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*Research Article*

# An Algebraic Approach to Modular Inequalities Based on Interval-Valued Fuzzy Hypersoft Sets via Hypersoft Set-Inclusions

**Atiqe Ur Rahman**,[1] **Muhammad Saeed**,[1] **Khuram Ali Khan**,[2] **Ammara Nosheen**,[2] **and Rostin Matendo Mabela**[3]

[1]*Department of Mathematics, University of Management and Technology, Lahore 54000, Pakistan*
[2]*Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan*
[3]*Department of Maths and Computer Science, Faculty of Science, University of Kinshasa, Congo*

Correspondence should be addressed to Rostin Matendo Mabela; rostin.mabela@unikin.ac.cd

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Interval-valued fuzzy hypersoft set is an emerging field of study which is projected to address the limitations of interval-valued fuzzy soft set for the entitlement of multiargument approximate function. This kind of function maps the subparametric tuples to power set of universe. It emphasizes on the partitioning of attributes into their respective subattribute values in the form of disjoint sets. These features make it a completely new mathematical tool for solving problems dealing with uncertainties. In this study, after characterization of essential properties, operations, and set-inclusions ($\mathscr{L}$-inclusion and $\mathscr{J}$-inclusion) of interval-valued fuzzy hypersoft set, some of its modular inequalities are discussed via set-inclusions. It is proved that all set-inclusion-based properties and inequalities are preserved when ordinary approximate function of interval-valued fuzzy soft set is replaced with multiargument approximate function of interval-valued fuzzy hypersoft set.

## 1. Introduction

Molodtsov [1] initiated the concept of soft set (s-set) to equip fuzzy set-like models [2–4] with parameterization tool. This set employs the concept of approximate function which maps single set of parameters to initial set of alternatives. This function is also known as single-argument approximate function (SAAF) due to consideration of single set of parameters as its domain. Many researchers contributed towards the characterization of rudiments of s-sets but works of Maji et al. [5], Ali et al. [6], and Ge and Yang [7] for the investigation on set-theoretic operations and Babitha and Sunil [8, 9] for the introduction of relations and functions are more significant. Pei and Miao [10] introduced information system based on s-sets to handle the informational vagueness. Li [11] extended the previous work on soft operations and introduced some new operations. Feng and Li [12] investigated in detail the soft subset and soft product operations. Liu et al. [13] made discussion on generalized soft equal relations. Maji et al. [14] developed fuzzy soft set (fs-set) by combining fuzzy set (f-set) and s-set to deal uncertainties with parameterization tools. Yang et al. [15] hybridized interval-valued fuzzy set (ivf-set) [16] with s-set and developed interval-valued fuzzy soft set (ivfs-set) to tackle uncertain scenarios having interval nature of information and data. Jun and Yang [17] rectified some results on ivfs-sets presented by Yang et al. Chetia and Das [18] applied the notions of ivfs-sets in decision-making for medical diagnosis, Jiang et al. [19] calculated the entropy of ivfs-sets, and Feng et al. [20] characterized level soft sets based on ivfs-sets and applied them in decision-making. Liu et al. [21, 22] discussed some nonclassical properties of ivfs-sets and their modular inequalities based on soft $\mathscr{J}$-inclusion.

In many real-world decision-making scenarios, the classification of parameters into their respective subparametric-valued disjoint sets is considered necessary for having reliable and precise decisions. Soft set-like structures (hybridized structures of soft set) are inadequate to tackle such scenarios. Smarandache [23] conceptualized hypersoft set (hs-set) to address the limitations of soft set-like models. In hs-set, set of parameters is further partitioned into disjoint sets having subparametric values. It employs an approximate function which maps the cartesian product of attribute-valued nonoverlapping sets to collection of alternatives. In this way, this function is also called multiargument approximate function (MAAF). Saeed et al. [24] discussed some elementary properties and set-theoretic operations of hs-set with numerical examples. Abbas et al. [25] characterized the notions of hs-points and hs-function for their utilization in the development of hs-function spaces. Ihsan et al. [26] and Rahman et al. [27] developed hs-expert set and bijective hs-set respectively and discussed their applications in multiattribute decision-making (MADM). Rahman et al. [28] introduced a conceptual framework for classical convexity cum concavity under hs-set environment. The researchers Yolcu and Ozturk [29], Jafar and Saeed [30], and Debnath [31] discussed decision-making applications based on fuzzy hypersoft set (fhs-set) (a hybridized structure of f-set and hs-set). Rahman et al. [32] investigated the parameterization of hs-set under fuzzy setting and discussed its utilization in decision-making. The authors Saeed et al. [33] and Rahman et al. [34, 35] developed hybridized structures of fhs-set with complex set in order to tackle periodic nature of data.

The existing literature on soft inclusions and modular inequalities for ivfs-sets is suitable for SAAF only but it is incapable to manage MAAF-settings. In other words, it can be viewed that the existing literature on fuzzy soft set is unable to provide a mathematical model which may tackle the following real-world situations collectively as a single model:

(1) The situation where uncertain nature of alternatives (entities in universal set) is required to be judged by assigning fuzzy membership grades to each entity corresponding to each parameter

(2) The scenarios where classification of parameters into their respective parametric valued subcollections is necessary to be considered

(3) The scenarios which has a big collection of interval-base information which is required to be tackled with the help of its interval-valued approximate setting

Therefore, motivating from the above described shortcoming of literature, this study is aimed at developing a new structure ivfhs-sets which is more flexible as compared to existing models because it is capable to manage their limitations and is useful for having reliable and unbiased decisions due to deep focusing on parameters and their subparametric tuples. Some contributions of this research are (i) basic notions of ivfhs-set are characterized; (ii) the notions of soft inclusions discussed in [13, 17] are generalized for ivfhs-sets; and (iii) modular inequalities for ivfhs-sets based on hs-inclusions are explored by extending the concepts of Liu et al. [21, 22] and Jun and Yang [17]. The rest of the paper is structured as follows: Section 2 recalls some essential basic definitions, properties, and results relating to ivfs-set, hs-set, and fhs-set to support the main results. Section 3 presents the basic notions, properties, and inclusions of ivfhs-sets with discussion on some particular cases of ivfhs-sets. In Section 4, modular inequalities of ivfhs-sets via $\mathscr{L}$-inclusion are discussed. In Section 5, modular inequalities of ivfhs-sets via $\mathscr{J}$-inclusion are discussed. Section 6 summarizes the paper with some future directions.

## 2. Preliminaries

This section reviews few elementary terminologies and properties from literature for proper understanding of main results. Throughout the paper, $\check{X}$, $\Gamma^{\check{X}}$, $\mathbb{I}$, and $\mathscr{P}$ denote initial universe, power set of $\check{X}$, unit closed interval, and set of parameters, respectively.

*Definition 1* (see [16]). Assume the set $\mathbb{Z}^{\mathbb{I}} = \{[\widehat{u}, \widehat{v}]: \widehat{u} \leq \widehat{v} \forall \widehat{u}, \widehat{v} \in \mathbb{I}\}$ and the order relation $\leq_{\mathbb{Z}^{\mathbb{I}}}$ stated by $[\widehat{u}_1, \widehat{v}_1] \leq_{\mathbb{Z}^{\mathbb{I}}} [\widehat{u}_2, \widehat{v}_2]$ if and only if $\widehat{u}_1 \leq \widehat{u}_2, \widehat{v}_1 \leq \widehat{v}_2$ for all $[\widehat{u}_1, \widehat{v}_1], [\widehat{u}_2, \widehat{v}_2] \in \mathbb{Z}^{\mathbb{I}}$, then $\mathscr{Z}^{\mathbb{I}} = (\mathbb{Z}^{\mathbb{I}}, \leq_{\mathbb{Z}^{\mathbb{I}}})$ forms a complete lattice. An ivf-set $\mathscr{F}^{iv}$ over $\check{X}$ is characterized by mapping $\widehat{\mu} : \check{X} \longrightarrow \mathbb{Z}^{\mathbb{I}}$, where $\widehat{\mu}$ is called membership function of $\mathscr{F}^{iv}$. The collection of all ivf-sets over $\check{X}$ is represented by $\Omega_{ivfs}$.

*Definition 2* (see [1]). Let $\check{X} = \{\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n\}$ be an initial universe and $\mathscr{P} = \{\widehat{p}_1, \widehat{p}_2, \cdots, \widehat{p}_n\}$ be a set of parameters then a SAAF is a mapping $\mathfrak{F}_{\breve{\mathfrak{S}}} : \mathcal{Q} \longrightarrow \Gamma^{\check{X}}$ and defined as $\mathfrak{F}_{\breve{\mathfrak{S}}}(\{\widehat{p}_1, \widehat{p}_2, \cdots, \widehat{p}_k\}) = \Gamma^{\{\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n\}}$, where $\Gamma^{\check{X}}$ denotes the power set of $\check{X}$, $\mathcal{Q} \subseteq \mathscr{P}$ with $k \leq n$. The pair $(\mathfrak{F}_{\breve{\mathfrak{S}}}, \mathcal{Q})$ is known as s-set and represented by $\breve{\mathfrak{S}}$. The subsets $\mathfrak{F}_{\breve{\mathfrak{S}}}(\widehat{p}_i) \subseteq \check{X}$ are known as $\widehat{p}_i$ -approximate sets having all $\widehat{p}_i$ -approximate elements. The pair $(\check{X}, \mathscr{P})$ is called soft-universe. The collection of s-sets is denoted by $\Omega_{ss}$.

*Definition 3* (see [17]). For any soft-universe $(\check{X}, \mathscr{P})$ with $\mathcal{Q} \subseteq \mathscr{P}$, an ivfs-set $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}} = (\mathscr{F}_{\breve{\mathfrak{S}}}, \mathcal{Q})$ is characterized by mapping $\mathscr{F}_{\breve{\mathfrak{S}}} : \mathcal{Q} \longrightarrow \Omega_{ivfs}$, where $\mathcal{Q}$ is the same as stated in Definition 2, and $\mathscr{F}_{\breve{\mathfrak{S}}}$ is known as SAAF of $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}$. The collection of ivfs-sets is denoted by $\Omega_{ivfss}$.

*Definition 4* (see [15]). Let $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}} = (\mathscr{F}_{\breve{\mathfrak{S}}}, \mathcal{Q}_1) \& \mathscr{G}^{iv}_{\breve{\mathfrak{S}}} = (\mathscr{G}_{\breve{\mathfrak{S}}}, \mathcal{Q}_2) \in \Omega_{ivfss}$, then their soft product operations, i.e., $\wedge \& \vee$ are given as:

(1) The $\wedge$-product (AND-operation) of $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}$ and $\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$ is an ivfs-set defined by

$$\mathscr{F}^{iv}_{\breve{\mathfrak{S}}} \wedge \mathscr{G}^{iv}_{\breve{\mathfrak{S}}} = (\mathscr{H}_{\breve{\mathfrak{S}}}, \mathcal{Q}_1 \times \mathcal{Q}_2), \tag{1}$$

such that

$$\mathscr{H}_{\breve{\mathfrak{S}}}(\widehat{p}_1,\widehat{p}_2)=\mathscr{F}_{\breve{\mathfrak{S}}}(\widehat{p}_1)\cap\mathscr{G}_{\breve{\mathfrak{S}}}(\widehat{p}_2),\quad \forall(\widehat{p}_1,\widehat{p}_2)\in\mathcal{Q}_1\times\mathcal{Q}_2. \tag{2}$$

(2) The $\vee$-product (OR-operation) of $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}$ and $\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$ is an ivfs-set defined by

$$\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}\vee\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}=(\mathscr{H}_{\breve{\mathfrak{S}}},\mathcal{Q}_1\times\mathcal{Q}_2), \tag{3}$$

such that

$$\mathscr{H}_{\breve{\mathfrak{S}}}(\widehat{p}_1,\widehat{p}_2)=\mathscr{F}_{\breve{\mathfrak{S}}}(\widehat{p}_1)\cup\mathscr{G}_{\breve{\mathfrak{S}}}(\widehat{p}_2),\quad \forall(\widehat{p}_1,\widehat{p}_2)\in\mathcal{Q}_1\times\mathcal{Q}_2. \tag{4}$$

The following two soft-inclusions relations Jun's inclusion $\widehat{\subseteq}_{\mathscr{J}}$ in [17] and Liu's inclusion $\widehat{\subseteq}_{\mathscr{L}}$ in [13] are prominent in literature for understanding the set-theoretic operations of ivfs-sets.

*Definition 5* (see [17]). Let $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}=(\mathscr{F}_{\breve{\mathfrak{S}}},\mathcal{Q}_1)\&\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}=(\mathscr{G}_{\breve{\mathfrak{S}}},\mathcal{Q}_2)\in\Omega_{ivfss}$, then

(1) $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}$ is said to be ivfs $\mathscr{J}$-subset of $\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$, denoted by $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}\widehat{\subseteq}_{\mathscr{J}}\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$, if for every $\widehat{p}_1\in\mathcal{Q}_1\exists\widehat{p}_2\in\mathcal{Q}_2$ such that $\mathscr{F}_{\breve{\mathfrak{S}}}(\widehat{p}_1)\subseteq\mathscr{G}_{\breve{\mathfrak{S}}}(\widehat{p}_2)$

(2) $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}$ and $\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$ are said to be ivfs $\mathscr{J}$-equal, denoted by $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}\widehat{=}_{\mathscr{J}}\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$, if $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}\widehat{\subseteq}_{\mathscr{J}}\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$ and $\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}\widehat{\subseteq}_{\mathscr{J}}\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}$

Liu et al. [13] introduced the following soft inclusions by modifying the soft inclusion of Jun and Yang [17].

*Definition 6* (see [13]). Let $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}=(\mathscr{F}_{\breve{\mathfrak{S}}},\mathcal{Q}_1)\&\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}=(\mathscr{G}_{\breve{\mathfrak{S}}},\mathcal{Q}_2)\in\Omega_{ivfss}$, then

(1) $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}$ is said to be ivfs $\mathscr{L}$-subset of $\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$, denoted by $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}\widehat{\subseteq}_{\mathscr{L}}\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$, if for every $\widehat{p}_1\in\mathcal{Q}_1\exists\widehat{p}_2\in\mathcal{Q}_2$ such that $\mathscr{F}_{\breve{\mathfrak{S}}}(\widehat{p}_1)=\mathscr{G}_{\breve{\mathfrak{S}}}(\widehat{p}_2)$

(2) $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}$ and $\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$ are said to be ivfs $\mathscr{L}$-equal, denoted by $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}\widehat{=}_{\mathscr{L}}\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$, if $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}\widehat{\subseteq}_{\mathscr{L}}\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$ and $\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}\widehat{\subseteq}_{\mathscr{L}}\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}$

Note: both $\widehat{\subseteq}_{\mathscr{J}}$ and $\widehat{\subseteq}_{\mathscr{L}}$ are termed as ivfs $\mathscr{J}$-inclusion and ivfs $\mathscr{L}$-inclusion respectively.

**Proposition 7.** *If $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}\widehat{\subseteq}_{\mathscr{L}}\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$, then it implies $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}\widehat{\subseteq}_{\mathscr{J}}\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$.*

*Definition 8* (see [13]). $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}$ is said to be identical to $\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$, denoted by $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}\widehat{\equiv}\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$, if $\mathcal{Q}_1=\mathcal{Q}_2$ and $\mathscr{F}_{\breve{\mathfrak{S}}}(\widehat{p}_1)=\mathscr{G}_{\breve{\mathfrak{S}}}(\widehat{p}_2)$ for every $\widehat{p}_1\in\mathcal{Q}_1\exists\widehat{p}_2\in\mathcal{Q}_2$.

**Proposition 9.** *If $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}\widehat{\equiv}\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$, then it implies $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}\widehat{\subseteq}_{\mathscr{L}}\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$ which further implies $\mathscr{F}^{iv}_{\breve{\mathfrak{S}}}\widehat{\subseteq}_{\mathscr{J}}\mathscr{G}^{iv}_{\breve{\mathfrak{S}}}$.*

Propositions 7 and 9 are not valid in general. Please refer to [12, 13] for detailed discussion regarding the generalization of these results.

