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Multi-Attribute Decision Support Model Based on Bijective Hypersoft Expert Set

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Abstract. Soft set tackles a single set of attributes while its extension hypersoft set is projected for dealing attribute-valued disjoint sets corresponding to distinct attributes with entitlement of multi-argument approximate function. In order to furnish soft set-like models with multi-decisive opinions of multi-experts, a new model i.e. soft expert set has been developed but this is inadequate for handling the scenario where partitioning of attributes into their respective attribute-valued sets, is required. Hence hypersoft expert set has made its place to be developed. This article intends to develop a new type of hypersoft set called bijective hypersoft expert set which is more flexible and effective. After characterization of its essential properties and set-theoretic operations like union, relaxed and restricted AND, a decision-support system is designed which is characterized by new operations such as decision system, reduced decision system, etc. with illustrated examples. The proposed decision-support system is applied in multi-attribute decision-making process to manage a real-life application.

AMS (MOS) Subject Classification Codes: 91B06; 93E25
Key Words: Soft set, Soft expert set, Bijective soft set, Hypersoft set, Hypersoft expert set, Bijective hypersoft set, Bijective hypersoft expert set.

1. Introduction

In 1999, Molodtsov [1] introduced the structure of soft set to explain problems of vague data and uncertain environment. After this, Maji et al. [2] presented its some basic properties, operations, laws and used this structure in different fields to explain different situations. In 2005, Pei et al. [3] explained the relationship between soft set and information system. Many other researchers [4]-[9] worked on this theory and introduced some new

operations, properties, laws and used them in decision-making problems. Saeed et al. [10] made use of this theory and introduced soft elements and members. Rahman et al. [11, 12] conceptualized m-convexity (m-concavity) and (m, n)-convexity ((m, n)-concavity) on soft sets with some properties. Gong et al. [27] conceptualized new structure of bijective soft set and discussed its essential characteristics. Kamacı et al. [13, 14, 15, 16] presented bijective soft matrix theory and reviewed different operations by making its use in decision making system. He also developed the structures of N-soft set, bipolar N-soft set and studied their applications. Many researchers [17, 18, 19, 20, 21, 22, 23, 24, 25, 26] broadened soft set theory, and developed different soft-like hybrids to make its use in many different fields like decision making and information system.
Alkhazalah [28] introduced the concept of soft expert set to solve the problem of different opinions of different experts in a single model. He used this structure in different areas like medical diagnosis and decision making. Ihsan et al. [29, 30] introduced convexity (concavity) on soft expert sets and fuzzy soft expert sets respectively and proved their certain properties. Soft set is used only for the single set of attributes whereas hypersoft set, developed by Smarandache [31], is useful for tackling further partitioning of attributes into their respective sub-attributive values in the form of disjoint sets. In 2020, Saeed et al. [32, 33] studied basic properties, operations of hypersoft set and explained with different examples. In 2020, Rahman et al. [34, 35] developed some new structures of hypersoft set like complex fuzzy hypersoft set and introduced the concept of convexity in hypersoft set. In 2021, Rahman et al. [36, 42, 43] presented some new structures like rough, fuzzy parameterized, neutrosophic hypersoft set and used them in decision-making problems. Saeed et al. [37, 38, 39, 41] worked on hypersoft classes, complex multi-fuzzy hypersoft set and explained some new methods of decision making for mappings. They also presented hypersoft graphs, a new class of hypersoft set with some characteristics. Working on hypersoft set, Yolcu et al. [44] gave the idea of fuzzy, intuitionistic fuzzy hypersoft sets and made use of them in decision making. Saqlain et al. [47, 48, 49] made their contributions in hypersoft set by introducing single and multi-valued neutrosophic hypersoft sets and calculated their tangent similarity measures. They described aggregate operators and TOPSIS method for neutrosophic hypersoft sets. Ihsan et al. [40] extended hypersoft set to hypersoft expert set and used it in decision-making problems. Ihsan et al. [45] introduced the structure of fuzzy hypersoft expert set with application in decision making problem. Ali et al. [63] developed the Einstein geometric aggregation operators using a novel complex interval-valued Pythagorean Fuzzy setting with application in green supplier chain management. Riaz et al. [64] worked on decision-making problems and described certain properties of soft multi set topologies with applications in decision-making problems.

1.1. **Research Gap and Motivation.** Following points will explain the research gap and motivations behind the choice of proposed structure:

(1) Gong et al. [27] introduced the concept of bijective soft set and discussed its necessary operations with an application in decision-making problems. After this, Gong et al. [53] extended their work in fuzzy environment and introduced the concept of bijective fuzzy soft set to deal with more uncertain problems. Kumar et al. [54] applied this concept of bijective soft set-in classification of medical data. Tiwari et al. [55] used an integrated Shannon entropy and TOPSIS for product

design concept evaluation based on bijective soft set. Gong et al. [56] worked on bijective soft set and used it in mining data from soft set environments, and applied it in some fields. Tiwari et al. [57] made a bijective soft set theoretic approach for concept selection in design process. Inbarani et al. [58] introduced the idea of rough bijective soft set and applied in medical field. Kumar et al. [59] gave the idea of improved bijective soft set and proposed a model for the classification of cancer based on gene expression profiles. Kumar et al. [60] discussed the structure of hybrid bijective soft set to construct novel automatic classification system for analysis of ECG signal and decision making purposes. Kumar et al. [61] also applied this structure for the identification of heart valve disease. Kamac et al. [62] introduced the concept of bijective soft matrix and applied for decision-making problems.

(2) It can be seen that the above bijective soft set like models deal with opinion of only single expert. But in real life, there are certain situations where we need different opinions of different experts in one model. To tackle this situation, soft expert set has been developed. However, there are also certain situations when attributes are further classified into their respective attribute-valued disjoint sets. Therefore, there is a need of new structure to handle such situations with multi-decisive opinions under multi-argument soft set like environment. So hypersoft expert set is developed.

(3) Having motivation from the above literature in general and specifically from [27] and [40], a novel structure bijective hypersoft expert set (BHSES) is developed with certain properties. By using the aggregate operations of BHSES, a new decision system is proposed and is used in multi-attribute decision-making problems.

1.2. **Main Contributions.** The following are the main contributions of the proposed study:

(1) Some basic definitions of soft set, soft expert, hypersoft set, bijective soft set, bijective hypersoft set are reviewed from literature.
(2) Theory of bijective hypersoft expert set (i.e., axiomatic properties, set-theoretic operations and laws) is conceptualized with the support of numerical illustrative examples.
(3) An algorithm is proposed and then validated by applying it in decision-making based daily-life problem.
(4) The proposed study is compared with existing relevant models to judge the advantageous aspects of proposed study.
(5) Paper is summarized with description of its scope and future directions to motivate the readers for further extensions.

1.3. **Paper Organization.** The remaining paper is organized as under:
Section 2 recalls some basic definitions and terms from existing literature to support main results. Section 3 describes the theory of bijective hypersoft expert set with the description of its decision support system. Section 4 proposes an algorithm based of BHSES with utilization in daily-life multi-attribute decision-making problem and section 5 summarizes the paper with more future directions.

2. Preliminaries

The following portion describes some basic definitions related to the literature and suggested work. In this article, universe of discourse will be shown by $\mathfrak{Z}$, $\mathfrak{S}$ will be used as an experts set and $\mathfrak{O}$ as an opinions set, $\mathfrak{R} = \mathfrak{P} \times \mathfrak{S} \times \mathfrak{O}$ with $\mathfrak{V} \subseteq \mathfrak{R}$, while $\mathfrak{P}$ a set of parameters. The symbol $\mathcal{P}(\mathfrak{Z})$ will denote the power set of universe of discourse.

Definition 1. [1]
A soft set S is defined by an approximate function $\beta_S : \theta \to \mathcal{P}(\mathfrak{Z})$ which is defined by approximate elements $\beta_S(\hat{s})$ for all members $\hat{s}$ of θ, a subset of parameters.

Definition 2. [2]
A soft expert set ϖ is characterized by an approximate function $\Phi_\varpi : \mathfrak{H} \to \mathcal{P}(\mathfrak{Z})$ which is defined by approximate elements $\Phi_\varpi(\hat{v})$ for all members $\hat{v}$ of $\mathfrak{H}$ where $\mathfrak{H} \subseteq \mathfrak{P}$.

Definition 3. [31]
Let $\phi_1, \phi_2, \phi_3, ..., \phi_\pi$, for $\pi \geq 1$, be π distinct attributes having $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3, ..., \mathfrak{H}_\pi$ as their respective attribute-valued sets with $\mathfrak{H}_\alpha \cap \mathfrak{H}_\beta = \emptyset$, for $\alpha \neq \beta$, and $\alpha, \beta \in \{1, 2, 3, ..., \pi\}$. The pair $(\Psi, \mathfrak{G})$ is named as a *hypersoft set* over $\mathfrak{Z}$ where $\mathfrak{G} = \mathfrak{H}_1 \times \mathfrak{H}_2 \times \mathfrak{H}_3 \times ..... \times \mathfrak{H}_\pi$ and $\Psi_\Upsilon : \mathfrak{G} \to \mathcal{P}(\mathfrak{Z})$ is its multi-argument approximate function characterized by approximate elements $\Psi_\Upsilon(\hat{g})$ for all $\hat{g} \in \mathfrak{G}$.

