_2	(0.9,0.7,0.1)	(0.2,0.6,0.8)	(0.2,.0.9,0.8)	(0.34,0.7,0.64)
	b_3	(0.1,.0.9,0.2)	(0.2,0.1,0.8)	(0.25,0.1,0.75)	(0.11,0.2,0.89)
$\mathfrak{U}_3$					
	c_1	(0.9,0.1,0.1)	(0.2,0.8,0.8)	(1,0.2,0)	(0.23,0.5.,0.77)
	c_2	(0.1,0.8.0.9)	(0.5,0.25,0.5)	(0.3,0.17,0.7)	(0.8,0.37,0.2)
	c_3	(0.1,0,0.9)	(0,0.6,1)	(0.3,0.1,0.7)	(0.2,0.24,0.8)
	c_4	(0.1,1,0.2)	(0.2,0,0.8)	(0.25,0.6,0.75)	(0.11,0.24,0.89)

Proposition 28. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *be WNSMSs over* $\mathfrak{U}$. *Then, the following statements are satisfied:*

1. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1) = (\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$,
2. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)_{\emptyset} = (\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)_{\emptyset}$, *[* $(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$ *defined in 26],*
3. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)_{\mathfrak{U}} = (\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$, *[* $(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$ *defined in 26],*
4. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2) = (\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$.

Proof. The proof follows from Definition 26. □

Proposition 29. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$, $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *and* $(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$ *be WNSMSs over* $\mathfrak{U}$. *Then, the following statements holds:*

1. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}((\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)) = ((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2))\widetilde{\cap}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$.
2. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}((\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)) = ((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2))\widetilde{\cup}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$.
3. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}((\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)) = ((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1))\widetilde{\cup}$ $((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3))$.
4. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}((\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)) = ((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1))\widetilde{\cap}$ $((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3))$.

Proof. In the following, we just prove (1); (2), (3) and (4) are proved analogously. Let's suppose that $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3) = (\mathfrak{N}_{\mathfrak{D}}, \mathfrak{D}, \omega_4)$, where $\mathfrak{D} = \mathfrak{B} \cap \mathfrak{C}$ and for all $e \in \mathfrak{D}$, $\omega_4 = \min\{\omega_2, \omega_3\}$, $\mathfrak{T}_{\mathfrak{N}_{\mathfrak{D}}} = \min\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{C}}}\}$, $\mathfrak{I}_{\mathfrak{N}_{\mathfrak{D}}} = \min\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{C}}}\}$ and $\mathfrak{F}_{\mathfrak{N}_{\mathfrak{D}}} = \max\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{C}}}\}$ for all $u \in \mathfrak{U}_i$, $i\Omega$.

Since $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}((\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)) = ((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{D}}, \mathfrak{D}, \omega_4)$, we consider that $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{D}}, \mathfrak{D}, \omega_4) = (\mathfrak{N}_{\mathfrak{M}}, \mathfrak{M}, \omega_5)$ where $\mathfrak{M} = \mathfrak{A} \cap \mathfrak{D} = \mathfrak{A} \cap \mathfrak{B}\mathfrak{C}$ and for all $e \in \mathfrak{M}$, $\omega_5 = \min\{\omega_1(e), \omega_4(e)\} = \min\{\omega_1(e), \omega_2(e), \omega_3(e)\}$,

$$\begin{aligned}\mathfrak{T}_{\mathfrak{N}_{\mathfrak{M}}} &= \min\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{D}}}\}\\ &= \min\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}}}, \min\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{C}}}\}\}\\ &= \min\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{C}}}\}\end{aligned}$$

$$\begin{aligned}\mathfrak{I}_{\mathfrak{N}_{\mathfrak{M}}} &= \min\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{D}}}\} \\ &= \min\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}}}, \min\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{C}}}\}\} \\ &= \min\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{C}}}\}\end{aligned}$$

$$\begin{aligned}\mathfrak{F}_{\mathfrak{N}_{\mathfrak{M}}} &= \max\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{D}}}\} \\ &= \max\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}}}, \max\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{C}}}\}\} \\ &= \max\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{C}}}\}\end{aligned}$$