*Definition 10* (see [23]). Let $\check{X}=\{\widehat{x}_1,\widehat{x}_2,\cdots,\widehat{x}_n\}$ be an initial universe and $\mathscr{P}=\{\widehat{p}_1,\widehat{p}_2,\cdots,\widehat{p}_n\}$ be a set of parameters. The respective attribute-valued nonoverlapping sets of each element of $\mathscr{P}$ are $\mathcal{Q}_1=\{\widehat{q}_{11},\widehat{q}_{12},\cdots,\widehat{q}_{1n}\}$, $\mathcal{Q}_2=\{\widehat{q}_{21},\widehat{q}_{22},\cdots,\widehat{q}_{2n}\}$, $\mathcal{Q}_3=\{\widehat{q}_{31},\widehat{q}_{32},\cdots,\widehat{q}_{3n}\}$ , ….., $\mathcal{Q}_n=\{\widehat{q}_{n1},\widehat{q}_{n2},\cdots,\widehat{q}_{nn}\}$ and $\mathcal{Q}=\mathcal{Q}_1\times\mathcal{Q}_2\times\mathcal{Q}_3\times\cdots.\times\mathcal{Q}_n=\{\widehat{q}_1,\widehat{q}_2,\widehat{q}_3,\cdots.,\widehat{q}_r\}$, where each $\widehat{q}_i(i=1,2,\cdots.,r)$ is a $n$-tuple element of $\mathcal{Q}$ and $r=\prod_{i=1}^n|\mathcal{Q}_i|$, $|\bullet|$ denotes set cardinality, then a MAAF is a mapping $\mathfrak{F}_{\breve{\mathfrak{H}}}:\mathscr{V}\longrightarrow\Gamma^{\check{X}}$ and defined as $\mathfrak{F}_{\breve{\mathfrak{H}}}(\{\widehat{q}_1,\widehat{q}_2,\cdots,\widehat{q}_k\})=\Gamma^{\{\widehat{x}_1,\widehat{x}_2,\cdots,\widehat{x}_n\}}$, where $\Gamma^{\check{X}}$ denotes the power set of $\check{X}$, $\mathscr{V}\subseteq\mathcal{Q}$ with $k\leq r$. The pair $(\mathfrak{F}_{\breve{\mathfrak{H}}},\mathscr{V})$ is known as hs-set and represented by $\breve{\mathfrak{H}}$. The collection of all hs-sets is symbolized as $\Omega_{hss}$.

*Definition 11* (see [23]). If $\Gamma^{\check{X}}_{\mathscr{F}}$ be the collection of all fuzzy sets, then a hs-set $\breve{\mathfrak{H}}=(\mathfrak{F}_{\breve{\mathfrak{H}}},\mathscr{V})$ is said to be fhs-set if $\mathfrak{F}_{\breve{\mathfrak{H}}}:\mathscr{V}\longrightarrow\Gamma^{\check{X}}_{\mathscr{F}}$, where $\mathscr{V}$ is same as discussed in Definition 10, and $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{v})$ is an approximate element of fhs-set for $\widehat{v}\in\mathscr{V}$.

## 3. Properties of ivfhs-Sets

In this section, novel notions of ivfhs-sets are characterized. During this characterization, focus is laid on those operations and properties which are essential to proceed further for the development of modular inequalities.

*Definition 12.* Let $\check{X}=\{\widehat{y}_1,\widehat{y}_2,\cdots,\widehat{y}_n\}$ be an initial universe and $\mathscr{P}=\{\widehat{p}_1,\widehat{p}_2,\cdots,\widehat{p}_n\}$ be a set of parameters. The respective subparametric-valued disjoint sets are $\mathscr{R}_1=\{\widehat{r}_{11},\widehat{r}_{12},\cdots,\widehat{r}_{1n}\}$, $\mathscr{R}_2=\{\widehat{r}_{21},\widehat{r}_{22},\cdots,\widehat{r}_{2n}\}$, $\mathscr{R}_3=\{\widehat{r}_{31},\widehat{r}_{32},\cdots,\widehat{r}_{3n}\}$ ,….., $\mathscr{R}_n=\{\widehat{r}_{n1},\widehat{r}_{n2},\cdots,\widehat{r}_{nn}\}$,and $\mathscr{R}=\mathscr{R}_1\times\mathscr{R}_2\times\mathscr{R}_3\times\cdots.\times\mathscr{R}_n=\{\widehat{r}_1,\widehat{r}_2,\widehat{r}_3,\cdots.,\widehat{r}_s\}$, where each $\widehat{r}_i(i=1,2,\cdots.,s)$ is a $n$-tuple element of $\mathscr{R}$ and $s=\prod_{i=1}^n|\mathscr{R}_i|$, $|\bullet|$ denotes set cardinality then the pair $(\Psi_{\text{FHS}},\mathscr{W})$ is known as ivfhs-set, where $\Psi_{FHS}:\mathscr{W}\longrightarrow\Omega_{ivfs}$ and defined as $\Psi_{\text{FHS}}(\{\widehat{r}_1,\widehat{r}_2,\cdots,\widehat{r}_k\})=\Omega_{ivfs}(\{\widehat{y}_1,\widehat{y}_2,\cdots,\widehat{y}_n\})$, and $\mathscr{W}\subseteq\mathscr{R}$ with $k\leq s$. The collection of all ivfhs-sets is symbolized as $\mho_{\text{ivfhss}}(\check{X},\mathscr{P})$.

*Example 1.* Suppose an organization plans to recruit a candidate to fill a vacant post of assistant manager. There are six candidates forming an initial universe of discourse $\check{X}=\{\mathfrak{C}_1,\mathfrak{C}_2,\mathfrak{C}_3,\mathfrak{C}_4,\mathfrak{C}_5,\mathfrak{C}_6\}$ and have been scrutinized by recruitment committee. The committee further requires evaluation to select one of these candidates. The evaluation indicators are qualification $(\widehat{p}_1)$, relevant experience in years $(\widehat{p}_2)$, and computer skill $(\widehat{p}_3)$. Their subparametric disjoint sets are $\mathscr{R}_1=\{\widehat{r}_{11}=\text{MBA}\}$, $\mathscr{R}_2=\{\widehat{r}_{21}=5,\widehat{r}_{22}=7,\widehat{r}_{23}=10\}$, and $\mathscr{R}_3$

$= \{\widehat{r}_{31} = MS - \text{office}\}$ respectively such that $\mathscr{R} = \mathscr{R}_1 \times \mathscr{R}_2 \times \mathscr{R}_3 = \{\widehat{r}_1, \widehat{r}_2, \widehat{r}_3\}$. Then, an ivfhs-set $(\Psi_{\text{FHS}}, \mathscr{R})$ is structured as $(\Psi_{\text{FHS}}, \mathscr{R}) = \{(\Psi_{\text{FHS}}(\widehat{r}_1), \widehat{r}_1), (\Psi_{\text{FHS}}(\widehat{r}_2), \widehat{r}_2), (\Psi_{\text{FHS}}(\widehat{r}_3), \widehat{r}_3)\}$, where

$$\begin{aligned}\Psi_{\text{FHS}}(\widehat{r}_{11}, \widehat{r}_{21}, \widehat{r}_{31}) = \Psi_{\text{FHS}}(\widehat{r}_1) &= \left\{\frac{\mathfrak{C}_1}{[0.2,0.4]}, \frac{\mathfrak{C}_2}{[0.3,0.5]}, \frac{\mathfrak{C}_3}{[0.4,0.6]}, \frac{\mathfrak{C}_4}{[0.5,0.7]}\right\},\\ \Psi_{\text{FHS}}(\widehat{r}_{11}, \widehat{r}_{22}, \widehat{r}_{31}) = \Psi_{\text{FHS}}(\widehat{r}_2) &= \left\{\frac{\mathfrak{C}_1}{[0.3,0.6]}, \frac{\mathfrak{C}_2}{[0.4,0.7]}, \frac{\mathfrak{C}_3}{[0.5,0.8]}, \frac{\mathfrak{C}_4}{[0.6,0.9]}\right\},\\ \Psi_{\text{FHS}}(\widehat{r}_{11}, \widehat{r}_{23}, \widehat{r}_{31}) = \Psi_{\text{FHS}}(\widehat{r}_3) &= \left\{\frac{\mathfrak{C}_1}{[0.1,0.5]}, \frac{\mathfrak{C}_2}{[0.2,0.6]}, \frac{\mathfrak{C}_3}{[0.3,0.7]}, \frac{\mathfrak{C}_4}{[0.4,0.8]}\right\}.\end{aligned} \tag{5}$$

Hence,

$$(\Psi_{\text{FHS}}, \mathscr{R}) = \left\{\begin{array}{l}\left(\left\{\frac{\mathfrak{C}_1}{[0.2,0.4]}, \frac{\mathfrak{C}_2}{[0.3,0.5]}, \frac{\mathfrak{C}_3}{[0.4,0.6]}, \frac{\mathfrak{C}_4}{[0.5,0.7]}\right\}, \widehat{r}_1\right),\\ \left(\left\{\frac{\mathfrak{C}_1}{[0.3,0.6]}, \frac{\mathfrak{C}_2}{[0.4,0.7]}, \frac{\mathfrak{C}_3}{[0.5,0.8]}, \frac{\mathfrak{C}_4}{[0.6,0.9]}\right\}, \widehat{r}_2\right),\\ \left(\left\{\frac{\mathfrak{C}_1}{[0.1,0.5]}, \frac{\mathfrak{C}_2}{[0.2,0.6]}, \frac{\mathfrak{C}_3}{[0.3,0.7]}, \frac{\mathfrak{C}_4}{[0.4,0.8]}\right\}, \widehat{r}_3\right)\end{array}\right\}. \tag{6}$$

Its tabular representation is given in Table 1.

*Definition 13.* The complement of an ivfhs-set $(\Psi_{\text{FHS}}, \mathscr{R})$, denoted by $(\Psi_{\text{FHS}}, \mathscr{R})^c$, is defined by $(\Psi_{\text{FHS}}, \mathscr{R})^c = (\Psi^c_{\text{FHS}}, \sim \mathscr{R})$ where $\Psi^c_{\text{FHS}} : \sim \mathscr{R} \longrightarrow \Omega_{\text{ivfs}}$ and $\sim \mathscr{R}$ stands for "not $\mathscr{R}$.

*Example 2.* Considering data from Example 1, we have

$$(\Psi_{\text{FHS}}, \mathscr{R})^c = \left\{\begin{array}{l}\left(\left\{\frac{\mathfrak{C}_1}{[0.6,0.8]}, \frac{\mathfrak{C}_2}{[0.5,0.7]}, \frac{\mathfrak{C}_3}{[0.4,0.6]}, \frac{\mathfrak{C}_4}{[0.3,0.5]}\right\}, \sim \widehat{r}_1\right),\\ \left(\left\{\frac{\mathfrak{C}_1}{[0.4,0.7]}, \frac{\mathfrak{C}_2}{[0.3,0.6]}, \frac{\mathfrak{C}_3}{[0.2,0.5]}, \frac{\mathfrak{C}_4}{[0.1,0.4]}\right\}, \sim \widehat{r}_2\right),\\ \left(\left\{\frac{\mathfrak{C}_1}{[0.5,0.9]}, \frac{\mathfrak{C}_2}{[0.4,0.8]}, \frac{\mathfrak{C}_3}{[0.3,0.7]}, \frac{\mathfrak{C}_4}{[0.2,0.6]}\right\}, \sim \widehat{r}_3\right)\end{array}\right\}. \tag{7}$$

Its tabular representation is given in Table 2.

*Definition 14.* Let $\mathfrak{G}_1 = (\Psi^1_{\text{FHS}}, \mathscr{R}_1) \& \mathfrak{G}_2 = (\Psi^2_{\text{FHS}}, \mathscr{R}_2)$ be two ivfhs-sets then their hypersoft aggregation operations, i.e., $\tilde{\sqcup} \& \tilde{\sqcap}$ are given as:

(1) Their union is an ivfhs-set defined by $\mathfrak{G}_1 \tilde{\sqcup} \mathfrak{G}_2 = (\Psi^3_{\text{FHS}}, \mathscr{R}_3)$ such that $\Psi^3_{\text{FHS}}(\widehat{p}) = \Psi^1_{\text{FHS}}(\widehat{p}) \cup \Psi^2_{\text{FHS}}(\widehat{p}_2) \forall \widehat{p} \in \mathscr{R}_1 \cup \mathscr{R}_2$ with maximum interval-valued fuzzy degrees respective to $\Psi^1_{\text{FHS}}(\widehat{p})$ and $\Psi^2_{\text{FHS}}(\widehat{p})$

(2) Their intersection is an ivfhs-set defined by $\mathfrak{G}_1 \tilde{\sqcap} \mathfrak{G}_2 = (\Psi^4_{\text{FHS}}, \mathscr{R}_4)$ such that $\Psi^4_{\text{FHS}}(\widehat{p}) = \Psi^1_{\text{FHS}}(\widehat{p}) \cap \Psi^2_{\text{FHS}}(\widehat{p}_2) \forall \widehat{p} \in \mathscr{R}_1 \cap \mathscr{R}_2$ with minimum interval-valued fuzzy degrees respective to $\Psi^1_{\text{FHS}}(\widehat{p})$ and $\Psi^2_{\text{FHS}}(\widehat{p})$

Table 1: Interval-valued fuzzy hypersoft set $(\Psi_{\text{FHS}}, \mathscr{R})$.

| $\mathscr{R}\downarrow\backslash\check{X}\longrightarrow$ | $\mathfrak{C}_1$ | $\mathfrak{C}_2$ | $\mathfrak{C}_3$ | $\mathfrak{C}_4$ |
|---|---|---|---|---|
| $\widehat{r}_1$ | [0.2,0.4] | [0.3,0.5] | [0.4,0.6] | [0.5,0.7] |
| $\widehat{r}_2$ | [0.3,0.6] | [0.4,0.7] | [0.5,0.8] | [0.6,0.9] |
| $\widehat{r}_3$ | [0.1,0.5] | [0.2,0.6] | [0.3,0.7] | [0.4,0.8] |

*Example 3.* Considering Example 1, we have following two ivfhs-sets $\mathfrak{G}_1 = (\Psi^1_{\text{FHS}}, \mathscr{R}_1) \& \mathfrak{G}_2 = (\Psi^2_{\text{FHS}}, \mathscr{R}_2)$.