Definition 4. [27]
A bijective soft set is a soft set (β_S, θ) which satisfies the following conditions:
(i) $\cup_{\epsilon \in \theta} \beta_S(\epsilon) = \mathfrak{Z}$
(ii) $\beta_S(\epsilon_i) \cap \beta_S(\epsilon_j) = \emptyset$, for any two parameters $\epsilon_i, \epsilon_j \in \theta,\ \epsilon_i \neq \epsilon_j$.

For the sake of collection of abstracts cum statistical data and information, scholars design questionnaires having description of parametric statements in the form of questions and then it is circulated to individuals with relevant field of expertise. In this scenario, the approached evaluators for such questionnaires are directed to provide their expert opinions regarding the relevance of certain elements of universal set (topics/areas under consideration for the study) with parameters. After the completion of this data collection, the submissions are further reviewed by internal experts (scholars, supervisors, co-supervisors etc.) who classify the collected data on agree and dis-agree basis and then evaluate with their weightage subject to the condition that two opinions collected from field should not overlap. It is pertinent to mention here that the questions of questionnaires are of parametric nature i.e. each question is parameterized with sub-parametric values. The existing literature is inadequate to provide a uncertain model to tackle such scenario so bijective hypersoft expert set is being characterized in the following section to address the scarcity of literature.

3. Bijective Hypersoft Expert Set (BHSES-set)

The following portion contains the definition of hypersoft expert set with example and the theory of bijective hypersoft expert set.

Definition 5. [52]
Let $\hat{p}_1, \hat{p}_2, \hat{p}_3, ..., \hat{p}_n$, for $n \geq 1$, be n distinct attributes having $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, ..., \mathcal{P}_n$ as

their respective attribute-valued sets with $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$, for $i \neq j$. The pair $(\Psi, \mathfrak{G})$ is said to be *bijective hypersoft set* over $\mathfrak{Z}$ where $\mathfrak{G} = \mathfrak{H}_1 \times \mathfrak{H}_2 \times \mathfrak{H}_3 \times ..... \times \mathfrak{H}_\pi$ and $\Psi_\Upsilon : \mathfrak{G} \to \mathcal{P}(\mathfrak{Z})$ is its multi-argument approximate function characterized by approximate elements $\Psi_\Upsilon(\hat{g})$ for all $\hat{g} \in \mathfrak{G}$ which satisfy the following conditions:

(1) $\bigcup_{\hat{g} \in \mathfrak{G}} \Psi_\Upsilon(\hat{g}) = \mathfrak{Z}$

(2) $\Psi_\Upsilon(\hat{g}_i) \cap \Psi_\Upsilon(\hat{g}_j) = \emptyset$, for $\hat{g}_i, \hat{g}_j \in \mathfrak{S}, \hat{g}_i \neq \hat{g}_j$.

Definition 6. [40]
A pair$(\xi, \mathbb{S})$ is known as a *hypersoft expert set* over $\mathfrak{Z}$, where $\xi : \mathbb{S} \to \mathcal{P}(\mathfrak{Z})$ where $\mathcal{P}(\mathfrak{Z})$ is collection of all fuzzy subsets of $\mathfrak{Z}$, $\mathbb{S} \subseteq \mathcal{H} = \mathcal{G} \times \mathcal{D} \times \mathbb{C}$ and $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \mathcal{G}_3 \times .... \times \mathcal{G}_p$ here $\mathcal{G}_i$ are disjoint attributive-valued sets corresponding to distinct attributes $g_i, i = 1, 2, 3, ..., p$, $\mathcal{D}$ be a set of specialists (operators) and $\mathbb{C}$ be a set of conclusions. For simplicity, $\mathbb{C} = \{0 = disagree, 1 = agree\}$.

Example 3.1. Suppose that Mr. John intends to purchase a mask from a medical store. There are four types of masks available in market forming initial universe $\mathfrak{Z} = \{\Bbbk_1, \Bbbk_2, \Bbbk_3, \Bbbk_4\}$. The choice of mask may be carried out by keeping in mind the following attributes i.e. γ_1 = Colour, γ_2 = Size, and γ_3 = Price. Following are the attribute-valued sets corresponding to these attributes are:
$\tau_{\gamma_1} = \{\gamma_{11}, \gamma_{12}\}$
$\tau_{\gamma_2} = \{\gamma_{21}, \gamma_{22}\}$
$\tau_{\gamma_3} = \{\gamma_{31}, \gamma_{32}\}$
then $\tau_\gamma = \tau_{\gamma_1} \times \tau_{\gamma_2} \times \tau_{\gamma_3}$

$$\tau_\gamma = \left\{ \begin{array}{l} (\varpi_1, \{\gamma_{11}, \gamma_{21}, \gamma_{31}\}), (\varpi_2, \{\gamma_{11}, \gamma_{21}, \gamma_{32}\}), \\ (\varpi_3, \{\gamma_{11}, \gamma_{22}, \gamma_{31}\}), (\varpi_4, \{\gamma_{11}, \gamma_{22}, \gamma_{32}\}), \\ (\varpi_5, \{\gamma_{12}, \gamma_{21}, \gamma_{31}\}), (\varpi_6, \{\gamma_{12}, \gamma_{21}, \gamma_{32}\}), \\ (\varpi_7, \{\gamma_{12}, \gamma_{22}, \gamma_{31}\}), (\varpi_8, \{\gamma_{12}, \gamma_{22}, \gamma_{32}\}) \end{array} \right\}$$

Now $\mathfrak{H} = \tau_\gamma \times \mathfrak{D} \times \mathfrak{C}$

$$\mathfrak{H} = \left\{ \begin{array}{l} (\varpi_1, \phi_1, 0), (\varpi_1, \phi_1, 1), (\varpi_1, \phi_2, 0), (\varpi_1, \phi_2, 1), (\varpi_1, \phi_3, 0), (\varpi_1, \phi_3, 1), \\ (\varpi_2, \phi_1, 0), (\varpi_2, \phi_1, 1), (\varpi_2, \phi_2, 0), (\varpi_2, \phi_2, 1), (\varpi_2, \phi_2, 0), (\varpi_2, \phi_3, 1), \\ (\varpi_3, \phi_1, 0), (\varpi_3, \phi_1, 1), (\varpi_3, \phi_2, 0), (\varpi_3, \phi_2, 1), (\varpi_3, \phi_3, 0), (\varpi_3, \phi_3, 1), \\ (\varpi_4, \phi_1, 0), (\varpi_4, \phi_1, 1), (\varpi_4, \phi_2, 0), (\varpi_4, \phi_2, 1), (\varpi_4, \phi_3, 0), (\varpi_4, \phi_3, 1), \\ (\varpi_5, \phi_1, 0), (\varpi_5, \phi_1, 1), (\varpi_5, \phi_2, 0), (\varpi_5, \phi_2, 1), (\varpi_5, \phi_3, 0), (\varpi_5, \phi_3, 1), \\ (\varpi_6, \phi_1, 0), (\varpi_6, \phi_1, 1), (\varpi_6, \phi_2, 0), (\varpi_6, \phi_2, 1), (\varpi_6, \phi_3, 0), (\varpi_6, \phi_3, 1), \\ (\varpi_7, \phi_1, 0), (\varpi_7, \phi_1, 1), (\varpi_7, \phi_2, 0), (\varpi_7, \phi_2, 1), (\varpi_7, \phi_3, 0), (\varpi_7, \phi_3, 1), \\ (\varpi_8, \phi_1, 0), (\varpi_8, \phi_1, 1), (\varpi_8, \phi_2, 0), (\varpi_8, \phi_2, 1), (\varpi_8, \phi_3, 0), (\varpi_8, \phi_3, 1) \end{array} \right\}$$

let

$$\phi_{\mathfrak{W}} = \left\{ \begin{array}{l} (\varpi_1, \phi_1, 0), (\varpi_1, \phi_1, 1), (\varpi_1, \phi_2, 0), (\varpi_1, \phi_2, 1), (\varpi_1, \phi_3, 0), (\varpi_1, \phi_3, 1), \\ (\varpi_2, \phi_1, 0), (\varpi_2, \phi_1, 1), (\varpi_2, \phi_2, 0), (\varpi_2, \phi_2, 1), (\varpi_2, \phi_3, 0), (\varpi_2, \phi_3, 1) \\ (\varpi_3, \phi_1, 0), (\varpi_3, \phi_1, 1), (\varpi_3, \phi_2, 0), (\varpi_3, \phi_2, 1), (\varpi_3, \phi_3, 0), (\varpi_3, \phi_3, 1), \end{array} \right\}$$

be a subset of $\mathfrak{H}$ and ϖ_1= Price ϖ_2= Colour ϖ_3= Reliability
$\mathfrak{D} = \{\phi_1, \phi_2, \phi_3, \}$ be a set of specialists.
Following survey depicts choices of three specialists:

