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# Hypersoft game theory models and their applications in multi-criteria decision making

## Hiperesnek oyun teorisi modelleri ve çok kriterli karar vermede uygulamaları

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**Abstract**

*The classical game theory has been extended for soft set structures, and thus, soft game theory, fuzzy soft game theory, intuitionistic fuzzy soft game theory, neutrosophic soft game theory have been introduced. The payoff function in the soft game approaches is the set-valued function and allows the use of set operations to obtain solution, which makes it very convenient and easily applicable in practice. Also, in these game approaches, the strategies can be determined as attributes/parameters. That is, all these soft game theories are designed to manipulate parametric information using a single-attribute function. However, another powerful tool is needed to process parametric information obtained using multi-attribute function. To model such problems mathematically, the concept of hypersoft set has proposed. In this paper, a game theory model based on hypersoft set called hypersoft game theory is constructed. In this game theory, payoff function is the set-valued function and the strategies are chosen as multi-attributes. A two-person hypersoft game is developed and different solution methods (such as hypersoft saddle point method, hypersoft elimination method, hypersoft Nash equilibrium method) are produced for such games. Also, the proposed methods are successfully applied to game theory-based decision making problems that may be encountered in real life. Finally, the two-person hypersoft game is extended to the n-person hypersoft game. Nash equilibrium of an n-person hypersoft game is described and an application for this solution method is presented.*

**Keywords:** Hypersoft sets, Hyperpayoffs, two-person hypersoft games, n-person hypersoft games

**Öz**

*Klasik oyun teorisi esnek küme yapıları için genişletilmiştir ve böylece esnek oyun teorisi, bulanık esnek oyun teorisi, sezgisel bulanık esnek oyun teorisi, nötrosofik esnek oyun teorisi tanıtılmıştır. Esnek oyun yaklaşımlarında getiri fonksiyonu, küme-değerli fonksiyondur ve çözüm elde etmek için küme işlemlerinin kullanılmasına izin verir, bu da onu pratikte çok uygun ve kolay uygulanabilir kılar. Ayrıca bu oyun yaklaşımlarında stratejiler nitelikler/parametreler olarak belirlenebilir. Yani, tüm bu esnek oyun teorileri, tek-nitelikli fonksiyonları kullanarak parametrik bilgileri işlemek için tasarlanmıştır. Ancak, çok-nitelikli fonksiyon kullanılarak elde edilen parametrik bilgileri işlemek için başka bir güçlü araca ihtiyaç vardır. Bu tür problemleri matematiksel olarak modellemek için hiperesnek küme kavramı önerilmiştir. Bu makalede, hiperesnek oyun teorisi adı verilen hiperesnek kümeye dayalı bir oyun teorisi modeli oluşturulmuştur. Bu oyun teorisinde, getiri fonksiyonu küme-değerli fonksiyondur ve stratejiler çok-nitelikli olarak seçilir. İki kişilik bir hyperesnek oyun geliştirilmiş ve bu tür oyunlar için farklı çözüm yöntemleri (hiperesnek eyer noktası yöntemi, hiperesnek eliminasyon yöntemi, hiperesnek Nash dengesi yöntemi gibi) üretilmiştir. Ayrıca önerilen yöntemler, gerçek hayatta karşılaşılabilecek oyun teorisi tabanlı karar verme problemlerine başarılı bir şekilde uygulanmıştır. Son olarak, iki kişilik hiperesnek oyun n-kişilik hiperesnek oyuna genişletilmiştir. Bir n-kişilik hiperesnek oyunun Nash dengesi tanımlanmış ve bu çözüm yöntemi için bir uygulama sunulmuştur.*

**Anahtar kelimeler:** Hiperesnek kümeler, hiperesnek getiriler, iki kişilik hiperesnek oyunlar, n-kişilik hiperesnek oyunlar



## 1 Introduction

In the real world, more often the information that we collect from various sources is not precise. This happens due to a lack of knowledge, missing information, communication gap, vague concept, or some other related issues. With the progress of science and technology, the level of uncertainty of particular information increased and so an extensive study is needed to assess such kind of information with an effective mathematical tool. Various mathematical tools are introduced to encounter numerous types of uncertainty up to a certain level. A few of them are fuzzy set [1], intuitionistic fuzzy set [2], and neutrosophic set [3]. Furthermore, realizing the importance of mixing two or more tools, researchers introduced many hybrid models to handle uncertainty. Also, in recent years, fuzzy-based modifications of hierarchical process models used in decision making have been studied [4]-[7].

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The fuzzy sets are based on membership functions with some restricted criteria, but these sets do not provide parametric evaluations. To cope with these difficulties, Molodtsov [8] offered the soft sets theory in his research paper. The main advantage of introducing the soft set theory is that it is independent of choosing membership function and so it is flexible to handle any uncertain information parametrically. It provides an approximate description of a parameterized class of sets. Motivated by Molodtsov's soft set concept, Maji et al. [9] introduced new operations on soft sets. Çağman et al. [10] redefined the soft sets that are useful to improve the earlier results. Aygün et al. [11] and Riaz et al. [12] studied forms of soft set operations. In recent years, developments on the matrix representations of soft sets have attracted the attention of researchers because they provide convenience/practicality in calculations in some cases. [13]-[16].

Furthermore, the different extended types or hybridization of a soft set theory captures more attention from the researchers as it can accommodate more uncertain information than the ordinary soft set. Some of them are intuitionistic fuzzy soft sets [17], neutrosophic soft sets [18], interval-valued neutrosophic soft sets [19], Pythagorean fuzzy soft sets [20], three-valued soft sets [21], N-soft sets [22],[23], Pythagorean fuzzy N-soft sets [24], neutrosophic N-soft sets [25], VFP-soft sets [26], and etc.. Hypersoft set was introduced by Smarandache [27] as another extension of soft set. Hypersoft set is formed by changing single-attribute function in the sft set with a multi-attribute function and thus it looks more logical and handy to address complexity involved in some decision making problems. Also, the use of the multi-attribute function gives more options for the decision makers to assign as many attributes as they want to study more characteristics of an object. In recent years, considering the advantages of hypersoft sets, many studies have been carried out on these sets and seminal articles have been published [28]-[31].

Game theory is a versatile mathematical tool that is designed for not only scientific study of the actions or moves of players or teams when they interact, but to assess or analyze their strategies under certain restrictions, payoff the players, their proclivity, their knowledge, and some other associated factors. It is perceived a lot of attention to the researchers as it takes care of different conflicts of interest, which arise due to the hazy or noisy nature of strategies of the decision-makers. Here a player/decision-maker signifies an individual or a group; it depends on the physical nature of the game. Game theory has been applied successfully in social sciences, economics, applied sciences, operation research, political sciences, decision-making, etc. There may be cooperative/non-cooperative games and the strategies of the players may be pure /mixed. There exist zero-sum/non-zero-sum games. The main objective of introducing the theory of games is to control the unknown strategies that influence the outcome. Neumann and Morgenstern [32] are the pioneer of the origination of the theory of games and economic behavior and Nash [33] furthered it in the year 1954. A few years later, game theory earned its popularity among researchers and they developed it to a great extent. By embedding fuzzy sets in classical game theory, different types of fuzzy games are introduced [34]-[36]. In [37], the authors focused on the probabilistic linguistic matrix games based on the fuzzy theories. Some resarchers studied several methodologies on fuzzy cooperative games and fuzzy non-cooperative games [38], [39]. Li and Tu [40] developed a new method to solving general intuitionistic fuzzy matrix games, which takes into account the degree of acceptance that general intuitionistic fuzzy constraints can be violated. In [41]-[43], different approaches were proposed for solving matrix games with neutrosophic payoffs and their applications in environmental behavior were presented. Fuzzy/ intuistionistic/ neutrosophic matrix game models are based on fuzzy sets/ intuitionistic fuzzy sets/ neutrosophic sets resp. and these sets are characterized by the membership (M)/ non-mebership (NM)/indeterminacy (I) functions. We know that such game models are useful to address uncertainty but there exists difficulty to assign M/NM/I functions for each particular case. In the fuzzy/ intuistionistic/ neutrosophic games, the payoff functions are real valued and therefore the solutions of such games are obtained by using arithmetic operations. There exists a difficulty to set the M/NM/I functions in each particular case and also the fuzzy/intuitionistic fuzzy/neutrosophic set operations based on the arithmetic operations with M/NM/I functions seem unnatural due to the nature of the M/NM/I functions is generally individual. To remove such difficulty, Deli et al. introduced a new game called the soft game [44]. A soft game is associated with a soft set which makes this game flexible and convenient. In the soft games, both the strategy sets and soft payoffs are crisp. For further development of the soft games, the fuzzy soft games [45], intuitionistic fuzzy soft games [46], neutrosophic fuzzy soft games [47], linguistic single-valued neutrosophic soft game [48], simplified neutrosophic multiplicative soft game [49], and neutrosophic cubic soft expert sets based game [50] were introduced. All of these soft game theory models are based on the single-attribute (S-A) function (see Table 1).

Table 1. Games based on fuzzy sets and soft sets

| **Games** | **Membership Functions** | | | **Parameter(Strategy) Functions** | |
|---|---|---|---|---|---|
| | **M** | **NM** | **I** | **S-A** | **M-A** |
| Fuzzy games | ✓ | × | × | × | × |
| Intuitionistic fuzzy games | ✓ | ✓ | × | × | × |
| Neutrosophic games | ✓ | ✓ | ✓ | × | × |
| Soft games | × | × | × | ✓ | × |
| Fuzzy soft games | ✓ | × | × | ✓ | × |
| Intuitionistic fuzzy soft games | ✓ | ✓ | × | ✓ | × |
| Neutrosophic soft games | ✓ | ✓ | ✓ | ✓ | × |

✓ means Yes and × means No.

They are insufficient to model multi-attributed (M-A) game theory in the soft set environment. So far, we have not witnessed that the game theory has been promoted using a hypersoft set. That's why, the motivation of this paper is to develop soft game theory and apply it for decision making in a systematic manner.

In this paper, we initiate a model of multi-attributed game theory in the soft set environment, which is called hypersoft game theory. This game theory can consider as an extension of soft game to tackle uncertainty with more accuracy. The use of the hypersoft set in game theory surely provides numerous choices to the decision makers to attain optimal solutions by using hypersoft payoffs. Thus, payoff functions of the hypersoft games are multi-argument functions and the solution of the game is obtained by using the set-theoretic operations of hypersoft sets making this game more versatile and promising than the existing games. The proposed game theory models make it possible to deal with different complex types of strategic games (see Figure 1).