$$\begin{aligned}\mathfrak{G}_1 = (\Psi^1_{\text{FHS}}, \mathscr{R}_1) &= \left\{\begin{array}{l}\left(\left\{\frac{\mathfrak{C}_1}{[0.2,0.4]}, \frac{\mathfrak{C}_2}{[0.3,0.5]}, \frac{\mathfrak{C}_3}{[0.4,0.6]}, \frac{\mathfrak{C}_4}{[0.5,0.7]}\right\}, \widehat{r}_1\right),\\ \left(\left\{\frac{\mathfrak{C}_1}{[0.3,0.6]}, \frac{\mathfrak{C}_2}{[0.4,0.7]}, \frac{\mathfrak{C}_3}{[0.5,0.8]}, \frac{\mathfrak{C}_4}{[0.6,0.9]}\right\}, \widehat{r}_2\right),\\ \left(\left\{\frac{\mathfrak{C}_1}{[0.1,0.5]}, \frac{\mathfrak{C}_2}{[0.2,0.6]}, \frac{\mathfrak{C}_3}{[0.3,0.7]}, \frac{\mathfrak{C}_4}{[0.4,0.8]}\right\}, \widehat{r}_3\right)\end{array}\right\},\\ \mathfrak{G}_2 = (\Psi^2_{\text{FHS}}, \mathscr{R}_2) &= \left\{\begin{array}{l}\left(\left\{\frac{\mathfrak{C}_1}{[0.3,0.4]}, \frac{\mathfrak{C}_2}{[0.4,0.5]}, \frac{\mathfrak{C}_3}{[0.5,0.6]}, \frac{\mathfrak{C}_4}{[0.6,0.7]}\right\}, \widehat{r}_1\right),\\ \left(\left\{\frac{\mathfrak{C}_1}{[0.4,0.6]}, \frac{\mathfrak{C}_2}{[0.5,0.7]}, \frac{\mathfrak{C}_3}{[0.6,0.8]}, \frac{\mathfrak{C}_4}{[0.7,0.9]}\right\}, \widehat{r}_2\right),\\ \left(\left\{\frac{\mathfrak{C}_1}{[0.2,0.5]}, \frac{\mathfrak{C}_2}{[0.3,0.6]}, \frac{\mathfrak{C}_3}{[0.4,0.7]}, \frac{\mathfrak{C}_4}{[0.5,0.8]}\right\}, \widehat{r}_3\right)\end{array}\right\},\end{aligned} \tag{8}$$

then

$$\mathfrak{G}_1 \tilde{\sqcup} \mathfrak{G}_2 = (\Psi^3_{\text{FHS}}, \mathscr{R}_3) = \left\{\begin{array}{l}\left(\left\{\frac{\mathfrak{C}_1}{[0.3,0.4]}, \frac{\mathfrak{C}_2}{[0.4,0.5]}, \frac{\mathfrak{C}_3}{[0.5,0.6]}, \frac{\mathfrak{C}_4}{[0.6,0.7]}\right\}, \widehat{r}_1\right),\\ \left(\left\{\frac{\mathfrak{C}_1}{[0.4,0.6]}, \frac{\mathfrak{C}_2}{[0.5,0.7]}, \frac{\mathfrak{C}_3}{[0.6,0.8]}, \frac{\mathfrak{C}_4}{[0.7,0.9]}\right\}, \widehat{r}_2\right),\\ \left(\left\{\frac{\mathfrak{C}_1}{[0.2,0.5]}, \frac{\mathfrak{C}_2}{[0.3,0.6]}, \frac{\mathfrak{C}_3}{[0.4,0.7]}, \frac{\mathfrak{C}_4}{[0.5,0.8]}\right\}, \widehat{r}_3\right)\end{array}\right\}, \tag{9}$$

and

$$\mathfrak{G}_1 \tilde{\sqcap} \mathfrak{G}_2 = (\Psi^4_{\text{FHS}}, \mathscr{R}_4) = \left\{\begin{array}{l}\left(\left\{\frac{\mathfrak{C}_1}{[0.2,0.4]}, \frac{\mathfrak{C}_2}{[0.3,0.5]}, \frac{\mathfrak{C}_3}{[0.4,0.6]}, \frac{\mathfrak{C}_4}{[0.5,0.7]}\right\}, \widehat{r}_1\right),\\ \left(\left\{\frac{\mathfrak{C}_1}{[0.3,0.6]}, \frac{\mathfrak{C}_2}{[0.4,0.7]}, \frac{\mathfrak{C}_3}{[0.5,0.8]}, \frac{\mathfrak{C}_4}{[0.6,0.9]}\right\}, \widehat{r}_2\right),\\ \left(\left\{\frac{\mathfrak{C}_1}{[0.1,0.5]}, \frac{\mathfrak{C}_2}{[0.2,0.6]}, \frac{\mathfrak{C}_3}{[0.3,0.7]}, \frac{\mathfrak{C}_4}{[0.4,0.8]}\right\}, \widehat{r}_3\right)\end{array}\right\}. \tag{10}$$

Table 2: Complement of ivfhs-set $(\Psi_{\mathrm{FHS}}, \mathscr{R})$.

| $\sim \mathscr{R}\downarrow\backslash \check{X} \longrightarrow$ | $\mathfrak{C}_1$ | $\mathfrak{C}_2$ | $\mathfrak{C}_3$ | $\mathfrak{C}_4$ |
|---|---|---|---|---|
| $\sim \widehat{r}_1$ | [0.6,0.8] | [0.5,0.7] | [0.4,0.6] | [0.3,0.5] |
| $\sim \widehat{r}_2$ | [0.4,0.7] | [0.3,0.6] | [0.2,0.5] | [0.1,0.4] |
| $\sim \widehat{r}_3$ | [0.5,0.9] | [0.4,0.8] | [0.3,0.7] | [0.2,0.6] |

*Definition 15.* Let $\mathfrak{G}_1 = (\Psi^1_{\mathrm{FHS}}, \mathscr{R}_1) \& \mathfrak{G}_2 = (\Psi^2_{\mathrm{FHS}}, \mathscr{R}_2)$ be two ivfhs-sets, then their hypersoft product operations, i.e., $\tilde{\otimes}$ & $\tilde{\oplus}$ are given as:

(1) The $\tilde{\otimes}$-product (AND-operation) is an ivfhs-set defined by $\mathfrak{G}_1 \tilde{\otimes} \mathfrak{G}_2 = (\Psi^3_{\mathrm{FHS}}, \mathscr{R}_1 \times \mathscr{R}_2)$ such that $\Psi^3_{\mathrm{FHS}}(\widehat{p}_1, \widehat{p}_2) = \Psi^1_{\mathrm{FHS}}(\widehat{p}_1) \cap \Psi^2_{\mathrm{FHS}}(\widehat{p}_2) \forall (\widehat{p}_1, \widehat{p}_2) \in \mathscr{R}_1 \times \mathscr{R}_2$

(2) The $\tilde{\oplus}$-product (OR-operation) is an ivfhs-set defined by $\mathfrak{G}_1 \tilde{\oplus} \mathfrak{G}_2 = (\Psi^4_{\mathrm{FHS}}, \mathscr{R}_1 \times \mathscr{R}_2)$ such that $\Psi^4_{\mathrm{FHS}}(\widehat{p}_1, \widehat{p}_2) = \Psi^1_{\mathrm{FHS}}(\widehat{p}_1) \cup \Psi^2_{\mathrm{FHS}}(\widehat{p}_2) \forall (\widehat{p}_1, \widehat{p}_2) \in \mathscr{R}_1 \times \mathscr{R}_2$

*Example 4.* Considering the values of two ivfhs-sets $\mathfrak{G}_1 = (\Psi^1_{\mathrm{FHS}}, \mathscr{R}_1) \& \mathfrak{G}_2 = (\Psi^2_{\mathrm{FHS}}, \mathscr{R}_2)$ from Example 3, then

$$\mathfrak{G}_1 \tilde{\otimes} \mathfrak{G}_2 = (\Psi^3_{\mathrm{FHS}}, \mathscr{R}_1 \times \mathscr{R}_2) = \left\{ \begin{array}{l} \left(\left\{\frac{\mathfrak{C}_1}{[0.2,0.4]}, \frac{\mathfrak{C}_2}{[0.3,0.5]}, \frac{\mathfrak{C}_3}{[0.4,0.6]}, \frac{\mathfrak{C}_4}{[0.5,0.7]}\right\}, (\widehat{r}_1, \widehat{r}_1)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.3,0.6]}, \frac{\mathfrak{C}_2}{[0.4,0.7]}, \frac{\mathfrak{C}_3}{[0.5,0.8]}, \frac{\mathfrak{C}_4}{[0.6,0.9]}\right\}, (\widehat{r}_1, \widehat{r}_2)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.1,0.5]}, \frac{\mathfrak{C}_2}{[0.2,0.6]}, \frac{\mathfrak{C}_3}{[0.3,0.7]}, \frac{\mathfrak{C}_4}{[0.4,0.8]}\right\}, (\widehat{r}_1, \widehat{r}_3)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.3,0.4]}, \frac{\mathfrak{C}_2}{[0.4,0.5]}, \frac{\mathfrak{C}_3}{[0.5,0.6]}, \frac{\mathfrak{C}_4}{[0.6,0.7]}\right\}, (\widehat{r}_2, \widehat{r}_1)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.3,0.6]}, \frac{\mathfrak{C}_2}{[0.4,0.7]}, \frac{\mathfrak{C}_3}{[0.5,0.8]}, \frac{\mathfrak{C}_4}{[0.6,0.9]}\right\}, (\widehat{r}_2, \widehat{r}_2)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.2,0.5]}, \frac{\mathfrak{C}_2}{[0.3,0.6]}, \frac{\mathfrak{C}_3}{[0.4,0.7]}, \frac{\mathfrak{C}_4}{[0.5,0.8]}\right\}, (\widehat{r}_2, \widehat{r}_3)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.1,0.4]}, \frac{\mathfrak{C}_2}{[0.2,0.5]}, \frac{\mathfrak{C}_3}{[0.3,0.6]}, \frac{\mathfrak{C}_4}{[0.4,0.7]}\right\}, (\widehat{r}_3, \widehat{r}_1)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.1,0.5]}, \frac{\mathfrak{C}_2}{[0.2,0.6]}, \frac{\mathfrak{C}_3}{[0.3,0.7]}, \frac{\mathfrak{C}_4}{[0.4,0.8]}\right\}, (\widehat{r}_3, \widehat{r}_2)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.1,0.5]}, \frac{\mathfrak{C}_2}{[0.2,0.6]}, \frac{\mathfrak{C}_3}{[0.3,0.7]}, \frac{\mathfrak{C}_4}{[0.4,0.8]}\right\}, (\widehat{r}_3, \widehat{r}_3)\right) \end{array} \right\}, \tag{11}$$

and

$$\mathfrak{G}_1 \tilde{\oplus} \mathfrak{G}_2 = (\Psi^4_{\mathrm{FHS}}, \mathscr{R}_1 \times \mathscr{R}_2) = \left\{ \begin{array}{l} \left(\left\{\frac{\mathfrak{C}_1}{[0.3,0.4]}, \frac{\mathfrak{C}_2}{[0.4,0.5]}, \frac{\mathfrak{C}_3}{[0.5,0.6]}, \frac{\mathfrak{C}_4}{[0.6,0.7]}\right\}, (\widehat{r}_1, \widehat{r}_1)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.4,0.6]}, \frac{\mathfrak{C}_2}{[0.5,0.7]}, \frac{\mathfrak{C}_3}{[0.6,0.8]}, \frac{\mathfrak{C}_4}{[0.7,0.9]}\right\}, (\widehat{r}_1, \widehat{r}_2)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.2,0.5]}, \frac{\mathfrak{C}_2}{[0.3,0.6]}, \frac{\mathfrak{C}_3}{[0.4,0.7]}, \frac{\mathfrak{C}_4}{[0.5,0.8]}\right\}, (\widehat{r}_1, \widehat{r}_3)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.3,0.6]}, \frac{\mathfrak{C}_2}{[0.4,0.7]}, \frac{\mathfrak{C}_3}{[0.5,0.8]}, \frac{\mathfrak{C}_4}{[0.6,0.9]}\right\}, (\widehat{r}_2, \widehat{r}_1)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.4,0.6]}, \frac{\mathfrak{C}_2}{[0.5,0.7]}, \frac{\mathfrak{C}_3}{[0.6,0.8]}, \frac{\mathfrak{C}_4}{[0.7,0.9]}\right\}, (\widehat{r}_2, \widehat{r}_2)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.2,0.5]}, \frac{\mathfrak{C}_2}{[0.3,0.6]}, \frac{\mathfrak{C}_3}{[0.4,0.7]}, \frac{\mathfrak{C}_4}{[0.5,0.8]}\right\}, (\widehat{r}_2, \widehat{r}_3)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.3,0.5]}, \frac{\mathfrak{C}_2}{[0.4,0.6]}, \frac{\mathfrak{C}_3}{[0.5,0.7]}, \frac{\mathfrak{C}_4}{[0.6,0.8]}\right\}, (\widehat{r}_3, \widehat{r}_1)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.4,0.6]}, \frac{\mathfrak{C}_2}{[0.5,0.7]}, \frac{\mathfrak{C}_3}{[0.6,0.8]}, \frac{\mathfrak{C}_4}{[0.7,0.9]}\right\}, (\widehat{r}_3, \widehat{r}_2)\right), \\ \left(\left\{\frac{\mathfrak{C}_1}{[0.2,0.5]}, \frac{\mathfrak{C}_2}{[0.3,0.6]}, \frac{\mathfrak{C}_3}{[0.4,0.7]}, \frac{\mathfrak{C}_4}{[0.5,0.8]}\right\}, (\widehat{r}_3, \widehat{r}_3)\right) \end{array} \right\}. \tag{12}$$

Table 3: Tabular representation of $\mathfrak{G}_1 \otimes \tilde{} \mathfrak{G}_2$.

| $(\mathscr{R}_1 \times \mathscr{R}_2)\downarrow\backslash\check{X} \longrightarrow$ | $\mathfrak{C}_1$ | $\mathfrak{C}_2$ | $\mathfrak{C}_3$ | $\mathfrak{C}_4$ |
|---|---|---|---|---|
| $(\hat{r}_1, r_1)$ | [0.2,0.4] | [0.3,0.5] | [0.4,0.6] | [0.5,0.7] |
| $(\hat{r}_1, \hat{r}_2)$ | [0.3,0.6] | [0.4,0.7] | [0.5,0.8] | [0.6,0.9] |
| $(\hat{r}_1, \hat{r}_3)$ | [0.1,0.5] | [0.2,0.6] | [0.3,0.7] | [0.4,0.8] |
| $(\hat{r}_2, \hat{r}_1)$ | [0.3,0.4] | [0.4,0.5] | [0.5,0.6] | [0.6,0.7] |
| $(\hat{r}_2, \hat{r}_2)$ | [0.3,0.6] | [0.4,0.7] | [0.5,0.8] | [0.6,0.9] |
| $(\hat{r}_2, \hat{r}_3)$ | [0.2,0.5] | [0.3,0.6] | [0.4,0.7] | [0.5,0.8] |
| $(\hat{r}_3, \hat{r}_1)$ | [0.1,0.4] | [0.2,0.5] | [0.3,0.6] | [0.4,0.7] |
| $(\hat{r}_3, \hat{r}_2)$ | [0.1,0.5] | [0.2,0.6] | [0.3,0.7] | [0.4,0.8] |
| $(\hat{r}_3, \hat{r}_3)$ | [0.1,0.5] | [0.2,0.6] | [0.3,0.7] | [0.4,0.8] |

Table 4: Tabular representation of $\mathfrak{G}_1 \tilde{\oplus} \mathfrak{G}_2$.

| $(\mathscr{R}_1 \times \mathscr{R}_2)\downarrow\backslash\check{X} \longrightarrow$ | $\mathfrak{C}_1$ | $\mathfrak{C}_2$ | $\mathfrak{C}_3$ | $\mathfrak{C}_4$ |
|---|---|---|---|---|
| $(\hat{r}_1, \hat{r}_1)$ | [0.3,0.4] | [0.4,0.5] | [0.5,0.6] | [0.6,0.7] |
| $(\hat{r}_1, \hat{r}_2)$ | [0.4,0.6] | [0.5,0.7] | [0.6,0.8] | [0.7,0.9] |
| $(\hat{r}_1, \hat{r}_3)$ | [0.2,0.5] | [0.3,0.6] | [0.4,0.7] | [0.5,0.8] |
| $(\hat{r}_2, \hat{r}_1)$ | [0.3,0.6] | [0.4,0.7] | [0.5,0.8] | [0.6,0.9] |
| $(\hat{r}_2, \hat{r}_2)$ | [0.4,0.6] | [0.5,0.7] | [0.6,0.8] | [0.7,0.9] |
| $(\hat{r}_2, \hat{r}_3)$ | [0.2,0.5] | [0.3,0.6] | [0.4,0.7] | [0.5,0.8] |
| $(\hat{r}_3, \hat{r}_1)$ | [0.3,0.5] | [0.4,0.6] | [0.5,0.7] | [0.6,0.8] |
| $(\hat{r}_3, \hat{r}_2)$ | [0.4,0.6] | [0.5,0.7] | [0.6,0.8] | [0.7,0.9] |
| $(\hat{r}_3, \hat{r}_3)$ | [0.2,0.5] | [0.3,0.6] | [0.4,0.7] | [0.5,0.8] |

Their tabular representations are presented in Tables 3 and 4.