$\pi_1 = \pi(\varpi_1, \phi_1, 1) = \{\mathbb{k}_2, \mathbb{k}_4\}$, $\pi_2 = \pi(\varpi_1, \phi_2, 1) = \{\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3, \mathbb{k}_4\}$,
$\pi_3 = \pi(\varpi_1, \phi_3, 1) = \{\mathbb{k}_3\}$, $\pi_4 = \pi(\varpi_2, \phi_1, 1) = \{\mathbb{k}_2, \mathbb{k}_3\}$,
$\pi_5 = \pi(\varpi_2, \phi_2, 1) = \{\mathbb{k}_4\}$, $\pi_6 = \pi(\varpi_2, \phi_3, 1) = \{\mathbb{k}_1, \mathbb{k}_4\}$,
$\pi_7 = \pi(\varpi_3, \phi_1, 1) = \{\mathbb{k}_1\}$, $\pi_8 = \pi(\varpi_3, \phi_2, 1) = \{\mathbb{k}_1, \mathbb{k}_3\}$,
$\pi_9 = \pi(\varpi_3, \phi_3, 1) = \{\mathbb{k}_2, \mathbb{k}_3, \mathbb{k}_4\}$, $\pi_{10} = \pi(\varpi_1, \phi_1, 0) = \{\mathbb{k}_2\}$,
$\pi_{11} = \pi(\varpi_1, \phi_2, 0) = \{\mathbb{k}_1, \mathbb{k}_3, \mathbb{k}_4\}$, $\pi_{12} = \pi(\varpi_1, \phi_3, 0) = \{\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_4\}$,
$\pi_{13} = \pi(\varpi_2, \phi_1, 0) = \{\mathbb{k}_1, \mathbb{k}_2\}$, $\pi_{14} = \pi(\varpi_2, \phi_2, 0) = \{\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3, \mathbb{k}_4\}$,
$\pi_{15} = \pi(\varpi_2, \phi_3, 0) = \{\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_4\}$, $\pi_{16} = \pi(\varpi_3, \phi_1, 0) = \{\mathbb{k}_3, \mathbb{k}_4\}$,
$\pi_{17} = \pi(\varpi_3, \phi_2, 0) = \{\mathbb{k}_1, \mathbb{k}_2\}$, $\pi_{18} = \pi(\varpi_3, \phi_3, 0) = \{\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3\}$.

The hypersoft expert set can be written as

$$(\pi, \phi_{\mathfrak{W}}) = \left\{ \begin{array}{l} ((\varpi_1, \phi_1, 1), \{\mathbb{k}_2, \mathbb{k}_4\}), ((\varpi_1, \phi_2, 1), \{\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3, \mathbb{k}_4\}), ((\varpi_1, \phi_3, 1), \{\mathbb{k}_3\}), \\ ((\varpi_2, \phi_1, 1), \{\mathbb{k}_2, \mathbb{k}_3\}), ((\varpi_2, \phi_2, 1), \{\mathbb{k}_4\}), ((\varpi_2, \phi_3, 1), \{\mathbb{k}_1, \mathbb{k}_4\}), \\ ((\varpi_3, \phi_1, 1), \{\mathbb{k}_1\}), ((\varpi_3, \phi_2, 1), \{\mathbb{k}_1, \mathbb{k}_3\}), ((\varpi_3, \phi_3, 1), \{\mathbb{k}_2, \mathbb{k}_3, \mathbb{k}_4\}), \\ ((\varpi_1, \phi_1, 0), \{\mathbb{k}_2\}), ((\varpi_1, \phi_2, 0), \{\mathbb{k}_1, \mathbb{k}_3, \mathbb{k}_4\}), ((\varpi_1, \phi_3, 0), \{\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_4\}), \\ ((\varpi_2, \phi_1, 0), \{\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_4\}), ((\varpi_2, \phi_2, 0), \{\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3, \mathbb{k}_4\}), ((\varpi_2, \phi_3, 0), \{\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_4\}), \\ ((\varpi_3, \phi_1, 0), \{\mathbb{k}_3, \mathbb{k}_4\}), ((\varpi_3, \phi_2, 0), \{\mathbb{k}_1, \mathbb{k}_2\}), ((\varpi_3, \phi_3, 0), \{\mathbb{k}_1, \mathbb{k}_2, \mathbb{k}_3\}) \end{array} \right\}$$

Following table represents the hypersoft expert set $(\pi, \mathfrak{S})$, TABLE 1, with $\mathbb{k}_i \in \mathfrak{S}(\varpi_i)$ then $\widehat{\oplus}$ otherwise $\widehat{\otimes}$.

....	$\mathbb{k}_1$	$\mathbb{k}_2$	$\mathbb{k}_3$	$\mathbb{k}_4$
π_1	$\widehat{\otimes}$	$\widehat{\oplus}$	$\otimes$	$\widehat{\oplus}$
π_2	$\widehat{\oplus}$	$\widehat{\oplus}$	$\widehat{\oplus}$	$\widehat{\oplus}$
π_3	$\otimes$	$\otimes$	$\widehat{\oplus}$	$\otimes$
π_4	$\widehat{\otimes}$	$\widehat{\oplus}$	$\widehat{\oplus}$	$\widehat{\otimes}$
π_5	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\oplus}$
π_6	$\widehat{\oplus}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\oplus}$
π_7	$\widehat{\oplus}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$
π_8	$\widehat{\oplus}$	$\widehat{\otimes}$	$\widehat{\oplus}$	$\widehat{\otimes}$
π_9	$\widehat{\otimes}$	$\widehat{\oplus}$	$\widehat{\oplus}$	$\widehat{\oplus}$
π_{10}	$\widehat{\otimes}$	$\widehat{\oplus}$	$\widehat{\otimes}$	$\widehat{\otimes}$
π_{11}	$\widehat{\oplus}$	$\widehat{\otimes}$	$\widehat{\oplus}$	$\widehat{\oplus}$
π_{12}	$\widehat{\oplus}$	$\widehat{\oplus}$	$\widehat{\otimes}$	$\widehat{\oplus}$
π_{13}	$\widehat{\oplus}$	$\widehat{\oplus}$	$\widehat{\otimes}$	$\widehat{\otimes}$
π_{14}	$\widehat{\oplus}$	$\widehat{\oplus}$	$\widehat{\oplus}$	$\widehat{\oplus}$
π_{15}	$\widehat{\oplus}$	$\widehat{\oplus}$	$\widehat{\otimes}$	$\widehat{\oplus}$
π_{16}	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\oplus}$	$\widehat{\oplus}$
π_{17}	$\widehat{\oplus}$	$\widehat{\oplus}$	$\widehat{\otimes}$	$\widehat{\otimes}$
π_{18}	$\widehat{\oplus}$	$\widehat{\oplus}$	$\widehat{\oplus}$	$\widehat{\otimes}$

TABLE 1. $(\pi, \mathcal{S})$ shows the tabular representation of HSES

Definition 7. A hypersoft expert set $(\pi, \mathcal{S})$ is named as a bijective hypersoft expert set if

(1) $\bigcup_{\varpi \in \mathcal{S}} \pi(\varpi) = \mathfrak{Z}$

(2) $\pi(\varpi_i) \cap \pi(\varpi_j) = \emptyset$ for any two $\varpi_i, \varpi_j \in \mathfrak{S}, \varpi_i \neq \varpi_j$

The symbol Υ_{BHSES} denotes the collection of all BHSESs over $\mathfrak{Z}$.

Example 3.2. Considering Example 3.1, we get BHSES

$$(\pi, \mathfrak{S}) = \left\{ \ (\varpi_1, (\mathbb{k}_2, \mathbb{k}_4), (\varpi_2, \mathbb{k}_3), (\varpi_3, (\mathbb{k}_1) \ \right\}$$

and TABLE 2 shows the tabular form of bijective hypersoft expert set.

....	$\mathbb{k}_1$	$\mathbb{k}_2$	$\mathbb{k}_3$	$\mathbb{k}_4$
ϖ_1	$\widehat{\otimes}$	$\widehat{\oplus}$	$\widehat{\otimes}$	$\widehat{\oplus}$
ϖ_2	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\oplus}$	$\widehat{\otimes}$
ϖ_3	$\widehat{\oplus}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$
ϖ_4	$\widehat{\oplus}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$

TABLE 2. The table form of BHSES $(\pi, \mathfrak{S})$

Definition 8. The operation AND between two hypersoft expert sets $(\pi_1, \mathfrak{S}_1)$ and $(\pi_2, \mathfrak{S}_2)$, shown as $(\pi_1, \mathfrak{S}_1) \bigwedge (\pi_2, \mathfrak{S}_2)$, is a hypersoft expert set $(\pi_3, \mathfrak{S}_3))$ with $\mathfrak{S}_3 = \mathfrak{S}_1 \times \mathfrak{S}_2$ and for $\varpi \in \mathfrak{S}_3$,

$$\pi_3(\varpi) = \pi_1(\varpi) \cap \pi_2(\varpi)$$

Theorem 3.3. *Suppose $(\pi_1, \mathfrak{H}_1)$ and $(\pi_2, \mathfrak{H}_2)$ are BHSESs, then $(\pi_1, \mathfrak{H}_1) \bigwedge (\pi_2, \mathfrak{H}_2)$ is a BHSES.*

Proof. Using definition 7 , we have
$(\pi_1, \mathfrak{H}_1) \wedge (\pi_2, \mathfrak{H}_2) = (\pi_3, \mathfrak{H}_3)$, where $\mathfrak{H}_3 = \mathfrak{H}_1 \times \mathfrak{H}_2$ and $\pi_3(\mathfrak{h}_1, \mathfrak{h}_2) = \pi_1(\mathfrak{h}_1) \cap \pi_2(\mathfrak{h}_2), \ \forall \ (\mathfrak{h}_1, \mathfrak{h}_2) \in \mathfrak{H}_3$.
Consider $\varepsilon \in \mathfrak{H}_3$ is a parameter of $(\pi_3, \mathfrak{H}_3)$ then

$$\pi_3(\varepsilon) = \pi_1 \left(\mathfrak{h}_1\right) \cap \pi_2 \left(\mathfrak{h}_2\right)$$