For all $u \in \mathfrak{U}_i, i \in \Omega$. Now, consider that $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2) = (\mathfrak{N}_{\mathfrak{K}}, \mathfrak{K}, \omega_6)$, where $\mathfrak{K} = \mathfrak{A} \cap \mathfrak{B}$ and for all $e \in \mathfrak{K}$, $\omega_6 = \min\{\omega_1, \omega_2\}$, $\mathfrak{T}_{\mathfrak{N}_{\mathfrak{K}}} = \min\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}}}\}$, $\mathfrak{I}_{\mathfrak{N}_{\mathfrak{D}}} = \min\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}}}\}$ and $\mathfrak{F}_{\mathfrak{N}_{\mathfrak{D}}} = \max\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}}}\}$ for all $u \in \mathfrak{U}_i$, $i\Omega$. Since $((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2))\widetilde{\cap}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3) = ((\mathfrak{N}_{\mathfrak{K}}, \mathfrak{K}, \omega_6)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$, we consider that $(\mathfrak{N}_{\mathfrak{K}}, \mathfrak{K}, \omega_6)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3) = (\mathfrak{N}_{\mathfrak{L}}, \mathfrak{L}, \omega_7)$ where $\mathfrak{L} = \mathfrak{K} \cap \mathfrak{C}= \mathfrak{A} \cap \mathfrak{B}\mathfrak{C}$ and for all $e \in \mathfrak{L}$, $\omega_7 = \min\{\omega_6(e), \omega_3(e)\} = \min\{\omega_1(e), \omega_2(e), \omega_3(e)\} = \omega_5(e)$,

$$\begin{aligned}\mathfrak{T}_{\mathfrak{N}_{\mathfrak{L}}} &= \min\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{K}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{C}}}\} \\ &= \min\{\min\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}}}\}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{C}}}\}\} \\ &= \min\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{C}}}\} = \mathfrak{T}_{\mathfrak{N}_{\mathfrak{M}}}\end{aligned}$$

$$\begin{aligned}\mathfrak{I}_{\mathfrak{N}_{\mathfrak{L}}} &= \min\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{K}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{C}}}\} \\ &= \min\{\min\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}}}\}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{C}}}\}\} \\ &= \min\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{C}}}\} = \mathfrak{I}_{\mathfrak{N}_{\mathfrak{M}}}\end{aligned}$$

$$\begin{aligned}\mathfrak{F}_{\mathfrak{N}_{\mathfrak{L}}} &= \max\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{K}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{C}}}\} \\ &= \max\{\max\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}}}\}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{C}}}\}\} \\ &= \max\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{C}}}\} = \mathfrak{F}_{\mathfrak{N}_{\mathfrak{M}}}.\end{aligned}$$

For all $u \in \mathfrak{U}_i, i \in \Omega$. Therefore, it can be seen that $\mathfrak{M} = \mathfrak{L}$ and for all $e \in \mathfrak{M}$, $\omega_5(e) = \omega_7(e)$, $\mathfrak{T}_{\mathfrak{N}_{\mathfrak{L}}} = \mathfrak{T}_{\mathfrak{N}_{\mathfrak{M}}}$, $\mathfrak{I}_{\mathfrak{N}_{\mathfrak{L}}} = \mathfrak{I}_{\mathfrak{N}_{\mathfrak{M}}}$ and $\mathfrak{F}_{\mathfrak{N}_{\mathfrak{L}}} = \mathfrak{F}_{\mathfrak{N}_{\mathfrak{M}}}$ for all $u \in \mathfrak{U}_i, i \in \Omega$. Hence, $(\mathfrak{N}_{\mathfrak{M}}, \mathfrak{M}, \omega_5) = (\mathfrak{N}_{\mathfrak{L}}, \mathfrak{L}, \omega_7)$, and this proves that $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}((\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)) = ((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2))\widetilde{\cap}(\mathfrak{N}_{\mathfrak{C}}, \mathfrak{C}, \omega_3)$. □

Proposition 30. *Let* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)$ *and* $(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)$ *be WNSMSs over* $\mathfrak{U}$. *Then, the following statements holds:*

1. $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)^c\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)^c\widetilde{\subseteq}((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2))^c$.
2. $((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cup}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2))^c\widetilde{\subseteq}(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)^c\widetilde{\cup}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)^c$.