Now, we present the generalized version of $\mathscr{J}$-inclusion and $\mathscr{L}$-inclusion for ivfhs-sets with entitlement of multiargument approximate functions.

*Definition 16.* Let $\mathfrak{G}_1 = (\Psi^1_{\mathrm{FHS}}, \mathscr{R}_1) \& \mathfrak{G}_2 = (\Psi^2_{\mathrm{FHS}}, \mathscr{R}_2)$ be two ivfhs-sets then

(1) $\mathfrak{G}_1$ is said to be ivfhs $\mathscr{J}$-subset of $\mathfrak{G}_2$, denoted by $\mathfrak{A} \widehat{\subseteq}_{\mathscr{J}} \mathfrak{B}$, if for every $\hat{r}_1 \in \mathscr{R}_1 \exists \hat{r}_2 \in \mathscr{R}_2$ such that $\Psi^1_{\mathrm{FHS}}(\hat{r}_1) \subseteq \Psi^2_{\mathrm{FHS}}(\hat{r}_2)$

(2) $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are said to be ivfhs $\mathscr{J}$-equal, denoted by $\mathfrak{G}_1 \widehat{=}_{\mathscr{J}} \mathfrak{G}_2$, if $\mathfrak{G}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{G}_2$ and $\mathfrak{G}_2 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{G}_1$

*Definition 17.* Let $\mathfrak{G}_1 = (\Psi^1_{\mathrm{FHS}}, \mathscr{R}_1) \& \mathfrak{G}_2 = (\Psi^2_{\mathrm{FHS}}, \mathscr{R}_2)$ be two ivfhs-sets, then

(1) $\mathfrak{G}_1$ is said to be ivfs $\mathscr{L}$-subset of $\mathfrak{G}_2$, denoted by $\mathfrak{G}_1 \widehat{\subseteq}_{\mathscr{L}} \mathfrak{G}_2$, if for every $\hat{r}_1 \in \mathscr{R}_1 \exists \hat{r}_2 \in \mathscr{R}_2$ such that $\Psi^1_{\mathrm{FHS}}(\hat{r}_1) = \Psi^2_{\mathrm{FHS}}(\hat{r}_2)$

(2) $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are said to be ivfs $\mathscr{L}$-equal, denoted by $\mathfrak{G}_1 \widehat{=}_{\mathscr{L}} \mathfrak{G}_2$, if $\mathfrak{G}_1 \widehat{=}_{\mathscr{L}} \mathfrak{G}_2$ and $\mathfrak{G}_2 \widehat{=}_{\mathscr{L}} \mathfrak{G}_1$

Note: both $\widehat{\subseteq}_{\mathscr{J}}$ and $\widehat{\subseteq}_{\mathscr{L}}$ are named as ivfhs $\mathscr{J}$-inclusion and ivfhs $\mathscr{L}$-inclusion, respectively.

**Proposition 18.** *If $\mathfrak{G}_1 \widehat{\subseteq}_{\mathscr{L}} \mathfrak{G}_2$, then it implies $\mathfrak{G}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{G}_2$.*

*Definition 19.* Let $\mathfrak{G}_1 = (\Psi^1_{\mathrm{FHS}}, \mathscr{R}_1) \& \mathfrak{G}_2 = (\Psi^2_{\mathrm{FHS}}, \mathscr{R}_2)$ be two ivfhs-sets then $\mathfrak{G}_1$ is said to be identical to $\mathfrak{G}_2$, denoted by $\mathfrak{G}_1 \widehat{\equiv} \mathfrak{G}_2$, if $\mathscr{R}_1 = \mathscr{R}_2$ and $\Psi^1_{\mathrm{FHS}}(\hat{r}_1) = \Psi^2_{\mathrm{FHS}}(\hat{r}_2)$ for every $\hat{r}_1 \in \mathscr{R}_1 \exists \hat{r}_2 \in \mathscr{R}_2$.

**Proposition 20.** *If $\mathfrak{G}_1 \widehat{\equiv} \mathfrak{G}_2$, then it implies $\mathfrak{G}_1 \widehat{\subseteq}_{\mathscr{L}} \mathfrak{G}_2$ which further implies $\mathfrak{G}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{G}_2$.*

*Proof.* Let $\mathfrak{G}_1 \widehat{\equiv} \mathfrak{G}_2$, then by Definition 19, we have $\mathscr{R}_1 = \mathscr{R}_2$ and $\Psi^1_{\mathrm{FHS}}(\hat{r}_1) = \Psi^2_{\mathrm{FHS}}(\hat{r}_2)$ for every $\hat{r}_1 \in \mathscr{R}_1 \exists \hat{r}_2 \in \mathscr{R}_2$. By Definition 17 part (1), we get $\mathfrak{G}_1 \widehat{\subseteq}_{\mathscr{L}} \mathfrak{G}_2$. Now, applying Proposition 18, we obtain $\mathfrak{G}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{G}_2$. □

It is pertinent to mention here that the results presented in Propositions 18 and 20 are not legitimate in general. Both $\widehat{\subseteq}_{\mathscr{J}}$ and $\widehat{\subseteq}_{\mathscr{L}}$ are preorder for $\mho_{\mathrm{ivfhss}}(\check{X}, \mathscr{P})$.

**Proposition 21.** *The relations $\widehat{=}_{\mathscr{J}}$ and $\widehat{=}_{\mathscr{L}}$ satisfy the properties of equivalence relation on $\mho_{ivfhss}(\check{X}, \mathscr{P})$.*

*Proof.* Applying the concept stated in Definitions 16 and Definitions 17, it is clear that both $\widehat{=}_{\mathscr{J}}$ and $\widehat{=}_{\mathscr{L}}$ satisfy reflexive property as $\mathfrak{G}_1 \widehat{=}_{\mathscr{J}} \mathfrak{G}_1$ and $\mathfrak{G}_1 \widehat{=}_{\mathscr{L}} \mathfrak{G}_1$. Their symmetric and transitive nature can also be deduced from these mentioned definitions. These properties collectively conclude that both $\widehat{=}_{\mathscr{J}}$ and $\widehat{=}_{\mathscr{L}}$ are equivalence relations.

We know from classical set theory, for a set $D \neq \varnothing$ with a preorder $\leq$, an upward directed set is a set $(D, \leq)$ in which every pair of elements in $D$ has an upper bound, i.e., for $d_1, d_2 \in D$, there exists $d_3$ such that $d_1 \leq d_3$ and $d_2 \leq d_3$. The following definition is the generalized set theoretic version of upward directed set under hypersoft set environment. □

*Definition 22.* An ivfhs-set $(\Psi_{\mathrm{FHS}}, \mathscr{R})$ with $\mathscr{R} \neq \varnothing$ is called an upward directed ivfhs-set (UD-ivfhss) if for $\hat{r}_1, \hat{r}_2 \in \mathscr{R}$, there exists $\hat{r}_3 \in \mathscr{R}$ such that

$$\{\Psi_{FHS}(\hat{r}_1) \cup \Psi_{FHS}(\hat{r}_2)\} \subseteq \Psi_{FHS}(\hat{r}_3). \tag{13}$$

*Example 5.* Considering data from Example 1, we have ivfhs-set $\mathfrak{G} = (\Psi_{\mathrm{FHS}}, \mathscr{R})$ as given below

$$(\Psi_{\mathrm{FHS}}, \mathscr{R}) = \left\{ \begin{array}{l} \left( \left\{ \frac{\mathfrak{G}_1}{[0.2,0.4]}, \frac{\mathfrak{G}_2}{[0.3,0.5]}, \frac{\mathfrak{G}_3}{[0.4,0.6]}, \frac{\mathfrak{G}_4}{[0.5,0.7]} \right\}, \widehat{r}_1 \right), \\ \left( \left\{ \frac{\mathfrak{G}_1}{[0.1,0.5]}, \frac{\mathfrak{G}_2}{[0.2,0.6]}, \frac{\mathfrak{G}_3}{[0.3,0.7]}, \frac{\mathfrak{G}_4}{[0.4,0.8]} \right\}, \widehat{r}_2 \right), \\ \left( \left\{ \frac{\mathfrak{G}_1}{[0.3,0.6]}, \frac{\mathfrak{G}_2}{[0.4,0.7]}, \frac{\mathfrak{G}_3}{[0.5,0.8]}, \frac{\mathfrak{G}_4}{[0.6,0.9]} \right\}, \widehat{r}_3 \right) \end{array} \right\}, \tag{14}$$

with tabular representation as given below in Table 5. It can easily be seen that $\{\Psi_{\mathrm{FHS}}(\widehat{r}_1) \cup \Psi_{\mathrm{FHS}}(\widehat{r}_2)\} \subseteq \Psi_{\mathrm{FHS}}(\widehat{r}_3)$ which shows that $(\Psi_{\mathrm{FHS}}, \mathscr{R})$ is an UD-ivfhss.

**Proposition 23.** *An ivfhs-set $\mathfrak{G} = (\Psi_{FHS}, \mathscr{R})$ with $\mathscr{W}_{\mathfrak{G}} = \{\Psi_{FHS}(\widehat{r}): \widehat{r} \in \mathscr{R}\}$ is an UD-ivfhss if and only if $(\mathscr{W}_{\mathfrak{G}}, \subseteq)$ is an UD-ivfhss.*

*Proof.* Let $\mathfrak{G} = (\Psi_{\mathrm{FHS}}, \mathscr{R})$ be an UD-ivfhss then by definition of UD-ivfhss, then following conditions hold:

(i) $\mathscr{R} \neq \varnothing$ and $\mathscr{W}_{\mathfrak{G}} = \{\Psi_{\mathrm{FHS}}(\widehat{r}): \widehat{r} \in \mathscr{R}\} \subseteq \Omega_{\mathrm{ivfs}}$ ; $\mathscr{W}_{\mathfrak{G}} \neq \varnothing$

(ii) for $\widehat{r}_1, \widehat{r}_2 \in \mathscr{R} \exists \widehat{r}_3 \in \mathscr{R}$ such that $\{\Psi_{\mathrm{FHS}}(\widehat{r}_1) \cup \Psi_{\mathrm{FHS}}(\widehat{r}_2)\} \subseteq \Psi_{\mathrm{FHS}}(\widehat{r}_3)$

The second condition implies that $\Psi_{\mathrm{FHS}}(\widehat{r}_1) \subseteq \Psi_{\mathrm{FHS}}(\widehat{r}_3)$ and $\Psi_{\mathrm{FHS}}(\widehat{r}_2) \subseteq \Psi_{\mathrm{FHS}}(\widehat{r}_3)$ which proves that $(\mathscr{W}_{\mathfrak{G}}, \subseteq)$ is an UD-ivfhss.

Conversely, let $(\mathscr{W}_{\mathfrak{G}}, \subseteq)$ is an UD-ivfhss, then the below given clauses hold due to definition of UD-ivfhss:

(i) both $\mathscr{R}$ and $\mathscr{W}_{\mathfrak{G}}$ are nonempty sets

(ii) for $\widehat{r}_1, \widehat{r}_2 \in \mathscr{R}$, there exists an upper bound for $\Psi_{\mathrm{FHS}}(\widehat{r}_1)$ and $\Psi_{\mathrm{FHS}}(\widehat{r}_2)$ in $\mathscr{W}_{\mathfrak{G}}$ which means $\exists \widehat{r}_3 \in \mathscr{R}$ such that $\Psi_{\mathrm{FHS}}(\widehat{r}_1) \subseteq \Psi_{\mathrm{FHS}}(\widehat{r}_3)$ and $\Psi_{\mathrm{FHS}}(\widehat{r}_2) \subseteq \Psi_{\mathrm{FHS}}(\widehat{r}_3)$

The clause (ii) further implies $\{\Psi_{\mathrm{FHS}}(\widehat{r}_1) \cup \Psi_{\mathrm{FHS}}(\widehat{r}_2)\} \subseteq \Psi_{\mathrm{FHS}}(\widehat{r}_3)$ which shows that $\mathfrak{G} = (\Psi_{\mathrm{FHS}}, \mathscr{R})$ is an UD-ivfhss. □

**Proposition 24.** *An ivfhs-set $\mathfrak{G} = (\Psi_{FHS}, \mathscr{R})$ with $\mathscr{R} \neq \varnothing$ is an UD-ivfhss if and only if $\mathfrak{G} \widehat{=}_{\mathscr{J}} \mathfrak{G} \tilde{\oplus} \mathfrak{G}$.*

*Proof.* Let $\mathfrak{G} \tilde{\oplus} \mathfrak{G} = (Y_{\mathrm{FHS}}, \mathscr{R} \times \mathscr{R})$ with $\mathfrak{G} = (\Psi_{\mathrm{FHS}}, \mathscr{R})$ be an UD-ivfhss. Therefore, there exists $\widehat{r}_3 \in \mathscr{R}$ corresponding to pair $(\widehat{r}_1, \widehat{r}_2) \in \mathscr{R} \times \mathscr{R}$ s.t.

$$Y_{\mathrm{FHS}}(\widehat{r}_1, \widehat{r}_2) = \{\Psi_{\mathrm{FHS}}(\widehat{r}_1) \cup \Psi_{\mathrm{FHS}}(\widehat{r}_2)\}, \tag{15}$$

and

$$\{\Psi_{\mathrm{FHS}}(\widehat{r}_1) \cup \Psi_{\mathrm{FHS}}(\widehat{r}_2)\} \subseteq \Psi_{\mathrm{FHS}}(\widehat{r}_3). \tag{16}$$

Combining above equations, we obtain $Y_{\mathrm{FHS}}(\widehat{r}_1, \widehat{r}_2) \subseteq \Psi_{\mathrm{FHS}}(\widehat{r}_3)$ which shows that $\mathfrak{G} \tilde{\oplus} \mathfrak{G} \widehat{\subseteq}_{\mathscr{J}} \mathfrak{G}$ but we know that $\mathfrak{G} \widehat{\subseteq}_{\mathscr{J}} \mathfrak{G} \tilde{\oplus} \mathfrak{G}$; hence, $\mathfrak{G} \widehat{=}_{\mathscr{J}} \mathfrak{G} \tilde{\oplus} \mathfrak{G}$.