$\therefore \bigcup_{\varepsilon \in \mathfrak{H}_2} \pi_3(\varepsilon) = \bigcup_{\mathfrak{h}_1 \in \mathfrak{H}_1} \bigcup_{\mathfrak{h}_2 \in \mathfrak{H}_2} \pi_1 \left(\mathfrak{h}_1\right) \cap \pi_2 \left(\mathfrak{h}_2\right) = \bigcup_{\mathfrak{h}_1 \in \mathfrak{H}_1} \pi_1 \left(\mathfrak{h}_1\right) \cap \left(\bigcup_{\mathfrak{h}_2 \in \mathfrak{H}_2} \pi_2 \left(\mathfrak{h}_2\right) \right) = \bigcup_{\mathfrak{h}_1 \in \mathfrak{H}_1} \pi_1 \left(\mathfrak{h}_1\right) \cap \mathfrak{Z} = \mathfrak{Z}.$
Suppose $\varepsilon_i, \varepsilon_j \in \mathfrak{H}_3, \varepsilon_i \neq \varepsilon_j, \varepsilon_i = \alpha_1 \times \beta_1, \alpha_1 \in \mathfrak{H}_1, \beta_1 \in \mathfrak{H}_2, \varepsilon_j = \alpha_2 \times \beta_2, \alpha_2 \in \mathfrak{H}_1, \beta_2 \in \mathfrak{H}_2$. Then

$$\pi_3(\varepsilon_i) \cap \pi_3(\varepsilon_j) = (\pi_1(\alpha_1) \cap \pi_2(\beta_1)) \cap (\pi_1(\alpha_2) \cap \pi_2(\beta_2)) = \emptyset.$$

Hence $(\pi_3, \mathfrak{H}_3) = (\pi_1, \mathfrak{H}_1) \bigwedge (\pi_2, \mathfrak{H}_2)$ is a bijective hypersoft expert set. □

Definition 9. A HSES $(\pi, \mathfrak{S})$ is named as a *null HSES*, shown as $(\pi, \mathfrak{S})_\Phi$, if $\pi(\varpi) = \emptyset, \forall \ \varpi \in \mathfrak{S}$.

Definition 10. The operation union between two HSESs is a HSES $(\pi_3, \mathfrak{S}_3)$ with $\mathfrak{S}_3 = \mathfrak{S}_1 \cup \mathfrak{S}_2$

$$\pi_3(\varpi) = \begin{cases} \pi_1(\varpi) & \varpi \in (\mathfrak{S}_1 \setminus \mathfrak{S}_2) \\ \pi_2(\varpi) & \varpi \in (\mathfrak{S}_2 \setminus \mathfrak{S}_1) \\ \pi_1(\varpi) \cup \pi_2(\varpi) & \varpi \in (\mathfrak{S}_1 \cap \mathfrak{S}_2) \end{cases}$$

for $\varpi \in \mathfrak{S}_3$.

Theorem 3.4. *Suppose $(\pi, \mathfrak{H})$ is a BHSES. Then $(\pi, \mathfrak{H}) \cup (\pi, \mathfrak{H})_\Phi$ is a BHSES.*

Proof. Suppose $(\pi, \mathfrak{H})_\Phi = (\pi_\Phi, \mathfrak{H}_1)$,
by using the definitions of 9 and 10, we get
$(\pi_2, \mathfrak{H}_2) = (\pi, \mathfrak{H}) \cup (\pi_\Phi, \mathfrak{H}_1)$

$$= \begin{cases} \pi(\epsilon) & ; \epsilon \in \mathfrak{H} - \mathfrak{H}_1 \\ \pi_\Phi(\epsilon) = \emptyset & ; \epsilon \in \mathfrak{H}_1 - \mathfrak{H} = (\pi, \mathfrak{H} \cup \mathfrak{H}_1) \\ \pi(\epsilon) \cup \pi_\Phi(\epsilon) = \pi(\epsilon) \cup \emptyset & ; \epsilon \in \pi \cap \pi_1 \end{cases}$$

where $\epsilon \in \mathfrak{H}_2$ and $(\pi, \mathfrak{H}_1) \subset (\pi, \mathfrak{H} \cup \mathfrak{H}_1)$ is a Null hypersoft set, implies
$(\pi_2, \mathfrak{H}_2) = (\pi, \mathfrak{H} \cup \mathfrak{H}_1)$ is a bijective hypersoft expert set over $\mathfrak{Z}$.

□

Definition 11. Suppose $\mathfrak{Z}_1 \subset \mathfrak{Z}$ and $(\pi, \mathfrak{H})$ is a BHSES. The restricted AND operation is written as $(\pi, \mathfrak{H}) \bigwedge_{\mathcal{R}} \mathfrak{Z}_1$, and is described as

$$\bigcup_{\varpi \in \mathfrak{H}} \{\pi(\varpi) : \pi(\varpi) \subseteq \mathfrak{Z}_1\}.$$

Example 3.5. Suppose $\mathfrak{Z} = \{F_1, F_2, F_3, ..., F_6\}$ and $\mathfrak{Z}_1 = \{F_1, F_2, F_3, \}$. If $(\pi, \mathfrak{H}) \in \Upsilon_{BHSES}$ with

$$(\pi, \mathfrak{H}) = \{ (\varpi_1, \{F_1, F_2\}), (\varpi_2, \{F_3\}), (\varpi_3, \{F_5, F_6\}) \}$$

then

$$(\pi, \mathfrak{H}) \bigwedge_{\mathcal{R}} \mathfrak{Z}_1 = \{F_1, F_2\} \cup \{F_3\} = \{F_1, F_2, F_3\}$$

Definition 12. Suppose $\mathfrak{Z}_1 \subset \mathfrak{Z}$ and $(\pi, \mathfrak{H})$ are BHSES. The relaxed AND operation is written as

$$\bigcup_{\varpi \in \mathfrak{H}} \{\pi(\varpi) : \pi(\varpi) \cap \mathfrak{Z}_1 \neq \emptyset\}.$$

and it is represented by $(\pi, \mathfrak{H}) \bigwedge_{\mathcal{R}el} \mathfrak{Z}_1$.

Example 3.6. Let $\mathfrak{Z} = \{F_1, F_2, F_3, ..., F_6\}$ and $\mathfrak{Z}_1 = \{F_1, F_2, F_3\}$. Suppose $(\pi, \mathfrak{H})$ is a BHSES with

$$(\pi, \mathfrak{H}) = \{ (\varpi_1, \{F_1, F_6\}), (\varpi_2, \{F_3, F_5\}), (\varpi_3, \{F_2, F_4\}), \}$$

then

$$(\pi, \mathfrak{H}) \bigwedge_{\mathcal{R}el} \mathfrak{Z}_1 = \{F_1, F_6\} \cup \{F_3, F_5\} \cup \{F_2, F_4\} = \{F_1, F_2, F_3, F_4, F_5, F_6\} = \mathfrak{Z}$$

Definition 13. Suppose $(\pi, \mathfrak{H})$ is a BHSES, a boundary region of BHSES w.r.t $\mathfrak{Z}_1 \subset \mathfrak{Z}$, written by $(\pi, \mathfrak{H})_\bullet$, is presented as

$$(\pi, \mathfrak{H})_\bullet = \left((\pi, \mathfrak{H}) \bigwedge_{\mathcal{R}el} \mathfrak{Z}_1 \right) \setminus \left((\pi, \mathfrak{H}) \bigwedge_{\mathcal{R}} \mathfrak{Z}_1 \right)$$

Example 3.7. Taking 3.6 for $\mathfrak{Z}$ and $\mathfrak{Z}_1$, we have

$$(\pi, \mathfrak{H}) = \left\{ \ (\varpi_1, \{F_1, F_6\}), (\varpi_2, \{F_3, F_5\}), (\varpi_3, \{F_2, F_4\}) \ \right\}$$

. Now

$$(\pi, \mathfrak{H}) \bigwedge_{\mathcal{R}el} \mathfrak{Z}_1 = \{F_1, F_6\} \cup \{F_3, F_5\} \cup \{F_2, F_4\} = \{F_1, F_2, F_3, F_4, F_5 F_6\}$$

and

$$(\pi, \mathfrak{H}) \bigwedge_{\mathcal{R}} \mathfrak{Z}_1 = \{F_1, F_6\} \cup \{F_3, F_5\} = \{F_1, F_3, F_5, F_6\}$$

therefore

$$(\pi, \mathfrak{H})_{\bullet} = \{F_2, F_6\}$$

Definition 14. Suppose $(\pi_1, \mathfrak{H}_1)$ and $(\pi_2, \mathfrak{H}_2)$ are two BHSES with $\mathfrak{H}_1 \cap \mathfrak{H}_2 = \emptyset$, then $(\pi_1, \mathfrak{H}_1)$ is said to depend on $(\pi_2, \mathfrak{H}_2)$ with degree $\chi \in [0, 1]$, shown by $(\pi_1, \mathfrak{H}_1) \underset{\chi}{\Rightarrow} (\pi_2, \mathfrak{H}_2)$, if

$$\chi = \daleth((\pi_1, \mathfrak{H}_1), (\pi_2, \mathfrak{H}_2)) = \frac{\left| \bigcup_{\varpi \in \mathfrak{H}_2} \{(\pi_1, \mathfrak{H}_1) \bigwedge_{\mathcal{R}} \pi_2(\varpi)\} \right|}{|\mathfrak{Z}|}$$

such that $| \cdot | =$ shows the cardinality of a set.
Note:

(i) If $\chi = 1$ so $(\pi_1, \mathfrak{H}_1)$ is full depended on $(\pi_2, \mathfrak{H}_2)$.
(ii) If $\chi = 0$ so $(\pi_1, \mathfrak{H}_1)$ is not depended on $(\pi_2, \mathfrak{H}_2)$.