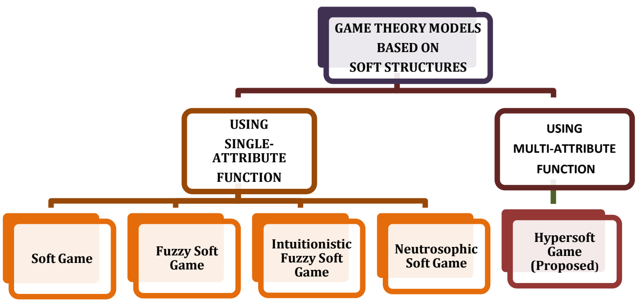


Figure 1. Relation between soft game theory models and hypersoft game theory

The other parts of this article are organized as follows. In Section 2, the cocepts of soft set, soft payoff, hypersoft set and inverse hypersoft set are recalled. In Section 3, two-person hypersoft game is described and the related properties are discussed. Section 4 introduces n-person hypersoft games. Section 5 is devoted to the conclusion of the paper.

## 2 Preliminaries

In this section, we discuss some fundamental concepts that are relevant for the subsequent sections.

### 2.1 Soft set

**Definition 2.1.** ([8]) Let $\mathfrak{R}$ be a set of universe and $Pow(\mathfrak{R})$ be a power set of $\mathfrak{R}$. Also, $\mathfrak{X}$ be a set of attributes (or criteria, parameters) and $\mathfrak{A} \subseteq \mathfrak{X}$. Then, a pair $(\dot{\mathfrak{f}}, \mathfrak{A})$ is said to be a soft set over $\mathfrak{R}$, where $\dot{\mathfrak{f}}$ is a mapping given by $\dot{\mathfrak{f}}: \mathfrak{A} \rightarrow Pow(\mathfrak{R})$.

**Example 2.2.** Let $\mathfrak{R} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$be the set of televisions under consideration, $\mathfrak{X} = \{\varkappa_1 = brand, \varkappa_2 = color, \varkappa_3 = size, \varkappa_4 = style\}$ be the set of attributes and $\mathfrak{A} = \mathfrak{X}$. Suppose that the following is obtained by evaluating televisions according to the specified attributes.

$\dot{\mathfrak{f}}(\varkappa_1) = \{\sigma_1, \sigma_2, \sigma_4\}$, $\dot{\mathfrak{f}}(\varkappa_2) = \{\sigma_2, \sigma_5\}$, $\dot{\mathfrak{f}}(\varkappa_3) = \{\sigma_1, \sigma_4\}$, and $\dot{\mathfrak{f}}(\varkappa_4) = \{\sigma_1, \sigma_3, \sigma_5\}$.

Then, the soft set $(\dot{\mathfrak{f}}, \mathfrak{A})$ is given as

$$(\dot{\mathfrak{f}}, \mathfrak{A}) = \left\{ \begin{matrix} (\varkappa_1, \{\sigma_1, \sigma_2, \sigma_4\}), (\varkappa_2, \{\sigma_2, \sigma_5\}), \\ (\varkappa_3, \{\sigma_1, \sigma_4\}), (\varkappa_4, \{\sigma_1, \sigma_3, \sigma_5\}) \end{matrix} \right\}.$$

The tabular structure of this soft set $(\dot{\mathfrak{f}}, \mathfrak{A})$ is shown in Table 2.

Table 2. Tabular structure of the soft set $(\dot{\mathfrak{f}}, \mathfrak{A})$.

| $(\dot{\mathfrak{f}}, \mathfrak{A})$ | $\varkappa_1$(brand) | $\varkappa_2$ (color) | $\varkappa_3$(size) | $\varkappa_4$ (style) |
|---|---|---|---|---|
| $\sigma_1$ | 1 | 0 | 1 | 1 |
| $\sigma_2$ | 1 | 1 | 0 | 0 |
| $\sigma_3$ | 0 | 0 | 0 | 1 |
| $\sigma_4$ | 1 | 0 | 1 | 0 |
| $\sigma_5$ | 0 | 1 | 0 | 1 |

### 2.2. Soft payoff

**Definition 2.3.** ([44]) Let $\mathfrak{M}$ and $\mathfrak{N}$ are the sets of strategies. A choice of behavior is termed to be an action. The elements of $\mathfrak{M} \times \mathfrak{N}$ are said to be action pairs. That is, $\mathfrak{M} \times \mathfrak{N}$ is the set of available actions.

**Definition 2.4.** ([44]) Suppose $\mathfrak{R}$ is a set of alternatives and its power set $Pow(\mathfrak{R})$. Also, let $\mathfrak{M}$ and $\mathfrak{N}$ be the strategy sets of Player 1 and Player 2, resp. Then a function $\lambda: \mathfrak{M} \times \mathfrak{N} \rightarrow Pow(\mathfrak{R})$ is known as a soft payoff function and $\forall (\mathfrak{m}, \mathfrak{n}) \in \mathfrak{M} \times \mathfrak{N}$, $\lambda(\mathfrak{m}, \mathfrak{n})$ is called a soft payoff.

### 2.3. Hypersoft set

**Definition 2.5.** ([27]) Let $\mathfrak{R}$ be a set of the universe and its power set $Pow(\mathfrak{R})$. Also, we assume $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n (n \geq 1)$ is $n$ attributes which the related attribute values are denoted by the sets $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n$ with $\mathfrak{A}_s \cap \mathfrak{A}_t = \emptyset$ for $s, t \in \{1,2, \ldots, n\}$ and $s \neq t$. Then, the pair $(\Gamma, \mathfrak{P}) = \{((\varkappa^1, \varkappa^2, \ldots, \varkappa^n), \gamma(\varkappa^1, \varkappa^2, \ldots, \varkappa^n)) : (\varkappa^1, \varkappa^2, \ldots, \varkappa^n) \in \mathfrak{P}$ and $\gamma(\varkappa^1, \varkappa^2, \ldots, \varkappa^n) \in Pow(\mathfrak{R})\}$ is called the hypersoft set over $\mathfrak{R}$, where $\mathfrak{P} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n$ and $\gamma: \mathfrak{P} \rightarrow Pow(\mathfrak{R})$.

**Example 2.6.** Let $\mathfrak{R} = \{\hat{\mathfrak{t}}_1, \hat{\mathfrak{t}}_2, \hat{\mathfrak{t}}_3, \hat{\mathfrak{t}}_4, \hat{\mathfrak{t}}_5\}$ be a set of motorcycles under consideration. Also, we consider the attributes as $\mathfrak{a}_1$=color, $\mathfrak{a}_2$=comfort-performance, $\mathfrak{a}_3$=weight, $\mathfrak{a}_4$=type. Their corresponding attribute value sets are$\mathfrak{X}_1 = \{white, black, red\}$, $\mathfrak{X}_2 = \{comfort, performance\}$, $\mathfrak{X}_3 = \{heavy, light, medium\}$ and $\mathfrak{X}_4 = \{crusier, sport, touring\}$. Assume that $\mathfrak{A}_i \subseteq \mathfrak{X}_i (i = 1,2,3,4)$ such that $\mathfrak{A}_1 = \{black, red\}$, $\mathfrak{A}_2 = \{comfort, performance\}$, $\mathfrak{A}_3 = \{heavy\}$ and $\mathfrak{A}_4 = \{sport, touring\}$. Then, by using multi-attribute function, the hypersoft set $(\Gamma, \mathfrak{P}) = (\gamma, \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times \mathfrak{A}_4)$ can be presented as

$$(\Gamma, \mathfrak{P}) = \left\{ \begin{matrix} ((black, comfort, heavy, sport), \{\sigma_1, \sigma_2, \sigma_3\}), \\ ((black, comfort, heavy, touring), \{\sigma_1, \sigma_3, \sigma_4\}), \\ ((black, performance, heavy, sport), \{\sigma_3, \sigma_5\}), \\ ((black, performance, heavy, touring), \{\sigma_1, \sigma_2\}), \\ ((red, comfort, heavy, sport), \emptyset), \\ ((red, comfort, heavy, touring), \{\sigma_2, \sigma_3, \sigma_5\}), \\ ((red, performance, heavy, sport), \emptyset), \\ ((red, performance, heavy, touring), \{\sigma_2, \sigma_3\}) \end{matrix} \right\}$$

**Definition 2.7.** Suppose that $(\gamma_1, \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n) = (\Gamma_1, \mathfrak{P}_1)$ and $(\gamma_2, \mathfrak{B}_1 \times \mathfrak{B}_2 \times \ldots \times \mathfrak{B}_n) = (\Gamma_2, \mathfrak{P}_2)$ is two hypersoft sets over the common universe $\mathfrak{R}$. Then,

**a)** The hypersoft set $(\gamma_1, \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n)$ is a subset of $(\gamma_2, \mathfrak{B}_1 \times \mathfrak{B}_2 \times \ldots \times \mathfrak{B}_n)$, denoted by $(\gamma_1, \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n) \subseteq (\gamma_2, \mathfrak{B}_1 \times \mathfrak{B}_2 \times \ldots \times \mathfrak{B}_n)$, if $\mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n \subseteq \mathfrak{B}_1 \times \mathfrak{B}_2 \times \ldots \times \mathfrak{B}_n$ and $\gamma_1(\alpha) \subseteq \gamma_2(\alpha)$ for all $\alpha = (\varkappa^1, \varkappa^2, \ldots, \varkappa^n) \in \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n$. [51]

**b)** The hypersoft sets $(\gamma_1, \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n)$ and $(\gamma_2, \mathfrak{B}_1 \times \mathfrak{B}_2 \times \ldots \times \mathfrak{B}_n)$ are equal, denoted by $(\gamma_1, \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n) = (\gamma_2, \mathfrak{B}_1 \times \mathfrak{B}_2 \times \ldots \times \mathfrak{B}_n)$, iff$(\gamma_1, \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n) \subseteq (\gamma_2, \mathfrak{B}_1 \times \mathfrak{B}_2 \times \ldots \times \mathfrak{B}_n)$ and $(\gamma_2, \mathfrak{B}_1 \times \mathfrak{B}_2 \times \ldots \times \mathfrak{B}_n) \widetilde{\subseteq} (\gamma_1, \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n)$. [51]

**c)** The complement of hypersoft set $(\gamma_1, \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n)$, denoted by $(\gamma_1, \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n)^c = (\gamma_1^c, \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n)$, if $\gamma_1^c(\alpha) = \mathfrak{R} - \gamma_1(\alpha)$ for all $\alpha = (\varkappa^1, \varkappa^2, \ldots, \varkappa^n) \in \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n$. [31]

**d)** The restricted intersection of hypersoft sets $(\Gamma_1, \mathfrak{P}_1)$ and $(\Gamma_2, \mathfrak{P}_2)$ is defined and denoted by $(\Gamma_3, \mathfrak{P}_3) = (\Gamma_1, \mathfrak{P}_1) \cap (\Gamma_2, \mathfrak{P}_2)$ where $\mathfrak{P}_3 = \mathfrak{P}_1 \cap \mathfrak{P}_2$ and $\gamma_3(\alpha)$=$\gamma_1(\alpha) \cap \gamma_2(\alpha)$ for all $\alpha = (\varkappa^1, \varkappa^2, \ldots, \varkappa^n) \in \mathfrak{P}_3$. [31,51]