Proof. In the following, we just prove (1); (2) is proved analogously.
Let's suppose that $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2) = (\mathfrak{N}_{\mathfrak{D}}, \mathfrak{D}, \omega_3)$ where $\mathfrak{D} = \mathfrak{A} \cap \mathfrak{B}$ and for all $e \in \mathfrak{D}$, $\omega_3(e) = \min\{\omega_1(e), \omega_2(e)\}$, $\mathfrak{T}_{\mathfrak{N}_{\mathfrak{D}}} = \min\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}}}\}$, $\mathfrak{I}_{\mathfrak{N}_{\mathfrak{D}}} = \min\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}}}\}$ and $\mathfrak{F}_{\mathfrak{N}_{\mathfrak{D}}} = \max\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}}}\}$ for all $u \in \mathfrak{U}_i$, $i\Omega$. Thus, $((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)\widetilde{\cap}(\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2))^c = (\mathfrak{N}_{\mathfrak{D}}, \mathfrak{D}, \omega_3)^c = (\mathfrak{N}^c_{\mathfrak{B}}, \mathfrak{B}, \omega^3_3)$ where for all $e \in \mathfrak{D}$, $\omega^c_3(e) = 1 - \min\{\omega_1(e), \omega_2(e)\}$ and $\mathfrak{T}^c_{\mathfrak{N}_{\mathfrak{D}}} = 1 - \min\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}}}\}$, $\mathfrak{I}^c_{\mathfrak{N}_{\mathfrak{D}}} = 1 - \min\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}}}\}$ and

$\mathfrak{F}^c_{\mathfrak{N}_{\mathfrak{D}}} = 1 - \max\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}}}\}$ for all $u \in \mathfrak{U}_i$, $i\Omega$. Again, $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)^c \widetilde{\cap} (\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)^c = (\mathfrak{N}_{\mathfrak{K}}, \mathfrak{K}, \omega_4)$, where $\mathfrak{K} = \mathfrak{A} \cap \mathfrak{B}$ and for all $e \in \mathfrak{K}$, $\omega_4(e) = \min\{\omega_1^c(e), \omega_2^c(e)\} = \min\{1 - \omega_1(e), 1 - \omega_2(e)\} = 1 - \max\{\omega_1(e), \omega_2(e)\} \leq \omega_3^c$;

$$\begin{aligned}\mathfrak{T}_{\mathfrak{N}_{\mathfrak{K}}} &= \min\{\mathfrak{T}^c_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{T}^c_{\mathfrak{N}_{\mathfrak{B}}}\} \\ &= \min\{1 - \mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}}}, 1 - \mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}}}\} \\ &= 1 - \max\{\mathfrak{T}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{T}_{\mathfrak{N}_{\mathfrak{B}}}\} \leq \mathfrak{T}_{\mathfrak{N}_{\mathfrak{D}}}\end{aligned}$$

$$\begin{aligned}\mathfrak{I}_{\mathfrak{N}_{\mathfrak{K}}} &= \min\{\mathfrak{I}^c_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{I}^c_{\mathfrak{N}_{\mathfrak{B}}}\} \\ &= \min\{1 - \mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}}}, 1 - \mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}}}\} \\ &= 1 - \max\{\mathfrak{I}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{I}_{\mathfrak{N}_{\mathfrak{B}}}\} \leq \mathfrak{I}_{\mathfrak{N}_{\mathfrak{D}}}\end{aligned}$$

$$\begin{aligned}\mathfrak{F}_{\mathfrak{N}_{\mathfrak{K}}} &= \max\{\mathfrak{F}^c_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{F}^c_{\mathfrak{N}_{\mathfrak{B}}}\} \\ &= \max\{1 - \mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}}}, 1 - \mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}}}\} \\ &= 1 - \min\{\mathfrak{F}_{\mathfrak{N}_{\mathfrak{A}}}, \mathfrak{F}_{\mathfrak{N}_{\mathfrak{B}}}\} \leq \mathfrak{F}_{\mathfrak{N}_{\mathfrak{D}}},\end{aligned}$$

for all $u \in \mathfrak{U}_i$, $i\Omega$. Therefore, this proves that
$(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1)^c \widetilde{\cap} (\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2)^c \widetilde{\subseteq} ((\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega_1) \widetilde{\cap} (\mathfrak{N}_{\mathfrak{B}}, \mathfrak{B}, \omega_2))^c$. □

4. APPLICATION OF WEIGHTED NEUTROSOPHIC SOFT MULTISET IN DECISION-MAKING

In this section, we procure a new approach to WNSMS based on decision-making, for solving decision-making in an uncertain situation under an indeterminacy. Kumar [15] defined an algorithm to determinate weighted neutrosophic soft set (WNSS), for more idea related to this algorithm, we refer the reader to page 6 in [15].