Table 5: An upward directed interval-valued fuzzy hypersoft set $(\Psi_{\mathrm{FHS}}, \mathscr{R})$.

| $\mathscr{R} \downarrow \backslash \check{X} \longrightarrow$ | $\mathfrak{G}_1$ | $\mathfrak{G}_2$ | $\mathfrak{G}_3$ | $\mathfrak{G}_4$ |
|---|---|---|---|---|
| $\widehat{r}_1$ | [0.2,0.4] | [0.3,0.5] | [0.4,0.6] | [0.5,0.7] |
| $\widehat{r}_2$ | [0.1,0.5] | [0.2,0.6] | [0.3,0.7] | [0.4,0.8] |
| $\widehat{r}_3$ | [0.3,0.6] | [0.4,0.7] | [0.5,0.8] | [0.6,0.9] |

Conversely, let $\mathfrak{G} \widehat{=}_{\mathscr{J}} \mathfrak{G} \tilde{\oplus} \mathfrak{G}$ then $\mathfrak{G} \tilde{\oplus} \mathfrak{G} \widehat{\subseteq}_{\mathscr{J}} \mathfrak{G}$ implies $(Y_{\mathrm{FHS}}, \mathscr{R} \times \mathscr{R}) \widehat{\subseteq}_{\mathscr{J}} \mathfrak{G}$, that is, there exists $\widehat{r}_3 \in \mathscr{R}$ corresponding to pair $(\widehat{r}_1, \widehat{r}_2) \in \mathscr{R} \times \mathscr{R}$ s.t.

$$Y_{\mathrm{FHS}}(\widehat{r}_1, \widehat{r}_2) = \{\Psi_{\mathrm{FHS}}(\widehat{r}_1) \cup \Psi_{\mathrm{FHS}}(\widehat{r}_2)\} \subseteq \Psi_{\mathrm{FHS}}(\widehat{r}_3), \tag{17}$$

which proves that $\mathfrak{G} = (\Psi_{\mathrm{FHS}}, \mathscr{R})$ be an UD-ivfhss. □

**Corollary 25.** *Let $\mathfrak{G} = (\Psi_{FHS}, \mathscr{R})$ be an ivfhs-set with $\mathscr{R} \neq \varnothing$, then the given below statements are equivalent:*

*(1) $\mathfrak{G}$ is an UD-ivfhss over $\check{X}$*

*(2) $\mathscr{W}_{\mathfrak{G}} = \{\Psi_{FHS}(\widehat{r}): \widehat{r} \in \mathscr{R}\}$ is an UD-ivfhss w.r.t $\subseteq$*

$$\begin{aligned} &\mathfrak{G} \widehat{=}_{\mathscr{J}} \mathfrak{G} \tilde{\oplus} \mathfrak{G}, \\ &\mathfrak{G} \tilde{\oplus} \mathfrak{G} \widehat{\subseteq}_{\mathscr{J}} \mathfrak{G}. \end{aligned} \tag{18}$$

*Proof.* These can easily be verified by considering the consequences of Proposition 23 and Proposition 24. □

## 4. Modular Inequalities of ivfhs-Sets via $\mathscr{L}$-Inclusion

Liu et al. [22] discussed some modular inequalities for ivfs-sets which employs approximate function that is unable to tackle multiargument settings (i.e., cartesian product of sub-parametric valued disjoint sets); therefore, in this section, such modular inequalities are generalized to manage such kind of settings.

Let $\check{X} = \{\widehat{o}_1, \widehat{o}_2, \cdots, \widehat{o}_n\}$ be an initial universe and $\mathscr{C} = \{\widehat{c}_1, \widehat{c}_2, \cdots, \widehat{c}_n\}$ be a set of parameters. The respective subparametric-valued disjoint sets are $\mathscr{K}_1 = \{\widehat{k}_{11}, \widehat{k}_{12}, \cdots, \widehat{k}_{1n}\}$, $\mathscr{K}_2 = \{\widehat{k}_{21}, \widehat{k}_{22}, \cdots, \widehat{k}_{2n}\}$, $\mathscr{K}_3 = \{\widehat{k}_{31}, \widehat{k}_{32}, \cdots, \widehat{k}_{3n}\}$,......, $\mathscr{K}_n = \{\widehat{k}_{n1}, \widehat{k}_{n2}, \cdots, \widehat{k}_{nn}\}$,and $\mathscr{K} = \mathscr{K}_1 \times \mathscr{K}_2 \times \mathscr{K}_3 \times . \cdots \times \mathscr{K}_n = \{\widehat{k}_1, \widehat{k}_2, \widehat{k}_3, . \cdots, \widehat{k}_\alpha\}$, where each $\widehat{k}_i (i = 1, 2, . \cdots, \alpha)$ is a $n$-tuple element of $\mathscr{K}$, and $\alpha = \prod_{i=1}^{n} |\mathscr{K}_i|$, $|\bullet|$ denotes set cardinality, then the pair $(\psi_{\mathrm{FHS}}, \mathscr{K})$ is known as ivfhs-set, where $\psi_{\mathrm{FHS}} : \mathscr{K} \longrightarrow \Omega_{\mathrm{ivfs}}$ and defined as $\psi_{\mathrm{FHS}}(\{\widehat{k}_1, \widehat{k}_2, \cdots, \widehat{k}_p\}) = \Omega_{\mathrm{ivfs}}(\{\widehat{o}_1, \widehat{o}_2, \cdots, \widehat{o}_n\})$, and $\mathscr{K} \subseteq \mathscr{C}$ with $p \leq \alpha$.

**Theorem 26.** *Let* $\widehat{\Theta}_1 = (\psi^1_{FHS}, \mathscr{K}_1)\&\widehat{\Theta}_2 = (\psi^2_{FHS}, \mathscr{K}_2)$ *be two ivfhs-sets, then*

$$\begin{aligned}&\widehat{\Theta}_1 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_1 \tilde{\oplus} \widehat{\Theta}_2,\\&\widehat{\Theta}_2 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_1 \tilde{\oplus} \widehat{\Theta}_2,\\&\widehat{\Theta}_1 \tilde{\otimes} \widehat{\Theta}_2 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_1,\\&\widehat{\Theta}_1 \tilde{\otimes} \widehat{\Theta}_2 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_1.\end{aligned} \tag{19}$$

**Theorem 27** (Generalized commutativity of ivfhs-sets). *Let* $\widehat{\Theta}_1 = (\psi^1_{FHS}, \mathscr{K}_1)\&\widehat{\Theta}_2 = (\psi^2_{FHS}, \mathscr{K}_2)$ *be two ivfhs-sets, then*

$$\begin{aligned}&\widehat{\Theta}_1 \tilde{\otimes} \widehat{\Theta}_2 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_2 \tilde{\otimes} \widehat{\Theta}_1,\\&\widehat{\Theta}_1 \tilde{\oplus} \widehat{\Theta}_2 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_2 \tilde{\oplus} \widehat{\Theta}_1.\end{aligned} \tag{20}$$

**Theorem 28.** *Let* $\widehat{\Theta}_1 = (\psi^1_{FHS}, \mathscr{K}_1)$, $\widehat{\Theta}_2 = (\psi^2_{FHS}, \mathscr{K}_2)\&\widehat{\Theta}_3 = (\psi^3_{FHS}, \mathscr{K}_3)$ *be three ivfhs-sets with* $\widehat{\Theta}_1 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_2$, *then*

$$\begin{aligned}&\widehat{\Theta}_3 \tilde{\oplus} \widehat{\Theta}_1 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_3 \tilde{\oplus} \widehat{\Theta}_2,\\&\widehat{\Theta}_3 \tilde{\oplus} \widehat{\Theta}_1 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_2 \tilde{\oplus} \widehat{\Theta}_3,\\&\widehat{\Theta}_1 \tilde{\oplus} \widehat{\Theta}_3 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_2 \tilde{\oplus} \widehat{\Theta}_3,\\&\widehat{\Theta}_1 \tilde{\oplus} \widehat{\Theta}_3 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_3 \tilde{\oplus} \widehat{\Theta}_2.\end{aligned} \tag{21}$$

*Proof.*

(1) Let $\widehat{\Theta}_3 \tilde{\oplus} \widehat{\Theta}_1 = (\psi^3_{\mathrm{FHS}}, \mathscr{K}_3) \tilde{\oplus} (\psi^1_{\mathrm{FHS}}, \mathscr{K}_1) = (\zeta^1_{\mathrm{FHS}}, \mathscr{K}_3 \times \mathscr{K}_1)$ and $\widehat{\Theta}_3 \tilde{\oplus} \widehat{\Theta}_2 = (\psi^3_{\mathrm{FHS}}, \mathscr{K}_3) \tilde{\oplus} (\psi^2_{\mathrm{FHS}}, \mathscr{K}_2) = (\zeta^2_{\mathrm{FHS}}, \mathscr{K}_3 \times \mathscr{K}_2)$. Since given that $\widehat{\Theta}_1 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_2$ which implies that there exists $\hat{k}_2 \in \mathscr{K}_2$ for every $\hat{k}_1 \in \mathscr{K}_1$ such that

$\psi^1_{\mathrm{FHS}}$

$(\hat{k}_1) = \psi^2_{\mathrm{FHS}}(\hat{k}_2)$. Let $(\hat{k}_3, \hat{k}_1) \in \mathscr{K}_3 \times \mathscr{K}_1$, then by definition of $\tilde{\oplus}$, we have

$$\zeta^1_{\mathrm{FHS}}\left(\hat{k}_3, \hat{k}_1\right) = \psi^3_{\mathrm{FHS}}\left(\hat{k}_3\right) \cup \psi^1_{\mathrm{FHS}}\left(\hat{k}_1\right). \tag{23}$$

By combining Equation (22) and Equation (23), we get

$$\zeta^1_{\mathrm{FHS}}\left(\hat{k}_3, \hat{k}_1\right) = \psi^3_{\mathrm{FHS}}\left(\hat{k}_3\right) \cup \psi^2_{\mathrm{FHS}}\left(\hat{k}_2\right). \tag{24}$$

Similarly for $(\hat{k}_3, \hat{k}_2) \in \mathscr{K}_3 \times \mathscr{K}_2$, we get

$$\zeta^2_{\mathrm{FHS}}\left(\hat{k}_3, \hat{k}_2\right) = \psi^3_{\mathrm{FHS}}\left(\hat{k}_3\right) \cup \psi^2_{\mathrm{FHS}}\left(\hat{k}_2\right). \tag{25}$$

From Equation (24) and Equation (25), we get $\zeta^1_{\mathrm{FHS}}(\hat{k}_3, \hat{k}_1) = \zeta^2_{\mathrm{FHS}}(\hat{k}_3, \hat{k}_2)$, which shows that $\widehat{\Theta}_3 \tilde{\oplus} \widehat{\Theta}_1 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_3 \tilde{\oplus} \widehat{\Theta}_2$.

(2) According to second part of Theorem 27, we have $\widehat{\Theta}_2 \tilde{\oplus} \widehat{\Theta}_3 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_3 \tilde{\oplus} \widehat{\Theta}_2$ which implies that

$$\psi^2_{\mathrm{FHS}}\left(\hat{k}_2\right) \cup \psi^3_{\mathrm{FHS}}\left(\hat{k}_3\right) = \psi^3_{\mathrm{FHS}}\left(\hat{k}_3\right) \cup \psi^2_{\mathrm{FHS}}\left(\hat{k}_2\right). \tag{26}$$

Therefore, from Equations (24), (25), and (26), it is vivid that $\zeta^1_{\mathrm{FHS}}(\hat{k}_3, \hat{k}_1) = \zeta^2_{\mathrm{FHS}}(\hat{k}_2, \hat{k}_3)$ which shows that $\widehat{\Theta}_3 \tilde{\oplus} \widehat{\Theta}_1 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_2 \tilde{\oplus} \widehat{\Theta}_3$.

Part 3 and part 4 can easily be verified with the help of Theorem 27(2) and above results. □

**Theorem 29.** *Let* $(\psi^i_{FHS}, \mathscr{K}_i)\&(\Phi^i_{FHS}, \mathscr{S}_i)$ *be two ivfhs-sets with* $(\psi^1_{FHS}, \mathscr{K}_1) \widehat{\subseteq}_{\mathscr{L}} (\Phi^1_{FHS}, \mathscr{S}_1)$ *and* $(\psi^2_{FHS}, \mathscr{K}_2) \widehat{\subseteq}_{\mathscr{L}} (\Phi^2_{FHS}, \mathscr{S}_2)$, *then*

$$\left(\psi^1_{FHS}, \mathscr{K}_1\right) \tilde{\oplus} \left(\psi^2_{FHS}, \mathscr{K}_2\right) \widehat{\subseteq}_{\mathscr{L}} \left(\Phi^1_{FHS}, \mathscr{S}_1\right) \tilde{\oplus} \left(\Phi^2_{FHS}, \mathscr{S}_2\right). \tag{27}$$

**Theorem 30.** *Let* $\widehat{\Theta}_1 = (\psi^1_{FHS}, \mathscr{K}_1)$, $\widehat{\Theta}_2 = (\psi^2_{FHS}, \mathscr{K}_2)\&\widehat{\Theta}_3 = (\psi^3_{FHS}, \mathscr{K}_3)$ *be three ivfhs-sets with* $\widehat{\Theta}_1 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_2$ *then*

$$\begin{aligned}&\widehat{\Theta}_3 \tilde{\otimes} \widehat{\Theta}_1 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_3 \tilde{\otimes} \widehat{\Theta}_2,\\&\widehat{\Theta}_3 \tilde{\otimes} \widehat{\Theta}_1 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_2 \tilde{\otimes} \widehat{\Theta}_3,\\&\widehat{\Theta}_1 \tilde{\otimes} \widehat{\Theta}_3 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_2 \tilde{\otimes} \widehat{\Theta}_3,\\&\widehat{\Theta}_1 \tilde{\otimes} \widehat{\Theta}_3 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_3 \tilde{\otimes} \widehat{\Theta}_2.\end{aligned} \tag{28}$$

*Proof.*

(1) Let $\widehat{\Theta}_3 \tilde{\otimes} \widehat{\Theta}_1 = (\psi^3_{\mathrm{FHS}}, \mathscr{K}_3) \tilde{\otimes} (\psi^1_{\mathrm{FHS}}, \mathscr{K}_1) = (\xi^1_{\mathrm{FHS}}, \mathscr{K}_3 \times \mathscr{K}_1)$ and $\widehat{\Theta}_3 \tilde{\otimes} \widehat{\Theta}_2 = (\psi^3_{\mathrm{FHS}}, \mathscr{K}_3) \tilde{\otimes} (\psi^2_{\mathrm{FHS}}, \mathscr{K}_2) = (\xi^2_{\mathrm{FHS}}, \mathscr{K}_3 \times \mathscr{K}_2)$. Since given that $\widehat{\Theta}_1 \widehat{\subseteq}_{\mathscr{L}} \widehat{\Theta}_2$ which implies that there exists $\hat{k}_2 \in \mathscr{K}_2$ for every $\hat{k}_1 \in \mathscr{K}_1$ such that

$\psi^1_{\mathrm{FHS}}(\hat{k}$

$_1) = \psi^2_{\mathrm{FHS}}(\hat{k}_2)$. Let $(\hat{k}_3, \hat{k}_1) \in \mathscr{K}_3 \times \mathscr{K}_1$, then by definition of $\tilde{\otimes}$, we have

$$\xi^1_{\mathrm{FHS}}\left(\hat{k}_3, \hat{k}_1\right) = \psi^3_{\mathrm{FHS}}\left(\hat{k}_3\right) \cap \psi^1_{\mathrm{FHS}}\left(\hat{k}_1\right). \tag{30}$$

By combining Equation (29) and Equation (30), we get

$$\xi^1_{\mathrm{FHS}}\left(\hat{k}_3, \hat{k}_1\right) = \psi^3_{\mathrm{FHS}}\left(\hat{k}_3\right) \cap \psi^2_{\mathrm{FHS}}\left(\hat{k}_2\right). \tag{31}$$

Similarly, for $(\hat{k}_3, \hat{k}_2) \in \mathscr{K}_3 \times \mathscr{K}_2$, we get

$$\xi^2_{\mathrm{FHS}}\left(\hat{k}_3, \hat{k}_2\right) = \psi^3_{\mathrm{FHS}}\left(\hat{k}_3\right) \cap \psi^2_{\mathrm{FHS}}\left(\hat{k}_2\right). \tag{32}$$

From Equation (31) and Equation (32), we get $\xi^1_{\mathrm{FHS}}(\widehat{k}_3, \widehat{k}_1) = \xi^2_{\mathrm{FHS}}(\widehat{k}_3, \widehat{k}_2)$ which shows that $\widehat{\Theta}_3 \tilde{\otimes} \widehat{\Theta}_1 \mathrel{\widehat{\subseteq}_{\mathscr{L}}} \widehat{\Theta}_3 \tilde{\otimes} \widehat{\Theta}_2$.