**e)** The extended intersection of hypersoft sets $(\Gamma_1, \mathfrak{P}_1)$ and $(\Gamma_2, \mathfrak{P}_2)$ is defined and denoted by $(\Gamma_3, \mathfrak{P}_3) = (\Gamma_1, \mathfrak{P}_1) \sqcap (\Gamma_2, \mathfrak{P}_2)$, where $\mathfrak{P}_3 = \mathfrak{P}_1 \cup \mathfrak{P}_2$ and for all $\alpha = (\varkappa^1, \varkappa^2, \ldots, \varkappa^n) \in \mathfrak{P}_3$

$$\gamma_3(\alpha) = \begin{cases} \gamma_1(\alpha), & \text{if } \alpha \in \mathfrak{P}_1 - \mathfrak{P}_2 \\ \gamma_2(\alpha), & \text{if } \alpha \in \mathfrak{P}_2 - \mathfrak{P}_1 \\ \gamma_1(\alpha) \cap \gamma_2(\alpha), & \text{if } \alpha \in \mathfrak{P}_1 \cap \mathfrak{P}_2 \end{cases} \quad [31,51]$$

**f)** The restricted union of hypersoft sets $(\Gamma_1, \mathfrak{P}_1)$ and $(\Gamma_2, \mathfrak{P}_2)$ is defined and denoted by $(\Gamma_3, \mathfrak{P}_3) = (\Gamma_1, \mathfrak{P}_1) \cup (\Gamma_2, \mathfrak{P}_2)$, where $\mathfrak{P}_3 = \mathfrak{P}_1 \cap \mathfrak{P}_2$ and $\gamma_3(\alpha) = \gamma_1(\alpha) \cup \gamma_2(\alpha)$ for all $\alpha = (\varkappa^1, \varkappa^2, \ldots, \varkappa^n) \in \mathfrak{P}_3$. [31,51]

**g)** The extended union of hypersoft sets $(\Gamma_1, \mathfrak{P}_1)$ and $(\Gamma_2, \mathfrak{P}_2)$ is defined and denoted by $(\Gamma_3, \mathfrak{P}_3) = (\Gamma_1, \mathfrak{P}_1) \sqcup (\Gamma_2, \mathfrak{P}_2)$, where $\mathfrak{P}_3 = \mathfrak{P}_1 \cup \mathfrak{P}_2$ and for all $\alpha = (\varkappa^1, \varkappa^2, \ldots, \varkappa^n) \in \mathfrak{P}_3$

$$\gamma_3(\alpha) = \begin{cases} \gamma_1(\alpha), & \text{if } \alpha \in \mathfrak{P}_1 - \mathfrak{P}_2 \\ \gamma_2(\alpha), & \text{if } \alpha \in \mathfrak{P}_2 - \mathfrak{P}_1 \\ \gamma_1(\alpha) \cup \gamma_2(\alpha), & \text{if } \alpha \in \mathfrak{P}_1 \cap \mathfrak{P}_2 \end{cases} \quad [31,51]$$

**h)** The restricted difference of hypersoft sets $(\Gamma_1, \mathfrak{P}_1)$ and $(\Gamma_2, \mathfrak{P}_2)$ is defined and denoted by $(\Gamma_3, \mathfrak{P}_3) = (\Gamma_1, \mathfrak{P}_1) \setminus (\Gamma_2, \mathfrak{P}_2)$, where $\mathfrak{P}_3 = \mathfrak{P}_1 \cap \mathfrak{P}_2$ and $\gamma_3(\alpha) = \gamma_1(\alpha) \setminus \gamma_2(\alpha)$ for all $\alpha = (\varkappa^1, \varkappa^2, \ldots, \varkappa^n) \in \mathfrak{P}_3$. [51]

**i)** The extended difference hypersoft sets $(\Gamma_1, \mathfrak{P}_1)$ and $(\Gamma_2, \mathfrak{P}_2)$ is defined and denoted by $(\Gamma_3, \mathfrak{P}_3) = (\Gamma_1, \mathfrak{P}_1) \tilde{\setminus} (\Gamma_2, \mathfrak{P}_2)$, where $\mathfrak{P}_3 = \mathfrak{P}_1 \cup \mathfrak{P}_2$ and for all $\alpha = (\varkappa^1, \varkappa^2, \ldots, \varkappa^n) \in \mathfrak{P}_3$

$$\gamma_3(\alpha) = \begin{cases} \gamma_1(\alpha), & \text{if } \alpha \in \mathfrak{P}_1 - \mathfrak{P}_2 \\ \emptyset, & \text{if } \alpha \in \mathfrak{P}_2 - \mathfrak{P}_1 \\ \gamma_1(\alpha) \setminus \gamma_2(\alpha), & \text{if } \alpha \in \mathfrak{P}_1 \cap \mathfrak{P}_2 \end{cases} \quad [29]$$

**Definition 2.8.** Let $(\Gamma, \mathfrak{P})$ be a hypersoft set such that $\gamma: \mathfrak{P} \to Pow(\mathfrak{R})$ where $\mathfrak{P} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n$. The strength of $l_i$ in the hypersoft set $(\Gamma, \mathfrak{P})$ is described and denoted by

$$\chi_{(\Gamma,\mathfrak{P})}(l_i) = \frac{|\gamma(l_i)|}{|\mathfrak{R}|}, \ \forall i \tag{1}$$

where |.| means the cardinality of set. By Eq. (1), it is obvious that $0 \leq \chi_{(\Gamma,\mathfrak{P})}(l_i) \leq 1 \ \forall i$.

**Example 2.9.** Let $\mathfrak{R} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ be a set of alternatives. Also, let $\mathfrak{A}_1 = \{\varkappa_1^1, \varkappa_2^1\}, \mathfrak{A}_2 = \{\varkappa_1^2\}$, $\mathfrak{A}_3 = \{\varkappa_1^3, \varkappa_2^3\}, \mathfrak{A}_4 = \{\varkappa_1^4\}$ such that $\mathfrak{P} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times \mathfrak{A}_4$, i.e.,

$$\mathfrak{P} = \begin{Bmatrix} l_1 = (\varkappa_1^1, \varkappa_1^2, \varkappa_1^3, \varkappa_1^4), l_2 = (\varkappa_1^1, \varkappa_1^2, \varkappa_2^3, \varkappa_1^4), \\ l_3 = (\varkappa_2^1, \varkappa_1^2, \varkappa_1^3, \varkappa_1^4), l_4 = (\varkappa_2^1, \varkappa_1^2, \varkappa_2^3, \varkappa_1^4) \end{Bmatrix}.$$

Assume that the hypersoft set $(\Gamma, \mathfrak{P})$ is described as $\gamma(l_1) = \{\sigma_1, \sigma_2, \sigma_3\}, \gamma(l_2) = \{\sigma_2, \sigma_3\}, \gamma(l_3) = \{\sigma_1, \sigma_3, \sigma_4, \sigma_5\}$ and $\gamma(l_4) = \{\sigma_1, \sigma_3\}$. That's, the hypersoft set $(\Gamma, \mathfrak{P})$ is

$(\Gamma, \mathfrak{P})$=
$\{(l_1, \{\sigma_1, \sigma_2, \sigma_3\}), (l_2, \{\sigma_2, \sigma_3\}), (l_3, \{\sigma_1, \sigma_3, \sigma_4, \sigma_5\}), (l_4, \{\sigma_1, \sigma_3\})\}$

Then, we calculate as $\chi_{(\Gamma,\mathfrak{P})}(l_1) = \frac{3}{5}, \chi_{(\Gamma,\mathfrak{P})}(l_2) = \frac{2}{5}, \chi_{(\Gamma,\mathfrak{P})}(l_3) = \frac{4}{5}$ and $\chi_{(\Gamma,\mathfrak{P})}(l_4) = \frac{2}{5}$.

**Definition 2.10.** Assume $(\Gamma, \mathfrak{P})$ is a hypersoft set such that $\gamma: \mathfrak{P} \to Pow(\mathfrak{R})$ where $\mathfrak{P} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n$. Then the inverse hypersoft set corresponding to $(\Gamma, \mathfrak{P})$ is denoted by $(\bar{\Gamma}, \mathfrak{R})$ where $\bar{\gamma}: \mathfrak{R} \to Pow(\mathfrak{P})$.

**Definition 2.11.** Let $(\bar{\Gamma}, \mathfrak{R})$ be an inverse hypersoft set such that $\bar{\gamma}$:$\mathfrak{R} \to Pow(\mathfrak{P})$ where $\mathfrak{P} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \ldots \times \mathfrak{A}_n$. The strength of $\sigma_i$ in the inverse hypersoft set $(\bar{\Gamma}, \mathfrak{R})$ is described and denoted by

$$\chi_{(\bar{\Gamma},\mathfrak{R})}(\sigma_i) = \frac{|\bar{\gamma}(\sigma_i)|}{|\mathfrak{P}|}, \ \forall i \tag{2}$$

where |.| means the cardinality of set. By Eq. (2), it is obvious that $0 \leq \chi_{(\bar{\Gamma},\mathfrak{R})}(\sigma_i) \leq 1 \ \forall i$.

**Example 2.12.** Let $\mathfrak{R} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ be a set of alternatives. Also, let $\mathfrak{A}_1 = \{\varkappa_1^1, \varkappa_2^1\}$, $\mathfrak{A}_2 = \{\varkappa_1^2\}$, $\mathfrak{A}_3 = \{\varkappa_1^3, \varkappa_2^3\}, \mathfrak{A}_4 = \{\varkappa_1^4\}$ such that $\mathfrak{P} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times \mathfrak{A}_4$, i.e.,

$$\mathfrak{P} = \begin{Bmatrix} l_1 = (\varkappa_1^1, \varkappa_1^2, \varkappa_1^3, \varkappa_1^4), l_2 = (\varkappa_1^1, \varkappa_1^2, \varkappa_2^3, \varkappa_1^4), \\ l_3 = (\varkappa_2^1, \varkappa_1^2, \varkappa_1^3, \varkappa_1^4), l_4 = (\varkappa_2^1, \varkappa_1^2, \varkappa_2^3, \varkappa_1^4) \end{Bmatrix}.$$

Assume that the inverse hypersoft set $(\bar{\Gamma}, \mathfrak{R})$ is described as $\bar{\gamma}(\sigma_1) = \{l_1, l_2\}, \bar{\gamma}(\sigma_2) = \{l_3\}, \bar{\gamma}(\sigma_3) = \{l_1, l_4\}, \bar{\gamma}(\sigma_4) = \{l_1, l_2, l_4\}$, and $\bar{\gamma}(\sigma_5) = \{l_1, l_2, l_3\}$.

Thus, the inverse hypersoft set $(\bar{\Gamma}, \mathfrak{R})$ is given by

$(\bar{\Gamma}, \mathfrak{R})$=
$\{(\sigma_1, \{l_1, l_2\}), (\sigma_2, \{l_3\}), (\sigma_3, \{l_1, l_4\}), (\sigma_4, \{l_1, l_2, l_4\}), (\sigma_5, \{l_1, l_2, l_3\})\}$

Then, we calculate as$\chi_{(\bar{\Gamma},\mathfrak{R})}(\sigma_1) = \frac{1}{2}, \chi_{(\bar{\Gamma},\mathfrak{R})}(\sigma_2) = \frac{1}{4}, \chi_{(\bar{\Gamma},\mathfrak{R})}(\sigma_3) = \frac{1}{2}, \chi_{(\bar{\Gamma},\mathfrak{R})}(\sigma_4) = \frac{3}{4}$ and $\chi_{(\bar{\Gamma},\mathfrak{R})}(\sigma_5) = \frac{3}{4}$.