To define the algorithm, we will need Kumar's algorithm which was defined in [15] as follows:

Kumar's algorithm

Step 1. input the neutrosophic soft sets ($\mathfrak{H}$, $\mathfrak{A}$), ($\mathfrak{G}$, $\mathfrak{B}$) and ($\mathfrak{H}$, $\mathfrak{C}$).
Step 2. input the weights (w_i) for the parameters $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$.
Step 3. Compute weighted neutrosophic soft sets ($\mathfrak{H}$, $\mathfrak{A}^w$), ($\mathfrak{G}$, $\mathfrak{B}^w$) and ($\mathfrak{H}$, $\mathfrak{C}^w$) corresponding to the neutrosophic soft sets ($\mathfrak{H}$, $\mathfrak{A}$), ($\mathfrak{G}$, $\mathfrak{B}$) and ($\mathfrak{H}$, $\mathfrak{C}$), respectively.
Step 4. Input the parameter set $\mathfrak{P}$ as preferred by the decision maker.
Step 5. Compute the corresponding neutrosophic soft set ($\mathfrak{S}$,$\mathfrak{P}$) from the weighted neutrosophic soft sets ($\mathfrak{H}$, $\mathfrak{A}^w$), ($\mathfrak{G}$, $\mathfrak{B}^w$) and ($\mathfrak{H}$, $\mathfrak{C}^w$) and place in tabular form.
Step 6. Compute the comparison matrix of the neutrosophic soft set ($\mathfrak{S}$,$\mathfrak{P}$).
Step 7. compute the score $\mathfrak{S}_i$ of o_i, for all $i = 1, 2, ..., n$.
Step 8. The decision making is o_k if $\mathfrak{S}_k = \max_i \mathfrak{S}_i$.
Step 9. If k has more than one values then any one of o_i may be chosen.

Then, we present the following algorithm:

Algorithm 1
Step 1. Input a NSMS $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A})$.
Step 2. Input a weighted ω for the NSMS $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A})$, where $\omega : \mathfrak{A} \to [0, 1]$.
Step 3. Compute the WNSMS $\varsigma = (\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega)$ with regard to a weighed ω.
Step 4. Apply Kumar's algorithm to the first WNSMS-part in ς to compute the decision S_{k_1}.
Step 5. Redefine the WNSMS by taking all values in each row where s_{k_1} is maximized and replacing the values in the other rows by zero, to get ς_1.
Step 6. Apply Kumar's algorithm to the second WNSMS-part in ς_1 to compute the decision S_{k_2}.
Step 7. Redefine the WNSMS ς_1 by taking the first and second WNSMS-parts in ς_1 and use the method in step 5 to find the third WNSMS-part in ς_1, to get ς_2.
Step 8. Apply Kumar's algorithm to the third WNSMS-part in ς_2 to compute the decision S_{k_3}.
Step 9. Continuing this way, we obtain the final decision $(S_{k_1}, S_{k_2}, ..., S_{k_n})$.

Remark 31. *The advantages of Algorithm 1 are:*
Firstly, we need not treat WNSMS directly in decision making, but only handle with the related WNSMS-parts and finally the crisp NSs of WNSMS-parts after choosing weighted vectors. This makes our algorithm simpler and easier for application in practical problems. Nevertheless, we also see that by using Algorithm 1 possibly we can get all the weighted choice values (WCVs) of objects for some WNSMS-parts are zero; this could be terrible for our decision. In this condition, we must adjust the weighted vectors in order to get a better choice. Besides, there are a large variety of weighted vectors that can be used to get the final optimal decision; consequently our algorithm has great flexibility and adjustable capability. As mentioned by [16], many decision making problems are basically humanistic and subjective in nature; henceforth there really does not exist a unique or uniform criterion for decision-making in an uncertain environment. This adjustable feature makes Algorithm 1 proficient, as well as more fitting for some reasonable applications. To illustrate the fundamental thought of Algorithm 1, let's consider the following example.