Example 3.8. Taking 3.6 for $\mathfrak{Z}$, we have

$$(\pi_1, \mathfrak{H}_1) = \left\{ \begin{array}{l} (\varpi_1, \{F_1\}), (\varpi_2, \{F_3\}), (\varpi_3, \{F_6\}), \\ (\varpi_4, \{F_5\}), (\varpi_5, \{F_2\}), (\varpi_6, \{F_4\}) \end{array} \right\}$$

and

$$(\pi_2, \mathfrak{H}_2) = \left\{ \ (\varpi_7, \{F_1, F_2\}), (\varpi_8, \{F_3, F_4\}), (\varpi_9, \{F_5, F_6\}) \ \right\}$$

Now

$$(\pi_1, \mathfrak{H}_1) \bigwedge_{\mathcal{R}} \pi_2(\varpi_7) = \{F_1\} \cup \{F_2\} = \{F_1, F_2\}$$

$$(\pi_1, \mathfrak{H}_1) \bigwedge_{\mathcal{R}} \pi_2(\varpi_8) = \{F_3\} \cup \{F_4\} = \{F_3, F_4\}$$

$$(\pi_1, \mathfrak{H}_1) \bigwedge_{\mathcal{R}} \pi_2(\varpi_9) = \{F_5\}$$

therefore

$$\bigcup_{\varpi \in \mathfrak{H}_2} \{(\pi_1, \mathfrak{H}_1) \bigwedge_{\mathcal{R}} \pi_2(\varpi)\} = \{F_1, F_2, F_3, F_4, F_5, \}$$

with

$$\chi = \frac{5}{6} = 0.833$$

Definition 15. Suppose $(\pi, \mathfrak{H})$ and $(\mathfrak{L}, \mathfrak{S})$ are two BHSESs. This $((\pi, \mathfrak{H}), (\mathfrak{L}, \mathfrak{S}), \mathfrak{Z})$ is named as BHSES decision system over $\mathfrak{Z}$, shown by $\mathfrak{D}_{BHE}$, if

(i) $\exists$ a property HSES $(\pi, \mathfrak{H}) = \bigcup\limits_{i=1}^{n} (\pi_i, \mathfrak{H}_i) \; \forall \, (\pi_i, \mathfrak{H}_i) \in \Upsilon_{BHSES}$ with $\mathfrak{H}_i \cap \mathfrak{H}_j = \emptyset, \;\; i \neq j$

(ii) $\exists$ a decision HSES $(\mathfrak{L}, \mathfrak{S})$ for which $\mathfrak{S} \cap \mathfrak{H}_i = \emptyset$.

Example 3.9. Consider 3.6 for $\mathfrak{Z}$, we have

$$(\pi_1, \mathfrak{H}_1) = \left\{ \; (\varpi_1, \{F_1\}), (\varpi_2, \{F_2\}), (\varpi_3, \{F_3\}) \; \right\}$$
$$(\pi_2, \mathfrak{H}_2) = \left\{ \; (\varpi_4, \{F_1, F_3\}), (\varpi_5, \{F_2, F_5\}), (\varpi_6, \{F_4, F_6\}) \; \right\}$$
$$(\pi_3, \mathfrak{H}_3) = \left\{ \; (\varpi_7, \{F_1, F_2, F_4\}), (\varpi_8, \{F_3, F_5, F_6\}) \; \right\}$$

and

$$(\mathfrak{L}, \mathfrak{S}) = \left\{ \; (\varpi_{10}, \{F_1, F_3, F_5\}), (\varpi_{11}, \{F_2, F_4\}), (\varpi_{12}, \{F_6\}) \; \right\}$$

therefore

$$\mathfrak{D}_{BHE} = (\bigcup_{i=1}^{3} (\pi_i, \mathfrak{H}_i), (\mathfrak{X}, \mathfrak{S}), \mathfrak{Z})$$

Definition 16. The BHSE dependency between $(\pi_1, \mathfrak{H}_1) \bigwedge (\pi_2, \mathfrak{H}_2) \bigwedge .... \bigwedge (\pi_m, \mathfrak{H}_m)$ and $(\mathfrak{X}, \mathfrak{S})$ is called BHSE decision system dependency of $\mathfrak{D}_{BHE}$ and represented by $\chi = \urcorner \left(\bigwedge\limits_{i=1}^{m} (\pi_i, \mathfrak{H}_i), (\mathfrak{X}, \mathfrak{S}) \right)$.

Example 3.10. Considering the $\mathfrak{Z}$ from Example 3.6, Let we have

$$(\pi_1, \mathfrak{H}_1) = \left\{ \; (\varpi_1, \{F_1, F_4, F_6\}), (\varpi_2, \{F_2, F_5\}) \; \right\}$$
$$(\pi_2, \mathfrak{H}_2) = \left\{ \; (\varpi_3, \{F_1, F_5, F_6\}), (\varpi_4, \{F_4\}) \; \right\}$$
$$(\pi_3, \mathfrak{H}_3) = \left\{ \; (\varpi_5, \{F_1, F_2, F_4\}), (\varpi_6, \{F_3, F_5, F_6\}) \; \right\}$$
$$(\mathfrak{X}, \mathfrak{S}) = \left\{ \; (\varpi_7, \{F_1, F_4, F_5, F_6\}), (\varpi_8, \{F_2, F_3\}) \; \right\}$$

then

$(\pi, \mathfrak{H}) = (\pi_1, \mathfrak{H}_1) \bigwedge (\pi_2, \mathfrak{H}_2) \bigwedge (\pi_3, \mathfrak{H}_3) =$

$$\left\{ \begin{array}{l} (\varepsilon_1 = (\varpi_1, \varpi_3, \varpi_5), \{F_1\}), (\varepsilon_2 = (\varpi_1, \varpi_3, \varpi_6), \{F_6\}), (\varepsilon_3 = (\varpi_1, \varpi_3, \varpi_7), \{F_1, F_6\}) \\ (\varepsilon_4 = (\varpi_1, \varpi_4, \varpi_5), \{F_4\}), (\varepsilon_5 = (\varpi_1, \varpi_4, \varpi_6), \emptyset), (\varepsilon_6 = (\varpi_1, \varpi_4, \varpi_7), \{F_4\}) \\ (\varepsilon_7 = (\varpi_2, \varpi_3, \varpi_5), \emptyset), (\varepsilon_8 = (\varpi_2, \varpi_3, \varpi_6), \{F_5\}), (\varepsilon_9 = (\varpi_2, \varpi_3, \varpi_7), \{F_4\}) \\ (\varepsilon_{10} = (\varpi_2, \varpi_4, \varpi_5), \emptyset), (\varepsilon_{11} = (\varpi_2, \varpi_4, \varpi_6), \emptyset), (\varepsilon_{12} = (\varpi_2, \varpi_4, \varpi_7), \emptyset) \\ (\varepsilon_{13} = (\varpi_2, \varpi_4, \varpi_8), \emptyset), (\varepsilon_{14} = (\varpi_1, \varpi_3, \varpi_8), \emptyset) \\ (\varepsilon_{15} = (\varpi_1, \varpi_4, \varpi_8), \emptyset), (\varepsilon_{16} = (\varpi_2, \varpi_3, \varpi_8), \emptyset) \end{array} \right\}.$$

The tabular form of $(\pi_1, \mathfrak{H}_1) \bigwedge (\pi_2, \mathfrak{H}_2) \bigwedge (\pi_3, \mathfrak{H}_3)$ is shown in TABLE 3.

Now

$\left(\bigwedge\limits_{i=1}^{3} (\pi_i, \mathfrak{H}_i) \bigwedge_{\mathcal{R}} (\mathfrak{X}, \mathfrak{S}) \right) = \{F_1, F_4, F_5, F_6\}$

therefore

$\chi = \urcorner \left(\bigwedge\limits_{i=1}^{3} (\pi_i, \mathfrak{H}_i), (\mathfrak{X}, \mathfrak{S}) \right) = \frac{4}{6} = 0.666$

....	F_1	F_2	F_3	F_4	F_5	F_6
$\pi(\varepsilon_1)$	$\oplus$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$
$\pi(\varepsilon_2)$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\oplus$
$\pi(\varepsilon_3)$	$\oplus$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\oplus$
$\pi(\varepsilon_4)$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\oplus$	$\widehat{\otimes}$	$\widehat{\otimes}$
$\pi(\varepsilon_5)$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$
$\pi(\varepsilon_6)$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\oplus$	$\widehat{\otimes}$	$\widehat{\otimes}$
$\pi(\varepsilon_7)$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$
$\pi(\varepsilon_8)$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\oplus$	$\widehat{\otimes}$
$\pi(\varepsilon_9)$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\oplus$	$\widehat{\otimes}$	$\widehat{\otimes}$
$\pi(\varepsilon_{10})$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$
$\pi(\varepsilon_{11})$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$
$\pi(\varepsilon_{12})$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$	$\widehat{\otimes}$

TABLE 3. $(\pi_1, \mathfrak{H}_1) \bigwedge (\pi_2, \mathfrak{H}_2) \bigwedge (\pi_3, \mathfrak{H}_3)$ in table form