## 3 Two-person hypersoft games

In this section, we introduce two-person hypersoft (tphs) game with hypersoft payoffs and present its application.

All of the soft game theory models in the literature are based on the single-attribute function. They are insufficient to model multi-attributed game theory in the soft set environment. However, in game theory problems encountered in many fields, the strategies based on multi-attributes are selected. In order to model such game theory problems and present solutions for them, we propose to extend soft game theory using the hypersoft sets (because the parameter set (domain) in the structure of hypersoft set is in the form of multi-attributes). Therefore, hypersotf payoff function (based on multi-attributes) and hypersoft game theory model are created as follows.

Payoff function is used for modeling human behavior. Payoff function for a player is a mapping from the cross-product of players strategy spaces to the players set of payoffs [52]. Now, we consider that player strategy spaces consist of multi-attributes and the players set of payoffs is the power set of the alternative set, i.e. the payoffs are subsets of the alternative set. Thus, the following definition is presented for the hypersoft payoff function.

**Definition 3.1.** Let $\mathfrak{R}$ be a set of alternatives and $\mathfrak{a}_1$, $\mathfrak{a}_2$ ,..., $\mathfrak{a}_n (n \geq 1)$ be $n$ attributes (strategies) which the related attribute values are denoted by the sets $\mathfrak{A}_1 = \{\varkappa_1^1, \varkappa_2^1, \dots, \varkappa_{m_1}^1\}$, $\mathfrak{A}_2 = \{\varkappa_1^2, \varkappa_2^2, \dots, \varkappa_{m_2}^2\}$,..., $\mathfrak{A}_n = \{\varkappa_1^n, \varkappa_2^n, \dots, \varkappa_{m_n}^n\}$with $\mathfrak{A}_s \cap \mathfrak{A}_t = \emptyset$ for $s, t \in \{1,2,\dots,n\}$ and $s \neq t$. Also, let $\mathfrak{P} = \mathfrak{A}_1 \times \mathfrak{A}_2 \times \dots \times \mathfrak{A}_n$ and $l_v = (\varkappa_{i_v^1}^1, \varkappa_{i_v^2}^2, \dots, \varkappa_{i_v^n}^n) \in \mathfrak{P}$ where$1 \leq v \leq n, 1 \leq i^k \leq m_k$. $\mathfrak{F} = \{l_1, l_2, \dots, l_{s_1}\}$ and $\mathfrak{G} = \{\breve{l}_1, \breve{l}_2, \dots, \breve{l}_{s_2}\}$ $(1 \leq s_1, s_2 \leq n)$ be the strategy sets (where, $l, \breve{l} \in \mathfrak{P}$). Then, the set-valued function

$$\gamma_H : \mathfrak{F} \times \mathfrak{G} \rightarrow Pow(\mathfrak{R}) \tag{3}$$

is called a hypersoft payoff function. For each $(l_p, \breve{l}_q) \in \mathfrak{F} \times \mathfrak{G}$, the value $\gamma_H(l_p, \breve{l}_q)$ is said to be a hypersoft payoff.

**Definition 3.2.** Let $\mathfrak{F} = \{l_1, l_2, \dots, l_{s_1}\}$ and $\mathfrak{G} = \{\breve{l}_1, \breve{l}_2, \dots, \breve{l}_{s_2}\}$ $(1 \leq s_1, s_2 \leq n)$ be two sets of strategies of Player 1 and Player 2 resp., $\mathfrak{R}$ be an alternative set and $\gamma_{H_\kappa} : \mathfrak{F} \times \mathfrak{G} \rightarrow Pow(\mathfrak{R})$ be a hypersoft payoff function for $\kappa = 1,2$.Then for each $\kappa$, a two-person hypersoft game (tphs-game) over $\mathfrak{R}$ is defined by a hypersoft set given as

$$(\Gamma_\kappa, \mathfrak{F} \times \mathfrak{G}) = \left\{ \left( (l_p, \breve{l}_q), \gamma_{H_\kappa}(l_p, \breve{l}_q) \right) : (l_p, \breve{l}_q) \in \mathfrak{F} \times \mathfrak{G},\ \gamma_{H_\kappa}(l_p, \breve{l}_q) \in Pow(\mathfrak{R}) \right\} \tag{4}$$

**Note 1.** Throughout this article, the tphs-game $(\Gamma_\kappa, \mathfrak{F} \times \mathfrak{G})$ can also be denoted as $\ddot{\Gamma}_\kappa$ for short.

**Definition 3.3.** The tphs-game is played in the following manner:

Assume that Player 1 chooses a strategy $l_p \in \mathfrak{F}$ and simultaneously Player 2 chooses a strategy $\breve{l}_q \in \mathfrak{G}$. When the two players interact, then each player $\kappa = 1,2$ receives the hypersoft payoffs $\gamma_{H_\kappa}(l_p, \breve{l}_q)$ of hypersoft game $\ddot{\Gamma}_\kappa$ and it can be shown in Table 3.

Table 3. Tabular structure of the tphs-game $\ddot{\Gamma}_\kappa$.

| $\ddot{\Gamma}_\kappa$ | $\breve{l}_1$ | $\breve{l}_2$ | ... | $\breve{l}_{s_2}$ |
|---|---|---|---|---|
| $l_1$ | $\gamma_{H_\kappa}(l_1, \breve{l}_1)$ | $\gamma_{H_\kappa}(l_1, \breve{l}_2)$ | ... | $\gamma_{H_\kappa}(l_1, \breve{l}_{s_2})$ |
| $l_2$ | $\gamma_{H_\kappa}(l_2, \breve{l}_1)$ | $\gamma_{H_\kappa}(l_2, \breve{l}_2)$ | ... | $\gamma_{H_\kappa}(l_2, \breve{l}_{s_2})$ |
| . | . | | | . |
| . | . | . | | . |
| . | . | | . | . |
| $l_{s_1}$ | $\gamma_{H_\kappa}(l_{s_1}, \breve{l}_1)$ | $\gamma_{H_\kappa}(l_{s_1}, \breve{l}_2)$ | ... | $\gamma_{H_\kappa}(l_{s_1}, \breve{l}_{s_2})$ |

**Example 3.4.** In recent years, with the development of technology, people's habits of accessing news and information have also changed. Written media such as newspapers and magazines lost their popularity in reaching the news. Instead, people started to follow internet news websites (or internet newspaper) from computers and smart phones. The internet news sites are in competition to reach more readers. Some factors (content, readibility, etc.) may make a type of news presented on an internet news site attract more attention from the reader.

Consider two internet news sites (Player 1 and Player 2) that offer the same types of news. Let $\mathfrak{R} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ be the types of news that these two sites offer to the reader, where $\sigma_i$ $(i = 1,2,\dots,5)$ are sports news, magazine news, economic news, politics news, and technology news, respectively. Suppose that the evaluation/analysis of these news sites is based on the following attributes.

$\mathfrak{a}_1 = general\ view$ and $\mathfrak{A}_1 = \{\varkappa_1^1 = page\ field, \varkappa_2^1 = photo\ (quality, size), \varkappa_3^1 = message/comment\ board\}$
$\mathfrak{a}_2 = readibility$ and $\mathfrak{A}_2 = \{\varkappa_1^2 = font\ type, \varkappa_2^2 = plain\ language, \varkappa_3^2 = text\ flow\}$
$\mathfrak{a}_3 = content$ and $\mathfrak{A}_3 = \{\varkappa_1^3 = competence, \varkappa_2^3 = scope, \varkappa_3^3 = impartiality, \varkappa_4^3 = accuracy\}$
$\mathfrak{a}_4 = accessibility$ and $\mathfrak{A}_4 = \{\varkappa_1^4 = usefulness, \varkappa_2^4 = load\ time, \varkappa_3^4 = flow\ map\}$

Now, Player 1 and Player 2 focus on a game to determine which the news types will appeal to more readers by considering different combinations of these attributes (i.e., by determining their own strategies and then considering the strategy moves of the opponent).

Player 1 determines the multiple attributes as $l_1 = (\varkappa_1^1, \varkappa_2^2, \varkappa_1^3, \varkappa_1^4)$, $l_2 = (\varkappa_1^1, \varkappa_2^2, \varkappa_3^3, \varkappa_1^4)$, $l_3 = (\varkappa_3^1, \varkappa_2^2, \varkappa_1^3, \varkappa_1^4)$, $l_4 = (\varkappa_3^1, \varkappa_2^2, \varkappa_3^3, \varkappa_1^4)$ and thus creates the strategy set as $\mathfrak{F} = \{l_1, l_2, l_3, l_4\}$.

Player 2 determines the multiple attributes as $\breve{l}_1 = (\varkappa_1^1, \varkappa_2^2, \varkappa_1^3, \varkappa_3^4)$ and $\breve{l}_2 = (\varkappa_1^1, \varkappa_2^2, \varkappa_4^3, \varkappa_3^4)$, and thus creates the strategy set as $\mathfrak{G} = \{\breve{l}_1, \breve{l}_2\}$.

Considering the analyzes, comments and surveys made under the determined strategies (of Player 1 and Player 2), if Player 1 constructs a tphs-game as follows:

$$\ddot{\Gamma}_1 = \left\{ \begin{array}{c} \big((l_1,\check{l}_1),\{\sigma_1,\sigma_3\}\big), \big((l_1,\check{l}_2),\{\sigma_3\}\big), \\ ((l_2,\check{l}_1),\{\sigma_1,\sigma_2\}), ((l_2,\check{l}_2),\{\sigma_1,\sigma_2,\sigma_3\}) \\ \big((l_3,\check{l}_1),\{\sigma_1,\sigma_5\}\big), \big((l_3,\check{l}_2),\{\sigma_1\}\big), \\ ((l_4,\check{l}_1),\{\sigma_2,\sigma_4,\sigma_5\}), ((l_4,\check{l}_2),\{\sigma_5\}) \end{array} \right\}$$

Then the hypersoft payoffs of the tphs-game are arranged in Table 4. (This table is created by considering $\ddot{\Gamma}_1$. For example, the first row of Table 4 represents $\big((l_1,\check{l}_1),\{\sigma_1,\sigma_3\}\big)$ and $\big((l_1,\check{l}_2),\{\sigma_3\}\big)$ in $\ddot{\Gamma}_1$. The other rows are created in a similar way.)