Example 32. *Let's consider the decision-making problem, including the WNSMS with its tabular representation given in Table 2. If we control this, we may utilize the information of* $\mathfrak{U}_1$*-WNSMS-part in* $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A})$ *and to establish the WCVs as can be seen in Table 12.*

Table 12: $\mathfrak{U}_1$-WNSMS-part in $(\mathfrak{N}_{\mathfrak{A}}, \mathfrak{A}, \omega) = \varsigma$ with WCVs

$\mathfrak{U}_1$	$x_1 0.3$	$x_2 0.5$	$x_3 0.7$	$x_4 0.4$	Choice Value	$WCV(S_k)$
a_1	(0.1,0.3.0.9)	(0.5,0.6,0.5)	(0.3,0.1,0.7)	(0.8,0.3,0.2)	4	$S_1 = 2.3$
a_2	(0.8,0.7,0.2)	(0.4,0.6,0.6)	(0.3,0.9,0.7)	(0.1,0.7,0.9)	4	$S_2 = 3.7$
a_3	(0.25,0.5,0.75)	(0.4,0.8,0.6)	(0.6,0.2,0.4)	(0.1,0.05,0.9)	4	$S_3 = 2.9$

From Table 12, we can see that the greatest WCV is 3.7, scored by a_2. Presently, we reclassify the WNSMS ς by keeping all values in every row where a_2 is maximized and changing the values in alternate rows by zero, to obtain ς_1 as in Table 13.

Table 13: The WNSMS ς_1

$\mathfrak{U}_i$		$x_1 0.3$	$x_2 0.5$	$x_3 0.7$	$x_4 0.4$
$\mathfrak{U}_1$					
	a_1	(0.1,0.3.0.9)	(0.5,0.6,0.5)	(0.3,0.1,0.7)	(0.8,0.3,0.2)
	a_2	(0.8,0.7,0.2)	(0.4,0.6,0.6)	(0.3,0.9,0.7)	(0.1,0.7,0.9)
	a_3	(0.25,0.5,0.75)	(0.4,0.8,0.6)	(0.6,0.2,0.4)	(0.1,0.05,0.9)
$\mathfrak{U}_2$					
	b_1	(0.1,1,0.9)	(0,0,1)	(0.3,0.5,0.7)	(0,0,1)
	b_2	(0.9,0.7,0.1)	(0,0,1)	(0.2,.0.9,0.8)	(0,0,1)
	b_3	(0.1,.0.9,0.2)	(0,0,1)	(0.25,0.1,0.75)	(0,0,1)
$\mathfrak{U}_3$					
	c_1	(0.9,0.1,0.1)	(0,0,1)	(1,0.2,0)	(0,0,1)
	c_2	(0.1,0.8.0.9)	(0,0,1)	(0.3,0.17,0.7)	(0,0,1)
	c_3	(0.1,0,0.9)	(0,0,1)	(0.3,0.1,0.7)	(0,0,1)
	c_4	(0.1,1,0.2)	(0,0,1)	(0.25,0.6,0.75)	(0,0,1)

Presently, we apply Kumar's Algorithm to the second WNSMS-part in ς_1 to take the choice from the accessibility set $\mathfrak{U}_2$.

Table 14: $\mathfrak{U}_2$-WNSMS-part in ς_2 with WCVs

$\mathfrak{U}_1$	$x_1 0.3$	$x_2 0.5$	$x_3 0.7$	$x_4 0.4$	Choice Value	$WCV(S_k)$
b_1	(0.1,1,0.9)	(0,0,1)	(0.3,0.5,0.7)	(0,0,1)	2	$S_1 = 2.1$
b_2	(0.9,0.7,0.1)	(0,0,1)	(0.2,.0.9,0.8)	(0,0,1)	2	$S_2 = 2.5$
b_3	(0.1,.0.9,0.2)	(0,0,1)	(0.25,0.1,0.75)	(0,0,1)	2	$S_3 = 1.3$

From Table 14, we can see that the greatest WCVs is 2.5, scored by b_2. Presently we reclassify the WNSMS ς_1 by keeping all values in every row where b_2 is maximized and replacing the values in alternate rows by zero, to obtain ς_2 as in Table 15.