(2) According to Theorem 27, we have $\widehat{\Theta}_2 \tilde{\otimes} \widehat{\Theta}_3 \mathrel{\widehat{\subseteq}_{\mathscr{L}}} \widehat{\Theta}_3 \tilde{\otimes} \widehat{\Theta}_2$ which implies that

$$\psi^2_{\mathrm{FHS}}\left(\widehat{k}_2\right) \cap \psi^3_{\mathrm{FHS}}\left(\widehat{k}_3\right) = \psi^3_{\mathrm{FHS}}\left(\widehat{k}_3\right) \cap \psi^2_{\mathrm{FHS}}\left(\widehat{k}_2\right). \tag{33}$$

Therefore from Equations (31), (32), and (33), it is vivid that $\xi^1_{\mathrm{FHS}}(\widehat{k}_3, \widehat{k}_1) = \xi^2_{\mathrm{FHS}}(\widehat{k}_2, \widehat{k}_3)$ which shows that $\widehat{\Theta}_3 \tilde{\otimes} \widehat{\Theta}_1 \mathrel{\widehat{\subseteq}_{\mathscr{L}}} \widehat{\Theta}_2 \tilde{\otimes} \widehat{\Theta}_3$.

Part 3 and part 4 can easily be verified with the help of Theorem 27(2) and above results. □

**Theorem 31.** *Let $(\psi^i_{FHS}, \mathscr{K}_i) \& (\Phi^i_{FHS}, \mathscr{S}_i)$ be two ivfhs-sets with $(\psi^1_{FHS}, \mathscr{K}_1) \mathrel{\widehat{\subseteq}_{\mathscr{L}}} (\Phi^1_{FHS}, \mathscr{S}_1)$ and $(\psi^2_{FHS}, \mathscr{K}_2) \mathrel{\widehat{\subseteq}_{\mathscr{L}}} (\Phi^2_{FHS}, \mathscr{S}_2)$, then $(\psi^1_{FHS}, \mathscr{K}_1) \tilde{\otimes} (\psi^2_{FHS}, \mathscr{K}_2) \mathrel{\widehat{\subseteq}_{\mathscr{L}}} (\Phi^1_{FHS}, \mathscr{S}_1) \tilde{\otimes} (\Phi^2_{FHS}, \mathscr{S}_2)$.*

**Theorem 32** (Generalized distributive inequalities of ivfhs-sets). *Let $\widehat{\Theta}_1 = (\psi^1_{FHS}, \mathscr{K}_1), \widehat{\Theta}_2 = (\psi^2_{FHS}, \mathscr{K}_2) \& \widehat{\Theta}_3 = (\psi^3_{FHS}, \mathscr{K}_3)$ be three ivfhs-sets then*

$$\begin{aligned}
&\left(\widehat{\Theta}_1 \tilde{\oplus} \widehat{\Theta}_2\right) \tilde{\otimes} \widehat{\Theta}_3 \mathrel{\widehat{\subseteq}_{\mathscr{L}}} \left(\widehat{\Theta}_1 \tilde{\otimes} \widehat{\Theta}_3\right) \tilde{\oplus} \left(\widehat{\Theta}_2 \tilde{\otimes} \widehat{\Theta}_3\right),\\
&\left(\widehat{\Theta}_1 \tilde{\otimes} \widehat{\Theta}_2\right) \tilde{\oplus} \widehat{\Theta}_3 \mathrel{\widehat{\subseteq}_{\mathscr{L}}} \left(\widehat{\Theta}_1 \tilde{\oplus} \widehat{\Theta}_3\right) \tilde{\otimes} \left(\widehat{\Theta}_2 \tilde{\oplus} \widehat{\Theta}_3\right),\\
&\left(\widehat{\Theta}_1 \tilde{\otimes} \widehat{\Theta}_2\right) \tilde{\oplus} \widehat{\Theta}_3 \mathrel{\widehat{\subseteq}_{\mathscr{L}}} \left(\widehat{\Theta}_1 \tilde{\otimes} \widehat{\Theta}_2\right) \tilde{\oplus} \left(\widehat{\Theta}_1 \tilde{\otimes} \widehat{\Theta}_3\right),\\
&\left(\widehat{\Theta}_1 \tilde{\oplus} \widehat{\Theta}_2\right) \tilde{\otimes} \widehat{\Theta}_3 \mathrel{\widehat{\subseteq}_{\mathscr{L}}} \left(\widehat{\Theta}_1 \tilde{\oplus} \widehat{\Theta}_2\right) \tilde{\otimes} \left(\widehat{\Theta}_1 \tilde{\oplus} \widehat{\Theta}_3\right).
\end{aligned} \tag{34}$$

## 5. Modular Inequalities of ivfhs-Sets via $\mathscr{J}$-Inclusion

Jun and Yang [17] discussed some modular inequalities for ivfs-sets by extending the concept presented by Liu et al. [22], and this concept too shows inadequacy regarding multiargument approximate settings (i.e., cartesian product of subparametric valued disjoint sets); therefore, in this section, such modular inequalities are generalized to manage such kind of settings.

Let $\check{X} = \{\widehat{u}_1, \widehat{u}_2, \cdots, \widehat{u}_n\}$ be an initial universe and $\mathscr{V} = \{\widehat{v}_1, \widehat{v}_2, \cdots, \widehat{v}_n\}$ be a set of parameters. The respective subparametric-valued disjoint sets are $\mathscr{D}_1 = \{\widehat{d}_{11}, \widehat{d}_{12}, \cdots, \widehat{d}_{1n}\}$, $\mathscr{D}_2 = \{\widehat{d}_{21}, \widehat{d}_{22}, \cdots, \widehat{d}_{2n}\}$, $\mathscr{D}_3 = \{\widehat{d}_{31}, \widehat{d}_{32}, \cdots, \widehat{d}_{3n}\}$,......, $\mathscr{D}_n = \{\widehat{d}_{n1}, \widehat{d}_{n2}, \cdots, \widehat{d}_{nn}\}$,and $\mathscr{D} = \mathscr{D}_1 \times \mathscr{D}_2 \times \mathscr{D}_3 \times . \cdots \times \mathscr{D}_n = \{\widehat{d}_1, \widehat{d}_2, \widehat{d}_3, . \cdots, \widehat{d}_s\}$, where each $\widehat{d}_i (i = 1, 2, \cdots, s)$ is an $n$-tuple element of $\mathscr{D}$ and $s = \prod_{i=1}^n |\mathscr{D}_i|$, $|\bullet|$ denotes set cardinality, then the pair $(\Psi_{\mathrm{FHS}}, \mathscr{D})$ is known as ivfhs-set where $\Psi_{\mathrm{FHS}} : \mathscr{D} \longrightarrow \Omega_{\mathrm{ivfs}}$ and defined as $\Psi_{\mathrm{FHS}}(\{\widehat{d}_1, \widehat{d}_2, \cdots, \widehat{d}_k\}) = \Omega_{\mathrm{ivfs}}(\{\widehat{u}_1, \widehat{u}_2, \cdots, \widehat{u}_n\})$, and $\mathscr{D} \subseteq \mathscr{V}$ with $k \leq s$.

**Theorem 33.** *Let $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1) \& \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2)$ be two ivfhs-sets, then*

$$\begin{aligned}
&\mathfrak{W}_1 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2,\\
&\mathfrak{W}_2 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2,\\
&\mathfrak{W}_1 \tilde{\otimes} \mathfrak{W}_2 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_1,\\
&\mathfrak{W}_1 \tilde{\otimes} \mathfrak{W}_2 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_1.
\end{aligned} \tag{35}$$

**Theorem 34.** *Let $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ be three ivfhs-sets with $\mathfrak{W}_1 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_2$, then*

$$\begin{aligned}
&\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_1 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_2,\\
&\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_1 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_3,\\
&\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_3 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_3,\\
&\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_3 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_2.
\end{aligned} \tag{36}$$

**Theorem 35.** *Let $(\Psi^i_{FHS}, \mathscr{D}_i) \& (\Phi^i_{FHS}, \mathscr{S}_i)$ be two ivfhs-sets with $(\Psi^1_{FHS}, \mathscr{D}_1) \mathrel{\widehat{\subseteq}_{\mathscr{J}}} (\Phi^1_{FHS}, \mathscr{S}_1)$ and $(\Psi^2_{FHS}, \mathscr{D}_2) \mathrel{\widehat{\subseteq}_{\mathscr{J}}} (\Phi^2_{FHS}, \mathscr{S}_2)$, then*

$$\left(\Psi^1_{FHS}, \mathscr{D}_1\right) \tilde{\oplus} \left(\Psi^2_{FHS}, \mathscr{D}_2\right) \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \left(\Phi^1_{FHS}, \mathscr{S}_1\right) \tilde{\oplus} \left(\Phi^2_{FHS}, \mathscr{S}_2\right). \tag{37}$$

**Theorem 36.** *Let $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ be three ivfhs-sets with $\mathfrak{W}_1 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_2$, then*

$$\begin{aligned}
&\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_1 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2,\\
&\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_1 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3,\\
&\mathfrak{W}_1 \tilde{\otimes} \mathfrak{W}_3 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3,\\
&\mathfrak{W}_1 \tilde{\otimes} \mathfrak{W}_3 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2.
\end{aligned} \tag{38}$$

**Theorem 37.** *Let $(\Psi^i_{FHS}, \mathscr{R}_i) \& (\Phi^i_{FHS}, \mathscr{S}_i)$ be two ivfhs-sets with*

$$\left(\Psi^1_{FHS}, \mathscr{R}_1\right) \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \left(\Phi^1_{FHS}, \mathscr{S}_1\right), \tag{39}$$

*and*

$$\left(\Psi^2_{FHS}, \mathscr{R}_2\right) \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \left(\Phi^2_{FHS}, \mathscr{S}_2\right), \tag{40}$$

*then*

$$\left(\Psi^1_{FHS}, \mathscr{R}_1\right) \tilde{\otimes} \left(\Psi^2_{FHS}, \mathscr{R}_2\right) \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \left(\Phi^1_{FHS}, \mathscr{S}_1\right) \tilde{\otimes} \left(\Phi^2_{FHS}, \mathscr{S}_2\right). \tag{41}$$

**Theorem 38.** *Let $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ be three ivfhs-sets, then $(\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3 \mathrel{\widehat{\subseteq}_{\mathscr{J}}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3)$.*

*Proof.* From Theorem 32(1), we have

$$(\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{L}} (\mathfrak{W}_1 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3), \quad (42)$$

and then, after applying Proposition 18, we get

$$(\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3), \quad (43)$$

and by Theorem 33, we obtain $\mathfrak{W}_1 \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1$; therefore,

$$(\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3), \quad (44)$$

implies

$$(\mathfrak{W}_1 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3), \quad (45)$$

which leads to following final result due to transitivity of $\widehat{\subseteq}_{\mathscr{J}}$

$$(\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3). \quad (46)$$

□

**Corollary 39.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets then*

$$\begin{aligned} &(\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3), \\ &(\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2), \\ &(\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2). \end{aligned} \quad (47)$$

*Proof.* From Theorem 27(2), we have

$$\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1 \widehat{=}_{\mathscr{L}} \mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2, \quad (48)$$

and then, after applying Proposition 18, we get

$$\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1 \widehat{=}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2, \quad (49)$$

implies

$$\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2. \quad (50)$$

Taking $\tilde{\otimes}$ on both sides of above inequality with $\mathfrak{W}_3$, we have

$$(\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3, \quad (51)$$

but by Theorem 38

$$(\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3), \quad (52)$$

which leads to following final result due to transitivity of $\widehat{\subseteq}_{\mathscr{J}}$

$$(\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3). \quad (53)$$

Other parts can easily be validated in the similar manner. □

**Corollary 40.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets then*

$$\begin{aligned} &\mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3), \\ &\mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2), \\ &\mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2), \\ &\mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3). \end{aligned} \quad (54)$$

**Corollary 41.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets, then*

$$\begin{aligned} &\mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} \mathfrak{W}_1, \\ &\mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \tilde{\oplus} \mathfrak{W}_1, \\ &\mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \tilde{\oplus} \mathfrak{W}_1, \\ &\mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} \mathfrak{W}_1. \end{aligned} \quad (55)$$

**Corollary 42.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets, then*

$$\begin{aligned} &(\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} \mathfrak{W}_1, \\ &(\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \tilde{\oplus} \mathfrak{W}_1, \\ &(\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \tilde{\oplus} \mathfrak{W}_1, \\ &(\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} \mathfrak{W}_1. \end{aligned} \quad (56)$$

**Theorem 43.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets. If* $\mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_3$, *then*

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3). \quad (57)$$

*Proof.* From Theorem 32(2), we have

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathscr{L}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_3), \quad (58)$$

and then, after applying Proposition 18, we get

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_3). \quad (59)$$

since given that,

$$\mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_3. \quad (60)$$

Taking $\tilde{\oplus}$ on both sides of above inequality with $\mathfrak{W}_3$, we have

$$\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3, \quad (61)$$

implies

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathcal{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3), \quad (62)$$

such that

$$(\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_3) \widehat{\subseteq}_{\mathcal{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3), \quad (63)$$

which leads to following final result due to transitivity of $\widehat{\subseteq}_{\mathcal{J}}$

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathcal{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3). \quad (64)$$

□

*Example 6.* Let $\check{X} = \{\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \hat{u}_5\}$ be an initial universe and $\mathscr{V} = \{\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4, \hat{v}_5, \hat{v}_6\}$ be a set of attributes. The respective attribute-valued disjoint sets are $\mathscr{D}_1 = \{\widehat{d}_{11}\}$, $\mathscr{D}_2 = \{\widehat{d}_{21}, \widehat{d}_{22}\}$, $\mathscr{D}_3 = \{\widehat{d}_{31}, \widehat{d}_{32}\}$, $\mathscr{D}_4 = \{\widehat{d}_{41}, \widehat{d}_{42}\}\mathscr{D}_5 = \{\widehat{d}_{51}\}$ $\mathscr{D}_6 = \{\widehat{d}_{61}\}$, and $\mathscr{D} = \mathscr{D}_1 \times \mathscr{D}_2 \times \mathscr{D}_3 \times . \cdots \times \mathscr{D}_n = \{\widehat{d}_1, \widehat{d}_2, \widehat{d}_3, \widehat{d}_4, \widehat{d}_5, \widehat{d}_6, \widehat{d}_7, \widehat{d}_8\}$. Let us take $\mathscr{E}_1 = \{\widehat{d}_1, \widehat{d}_2, \widehat{d}_3\}$, $\mathscr{E}_2 = \{\widehat{d}_4, \widehat{d}_5\}$, and $\mathscr{E}_3 = \{\widehat{d}_6, \widehat{d}_7, \widehat{d}_8\}$ as subsets of $\mathscr{D}$, then we have three ivfhs-sets $\mathfrak{W}_1 = (\Psi^1_{\text{FHS}}, \mathscr{E}_1)$, $\mathfrak{W}_2 = (\Psi^2_{\text{FHS}}, \mathscr{E}_2)$&$\mathfrak{W}_3 = (\Psi^3_{\text{FHS}}, \mathscr{E}_3)$ with

$$\begin{aligned}
\Psi^1_{\text{FHS}}(\widehat{d}_1) &= \left\{\frac{\hat{u}_1}{[0.2,0.7]}, \frac{\hat{u}_2}{[0.3,0.8]}, \frac{\hat{u}_3}{[0.4,0.8]}, \frac{\hat{u}_4}{[0.5,0.6]}, \frac{\hat{u}_5}{[0.6,0.7]}\right\},\\
\Psi^1_{\text{FHS}}(\widehat{d}_2) &= \left\{\frac{\hat{u}_1}{[0.3,0.7]}, \frac{\hat{u}_2}{[0.4,0.8]}, \frac{\hat{u}_3}{[0.5,0.8]}, \frac{\hat{u}_4}{[0.6,0.7]}, \frac{\hat{u}_5}{[0.7,0.8]}\right\},\\
\Psi^1_{\text{FHS}}(\widehat{d}_3) &= \left\{\frac{\hat{u}_1}{[0.4,0.7]}, \frac{\hat{u}_2}{[0.5,0.8]}, \frac{\hat{u}_3}{[0.6,0.8]}, \frac{\hat{u}_4}{[0.7,0.8]}, \frac{\hat{u}_5}{[0.5,0.8]}\right\},\\
\Psi^2_{\text{FHS}}(\widehat{d}_4) &= \left\{\frac{\hat{u}_1}{[0.5,0.8]}, \frac{\hat{u}_2}{[0.6,0.8]}, \frac{\hat{u}_3}{[0.7,0.8]}, \frac{\hat{u}_4}{[0.8,0.9]}, \frac{\hat{u}_5}{[0.6,0.8]}\right\},\\
\Psi^2_{\text{FHS}}(\widehat{d}_5) &= \left\{\frac{\hat{u}_1}{[0.6,0.8]}, \frac{\hat{u}_2}{[0.7,0.8]}, \frac{\hat{u}_3}{[0.8,0.9]}, \frac{\hat{u}_4}{[0.8,0.9]}, \frac{\hat{u}_5}{[0.7,0.9]}\right\},\\
\Psi^3_{\text{FHS}}(\widehat{d}_6) &= \left\{\frac{\hat{u}_1}{[0.3,0.8]}, \frac{\hat{u}_2}{[0.4,0.9]}, \frac{\hat{u}_3}{[0.5,0.9]}, \frac{\hat{u}_4}{[0.6,0.7]}, \frac{\hat{u}_5}{[0.7,0.8]}\right\},\\
\Psi^3_{\text{FHS}}(\widehat{d}_7) &= \left\{\frac{\hat{u}_1}{[0.4,0.8]}, \frac{\hat{u}_2}{[0.5,0.9]}, \frac{\hat{u}_3}{[0.6,0.9]}, \frac{\hat{u}_4}{[0.7,0.8]}, \frac{\hat{u}_5}{[0.8,0.9]}\right\},\\
\Psi^3_{\text{FHS}}(\widehat{d}_8) &= \left\{\frac{\hat{u}_1}{[0.5,0.8]}, \frac{\hat{u}_2}{[0.6,0.9]}, \frac{\hat{u}_3}{[0.7,0.9]}, \frac{\hat{u}_4}{[0.8,0.9]}, \frac{\hat{u}_5}{[0.6,0.9]}\right\}.
\end{aligned} \quad (65)$$