Table 4. Tabular form of hypersoft payoffs for $\ddot{\Gamma}_1$.

| $\ddot{\Gamma}_1$ | $\check{l}_1$ | $\check{l}_2$ |
|---|---|---|
| $l_1$ | $\{\sigma_1,\sigma_3\}$ | $\{\sigma_3\}$ |
| $l_2$ | $\{\sigma_1,\sigma_2\}$ | $\{\sigma_1,\sigma_2,\sigma_3\}$ |
| $l_3$ | $\{\sigma_1,\sigma_5\}$ | $\{\sigma_1\}$ |
| $l_4$ | $\{\sigma_2,\sigma_4,\sigma_5\}$ | $\{\sigma_5\}$ |

Table 4 can be explained as follows: if Player 1 selects the strategy $l_2 = (\varkappa_1^1,\varkappa_2^2,\varkappa_3^3,\varkappa_1^4)$ when Player 2 selects the strategy $\check{l}_2 = (\varkappa_1^1,\varkappa_2^2,\varkappa_4^3,\varkappa_3^4)$ then the value of the game is the hypersoft payoff $\{\sigma_1,\sigma_2,\sigma_3\}$, that is, $\gamma_{H_1}(l_2,\check{l}_2) = \{\sigma_1,\sigma_2,\sigma_3\}$. Thereby, Player 1 wins the set of news $\{\sigma_1,\sigma_2,\sigma_3\}$ and Player 2 loses the same set. Briefly, if Player 1 selects the strategy $l_2$ when Player 2 selects the strategy $\check{l}_2$ in this tphs-game then the sports news, magazine news, and economic news on website of Player 1 gain the attention of readers (i.e., more readers) in this competition. This causes Player 2 to lose readers for the same types of news. The other components in table can be interpreted similarly.

Similarly, Player 2 may construct another tphs-game and execute it in a tabular form given in Table 5.

Table 5. Tabular form of hypersoft payoffs for $\ddot{\Gamma}_2$.

| $\ddot{\Gamma}_2$ | $\check{l}_1$ | $\check{l}_2$ |
|---|---|---|
| $l_1$ | $\{\sigma_1,\sigma_2,\sigma_3\}$ | $\{\sigma_1,\sigma_2\}$ |
| $l_2$ | $\{\sigma_4\}$ | $\{\sigma_3,\sigma_4\}$ |
| $l_3$ | $\{\sigma_2\}$ | $\{\sigma_4,\sigma_5\}$ |
| $l_4$ | $\{\sigma_3,\sigma_5\}$ | $\{\sigma_2,\sigma_3\}$ |

From Table 5, if Player 2 selects the strategy $\check{l}_1 = (\varkappa_1^1,\varkappa_2^2,\varkappa_1^3,\varkappa_3^4)$ when Player 1 selects the strategy $l_4 = (\varkappa_3^1,\varkappa_2^2,\varkappa_3^3,\varkappa_1^4)$ then the value of the game is the hypersoft payoff $\{\sigma_3,\sigma_5\}$, that is, $\gamma_{H_2}(l_4,\check{l}_1) = \{\sigma_3,\sigma_5\}$. In this case, Player 2 wins the set of news $\{\sigma_3,\sigma_5\}$ and Player 1 loses the same set. That is, if Player 2 selects the strategy $\check{l}_1$ when Player 1 selects the strategy $l_4$ in this tphs-game then the economic news and technology news on website of Player 2 gain the attention of readers (i.e., more readers) in this competition. This causes Player 1 to lose readers for the same news types. The others can be interpreted similarly.

Note that in Tables 4 and 5, the payoffs should be considered as the gains of the first player (who responds to the opponent's strategy and thus seeks gains). Therefore, it is necessary to interpret each table according to the first player.

**Definition 3.5.** Let $(\Gamma_\kappa, \mathfrak{F}\times\mathfrak{G})$ be a tphs-game and $(l_p,\check{l}_q),(l_m,\check{l}_n) \in \mathfrak{F}\times\mathfrak{G}$. Then, a Player $\kappa$ is said to be a rational, if the player's hypersoft payoff provides the following cases:
1. either $\gamma_{H_\kappa}(l_p,\check{l}_q) \supseteq \gamma^\kappa_{\mathfrak{F}\times\mathfrak{G}}(l_m,\check{l}_n)$ or $\gamma_{H_\kappa}(l_m,\check{l}_n) \supseteq \gamma^\kappa_{\mathfrak{F}\times\mathfrak{G}}(l_p,\check{l}_q)$,
2. if $\gamma_{H_\kappa}(l_p,\check{l}_q) \supseteq \gamma^\kappa_{\mathfrak{F}\times\mathfrak{G}}(l_m,\check{l}_n)$ and $\gamma_{H_\kappa}(l_m,\check{l}_n) \supseteq \gamma^\kappa_{\mathfrak{F}\times\mathfrak{G}}(l_p,\check{l}_q)$ then $\gamma_{H_\kappa}(l_p,\check{l}_q) = \gamma^\kappa_{\mathfrak{F}\times\mathfrak{G}}(l_m,\check{l}_n)$.

**Definition 3.6.** Let $(\Gamma_\kappa, \mathfrak{F}\times\mathfrak{G})$ be a tphs-game. An action $(l^\#,\check{l}^\#) \in \mathfrak{F}\times\mathfrak{G}$ is called an optimal action if $\gamma_{H_\kappa}(l^\#,\check{l}^\#) \supseteq \gamma_{H_\kappa}(l,\check{l})$ for all $(l,\check{l}) \in \mathfrak{F}\times\mathfrak{G}$.

**Definition 3.7.** Let $(\Gamma_\kappa, \mathfrak{F}\times\mathfrak{G})$ be a tphs-game and $(l_p,\check{l}_q),(l_m,\check{l}_n) \in \mathfrak{F}\times\mathfrak{G}$. Then
1. if $\gamma_{H_\kappa}(l_m,\check{l}_n) \subset \gamma_{H_\kappa}(l_p,\check{l}_q)$ then the action $(l_p,\check{l}_q)$ is superior over action $(l_m,\check{l}_n)$,
2. if $\gamma_{H_\kappa}(l_m,\check{l}_n) = \gamma_{H_\kappa}(l_p,\check{l}_q)$ then there is no difference between the two actions,
3. if $\gamma_{H_\kappa}(l_m,\check{l}_n) \subseteq \gamma_{H_\kappa}(l_p,\check{l}_q)$, then either there is no difference between the two actions or the action $(l_p,\check{l}_q)$ is superior over action $(l_m,\check{l}_n)$.

**Definition 3.8.** Let $\ddot{\Gamma}_\kappa = (\Gamma_\kappa, \mathfrak{F}\times\mathfrak{G})$ be a tphs-game over $\mathfrak{R}$. Then
1. $\ddot{\Gamma}_\kappa$ is called an empty tphs-game if $\gamma_{H_\kappa}(l_p,\check{l}_q) = \emptyset$ for every $(l_p,\check{l}_q) \in \mathfrak{F}\times\mathfrak{G}$, and it is denoted by $\ddot{\Gamma}_\Phi$.
2. $\ddot{\Gamma}_\kappa$ is called a universal tphs-game if $\gamma_{H_\kappa}(l_p,\check{l}_q) = \mathfrak{R}$ for every $(l_p,\check{l}_q) \in \mathfrak{F}\times\mathfrak{G}$, and it is symbolized by $\ddot{\Gamma}_{\mathfrak{P}}$.

**Definition 3.9.** Let $\ddot{\Gamma}_\kappa = (\Gamma_\kappa, \mathfrak{F}\times\mathfrak{G})$ be a tphs-game over $\mathfrak{R}$.
1. $\ddot{\Gamma}_\kappa$ is said to be a two person disjoint hypersoft game (tpDhs-game) if for each action pair, the intersection of hypersoft payoff of players is empty set.
2. $\ddot{\Gamma}_\kappa$ is said to be a two person universal hypersoft game (tpUhs-game) if for each action pair, the union of hypersoft payoff of players is universal set.

**Proposition 3.10.** Let $\ddot{\Gamma}_\kappa = (\Gamma_\kappa, \mathfrak{F}\times\mathfrak{G})$ be a tpDhs-game (for $\kappa = 1,2$). Then, we consider the following:
**i.** $(\ddot{\Gamma}_\kappa^c)^c = \ddot{\Gamma}_\kappa$, $\kappa = 1,2$.
**ii.** $\ddot{\Gamma}_1 \setminus \ddot{\Gamma}_2 = \ddot{\Gamma}_1$ and $\ddot{\Gamma}_2 \setminus \ddot{\Gamma}_1 = \ddot{\Gamma}_2$.
**iii.** $\ddot{\Gamma}_1 \cap \ddot{\Gamma}_2 = \ddot{\Gamma}_\Phi$.

**Proof.** They are straightforward, therefore omitted.

**Proposition 3.11.** Let $\ddot{\Gamma}_\kappa = (\Gamma_\kappa, \mathfrak{F}\times\mathfrak{G})$ be a tpUhs-game (for $\kappa = 1,2$). Then, we have the following properties:
**i.** $\ddot{\Gamma}_1^c = \ddot{\Gamma}_2$ and $\ddot{\Gamma}_2^c = \ddot{\Gamma}_1$.
**ii.** $(\ddot{\Gamma}_\kappa^c)^c = \ddot{\Gamma}_\kappa$, $\kappa = 1,2$.
**iii.** $\ddot{\Gamma}_1 \cup \ddot{\Gamma}_2 = \ddot{\Gamma}_{\mathfrak{P}}$.

**Proof.** They are straightforward, therefore omitted.

**Proposition 3.12.** Let $\ddot{\Gamma}_\kappa = (\Gamma_\kappa, \mathfrak{F} \times \mathfrak{G})$ be both tpDhs-game and tpUhs-game (for $\kappa = 1,2$). Then, the following properties are satisfied:

**i.** $\ddot{\Gamma}_1 \setminus \ddot{\Gamma}_2 = \ddot{\Gamma}_1$ and $\ddot{\Gamma}_2 \setminus \ddot{\Gamma}_1 = \ddot{\Gamma}_2$.

**ii.** $\ddot{\Gamma}_1 \cap \ddot{\Gamma}_2 = \ddot{\Gamma}_\Phi$.

**iii.** $\ddot{\Gamma}_1 \cup \ddot{\Gamma}_2 = \ddot{\Gamma}_{\mathfrak{P}}$.

**Proof.** They are straightforward, therefore omitted.

### 3.1. Hypersoft Saddle Point

**Definition 3.13.** Let $\gamma_{H_\kappa}$ be the hypersoft payoff function of a tphs-game $\ddot{\Gamma}_\kappa = (\Gamma_\kappa, \mathfrak{F} \times \mathfrak{G})$. If the following conditions hold then $\gamma_{H_\kappa}(l, \check{l})$ is said to be a hypersoft saddle point value and $(l, \check{l})$ is said to be a hypersoft saddle point of Player $\kappa$ in this tphs-game.

1. $\bigcup_{p=1}^{S_1} \gamma_{H_\kappa}(l_p, \check{l}_q) = \gamma_{H_\kappa}(l, \check{l})$.
2. $\bigcap_{q=1}^{S_2} \gamma_{H_\kappa}(l_p, \check{l}_q) = \gamma_{H_\kappa}(l, \check{l})$.