Theorem 3.11. *Suppose $\mathfrak{D}_{BHE} = ((\pi, \mathfrak{H}), (\mathfrak{X}, \mathfrak{S}), \mathfrak{Z})$, where $(\pi, \mathfrak{H}) = \bigcup\limits_{i=1}^{n} (\pi_i, \mathfrak{H}_i)$ and $(\pi_i, \mathfrak{H}_i) \in \Upsilon_{BHSES}$. If $\chi = \urcorner \left(\bigwedge\limits_{i=1}^{n} (\pi_i, \mathfrak{H}_i), (\mathfrak{X}, \mathfrak{S}) \right)$ and $\chi_1 = \urcorner \left(\bigwedge\limits_{i=1}^{m} (\pi_i, \mathfrak{H}_i), (\mathfrak{X}, \mathfrak{S}) \right)$ with $m \leq n$ then $\chi_1 \leq \chi$.*

Proof. Suppose that $(\mathfrak{P}, \mathfrak{C}) = \bigwedge_{i=1}^{n} (\pi_i, \mathfrak{H}_i), (\mathfrak{J}, \mathcal{K}) = \bigwedge_{i=1}^{m} (\pi_i, \mathfrak{H}_i)$ then we have

$$\chi = \urcorner \left(\bigwedge_{i=1}^{n} (\pi_i, \mathfrak{H}_i), (\mathfrak{X}, \mathfrak{S}) \right) = \frac{\left| \bigcup\limits_{\varepsilon \in \mathfrak{S}} (\mathfrak{P}, \mathfrak{C}) \bigwedge_{\mathcal{R}} \mathfrak{X}(\varepsilon) \right|}{|\mathfrak{Z}|} = \frac{\left| \bigcup\limits_{\varepsilon \in \mathfrak{S}} \bigcup\limits_{\lambda \in \mathfrak{C}} \{\mathfrak{P}(\lambda) : \mathfrak{P}(\lambda) \subseteq \mathfrak{X}(\varepsilon)\} \right|}{|\mathfrak{Z}|}$$

$$\chi_1 = \urcorner \left(\bigwedge_{i=1}^{m} (\pi_i, \mathfrak{H}_i), (\mathfrak{X}, \mathfrak{S}) \right) = \frac{\left| \bigcup\limits_{\varepsilon \in \mathfrak{S}} (\mathfrak{J}, \mathcal{K}) \bigwedge_{\mathcal{R}} \mathfrak{X}(\varepsilon) \right|}{|\mathfrak{Z}|} = \frac{\left| \bigcup\limits_{\varepsilon \in \mathfrak{S}} \bigcup\limits_{\lambda \in \mathcal{K}} \{\mathfrak{J}(\lambda) : \mathfrak{J}(\lambda) \subseteq \mathfrak{X}(\varepsilon)\} \right|}{|\mathfrak{Z}|}.$$

From Definition 2.6,

$$\mathfrak{P}(\varepsilon_1, \phi_2, .., \phi_n) = \pi_1(\phi_1) \cap \pi_2(\phi_2) \cap \cdot\cdot \cap \pi_m(\phi_m) \cap \cdot\cdot \cap \pi_n(\phi_n), \forall (\phi_1, \phi_2, .., \phi_n) \in \mathfrak{H}_1 \times \mathfrak{H}_2 \times \cdots \times \mathfrak{H}_n$$

$$\mathfrak{J}(\phi_1, \phi_2, .., \phi_m) = \pi_1(\phi_1) \cap \pi_2(\phi_2) \cap \cdot\cdot \cap \pi_m(\phi_m), \ \forall (\phi_1, \phi_2, .., \phi_m) \in \mathfrak{H}_1 \times \mathfrak{H}_2 \times \cdots \times \mathfrak{H}_m$$

$n > m$

$$\mathfrak{P}(\phi_1, \phi_2, .., \phi_n) \supseteq \mathfrak{J}(\phi_1, \phi_2, .., \phi_m)$$

and

$$\bigcup_{\phi \in \mathfrak{C}} \mathfrak{P}(\phi) = \mathfrak{Z}, \ \bigcup_{\phi \in \mathcal{K}} \mathfrak{J}(\phi) = \mathfrak{Z}.$$

Therefore,

$$\left|\underset{\phi\in\mathfrak{C}}{\cup}\{\mathfrak{P}(\phi):\mathfrak{P}(\phi)\subseteq\mathfrak{S}(\phi)\}\right|\geq\left|\underset{\phi\in\mathcal{K}}{\cup}\{\mathfrak{I}(\phi):\mathfrak{I}(\phi)\subseteq\mathfrak{S}(\phi)\}\right|.$$

$$\daleth\left(\bigwedge_{i=1}^{m}(\pi_i,\mathfrak{H}_i),(\mathfrak{X},\mathfrak{S})\right)\leq\chi.$$

□

Definition 17. Suppose $\mathfrak{D}_{BHE}=((\pi,\mathfrak{H}),(\mathfrak{X},\mathfrak{S}),\mathfrak{Z})$, where $(\pi,\mathfrak{H})=\overset{n}{\underset{i=1}{\cup}}(\pi_i,\mathfrak{H}_i)$ and $\overset{m}{\underset{i=1}{\cup}}(\pi_i,\mathfrak{H}_i)\subset(\pi,\mathfrak{H})$. If $\daleth\left(\bigwedge_{i=1}^{n}(\pi_i,\mathfrak{H}_i),(\mathfrak{X},\mathfrak{S})\right)=\daleth\left(\bigwedge_{i=1}^{m}(\pi_i,\mathfrak{H}_i),(\mathfrak{X},\mathfrak{S})\right)=\chi$ then $\overset{m}{\underset{i=1}{\cup}}(\pi_i,\mathfrak{H}_i)$ is named as a reduct of $\mathfrak{D}_{BHE}$.

Example 3.12. Using 3.6 for $\mathfrak{Z}$ and 3.10 for sets , we see

$$(\pi_1,\mathfrak{H}_1)=\left\{\ (\varpi_1,\{F_1,F_4,F_6\}),(\varpi_2,\{F_2,F_5\})\ \right\}$$

$$(\pi_2,\mathfrak{H}_2)=\left\{\ (\varpi_3,\{F_1,F_5,F_6\}),(\varpi_4,\{F_4\})\ \right\}$$

$$(\pi_3,\mathfrak{H}_3)=\left\{\ (\varpi_5,\{F_1,F_2,F_4\}),(\varpi_6,\{F_3,F_5,F_6\})\ \right\}$$

$$(\mathfrak{X},\mathfrak{S})=\left\{\ (\varpi_7,\{F_1,F_4,F_5,F_6\}),(\varpi_8,\{F_2,F_3\})\ \right\}$$

then

$$(\pi_1,\mathfrak{H}_1)\bigwedge(\pi_2,\mathfrak{H}_2)=\left\{\begin{array}{l}(\phi_1=(\varpi_1,\varpi_3),\{F_1,F_6\}),(\phi_2=(\varpi_1,\varpi_4),\{F_4\}),\\(\phi_3=(\varpi_2,\varpi_3),\{F_5\})\,(\phi_4=(\varpi_2,\varpi_4),\emptyset)\end{array}\right\}.$$

Now
$\left(\bigwedge_{i=1}^{2}(\pi_i,\mathfrak{H}_i)\bigwedge_{\mathcal{R}}(\mathfrak{X},\mathfrak{S})\right)=\{F_1,F_4,F_5,F_6\}$
therefore
$\chi=\daleth\left(\bigwedge_{i=1}^{2}(\pi_i,\mathfrak{H}_i),(\mathfrak{X},\mathfrak{S})\right)=\frac{4}{6}=0.666$ which is similar to $\daleth\left(\bigwedge_{i=1}^{3}(\pi_i,\mathfrak{H}_i),(\mathfrak{X},\mathfrak{S})\right)$ measured in Example 3.10. So $(\pi_1,\mathfrak{H}_1)\cup(\pi_2,\mathfrak{H}_2)$ is a reduct of $\mathfrak{D}_{BHE}$.

Definition 18. Suppose $\mathfrak{D}_{BHE}=(\overset{n}{\underset{i=1}{\cup}}(\pi_i,\mathfrak{H}_i),(\mathfrak{X},\mathfrak{S}),\mathfrak{Z})$. The significance of BHSES to decision HSES, shown $\varpi((\pi_j,\mathfrak{H}_j),\overset{n}{\underset{i=1}{\cup}}(\pi_i,\mathfrak{H}_i),(\mathfrak{X},\mathcal{G}))$, is presented as

$$\varpi((\pi_j,\mathfrak{H}_j),\overset{n}{\underset{i=1}{\cup}}(\pi_i,\mathfrak{H}_i),(\mathfrak{X},\mathfrak{S}))=\chi-\daleth((\mathfrak{P},\mathfrak{C}),(\mathfrak{X},\mathfrak{S})),$$

where $(\mathfrak{P},\mathfrak{C})=\bigwedge_{i=1}^{n}(\pi_i,\mathfrak{H}_i)(i\neq j)$.