It is clear that $\Psi^1_{\text{FHS}}(\widehat{d}_1) \subseteq \Psi^3_{\text{FHS}}(\widehat{d}_6)$, $\Psi^1_{\text{FHS}}(\widehat{d}_2) \subseteq \Psi^3_{\text{FHS}}(\widehat{d}_7)$, and $\Psi^1_{\text{FHS}}(\widehat{d}_3) \subseteq \Psi^3_{\text{FHS}}(\widehat{d}_8)$; therefore, $\mathfrak{W}_1 \widehat{\subseteq}_{\mathcal{J}} \mathfrak{W}_3$.

Consider $\mathfrak{W}_4 = \mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_2$ or $(\Phi_{FHS}, \mathscr{E}_2 \times \mathscr{E}_3) = (\Psi^2_{\text{FHS}}, \mathscr{E}_2) \tilde{\otimes} (\Psi^3_{\text{FHS}}, \mathscr{E}_3)$ with

$$\begin{aligned}
\Phi_{\text{FHS}}(\widehat{d}_4, \widehat{d}_6) &= \left\{\frac{\hat{u}_1}{[0.3,0.8]}, \frac{\hat{u}_2}{[0.4,0.8]}, \frac{\hat{u}_3}{[0.5,0.8]}, \frac{\hat{u}_4}{[0.6,0.7]}, \frac{\hat{u}_5}{[0.6,0.8]}\right\},\\
\Phi_{\text{FHS}}(\widehat{d}_4, \widehat{d}_7) &= \left\{\frac{\hat{u}_1}{[0.4,0.8]}, \frac{\hat{u}_2}{[0.5,0.8]}, \frac{\hat{u}_3}{[0.6,0.8]}, \frac{\hat{u}_4}{[0.7,0.8]}, \frac{\hat{u}_5}{[0.6,0.8]}\right\},\\
\Phi_{\text{FHS}}(\widehat{d}_4, \widehat{d}_8) &= \left\{\frac{\hat{u}_1}{[0.5,0.8]}, \frac{\hat{u}_2}{[0.6,0.8]}, \frac{\hat{u}_3}{[0.7,0.8]}, \frac{\hat{u}_4}{[0.8,0.9]}, \frac{\hat{u}_5}{[0.6,0.8]}\right\},\\
\Phi_{\text{FHS}}(\widehat{d}_5, \widehat{d}_6) &= \left\{\frac{\hat{u}_1}{[0.3,0.8]}, \frac{\hat{u}_2}{[0.4,0.8]}, \frac{\hat{u}_3}{[0.5,0.9]}, \frac{\hat{u}_4}{[0.6,0.7]}, \frac{\hat{u}_5}{[0.7,0.8]}\right\},\\
\Phi_{\text{FHS}}(\widehat{d}_5, \widehat{d}_7) &= \left\{\frac{\hat{u}_1}{[0.4,0.8]}, \frac{\hat{u}_2}{[0.5,0.8]}, \frac{\hat{u}_3}{[0.6,0.9]}, \frac{\hat{u}_4}{[0.7,0.8]}, \frac{\hat{u}_5}{[0.7,0.9]}\right\},\\
\Phi_{\text{FHS}}(\widehat{d}_5, \widehat{d}_8) &= \left\{\frac{\hat{u}_1}{[0.5,0.8]}, \frac{\hat{u}_2}{[0.6,0.8]}, \frac{\hat{u}_3}{[0.7,0.9]}, \frac{\hat{u}_4}{[0.8,0.9]}, \frac{\hat{u}_5}{[0.6,0.9]}\right\}.
\end{aligned} \quad (66)$$

Now, let $\mathfrak{W}_5 = \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_2)$ or

$$(\Upsilon_{\text{FHS}}, \mathscr{E}_1 \times (\mathscr{E}_2 \times \mathscr{E}_3)) = (\Psi^1_{\text{FHS}}, \mathscr{E}_1) \tilde{\oplus} \left[(\Psi^2_{\text{FHS}}, \mathscr{E}_2) \tilde{\otimes} (\Psi^3_{\text{FHS}}, \mathscr{E}_3)\right], \quad (67)$$

with

$$\begin{aligned}
\Upsilon_{\text{FHS}}(\widehat{d}_1, \widehat{d}_4, \widehat{d}_6) &= \left\{\frac{\hat{u}_1}{[0.3,0.8]}, \frac{\hat{u}_2}{[0.4,0.8]}, \frac{\hat{u}_3}{[0.5,0.8]}, \frac{\hat{u}_4}{[0.6,0.7]}, \frac{\hat{u}_5}{[0.6,0.8]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_1, \widehat{d}_4, \widehat{d}_7) &= \left\{\frac{\hat{u}_1}{[0.4,0.8]}, \frac{\hat{u}_2}{[0.5,0.8]}, \frac{\hat{u}_3}{[0.6,0.8]}, \frac{\hat{u}_4}{[0.7,0.8]}, \frac{\hat{u}_5}{[0.6,0.8]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_1, \widehat{d}_4, \widehat{d}_8) &= \left\{\frac{\hat{u}_1}{[0.5,0.8]}, \frac{\hat{u}_2}{[0.6,0.8]}, \frac{\hat{u}_3}{[0.7,0.8]}, \frac{\hat{u}_4}{[0.8,0.9]}, \frac{\hat{u}_5}{[0.6,0.8]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_1, \widehat{d}_5, \widehat{d}_6) &= \left\{\frac{\hat{u}_1}{[0.3,0.8]}, \frac{\hat{u}_2}{[0.4,0.8]}, \frac{\hat{u}_3}{[0.5,0.9]}, \frac{\hat{u}_4}{[0.6,0.7]}, \frac{\hat{u}_5}{[0.7,0.8]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_1, \widehat{d}_5, \widehat{d}_7) &= \left\{\frac{\hat{u}_1}{[0.4,0.8]}, \frac{\hat{u}_2}{[0.5,0.8]}, \frac{\hat{u}_3}{[0.6,0.9]}, \frac{\hat{u}_4}{[0.7,0.8]}, \frac{\hat{u}_5}{[0.7,0.9]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_1, \widehat{d}_5, \widehat{d}_8) &= \left\{\frac{\hat{u}_1}{[0.5,0.8]}, \frac{\hat{u}_2}{[0.6,0.8]}, \frac{\hat{u}_3}{[0.7,0.9]}, \frac{\hat{u}_4}{[0.8,0.9]}, \frac{\hat{u}_5}{[0.6,0.9]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_2, \widehat{d}_4, \widehat{d}_6) &= \left\{\frac{\hat{u}_1}{[0.3,0.8]}, \frac{\hat{u}_2}{[0.4,0.8]}, \frac{\hat{u}_3}{[0.5,0.8]}, \frac{\hat{u}_4}{[0.6,0.7]}, \frac{\hat{u}_5}{[0.7,0.8]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_2, \widehat{d}_4, \widehat{d}_7) &= \left\{\frac{\hat{u}_1}{[0.4,0.8]}, \frac{\hat{u}_2}{[0.5,0.8]}, \frac{\hat{u}_3}{[0.6,0.8]}, \frac{\hat{u}_4}{[0.7,0.8]}, \frac{\hat{u}_5}{[0.7,0.8]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_2, \widehat{d}_4, \widehat{d}_8) &= \left\{\frac{\hat{u}_1}{[0.5,0.8]}, \frac{\hat{u}_2}{[0.6,0.8]}, \frac{\hat{u}_3}{[0.7,0.8]}, \frac{\hat{u}_4}{[0.8,0.9]}, \frac{\hat{u}_5}{[0.7,0.8]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_2, \widehat{d}_5, \widehat{d}_6) &= \left\{\frac{\hat{u}_1}{[0.3,0.8]}, \frac{\hat{u}_2}{[0.4,0.8]}, \frac{\hat{u}_3}{[0.5,0.9]}, \frac{\hat{u}_4}{[0.6,0.7]}, \frac{\hat{u}_5}{[0.7,0.8]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_2, \widehat{d}_5, \widehat{d}_7) &= \left\{\frac{\hat{u}_1}{[0.4,0.8]}, \frac{\hat{u}_2}{[0.5,0.8]}, \frac{\hat{u}_3}{[0.6,0.9]}, \frac{\hat{u}_4}{[0.7,0.8]}, \frac{\hat{u}_5}{[0.7,0.9]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_2, \widehat{d}_5, \widehat{d}_8) &= \left\{\frac{\hat{u}_1}{[0.5,0.8]}, \frac{\hat{u}_2}{[0.6,0.8]}, \frac{\hat{u}_3}{[0.7,0.9]}, \frac{\hat{u}_4}{[0.8,0.9]}, \frac{\hat{u}_5}{[0.7,0.9]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_3, \widehat{d}_4, \widehat{d}_6) &= \left\{\frac{\hat{u}_1}{[0.4,0.8]}, \frac{\hat{u}_2}{[0.5,0.8]}, \frac{\hat{u}_3}{[0.6,0.8]}, \frac{\hat{u}_4}{[0.7,0.8]}, \frac{\hat{u}_5}{[0.7,0.8]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_3, \widehat{d}_4, \widehat{d}_7) &= \left\{\frac{\hat{u}_1}{[0.4,0.8]}, \frac{\hat{u}_2}{[0.5,0.8]}, \frac{\hat{u}_3}{[0.6,0.8]}, \frac{\hat{u}_4}{[0.7,0.8]}, \frac{\hat{u}_5}{[0.6,0.8]}\right\},\\
\Upsilon_{\text{FHS}}(\widehat{d}_3, \widehat{d}_4, \widehat{d}_8) &= \left\{\frac{\hat{u}_1}{[0.5,0.8]}, \frac{\hat{u}_2}{[0.6,0.8]}, \frac{\hat{u}_3}{[0.7,0.8]}, \frac{\hat{u}_4}{[0.8,0.9]}, \frac{\hat{u}_5}{[0.6,0.8]}\right\},
\end{aligned}$$

$$\begin{aligned}
Y_{\mathrm{FHS}}\left(\widehat{d}_3, \widehat{d}_5, \widehat{d}_6\right) &= \left\{ \frac{\widehat{u}_1}{[0.4,0.8]}, \frac{\widehat{u}_2}{[0.5,0.8]}, \frac{\widehat{u}_3}{[0.6,0.9]}, \frac{\widehat{u}_4}{[0.7,0.8]}, \frac{\widehat{u}_5}{[0.7,0.8]} \right\},\\
Y_{\mathrm{FHS}}\left(\widehat{d}_3, \widehat{d}_5, \widehat{d}_7\right) &= \left\{ \frac{\widehat{u}_1}{[0.4,0.8]}, \frac{\widehat{u}_2}{[0.5,0.8]}, \frac{\widehat{u}_3}{[0.6,0.9]}, \frac{\widehat{u}_4}{[0.7,0.8]}, \frac{\widehat{u}_5}{[0.7,0.9]} \right\},\\
Y_{\mathrm{FHS}}\left(\widehat{d}_3, \widehat{d}_5, \widehat{d}_8\right) &= \left\{ \frac{\widehat{u}_1}{[0.5,0.8]}, \frac{\widehat{u}_2}{[0.6,0.8]}, \frac{\widehat{u}_3}{[0.7,0.9]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.6,0.9]} \right\}.
\end{aligned} \tag{68}$$

Now, we find $\mathfrak{W}_6 = \mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2 = (\Psi^1_{\mathrm{FHS}}, \mathscr{E}_1) \tilde{\oplus} (\Psi^2_{\mathrm{FHS}}, \mathscr{E}_2) = (\Phi^1_{\mathrm{FHS}}, \mathscr{E}_1 \times \mathscr{E}_2)$ with

$$\begin{aligned}
\Phi^1_{\mathrm{FHS}}\left(\widehat{d}_1, \widehat{d}_4\right) &= \left\{ \frac{\widehat{u}_1}{[0.5,0.8]}, \frac{\widehat{u}_2}{[0.6,0.8]}, \frac{\widehat{u}_3}{[0.7,0.8]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.6,0.8]} \right\},\\
\Phi^1_{\mathrm{FHS}}\left(\widehat{d}_1, \widehat{d}_5\right) &= \left\{ \frac{\widehat{u}_1}{[0.6,0.8]}, \frac{\widehat{u}_2}{[0.7,0.8]}, \frac{\widehat{u}_3}{[0.8,0.9]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.7,0.8]} \right\},\\
\Phi^1_{\mathrm{FHS}}\left(\widehat{d}_2, \widehat{d}_4\right) &= \left\{ \frac{\widehat{u}_1}{[0.5,0.8]}, \frac{\widehat{u}_2}{[0.6,0.8]}, \frac{\widehat{u}_3}{[0.7,0.8]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.7,0.8]} \right\},\\
\Phi^1_{\mathrm{FHS}}\left(\widehat{d}_2, \widehat{d}_5\right) &= \left\{ \frac{\widehat{u}_1}{[0.6,0.8]}, \frac{\widehat{u}_2}{[0.7,0.8]}, \frac{\widehat{u}_3}{[0.8,0.9]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.7,0.8]} \right\},\\
\Phi^1_{\mathrm{FHS}}\left(\widehat{d}_3, \widehat{d}_4\right) &= \left\{ \frac{\widehat{u}_1}{[0.5,0.8]}, \frac{\widehat{u}_2}{[0.6,0.8]}, \frac{\widehat{u}_3}{[0.7,0.8]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.6,0.8]} \right\},\\
\Phi^1_{\mathrm{FHS}}\left(\widehat{d}_3, \widehat{d}_5\right) &= \left\{ \frac{\widehat{u}_1}{[0.6,0.8]}, \frac{\widehat{u}_2}{[0.7,0.8]}, \frac{\widehat{u}_3}{[0.8,0.9]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.7,0.9]} \right\}.
\end{aligned} \tag{69}$$