**Example 3.14.** Let $\mathfrak{R} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$ be a set of six laptops (with different features) under consideration. We consider the attributes and their corresponding attribute values as

$\mathfrak{a}_1 = operating\ system$ and $\mathfrak{A}_1 = \{\varkappa_1^1 = Windows, \varkappa_2^1 = Linux, \varkappa_3^1 = Mac\}$

$\mathfrak{a}_2 = hard\ drive$ and $\mathfrak{A}_2 = \{\varkappa_1^2 = HDD, \varkappa_2^2 = SSD\}$

$\mathfrak{a}_3 = RAM$ and $\mathfrak{A}_3 = \{\varkappa_1^3 = low, \varkappa_2^3 = medium, \varkappa_3^3 = high\}$

$\mathfrak{a}_4 = processor$ and $\mathfrak{A}_4 = \{\varkappa_1^4 = Dual\ core, \varkappa_2^4 = AMD, \varkappa_3^4 = Quad\ core\}$

$\mathfrak{a}_5 = screen\ size$ and $\mathfrak{A}_5 = \{\varkappa_1^5 = small, \varkappa_2^5 = medium, \varkappa_3^5 = large\}$

Two experts (Player 1 and Player 2) are appointed by two companies that product each of the 6 laptops. Each of the players determines the strategies and then considering the other player's strategy, determines the payoffs for her/his own strategy. These payoffs indicate which laptops of the company (that appoints the player) will profit according to the determined strategy. Note that the players create payoffs as a result of evaluation of data, rating, survey, etc.

Assume that the strategy sets of Player 1 and Player 2 are respectively

$$\mathfrak{F} = \left\{\begin{matrix} l_1 = (\varkappa_1^1, \varkappa_1^2, \varkappa_2^3, \varkappa_1^4, \varkappa_1^5), l_2 = (\varkappa_1^1, \varkappa_1^2, \varkappa_2^3, \varkappa_1^4, \varkappa_3^5), \\ l_3 = (\varkappa_1^1, \varkappa_1^2, \varkappa_3^3, \varkappa_1^4, \varkappa_1^5), l_4 = (\varkappa_1^1, \varkappa_1^2, \varkappa_3^3, \varkappa_1^4, \varkappa_3^5), \\ l_5 = (\varkappa_2^1, \varkappa_1^2, \varkappa_2^3, \varkappa_1^4, \varkappa_1^5), l_6 = (\varkappa_2^1, \varkappa_1^2, \varkappa_2^3, \varkappa_1^4, \varkappa_3^5), \\ l_7 = (\varkappa_2^1, \varkappa_1^2, \varkappa_3^3, \varkappa_1^4, \varkappa_1^5), l_8 = (\varkappa_2^1, \varkappa_1^2, \varkappa_3^3, \varkappa_1^4, \varkappa_3^5) \end{matrix}\right\}$$

and

$$\mathfrak{G} = \left\{\begin{matrix} \check{l}_1 = (\varkappa_1^1, \varkappa_1^2, \varkappa_2^3, \varkappa_1^4, \varkappa_1^5), \check{l}_2 = (\varkappa_1^1, \varkappa_1^2, \varkappa_2^3, \varkappa_1^4, \varkappa_3^5), \\ \check{l}_3 = (\varkappa_1^1, \varkappa_1^2, \varkappa_3^3, \varkappa_1^4, \varkappa_1^5), \check{l}_4 = \varkappa_1^1, \varkappa_1^2, \varkappa_3^3, \varkappa_1^4, \varkappa_3^5 \end{matrix}\right\}.$$

Based on the above strategy sets, Table 6 presents the tphs-game of Player 1.

Table 6. Tabular structure of the tphs-game of Player 1.

| $\ddot{\Gamma}_\kappa$ | $\check{l}_1$ | $\check{l}_2$ | $\check{l}_3$ | $\check{l}_4$ |
|---|---|---|---|---|
| $l_1$ | $\{\sigma_2, \sigma_3\}$ | $\{\sigma_2, \sigma_6\}$ | $\{\sigma_3\}$ | $\{\sigma_2, \sigma_5\}$ |
| $l_2$ | $\{\sigma_2\}$ | $\{\sigma_3, \sigma_6\}$ | $\{\sigma_4\}$ | $\{\sigma_4\}$ |
| $l_3$ | $\{\sigma_3\}$ | $\{\sigma_3\}$ | $\{\sigma_3\}$ | $\{\sigma_5\}$ |
| $l_4$ | $\{\sigma_5\}$ | $\{\sigma_4\}$. | $\{\sigma_2, \sigma_4\}$ | $\{\sigma_4, \sigma_5, \sigma_6\}$ |
| $l_5$ | $\{\sigma_5\}$ | $\{\sigma_3\}$ | $\{\sigma_2\}$ | $\{\sigma_1, \sigma_2\}$ |
| $l_6$ | $\{\sigma_3, \sigma_5\}$ | $\{\sigma_6\}$ | $\{\sigma_6\}$ | $\{\sigma_4, \sigma_6\}$ |
| $l_7$ | $\{\sigma_2, \sigma_3, \sigma_5\}$ | $\{\sigma_1, \sigma_2, \sigma_3, \sigma_5\}$ | $\{\sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ | $\{\sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ |
| $l_8$ | $\{\sigma_2, \sigma_5\}$ | $\{\sigma_6\}$ | $\{\sigma_2\}$ | $\{\sigma_2, \sigma_3\}$ |

From Table 6 created for the tphs-game of Player 1, we can say that if Player 1 selects the strategy $l_1$ when Player 2 selects the startegy $\check{l}_1$ then the laptops $\sigma_2$ and $\sigma_3$ of company (of Player 1) will profit (according to that of the other company). In this case, it is obvious that the laptops $\sigma_2$ and $\sigma_3$ of the company (of Player 2) will cause loss (not profit). Other payoffs can be interpreted similarly.

Since the first player of this game (i.e. creating the Table 6) is Player 1, this player will consider the maximum profit. In contrast, Player 2 will also aim for a minimum loss when Player 1 makes (maximum) profit.

By Definition 3.13, we determine the following results

$\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_1) = \{\sigma_2, \sigma_3\} \cup \{\sigma_2\} \cup \{\sigma_3\} \cup \{\sigma_5\} \cup \{\sigma_5\} \cup \{\sigma_3, \sigma_5\} \cup \{\sigma_2, \sigma_3, \sigma_5\} \cup \{\sigma_2, \sigma_5\} = \{\sigma_2, \sigma_3, \sigma_5\}$,

$\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_2) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$, $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_3) = \{\sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$, $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_4) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$

and

$\bigcap_{q=1}^{4} \gamma_{H_\kappa}(l_1, \check{l}_q) = \{\sigma_2, \sigma_3\} \cap \{\sigma_2, \sigma_6\} \cap \{\sigma_3\} \cap \{\sigma_2, \sigma_5\} = \emptyset$,

$\bigcap_{q=1}^{4} \gamma_{H_\kappa}(l_2, \check{l}_q) = \emptyset$, $\bigcap_{q=1}^{4} \gamma_{H_\kappa}(l_3, \check{l}_q) = \emptyset$, $\bigcap_{q=1}^{4} \gamma_{H_\kappa}(l_4, \check{l}_q) = \emptyset$,

$\bigcap_{q=1}^{4} \gamma_{H_\kappa}(l_5, \check{l}_q) = \emptyset$, $\bigcap_{q=1}^{4} \gamma_{H_\kappa}(l_6, \check{l}_q) = \emptyset$, $\bigcap_{q=1}^{4} \gamma_{H_\kappa}(l_7, \check{l}_q) = \{\sigma_2, \sigma_3, \sigma_5\}$, $\bigcap_{q=1}^{4} \gamma_{H_\kappa}(l_8, \check{l}_q) = \emptyset$.

Clearly, it is seen that $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_1) = \bigcap_{q=1}^{4} \gamma_{H_\kappa}(l_7, \check{l}_q) = \{\sigma_2, \sigma_3, \sigma_5\}$.

Since the intersection of the seventh row is equal to the union of the first column, therefore $(l_7, \check{l}_1)$ is the hypersoft saddle point and $\{\sigma_2, \sigma_3, \sigma_5\}$ is the hypersoft saddle point value or simply value of this tphs-game. Consequently, Player 1 chooses the strategy $l_7$ for maximum profit and then Player 2 chooses $\check{l}_1$ for minimum loss (when Player 1 makes maximum profit). The laptops $\sigma_2, \sigma_3$ and $\sigma_5$ are the maximum profit for company (of Player 1) and the minimum loss for company (of Player 2) when the company (of Player 1) makes maximum profit.

**Note 2.** Every tphs-game has not a hypersoft saddle point. Let us explain some situations where hypersoft saddle point will not be found using Table 6.

(i) From Table 6, it is obvious that $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_1) = \gamma_{H_\kappa}(l_7, \check{l}_1)$, $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_1) \subset \gamma_{H_\kappa}(l_7, \check{l}_2)$, $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_1) = \gamma_{H_\kappa}(l_7, \check{l}_3)$ and $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_1) = \gamma_{H_\kappa}(l_7, \check{l}_4)$, but $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_2) \nsubseteq \gamma_{H_\kappa}(l_i, \check{l}_j)$ for all $i = 1, \ldots, 8$ and $j = 1, \ldots, 4$. Hence, if for all $q$ $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_q) \nsubseteq \gamma_{H_\kappa}(l_i, \check{l}_j)$ for all $i = 1, \ldots, 8$ and $j = 1, \ldots, 4$ then the hypersoft saddle point value cannot be found. That is, if $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_q)$ for all $q$ is not equal to or a subset of any of the components in Table 6 then the hypersoft saddle point value of this hypersoft game cannot be found. Now, we replace the first column-seventh row entry by $\{\sigma_2, \sigma_3, \sigma_5, \sigma_6\}$ then we have $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_1) = \{\sigma_2, \sigma_3, \sigma_5, \sigma_6\}$. Also, we know that $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_2) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$, $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_3) = \{\sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$ and $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_4) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$. It is clear that $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_q)$ for $q = 1,2,3,4$ are not equal to or a subset of any of the components in Table 6 (i.e., $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_q) \nsubseteq \gamma_{H_\kappa}(l_i, \check{l}_j)$ for $q = 1,2,3,4$ ). Thus, we can say that the hypersoft saddle point value of this hypersoft game cannot be found. On the other hand, it is easily seen that we have $\bigcup_{p=1}^{8} \gamma_{H_\kappa}(l_p, \check{l}_1) \neq \bigcap_{q=1}^{4} \gamma_{H_\kappa}(l_7, \check{l}_q)$ when we replace the first column-seventh row entry by $\{\sigma_2, \sigma_3, \sigma_5, \sigma_6\}$.