Example 3.13. Taking 3.10, we have
$\chi=\daleth\left(\bigwedge_{i=1}^{3}(\pi_i,\mathfrak{H}_i),(\mathfrak{X},\mathfrak{S})\right)=\frac{4}{6}=0.666$ and

$$(\pi_2,\mathfrak{H}_2)\bigwedge(\pi_3,\mathfrak{H}_3)=\left\{\begin{array}{l}(\phi_1=(\varpi_3,\varpi_5),\{F_1\}),(\phi_2=(\varpi_3,\varpi_6),\{F_5,F_6\}),\\(\phi_4=(\varpi_4,\varpi_5),\{F_4\}),(\phi_5=(\varpi_4,\varpi_6),\{\})\end{array}\right\}.$$

Now
$\left(\bigwedge_{i=2}^{3}(\pi_i,\mathfrak{H}_i)\bigwedge_{\mathcal{R}}(\mathfrak{X},\mathfrak{S})\right)=\{F_1,F_4,F_5,F_6\}$
therefore
$\daleth\left(\bigwedge_{i=2}^{3}(\pi_i,\mathfrak{H}_i),(\mathfrak{X},\mathfrak{S})\right)=\frac{4}{6}=0.666$ hence

$$\varpi((\pi_1,\mathfrak{H}_1),\widetilde{\bigcup}_{i=1}^{3}(\pi_i,\mathfrak{H}_i),(\mathfrak{X},\mathfrak{S}))=\chi-\daleth\left(\bigwedge_{i=2}^{3}(\pi_i,\mathfrak{H}_i),(\mathfrak{X},\mathfrak{S})\right)=0.666-0.666=0$$

Definition 19. A BHSES $(\mathfrak{P},\mathfrak{C})$ is named as a core BHSES of $\mathfrak{D}_{BHE}$ when it $\in$ reduct of $\mathfrak{D}_{BHE}$.

Definition 20. Suppose $\mathfrak{D}_{BHE}=((\pi,\mathfrak{H}),(\mathfrak{X},\mathfrak{S}),\mathfrak{Z})$, where $(\pi,\mathfrak{H})=\bigcup_{i=1}^{n}(\pi_i,\mathfrak{H}_i)$ and $\bigcup_{i=1}^{m}(\pi_i,\mathfrak{H}_i)\subset(\pi,\mathfrak{H})$ is a reduct of $\mathfrak{D}_{BHE}$. Let $(\mathfrak{P},\mathfrak{C})=\bigwedge_{i=1}^{m}(\pi_i,\mathfrak{H}_i)$. We say

$$if\ \ e_i\ \ ,then\ \ e_j\left(\frac{|\mathfrak{P}(e_i)|}{|\mathfrak{X}(e_j)|}\right)$$

a decision rule induced by $\bigcup_{i=1}^{m}(\pi_i,\mathfrak{H}_i)$ where $e_i\in\mathfrak{C}$, $\mathfrak{X}(e_j)\supseteq\mathfrak{P}(e_i)$, $e_j\in\mathfrak{S}$ and $\frac{|\mathfrak{P}(e_i)|}{|\mathfrak{X}(e_j)|}$ presents the coverage proportion rule.

Example 3.14. Taking 3.12, we have

$$(\pi_1,\mathfrak{H}_1)\bigwedge(\pi_2,\mathfrak{H}_2)=\left\{\begin{array}{l}(\phi_1=(\varpi_1,\varpi_3),\{F_1,F_6\}),(\phi_2=(\varpi_1,\varpi_4),\{F_4\}),\\(\phi_3=(\varpi_2,\varpi_3),\{F_5\})\,(\phi_4=(\varpi_2,\varpi_4),\emptyset)\end{array}\right\}.$$

Now

(i) If ϕ_1 then $\varpi_7(2/4)$
(ii) If ϕ_2 then $\varpi_7(1/4)$
(iii) If ϕ_3 then $\varpi_7(1/4)$
(iv) If ϕ_4 then $\varpi_7(0/4)$

4. An application of Bijective Hypersoft Expert set

This section presents an application of bijective hypersoft expert set to describe the decision rules.

Example 4.1. Suppose that one of the immediate selling organizations wishes to assess eight products from a producer and pick the most appropriate product for it to advertise. Let there are eight products which form the universe of discourse $Q=\{o_1,o_2,...,o_8\}$ with expert set $P=\{p_1,p_2,p_3\}$. Let $E=\{w_1,w_2,w_3,w_4,w_5\}$ be the set of parameters which stand for
w_1 = Price =$\{low=\tilde{\mathfrak{p}}_1,high=\tilde{\mathfrak{p}}_2\}$
w_2 = Effectiveness = $\{more=\tilde{\mathfrak{p}}_3,less=\tilde{\mathfrak{p}}_4\}$
w_3 = Date of expire = $\{ok=\tilde{\mathfrak{p}}_5,notok=\tilde{\mathfrak{p}}_6\}$
w_4 = Utilization = $\{more=\tilde{\mathfrak{p}}_7,less=\tilde{\mathfrak{p}}_8\}$
w_5 = Quality = $\{good=\tilde{\mathfrak{p}}_9,better=\tilde{\mathfrak{p}}_{10}\}$
and then

$\tilde{\mathfrak{p}} = \tilde{\mathfrak{p}}_1 \times \tilde{\mathfrak{p}}_2 \times \tilde{\mathfrak{p}}_3 \times \tilde{\mathfrak{p}}_4 \times \tilde{\mathfrak{p}}_5$

$$\tilde{\mathfrak{p}} = \left\{ \begin{array}{l} (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_9), (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_{10}), (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_9), (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_{10}), (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_9), \\ (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_{10}), (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_9), (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_{10}), (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_9), (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_{10}), \\ (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_9), (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_{10}), (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_9), (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_{10}), (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_9), \\ (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_{10}), (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_9), (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_{10}), (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_9), (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_{10}), \\ (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_9), (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_{10}), (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_9), (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_{10}), (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_9), \\ (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_{10}), (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_9), (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_{10}), (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_9), (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_{10}), \\ (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_9), (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_{10}) \end{array} \right\}$$

and now take $\mathfrak{K} \subseteq \mathfrak{L}$ as

$\mathfrak{K} = \{\varpi_1 = (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_5, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_9), \varpi_2 = (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_{10}), \varpi_3 = (\tilde{\mathfrak{p}}_1, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_9), \varpi_4 = (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_3, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_8, \tilde{\mathfrak{p}}_9), \varpi_5 = (\tilde{\mathfrak{p}}_2, \tilde{\mathfrak{p}}_4, \tilde{\mathfrak{p}}_6, \tilde{\mathfrak{p}}_7, \tilde{\mathfrak{p}}_{10})\}$

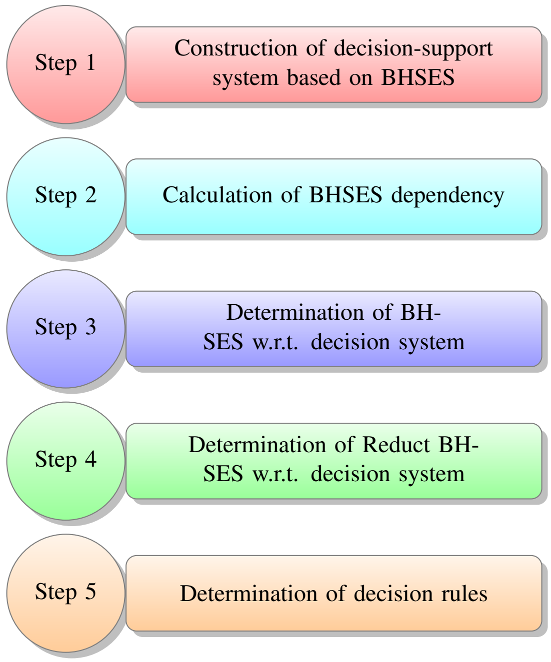
and

$$(\xi, \mathfrak{K}) = \left\{ \begin{array}{l} ((\varpi_1, P_1, 1), \{o_1, o_2, o_3\}), ((\varpi_1, P_2, 1), \{o_1, o_4, o_5, o_8\}), \\ ((\varpi_1, P_3, 1), \{o_1, o_3, o_4, o_5, o_6, o_7, o_8\}), ((\varpi_2, P_1, 1), \{o_1, o_3, o_5, o_8\}), \\ ((\varpi_2, P_2, 1), \{o_1, o_3, o_4, o_5, o_6, o_8\}), ((\varpi_2, P_3, 1), \{o_1, o_2, o_4, o_5, o_6, o_8\}), \\ ((\varpi_3, P_1, 1), \{o_1, o_4, o_5, o_7\}), ((\varpi_3, P_2, 1), \{o_1, o_2, o_5, o_8\}), \\ ((\varpi_3, P_3, 1), \{o_1, o_3, o_5, o_8\}), ((\varpi_4, P_1, 1), \{o_1, o_7, o_8\}), \\ ((\varpi_4, P_2, 1), \{o_1, o_4, o_5, o_8\}), ((\varpi_4, P_3, 1), \{o_1, o_6, o_7, o_8\}), \\ ((\varpi_5, P_1, 1), \{o_1, o_3, o_4, o_5, o_7, o_8\}), ((\varpi_5, P_2, 1), \{o_1, o_4, o_5, o_6, o_8\}), \\ ((\varpi_5, P_3, 1), \{o_1, o_3, o_4, o_5, o_6, o_7, o_8\}), ((\varpi_1, P_1, 0), \{o_1, o_6, o_7, o_8\}), \\ ((\varpi_1, P_2, 0), \{o_2, o_3, o_6, o_7, o_8\}), ((\varpi_1, P_3, 0), \{o_1, o_5\}), \\ ((\varpi_2, P_1, 0), \{o_1, o_2, o_4, o_5, o_6\}), ((\varpi_2, P_2, 0) o_1, o_7\}, \\ ((\varpi_2, P_3, 0), \{o_1, o_3, o_4, o_5, o_6, o_7\}), ((\varpi_3, P_1, 0), \{o_1, o_2, o_6, o_8\}), \\ ((\varpi_3, P_2, 0), \{o_3, o_4, o_6, o_7\}), ((\varpi_3, P_3, 0), \{o_1, o_3, o_4, o_5, o_7\}), \\ ((\varpi_4, P_1, 0), \{o_1, o_2, o_3, o_4, o_5, o_7\}), ((\varpi_4, P_2, 0), \{o_2, o_3, o_6, o_7\}), \\ ((\varpi_4, P_3, 0), \{o_1, o_3, o_4, o_5\}), ((\varpi_5, P_1, 0), \{o_1, o_6\}), \\ ((\varpi_5, P_2, 0), \{o_1, o_2, o_6, o_7\}), ((\varpi_5, P_3, 0), \{o_1, o_4, o_6\}), \end{array} \right\}$$

is a hypersoft expert set.