Similarly, $\mathfrak{W}_7 = \mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3 = (\Psi^3_{\mathrm{FHS}}, \mathscr{E}_3) \tilde{\oplus} (\Psi^3_{\mathrm{FHS}}, \mathscr{E}_3) = (\Phi^2_{\mathrm{FHS}}, \mathscr{E}_3 \times \mathscr{E}_3)$ with

$$\begin{aligned}
\Phi^2_{\mathrm{FHS}}\left(\widehat{d}_6, \widehat{d}_6\right) &= \left\{ \frac{\widehat{u}_1}{[0.3,0.8]}, \frac{\widehat{u}_2}{[0.4,0.9]}, \frac{\widehat{u}_3}{[0.5,0.9]}, \frac{\widehat{u}_4}{[0.6,0.7]}, \frac{\widehat{u}_5}{[0.7,0.8]} \right\},\\
\Phi^2_{\mathrm{FHS}}\left(\widehat{d}_6, \widehat{d}_7\right) &= \left\{ \frac{\widehat{u}_1}{[0.4,0.8]}, \frac{\widehat{u}_2}{[0.5,0.9]}, \frac{\widehat{u}_3}{[0.6,0.9]}, \frac{\widehat{u}_4}{[0.7,0.8]}, \frac{\widehat{u}_5}{[0.8,0.9]} \right\},\\
\Phi^2_{\mathrm{FHS}}\left(\widehat{d}_6, \widehat{d}_8\right) &= \left\{ \frac{\widehat{u}_1}{[0.5,0.8]}, \frac{\widehat{u}_2}{[0.6,0.9]}, \frac{\widehat{u}_3}{[0.7,0.9]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.7,0.9]} \right\},\\
\Phi^2_{\mathrm{FHS}}\left(\widehat{d}_7, \widehat{d}_6\right) &= \left\{ \frac{\widehat{u}_1}{[0.4,0.8]}, \frac{\widehat{u}_2}{[0.5,0.9]}, \frac{\widehat{u}_3}{[0.6,0.9]}, \frac{\widehat{u}_4}{[0.7,0.8]}, \frac{\widehat{u}_5}{[0.8,0.9]} \right\},\\
\Phi^2_{\mathrm{FHS}}\left(\widehat{d}_7, \widehat{d}_7\right) &= \left\{ \frac{\widehat{u}_1}{[0.4,0.8]}, \frac{\widehat{u}_2}{[0.5,0.9]}, \frac{\widehat{u}_3}{[0.6,0.9]}, \frac{\widehat{u}_4}{[0.7,0.8]}, \frac{\widehat{u}_5}{[0.8,0.9]} \right\},\\
\Phi^2_{\mathrm{FHS}}\left(\widehat{d}_7, \widehat{d}_8\right) &= \left\{ \frac{\widehat{u}_1}{[0.5,0.8]}, \frac{\widehat{u}_2}{[0.6,0.9]}, \frac{\widehat{u}_3}{[0.7,0.9]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.8,0.9]} \right\},\\
\Phi^2_{\mathrm{FHS}}\left(\widehat{d}_8, \widehat{d}_6\right) &= \left\{ \frac{\widehat{u}_1}{[0.5,0.8]}, \frac{\widehat{u}_2}{[0.6,0.9]}, \frac{\widehat{u}_3}{[0.7,0.9]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.7,0.9]} \right\},\\
\Phi^2_{\mathrm{FHS}}\left(\widehat{d}_8, \widehat{d}_7\right) &= \left\{ \frac{\widehat{u}_1}{[0.5,0.8]}, \frac{\widehat{u}_2}{[0.6,0.9]}, \frac{\widehat{u}_3}{[0.7,0.9]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.8,0.9]} \right\},\\
\Phi^2_{\mathrm{FHS}}\left(\widehat{d}_8, \widehat{d}_8\right) &= \left\{ \frac{\widehat{u}_1}{[0.5,0.8]}, \frac{\widehat{u}_2}{[0.6,0.9]}, \frac{\widehat{u}_3}{[0.7,0.9]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.6,0.9]} \right\}.
\end{aligned} \tag{70}$$

Lastly, $\mathfrak{W}_8 = \mathfrak{W}_6 \tilde{\otimes} \mathfrak{W}_7 = (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3) = (\Phi^1_{\mathrm{FHS}}, \mathscr{E}_1 \times \mathscr{E}_2) \tilde{\otimes} (\Phi^2_{\mathrm{FHS}}, \mathscr{E}_3 \times \mathscr{E}_3) = (\Phi^3_{\mathrm{FHS}}, (\mathscr{E}_1 \times \mathscr{E}_2) \times (\mathscr{E}_3 \times \mathscr{E}_3))$ with

$$\begin{aligned}
\Phi^3_{\mathrm{FHS}}\left(\left(\widehat{d}_1, \widehat{d}_4\right), \left(\widehat{d}_6, \widehat{d}_6\right)\right) &= \left\{ \frac{\widehat{u}_1}{[0.3,0.8]}, \frac{\widehat{u}_2}{[0.4,0.8]}, \frac{\widehat{u}_3}{[0.5,0.8]}, \frac{\widehat{u}_4}{[0.6,0.7]}, \frac{\widehat{u}_5}{[0.6,0.8]} \right\},\\
\Phi^3_{\mathrm{FHS}}\left(\left(\widehat{d}_1, \widehat{d}_4\right), \left(\widehat{d}_6, \widehat{d}_7\right)\right) &= \left\{ \frac{\widehat{u}_1}{[0.4,0.8]}, \frac{\widehat{u}_2}{[0.5,0.8]}, \frac{\widehat{u}_3}{[0.6,0.8]}, \frac{\widehat{u}_4}{[0.7,0.8]}, \frac{\widehat{u}_5}{[0.6,0.8]} \right\},\\
\Phi^3_{\mathrm{FHS}}\left(\left(\widehat{d}_1, \widehat{d}_4\right), \left(\widehat{d}_6, \widehat{d}_8\right)\right) &= \left\{ \frac{\widehat{u}_1}{[0.5,0.8]}, \frac{\widehat{u}_2}{[0.6,0.8]}, \frac{\widehat{u}_3}{[0.7,0.8]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.6,0.8]} \right\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\Phi^3_{\mathrm{FHS}}\left(\left(\widehat{d}_3, \widehat{d}_5\right), \left(\widehat{d}_8, \widehat{d}_8\right)\right) &= \left\{ \frac{\widehat{u}_1}{[0.5,0.8]}, \frac{\widehat{u}_2}{[0.6,0.8]}, \frac{\widehat{u}_3}{[0.7,0.9]}, \frac{\widehat{u}_4}{[0.8,0.9]}, \frac{\widehat{u}_5}{[0.6,0.9]} \right\}.
\end{aligned} \tag{71}$$

It can be seen that

$$\mathfrak{W}_5 \widehat{\subseteq}_{\mathcal{J}} \mathfrak{W}_8 = \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathcal{J}}, \tag{72}$$

$(\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3)$ is not valid in general.

**Corollary 44.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1)$, $\mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2)$&$\mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets. If* $\mathfrak{W}_1 \widehat{\subseteq}_{\mathcal{J}} \mathfrak{W}_3$, *then*

$$\begin{aligned}
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \widehat{\subseteq}_{\mathcal{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3),\\
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \widehat{\subseteq}_{\mathcal{J}} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3),\\
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathcal{J}} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3).
\end{aligned} \tag{73}$$

*Proof.* From Theorem 27(2), we have

$$\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3 \widehat{=}_{\mathscr{L}} \mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2, \tag{74}$$

and then, after applying Proposition 18, we get

$$\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3 \widehat{=}_{\mathcal{J}} \mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2, \tag{75}$$

implies

$$\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2 \widehat{\subseteq}_{\mathcal{J}} \mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3. \tag{76}$$

Taking $\tilde{\oplus}$ on both sides of above inequality with $\mathfrak{W}_1$, we have

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \widehat{\subseteq}_{\mathcal{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3), \tag{77}$$

but by Theorem 43, we get

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathcal{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3), \tag{78}$$

which leads to following final result due to transitivity of $\widehat{\subseteq}_{\mathcal{J}}$

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \widehat{\subseteq}_{\mathcal{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3). \tag{79}$$

Other parts can easily be validated in the similar manner. □

**Corollary 45.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets. If* $\mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_3$, *then*

$$\begin{aligned}
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3) \tilde{\otimes} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1),\\
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3) \tilde{\otimes} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1),\\
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3) \tilde{\otimes} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2),\\
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3) \tilde{\otimes} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2).
\end{aligned} \tag{80}$$

**Corollary 46.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets. If* $\mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_3$, *then*

$$\begin{aligned}
&(\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \tilde{\oplus} \mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3) \tilde{\otimes} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1),\\
&(\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} \mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3) \tilde{\otimes} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1),\\
&(\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} \mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3) \tilde{\otimes} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2),\\
&(\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \tilde{\oplus} \mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3) \tilde{\otimes} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2).
\end{aligned} \tag{81}$$

**Corollary 47.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets. If* $\mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_3$, *then*

$$\begin{aligned}
&(\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \tilde{\oplus} \mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3),\\
&(\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} \mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3),\\
&(\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} \mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3),\\
&(\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \tilde{\oplus} \mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3).
\end{aligned} \tag{82}$$

**Theorem 48.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets. If* $\mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_3$ *and* $\mathfrak{W}_3$ *is an UD-ivfhss, then we have*

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{=}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3. \tag{83}$$

*Proof.* Since we know from Theorem 43 that

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3), \tag{84}$$

As given that $\mathfrak{W}_3$ is an UD-ivfhss, therefore, $\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3 \widehat{=}_{\mathscr{J}} \mathfrak{W}_3$ which implies $\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_3$ such that

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3, \tag{85}$$

implies

$$(\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} (\mathfrak{W}_3 \tilde{\oplus} \mathfrak{W}_3) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3, \tag{86}$$

which leads to following final result due to transitivity of $\widehat{\subseteq}_{\mathscr{J}}$

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_3) \widehat{\subseteq}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3. \tag{87}$$

□

**Corollary 49.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets. If* $\mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_3$ *and* $\mathfrak{W}_3$ *is an UD-ivfhss, then*

$$\begin{aligned}
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \widehat{=}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3,\\
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \widehat{=}_{\mathscr{J}} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} \mathfrak{W}_3,\\
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{=}_{\mathscr{J}} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} \mathfrak{W}_3.
\end{aligned} \tag{88}$$

*Proof.* Since we know from Theorem 27 that $\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3 \widehat{=}_{\mathscr{L}} \mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2$ which further implies that $\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3 \widehat{=}_{\mathscr{J}} \mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2$, i.e.,

$$\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2, \tag{89}$$

and

$$\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3. \tag{90}$$

By applying Theorem 36, we have

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2), \tag{91}$$

and

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3), \tag{92}$$

so

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \widehat{=}_{\mathscr{J}} \mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3), \tag{93}$$

since by Theorem 48, we have

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{=}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3. \tag{94}$$

Hence,

$$\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \widehat{=}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3. \tag{95}$$

□

**Corollary 50.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets. If* $\mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_3$ *and* $\mathfrak{W}_3$ *is an UD-ivfhss, then*

$$\begin{aligned}
&(\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \tilde{\oplus} \mathfrak{W}_1 \widehat{=}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3,\\
&(\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \tilde{\oplus} \mathfrak{W}_1 \widehat{=}_{\mathscr{J}} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} \mathfrak{W}_3,\\
&(\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} \mathfrak{W}_1 \widehat{=}_{\mathscr{J}} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1) \tilde{\otimes} \mathfrak{W}_3,\\
&(\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} \mathfrak{W}_1 \widehat{=}_{\mathscr{J}} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2) \tilde{\otimes} \mathfrak{W}_3.
\end{aligned} \tag{96}$$

**Corollary 51.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1), \mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2) \& \mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets. If* $\mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_3$ *and* $\mathfrak{W}_3$ *is an UD-ivfhss, then*

Table 6: Structural comparison of proposed structure with existing relevant models.

| References | Structures | DoM | SAAF | MAAF | DFPT | IVTD |
|---|---|---|---|---|---|---|
| Molodtsov[1] | f-set | × | ✓ | × | × | × |
| Zadeh [2] | f-set | ✓ | × | × | × | × |
| Maji et al. [14] | fs-set | ✓ | ✓ | × | × | × |
| Yang et al. [15] | ivfs-set | ✓ | ✓ | × | × | ✓ |
| Zadeh [16] | ivf-set | ✓ | × | × | × | ✓ |
| Proposed model | ivfhs-set | ✓ | ✓ | ✓ | ✓ | ✓ |

$$\begin{aligned}
&(\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \tilde{\oplus} \mathfrak{W}_1 \widehat{=}_{\mathscr{J}} \mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_1 \tilde{\otimes} \mathfrak{W}_2), \\
&(\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \tilde{\oplus} \mathfrak{W}_1 \widehat{=}_{\mathscr{J}} \mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_1), \\
&(\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} \mathfrak{W}_1 \widehat{=}_{\mathscr{J}} \mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_1), \\
&(\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \tilde{\oplus} \mathfrak{W}_1 \widehat{=}_{\mathscr{J}} \mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_1 \tilde{\otimes} \mathfrak{W}_2).
\end{aligned} \tag{97}$$

**Corollary 52.** *Let* $\mathfrak{W}_1 = (\Psi^1_{FHS}, \mathscr{D}_1)$, $\mathfrak{W}_2 = (\Psi^2_{FHS}, \mathscr{D}_2)$&$\mathfrak{W}_3 = (\Psi^3_{FHS}, \mathscr{D}_3)$ *be three ivfhs-sets. If* $\mathfrak{W}_1 \widehat{\subseteq}_{\mathscr{J}} \mathfrak{W}_3$ *and* $\mathfrak{W}_3$ *is an UD-ivfhss, then*

$$\begin{aligned}
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \widehat{=}_{\mathscr{J}} \mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2), \\
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_3 \tilde{\otimes} \mathfrak{W}_2) \widehat{=}_{\mathscr{J}} \mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1), \\
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{=}_{\mathscr{J}} \mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_2 \tilde{\oplus} \mathfrak{W}_1), \\
&\mathfrak{W}_1 \tilde{\oplus} (\mathfrak{W}_2 \tilde{\otimes} \mathfrak{W}_3) \widehat{=}_{\mathscr{J}} \mathfrak{W}_3 \tilde{\otimes} (\mathfrak{W}_1 \tilde{\oplus} \mathfrak{W}_2).
\end{aligned} \tag{98}$$

*5.1. Discussion.* Now, we prove the flexibility of our presented model ivfhs-set through structural comparison based on some important evaluating features like DoM (degree of membership), SAAF (single-argument approximate function), MAAF (multiargument approximate function), DFPT (deep focus on parametric tuples), and IVTD (interval-valued type data). The Table 6 presents this comparison with some relevant existing studies. Some of the advantages of the proposed model are as under:

(1) It is capable to manage the uncertain nature of alternatives (entities in universal set) by assigning fuzzy membership grades to each entity corresponding to each parameter

(2) It has ability to tackle the scenarios where classification of parameters into their respective parametric-valued subcollections is necessary to be considered

(3) It is useful to manage big collection of interval-base information with the help of its interval-valued approximate setting

In short, the ivfhs-set tackle all the above three situations collectively in one model.

## 6. Conclusion

In this research, some essential elementary rudiments (i.e., properties, set-theoretic operations, and set-inclusions) of ivfhs-set are conceptualized, and then, some modular inequalities of ivfhs-set are established by employing the concept of $\mathscr{L}$-inclusion and $\mathscr{J}$-inclusion. It is observed that the transformation of approximate function from ivfs-set to ivfhs-set preserve all set-inclusion-based properties and inequalities. As this paper focuses on the fuzzy membership with interval setting under hs-set environment, so it is inadequate for the scenarios where the consideration of falsity degree and indeterminacy degree is mandatory. Therefore, the future work may include the extension of this study to tackle above said scenarios. This can also be extended to the development of algebraic structures based on fuzzy hypersoft set with interval-valued setting.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest.

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