(ii) If $\bigcap_{q=1}^{4} \gamma_{H_\kappa}(l_p, \check{l}_q) = \emptyset$ for all $p$ (i.e., if the intersection of the components in each row of Table 6 is the empty set) then the hypersoft saddle point value cannot be found. Now, we replace the first column-seventh row entry by $\{\sigma_1, \sigma_6\}$ then we have $\bigcap_{q=1}^{4} \gamma_{H_\kappa}(l_7, \check{l}_q) = \emptyset$. Also, we know that $\bigcap_{q=1}^{4} \gamma_{H_\kappa}(l_p, \check{l}_q) = \emptyset$ for $p = 1,2,3,4,5,6,8$. Then, we can say that the hypersoft saddle point value of this hypersoft game cannot be found.

**Definition 3.15.** Let $\ddot{\Gamma}_\kappa = (\Gamma_\kappa, \mathfrak{F} \times \mathfrak{G})$ be a tphs-game over a given universal set $\mathfrak{R}$, where $\gamma_{H_\kappa}$ be a hypersoft payoff function. Then, we define the following:

1. The hypersoft upper value of the tphs-game $\ddot{\Gamma}_\kappa$ is denoted by $\overline{\Lambda}$ and described as

$$\overline{\Lambda} = \bigcap_{\check{l} \in \mathfrak{G}} \left( \bigcup_{l \in \mathfrak{F}} (\gamma_{H_\kappa}(l, \check{l})) \right) \quad (5)$$

2. The hypersoft lower value of the tphs-game $\ddot{\Gamma}_\kappa$ is denoted by $\underline{\Lambda}$ and described as

$$\underline{\Lambda} = \bigcup_{l \in \mathfrak{F}} \left( \bigcap_{\check{l} \in \mathfrak{G}} (\gamma_{H_\kappa}(l, \check{l})) \right) \quad (6)$$

3. If hypersoft upper and lower values of the tphs-game $\ddot{\Gamma}_\kappa$ are equal, i.e. $\overline{\Lambda} = \underline{\Lambda}$, then they are termed to be the value of this tphs-game, denoted by $\Lambda$.

**Theorem 3.16.** Let $\ddot{\Gamma}_\kappa = (\Gamma_\kappa, \mathfrak{F} \times \mathfrak{G})$ be a tphs-game over a given universal set $\mathfrak{R}$, where $\gamma_{H_\kappa}$ be a hypersoft payoff function. Also, let $\overline{\Lambda}$ and $\underline{\Lambda}$ be considered as hypersoft lower value and hypersoft upper value of the tp-hs-game $\ddot{\Gamma}_\kappa$, resp. Then, $\underline{\Lambda} \subseteq \overline{\Lambda}$.

**Proof.** Suppose that $\ddot{\Gamma}_\kappa = (\Gamma_\kappa, \mathfrak{F} \times \mathfrak{G})$ is a tphs-game, where $\mathfrak{F} = \{l_1, l_2, \dots, l_{s_1}\}$ and $\mathfrak{G} = \{\check{l}_1, \check{l}_2, \dots, \check{l}_{s_2}\}$ $(1 \leq s_1, s_2 \leq n)$ be the strategy sets for Player 1 and 2, respectively. Also, we assume that $\overline{\Lambda}$ and $\underline{\Lambda}$ are hypersoft lower value and hypersoft upper value of this tp-hs-game, respectively.

We take $l^{\#} \in \mathfrak{F}$ and $\check{l}^{\#} \in \mathfrak{G}$. Then, we obtain that

$$\underline{\Lambda} = \bigcup_{l \in \mathfrak{F}} \left( \bigcap_{\check{l} \in \mathfrak{G}} \left( \gamma_{\mathfrak{F} \times \mathfrak{G}}(l, \check{l}) \right) \right) \subseteq \bigcap_{\check{l} \in \mathfrak{G}} \left( \gamma_{\mathfrak{F} \times \mathfrak{G}}(l^{\#}, \check{l}) \right) \subseteq \gamma_{\mathfrak{F} \times \mathfrak{G}}(l^{\#}, \check{l}^{\#}) \subseteq \bigcup_{l \in \mathfrak{F}} \left( \gamma_{\mathfrak{F} \times \mathfrak{G}}(l, \check{l}^{\#}) \right) = \bigcap_{\check{l} \in \mathfrak{G}} \left( \bigcup_{l \in \mathfrak{F}} \left( \gamma_{\mathfrak{F} \times \mathfrak{G}}(l, \check{l}) \right) \right) = \overline{\Lambda}$$

Thus, proof is completed.

**Theorem 3.17.** Let $\ddot{\Gamma}_\kappa = (\Gamma_\kappa, \mathfrak{F} \times \mathfrak{G})$ be a tphs-game. Also let $\gamma_{H_\kappa}(l, \check{l})$ be a hypersoft saddle point, $\overline{\Lambda}$ and $\underline{\Lambda}$ are hypersoft lower and upper values of this tp-hs-game. Then, $\underline{\Lambda} \subseteq \gamma_{H_\kappa}(l^{\#}, \check{l}^{\#}) \subseteq \overline{\Lambda}$.

**Proof.** It can be shown in a similar way to the proof of Theorem 3.16. So it is omitted.

### 3.2. Hypersoft Elimination Method

**Definition 3.18.** Let $\ddot{\Gamma}_\kappa$ be a tphs-game with its hypersoft payoff function $\gamma_{H_\kappa}$. Then, (for each $\kappa = 1,2$)

1. a strategy $l_p \in \mathfrak{F}$ is said to be a hypersoft dominated to another strategy $l_{p'} \in \mathfrak{F}$ if for all $\check{l} \in \mathfrak{G}$,

$$\gamma_{H_\kappa}(l_{p'}, \check{l}) \subseteq \gamma_{H_\kappa}(l_p, \check{l}) \quad (7)$$

2. a strategy $\check{l}_q \in \mathfrak{G}$ is said to be a hypersoft dominated to another strategy $\check{l}_{q'} \in \mathfrak{G}$ if for all $l \in \mathfrak{F}$,

$$\gamma_{H_\kappa}(l, \check{l}_q) \subseteq \gamma_{H_\kappa}(l, \check{l}_{q'}) \quad (8)$$

**Note 3.** By using Definition 3.18, we may delete the rows and columns that are insignificant for the tphs-games, and repeating this method leads to a solution of a tphs-game. This technique is termed as a hypersoft elimination method.

**Example 3.19.** Let us consider a problem for determining the best site where two building contractors can build their new apartment complex in the following: Suppose $\mathfrak{R} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}$ denotes the set of sites which can be selected by the building contractors to build the new apartment complex. The choice of site is important, both for the progress of the construction process of the apartment complex and for the future profit, and therefore two contractors engage in a strategic game to determine the best location for their new apartment complex. In other words, they act as actors in this evaluation process. By considering the following multi-valued attributes, two building contractors (Player 1 and Player 2) determine their own strategies in this evaluation (game) process.

$\mathfrak{a}_1 = physical\ feature$ and $\mathfrak{A}_1 = \{\varkappa_1^1 = regular\ in\ shape, \varkappa_2^1 = exact\ boundaries, \varkappa_3^1 = facing\ S-N\ direction\}$

$\mathfrak{a}_2 = soil\ condition$ and $\mathfrak{A}_2 = \{\varkappa_1^2 = clay, \varkappa_2^2 = sandy, \varkappa_3^2 = rocky\}$

$\mathfrak{a}_3 = locality$ and $\mathfrak{A}_3 = \{\varkappa_1^3 = near\ hospital, \varkappa_2^3 = crowded, \varkappa_3^3 = near\ bank, \varkappa_4^3 = good\ neighbour\}$

$\mathfrak{a}_4 = cost\ (in\ dollar)$ and $\mathfrak{A}_4 = \{\varkappa_1^4 = 5000 - 10000, \varkappa_2^4 = 3000 - 5000, \varkappa_3^4 = 10000 - 12000\}$

Player 1 determines the strategies as $l_1 = (\varkappa_1^1, \varkappa_3^2, \varkappa_1^3, \varkappa_2^4)$, $l_2 = (\varkappa_2^1, \varkappa_3^2, \varkappa_3^3, \varkappa_1^4)$, $l_3 = (\varkappa_3^1, \varkappa_2^2, \varkappa_1^3, \varkappa_1^4)$, $l_4 = (\varkappa_3^1, \varkappa_2^2, \varkappa_3^3, \varkappa_1^4)$, and thus creates the strategy set as $\mathfrak{F} = \{l_1, l_2, l_3, l_4\}$.

Player 2 determines the strategies as $\check{l}_1 = (\varkappa_1^1, \varkappa_2^2, \varkappa_1^3, \varkappa_3^4)$, $\check{l}_2 = (\varkappa_1^1, \varkappa_2^2, \varkappa_4^3, \varkappa_3^4)$, $\check{l}_3 = (\varkappa_2^1, \varkappa_3^2, \varkappa_2^3, \varkappa_3^4)$, and thus creates the strategy set as $\mathfrak{G} = \{\check{l}_1, \check{l}_2, \check{l}_3\}$.

Then, tphs-game of Player 1 is given as in Table 7.

Table 7. The tphs-game of first building contractor (Player 1).

| $\ddot{\Gamma}_1$ | $\check{l}_1$ | $\check{l}_2$ | $\check{l}_3$ |
|---|---|---|---|
| $l_1$ | $\{\sigma_1,\sigma_3\}$ | $\{\sigma_3\}$ | $\{\sigma_5\}$ |
| $l_2$ | $\{\sigma_1,\sigma_2\}$ | $\{\sigma_1\}$ | $\{\sigma_1,\sigma_3\}$ |
| $l_3$ | $\{\sigma_1,\sigma_4,\sigma_5\}$ | $\{\sigma_1,\sigma_5\}$ | $\{\sigma_6\}$ |
| $l_4$ | $\{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6\}$ | $\{\sigma_1,\sigma_3,\sigma_5\}$ | $\{\sigma_1,\sigma_3,\sigma_5,\sigma_6\}$ |

The second column is dominated by the first column. Therefore, the first column is deleted from Table 7. Then, the last row dominates each of first, second and third rows and so these rows are deleted. As a result, Table 8 is created as follows:

Table 8. The reduced form of tphs-game of Player 1.

| $\ddot{\Gamma}_1$ | $\check{l}_2$ | $\check{l}_3$ |
|---|---|---|
| $l_4$ | $\{\sigma_1,\sigma_3,\sigma_5\}$ | $\{\sigma_1,\sigma_3,\sigma_5,\sigma_6\}$ |

The first column is dominated by the second column in Table 8. By deleting the second column, we obtain the solution (optimal strategy) $(l_4,\check{l}_2)$ and the value of this tphs-game is $\{\sigma_1,\sigma_3,\sigma_5\}$.

**Note 4.** (i) In Example 3.19 (for Table 7), if we use hypersoft saddle point method then we obtain the solution (optimal strategy) $(l_4,\check{l}_2)$ and the value of this tphs-game is $\{\sigma_1,\sigma_3,\sigma_5\}$.
(ii) If we take Table 9 instead of Table 7.