Here an algorithm of bijective hypersoft expert sets is presented for the establishment of decision rules.

Proposed Algorithm for Optimal Selection of Surgical Mask



Step 1
suppose there are the following bijective hypersoft expert sets

$$(\mathcal{M}_1,\mathcal{N}_1)=\left\{\ (\varpi_1,\{o_1,o_2,\}),(\varpi_2,\{o_1,o_3,o_8\})\ \right\}$$
$$(\mathcal{M}_2,\mathcal{N}_2)=\left\{\ (\varpi_3,\{o_1,o_5,o_7\}),(\varpi_4,\{o_4,o_5,o_8\})\ \right\}$$
$$(\mathcal{M}_3,\mathcal{N}_3)=\left\{\ (\varpi_5,\{o_1,o_2,o_7\}),(\varpi_6,\{o_3,o_6\}),(\varpi_7,\{o_6,o_7\})\ \right\}$$
$$(\mathfrak{X},\Theta)=\left\{\ (\varpi_8,\{o_1,o_4,o_5,o_6\}),(\varpi_9,\{o_2,o_3,o_7,o_8\})\ \right\}$$

which form $\mathfrak{D}_{BHE}=(\bigcup_{i=1}^{3}(\mathcal{M}_i,\mathcal{N}_i),(\mathfrak{X},\Theta),\mathfrak{Z})$ and $\mathcal{N}_i,\Theta\subseteq\mathbb{L}$.

Step 2
Since

$$(\mathcal{M}_1,\mathcal{N}_1)\bigwedge(\mathcal{M}_2,\mathcal{N}_2)=\left\{\begin{array}{l}(F_1=(\varpi_1,\varpi_3),\{o_1\}),(F_2=(\varpi_1,\varpi_4),\{o_8\}),\\(F_3=(\varpi_2,\varpi_3),\{o_1\}),(F_4=(\varpi_2,\varpi_4),\{o_8\})\end{array}\right\}$$

and

$$(\mathcal{M}_2,\mathcal{N}_2)\bigwedge(\mathcal{M}_3,\mathcal{N}_3)=\left\{\begin{array}{l}(F_5=(\varpi_3,\varpi_5),\{o_1,o_7\}),(F_6=(\varpi_3,\varpi_6),\emptyset),(F_7=(\varpi_3,\varpi_7),\{o_7\})\\(F_8=(\varpi_4,\varpi_5),\emptyset),(F_9=(\varpi_4,\varpi_6),\emptyset),(F_{10}=(\varpi_4,\varpi_7),\emptyset)\end{array}\right\}$$

and

$$(\mathcal{M}_1,\mathcal{N}_1)\bigwedge(\mathcal{M}_3,\mathcal{N}_3)=\left\{\begin{array}{l}(F_{11}=(\varpi_1,\varpi_5),\{o_1,o_2\}),(F_{12}=(\varpi_1,\varpi_6),\emptyset),(F_{13}=(\varpi_1,\varpi_7),\emptyset)\\(F_{14}=(\varpi_2,\varpi_5),\{o_1\}),(F_{15}=(\varpi_2,\varpi_6),\{o_3\}),(F_{16}=(\varpi_2,\varpi_7),\{o_4\})\end{array}\right\}$$

Now
$\chi_1=\daleth((\mathcal{M}_1,\mathcal{N}_1),(\mathfrak{X},\Theta))=\frac{4}{8}=0.5$
$\chi_2=\daleth((\mathcal{M}_2,\mathcal{N}_2),(\mathfrak{X},\Theta))=\frac{5}{8}=0.625$

$\chi_3 = \urcorner ((\mathcal{M}_3, \mathcal{N}_3), (\mathfrak{X}, \Theta)) = \frac{5}{8} = 0.625$
$\chi_4 = \urcorner ((\mathcal{M}_1, \mathcal{N}_1) \bigwedge (\mathcal{M}_2, \mathcal{N}_2), (\mathfrak{X}, \Theta)) = \frac{2}{8} = 0.5$
$\chi_5 = \urcorner ((\mathcal{M}_2, \mathcal{N}_2) \bigwedge (\mathcal{M}_3, \mathcal{N}_3), (\mathfrak{X}, \Theta)) = \frac{3}{8} = 0.375$
$\chi_6 = \urcorner ((\mathcal{M}_1, \mathcal{N}_1) \bigwedge (\mathcal{M}_3, \mathcal{N}_3), (\mathfrak{X}, \Theta)) = \frac{4}{8} = 0.5$

Step 3

$(\mathcal{M}_1, \mathcal{N}_1) \bigwedge (\mathcal{M}_2, \mathcal{N}_2) \bigwedge (\mathcal{M}_3, \mathcal{N}_3) =$

$$\left\{ \begin{array}{l} (F_1 = (\varpi_1, \varpi_3, \varpi_5), \{o_1\}), (F_2 = (\varpi_1, \varpi_3, \varpi_6), \emptyset), (F_3 = (\varpi_1, \varpi_3, \varpi_7), \emptyset) \\ (F_4 = (\varpi_1, \varpi_4, \varpi_5), \emptyset), (F_5 = (\varpi_1, \varpi_4, \varpi_6), \emptyset), (F_6 = (\varpi_1, \varpi_4, \varpi_7), \emptyset) \\ (F_7 = (\varpi_2, \varpi_3, \varpi_5), \{o_1\}), (F_8 = (\varpi_2, \varpi_3, \varpi_6), \emptyset), (F_9 = (\varpi_2, \varpi_3, \varpi_7), \{o_3\}) \\ (F_{10} = (\varpi_2, \varpi_4, \varpi_5), \emptyset), (F_{11} = (\varpi_2, \varpi_4, \varpi_6), \emptyset), (F_{12} = (\varpi_2, \varpi_4, \varpi_7), \emptyset) \end{array} \right\}.$$

therefore

$\chi = \urcorner \left(\bigwedge_{i=1}^{3} (\mathcal{M}_i, \mathcal{N}_i), (\mathfrak{X}, \Theta) \right) = \frac{2}{8} = 0.5$

Step 4

As

$$\urcorner \Big((\mathcal{M}_1, \mathcal{N}_1) \bigwedge (\mathcal{M}_2, \mathcal{N}_2), (\mathfrak{X}, \Theta) \Big) = 0.5 = \urcorner \left(\bigwedge_{i=1}^{3} (\mathcal{M}_i, \mathcal{N}_i), (\mathfrak{X}, \Theta) \right)$$

therefore $(\mathcal{M}_1, \mathcal{N}_1) \cup (\mathcal{M}_2, \mathcal{N}_2)$ is a reduct of $\mathfrak{D}_{BHE}$.

Step 5

Since $(\mathcal{M}_1, \mathcal{N}_1) \cup (\mathcal{M}_2, \mathcal{N}_2)$ is a reduct of $\mathfrak{D}_{BHE}$ so, decision rules w.r.t. $\mathfrak{D}_{BHE}$

(i) If F_1 then $\varpi_8(1/4)$
(ii) If F_2 then $\varpi_9(1/4)$
(iii) If F_3 then $\varpi_8(1/4)$
(iv) If F_4 then $\varpi_9(1/4)$

Here ϖ_8 and ϖ_9 have same values, so both are valuable for further evaluation.

4.2. **Comparative study.** In this subsection, we compare our proposed structure with the existing studies.

5. CONCLUSION

The paper is summarized as under

(1) Axiomatic properties, set-theoretic operations and laws of bijective hypersoft expert set are conceptualized with the support of numerical illustrative examples.
(2) A novel decision-support system is constructed with the help of some special type of aggregation operations like relaxed and restricted AND, dependency etc.
(3) A decision-making based daily-life problem is discussed with the help of an algorithm based on aggregation of bijective hypersoft expert set.
(4) The advantageous aspects of proposed study are judged through comparison with existing relevant models.

Following models may be developed by extending this study:

- Bijective fuzzy hypersoft expert set
- Bijective intuitionistic fuzzy hypersoft expert set
- Bijective Pythagorean fuzzy hypersoft expert set

TABLE 4. Comparison of proposed study with existing relevant models

Authors	Structures	Remarks
H. Kamacı et al. [13]	Bijective soft matrix theory	• Single set of attributes is employed to develop decision system via bijection on matrix theory • Multi-bijective linguistic soft decision system is established
Gong et al. [27]	Bijective soft set	• Single set of attributes is employed to develop decision system
Rahman et al. [52]	Bijective hypersoft set	• Attributes are further classified into disjoint attribute-valued sets • Decision system is developed via employment of multi-argument approximate functions.
Proposed structure	Bijective hypersoft expert set	• Attributes are further classified into disjoint attribute-valued sets • Decision system is developed via employment of multi-argument approximate functions • Multi Decisive opinion is being used to get the required result.

- Bijective picture fuzzy hypersoft expert set
- Bijective neutrosophic hypersoft expert set and many other hybridized structures with their applications in decision-making, optimization and other fields of pure and applied mathematics.

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