Table 15: The WNSMS ς_2

$\mathfrak{U}_i$		$x_1 0.3$	$x_2 0.5$	$x_3 0.7$	$x_4 0.4$
$\mathfrak{U}_1$					
	a_1	(0.1,0.3.0.9)	(0.5,0.6,0.5)	(0.3,0.1,0.7)	(0.8,0.3,0.2)
	a_2	(0.8,0.7,0.2)	(0.4,0.6,0.6)	(0.3,0.9,0.7)	(0.1,0.7,0.9)
	a_3	(0.25,0.5,0.75)	(0.4,0.8,0.6)	(0.6,0.2,0.4)	(0.1,0.05,0.9)
$\mathfrak{U}_2$					
	b_1	(0.1,1,0.9)	(0,0,1)	(0.3,0.5,0.7)	(0,0,1)
	b_2	(0.9,0.7,0.1)	(0,0,1)	(0.2,.0.9,0.8)	(0,0,1)
	b_3	(0.1,.0.9,0.2)	(0,0,1)	(0.25,0.1,0.75)	(0,0,1)
$\mathfrak{U}_3$					
	c_1	(0,0,1)	(0,0,1)	(1,0.2,0)	(0,0,1)
	c_2	(0,0,1)	(0,0,1)	(0.3,0.17,0.7)	(0,0,1)
	c_3	(0,0,1)	(0,0,1)	(0.3,0.1,0.7)	(0,0,1)
	c_4	(0,0,1)	(0,0,1)	(0.25,0.6,0.75)	(0,0,1)

Presently, we apply Kumar's Algorithm to the third WNSMS-part in ς_2 to take the choice from $\mathfrak{U}_3$-WNSMS-part in ς_2. The tabular form of $\mathfrak{U}_3$-WNSM-part in ς_2 with WCVs as in Table 16.

Table 16: $\mathfrak{U}_3$-WNSMS-part in ς_2 with WCVs

$\mathfrak{U}_1$	$x_1 0.3$	$x_2 0.5$	$x_3 0.7$	$x_4 0.4$	Choice Value	$WCV(S_k)$
c_1	(0,0,1)	(0,0,1)	(1,0.2,0)	(0,0,1)	1	$S_1 = 1.25$
c_2	(0,0,1)	(0,0,1)	(0.3,0.17,0.7)	(0,0,1)	1	$S_2 = 2.6$
c_3	(0,0,1)	(0,0,1)	(0.3,0.1,0.7)	(0,0,1)	1	$S_3 = 1.9$

From the Table 16, it is clear that the greatest WCV is 2.6, by c_2. At that point from the above results the choice for Mr. Y is (a_2, b_2, c_2).

Remark 33. *From the above Example 32, we can see that in the Tables 12, 14 and 16, the greatest choice value is 4, 2 and 1; scored by a_1, a_2, a_3; b_1, b_2, b_3 and c_1, c_2, c_3, respectively. Thus, it is confusing for decision-maker to which one he has to select using choice value. But in case of WCV, the greatest WCV is 3.7, 2.5 and 2.6, scored by a_2, b_2 and c_2 respectively. So effectively we can obtain the ideal choice. Algorithms 1 is really an adjustable approach due to the fact that the final optimal decisions vary with the weighted vectors. This adjustable feature makes Algorithms 1 efficiently capture the uncertainty and subjectivity of some decision making problems.*

5. CONCLUSION

In this paper, we have presented the idea of WNSMS as a generalization of NSMS and its basic properties are studied. Furthermore, we define a new adjustable approach to WNSMS based on decision-making, for solving decision-making in an indeterminacy situation. The feasibility of our proposed WNSMS based on decision-making procedure in practical application is read by a mathematical example. The WNSMS is regarded as one of the neutrosophy generalized concepts of soft sets [1] and multi sets [17], the last one has more extensive applications in practice.

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