Table 9. The tphs-game of first building contractor (Player 1).

| $\ddot{\Gamma}_1$ | $\check{l}_1$ | $\check{l}_2$ | $\check{l}_3$ |
|---|---|---|---|
| $l_1$ | $\{\sigma_1,\sigma_3\}$ | $\{\sigma_3\}$ | $\{\sigma_2\}$ |
| $l_2$ | $\{\sigma_1,\sigma_2\}$ | $\{\sigma_1\}$ | $\{\sigma_6,\sigma_7\}$ |
| $l_3$ | $\{\sigma_1,\sigma_4,\sigma_5\}$ | $\{\sigma_1,\sigma_5\}$ | $\{\sigma_6\}$ |
| $l_4$ | $\{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6\}$ | $\{\sigma_1,\sigma_3,\sigma_5\}$ | $\{\sigma_2,\sigma_6,\sigma_7\}$ |

Then we cannot obtain a solution using hypersoft saddle point method (considering Note 2 (ii)). For this tphs-game, the hypersoft elimination method can use and the solution is obtained.
(iii) We consider the tphs-game in Table 6. We have a solution (optimal strategy) using hypersoft saddle point method. However, it is easily seen that the hypersoft elimination method cannot be used for this tphs-game. Because each column does not dominate any other column and each row does not dominate any other row.

### 3.3. Hypersoft Nash Equilibrium

In this part, we describe hypersoft Nash equilibrium, which is a different solution method for the tphs-games.

**Definition 3.20.** Let $\ddot{\Gamma}_\kappa$ be a tphs-game with its hypersoft payoff function $\gamma_{H_\kappa}$ for $\kappa = 1,2$. If the following features are satisfied then $(l^{\#},\check{l}^{\#}) \in \mathfrak{F} \times \mathfrak{G}$ is said to be a hypersoft Nash equilibrium of the tphs-game $\ddot{\Gamma}_\kappa$.
1. $\gamma_{H_1}(l,\check{l}^{\#}) \subseteq \gamma_{H_1}(l^{\#},\check{l}^{\#})$ for each $l \in \mathfrak{F}$
2. $\gamma_{H_2}(l^{\#},\check{l}) \subseteq \gamma_{H_2}(l^{\#},\check{l}^{\#})$ for each $\check{l} \in \mathfrak{G}$

Note that if $(l^{\#},\check{l}^{\#}) \in \mathfrak{F} \times \mathfrak{G}$ is a hypersoft Nash equilibrium of a tphs-game then Player 1 can gain at least $\gamma_{H_1}(l^{\#},\check{l}^{\#})$ by choosing the strategy $l^{\#} \in \mathfrak{F}$ and Player 2 can gain at least $\gamma_{H_2}(l^{\#},\check{l}^{\#})$ by choosing the strategy $\check{l}^{\#} \in \mathfrak{G}$. So the hypersoft Nash equilibrium is an optimal action for the tphs-game, and thus $\gamma_{H_\kappa}(l^{\#},\check{l}^{\#})$ is a solution of the tphs-game for Player $\kappa$ ($\kappa = 1,2$).

Note that the example of hypersoft Nash equilibrium will be given for multiplayer in the next section, so it is avoided in this subsection.

### 3.4. An Application for Combination of Solution Methods

Now, we give an application for the combined use of the hypersoft elimination method (or hypersoft dominated strategy) and hypersoft saddle point method.

**Example 3.21.** Let us consider the problem in Example 3.19. Also, we take the tphs-game of Player 1 in Table 10.

Table 10. The tphs-game of first contractor (Player 1).

| $\ddot{\Gamma}_1$ | $\check{l}_1$ | $\check{l}_2$ | $\check{l}_3$ |
|---|---|---|---|
| $l_1$ | $\{\sigma_1,\sigma_3\}$ | $\{\sigma_3\}$ | $\{\sigma_7\}$ |
| $l_2$ | $\{\sigma_1\}$ | $\{\sigma_1\}$ | $\{\sigma_1,\sigma_3\}$ |
| $l_3$ | $\{\sigma_1,\sigma_3,\sigma_5\}$ | $\{\sigma_1,\sigma_5\}$ | $\{\sigma_6\}$ |
| $l_4$ | $\{\sigma_1,\sigma_3,\sigma_5\}$ | $\{\sigma_1,\sigma_3,\sigma_5\}$ | $\{\sigma_1,\sigma_3,\sigma_5,\sigma_6\}$ |

We try to solve this problem using both hypersoft dominated strategy and hypersoft saddle point methods.

The first column is removed from Table 10 because this column dominates the second column. Then, since each of second and third rows is dominated by the fourth row, these rows are deleted. As aresult, Table 11 is constructed as follows:

Table 11. The reduced form of tphs-game (in Table 10).

| $\ddot{\Gamma}_1$ | $\check{l}_2$ | $\check{l}_3$ |
|---|---|---|
| $l_1$ | $\{\sigma_3\}$ | $\{\sigma_7\}$ |
| $l_4$ | $\{\sigma_1,\sigma_3,\sigma_5\}$ | $\{\sigma_1,\sigma_3,\sigma_5,\sigma_6\}$ |

There is no another hypersoft dominated strategy in Table 11. Now we can use hypersoft saddle point method.

From Table 11, we compute $\bigcup_{p=1,4} \gamma_{H_1}(l_p,\check{l}_2)$=$\gamma_{H_1}(l_1,\check{l}_2) \cup \gamma_{H_1}(l_4,\check{l}_2)$=$\{\sigma_3\} \cup \{\sigma_1,\sigma_3,\sigma_5\}$=$\{\sigma_1,\sigma_3,\sigma_5\}$,
$\bigcup_{p=1,4} \gamma_{H_1}(l_p,\check{l}_3)$=$\{\sigma_1,\sigma_3,\sigma_5,\sigma_6,\sigma_7\}$, $\bigcap_{q=2,3} \gamma_{H_1}(l_1,\check{l}_q)$=$\emptyset$, and $\bigcap_{q=2,3} \gamma_{H_1}(l_4,\check{l}_q)$=$\{\sigma_1,\sigma_3,\sigma_5\}$.

Then, it is seen that $\bigcup_{p=1,4} \gamma_{H_\kappa}(l_p,\check{l}_2) = \bigcap_{q=2,3} \gamma_{H_\kappa}(l_4,\check{l}_q) = \{\sigma_1,\sigma_3,\sigma_5\}$. Thus, we say that the optimal strategy of the tphs-game $\ddot{\Gamma}_1$ is $(l_4,\check{l}_2)$ and the value of the tphs-game is $\{\sigma_1,\sigma_3,\sigma_5\}$.

**Note 5.** In Example 3.21, if we use hypersoft saddle point method then we obtain the optimal strategies of the tphs-game $\ddot{\Gamma}_1$ are $(l_4,\check{l}_1)$ and $(l_4,\check{l}_2)$ and the value of the tphs-game is $\{\sigma_1,\sigma_3,\sigma_5\}$. But since the second column is dominated by the first column according to the dominated strategy method, it is obvious that it would make sense to choose $(l_4, \check{l}_2)$ as the optimal strategy. In such cases where the optimal strategy is more than one, changing the method or using them together will bring us closer to the real/convincing solution. This is how we can determine which solution method to use.

**Note 6.** From Notes 2,4, and 5, we can say that when the optimal strategy (solution) of the tphs-game cannot be found using any of the proposed solution methods, this does not mean that there is no optimal strategy. In such cases, the optimal strategy should be sought by using a different solution method or combining the solution methods. The same is true for reducing the number of optimal strategies if the number of optimal strategies is more than one, and for determining the convincing optimal strategy. The path to be followed for the selection of the solution method(s) should be determined according to the data presented in the tphs-game.

## 4 n-person hypersoft games

In many real-world applications, the hypersoft games can sometimes be played between three or more players. Therefore, tphs-games can be extended to $n$-person hypersoft games (in short $n$phs-games).

**Definition 4.1.** Let $\mathfrak{R}$ be an alternative set. Also, let $\mathfrak{F}_\kappa = \{l_1, l_2, \dots, l_{s_\kappa}\}$ $(1 \leq s_\kappa \leq n)$ be a set of strategies of Player $\kappa$ $(\kappa = 1,2,\dots,n)$ and $\gamma_{H_\kappa}^{\ n}\colon \otimes_{\kappa=1}^{n} \mathfrak{F}_\kappa \to Pow(\mathfrak{R})$ be a hypersoft payoff function of Player $\kappa$ where $\otimes_{\kappa=1}^{n} \mathfrak{F}_\kappa = \mathfrak{F}_1 \times \mathfrak{F}_2 \times \dots \times \mathfrak{F}_n$. For each Player $\kappa$, an $n$-person hypersoft game (nphs-game) over $\mathfrak{R}$ is defined by a hypersoft set given as

$$\begin{aligned}\ddot{\Gamma}_\kappa^n &= (\Gamma_\kappa^n, \otimes_{\kappa=1}^{n} \mathfrak{F}_\kappa)\\ &= \{((l_{p_1}, l_{p_2}, l_{p_3}, \dots, l_{p_n}), \gamma_{H_\kappa}^{\ n}(l_{p_1}, l_{p_2}, l_{p_3}, \dots, l_{p_n}))\colon\\ &\qquad (l_{p_1}, l_{p_2}, l_{p_3}, \dots, l_{p_n}) \in \otimes_{\kappa=1}^{n} \mathfrak{F}_\kappa\}\end{aligned} \tag{9}$$

The nphs-game is played as follows. Suppose that Player 1 chooses a strategy $l_{p_1} \in \mathfrak{F}_1$ and simultaneously Player $\mu$ $(\mu = 2,3,\dots,n)$ chooses the strategy $l_{p_\mu} \in \mathfrak{F}_\mu$ $(\mu = 2,3,\dots,n)$. When the $n$ players interact, then each Player $\kappa$ $(\kappa = 1,2,3,\dots,n)$ receives the hypersoft payoffs $\gamma_{H_\kappa}^{\ n}(l_{p_1}, l_{p_2}, l_{p_3}, \dots, l_{p_n})$ of hypersoft game $\ddot{\Gamma}_\kappa^n$.

**Definition 4.2.** Let

$$\ddot{\Gamma}_\kappa^n = \{((l_{p_1}, l_{p_2}, l_{p_3}, \dots, l_{p_n}), \gamma_{H_\kappa}^{\ n}(l_{p_1}, l_{p_2}, l_{p_3}, \dots, l_{p_n}))\colon (l_{p_1}, l_{p_2}, l_{p_3}, \dots, l_{p_n}) \in \otimes_{\kappa=1}^{n} \mathfrak{F}_\kappa\}$$

be an nphs-game. Then, a strategy $l_{p_\kappa} \in \mathfrak{F}_\kappa$ is called a hypersoft dominated to another strategy $l_p \in \mathfrak{F}_\kappa$ if

$$\begin{aligned}&\gamma_{H_\kappa}^{\ n}(l_{p_1}, \dots, l_{p_{\kappa-1}}, l_p, l_{p_{\kappa+1}}, \dots, l_{p_n})\\ &\subseteq \gamma_{H_\kappa}^{\ n}(l_{p_1}, \dots, l_{p_{\kappa-1}}, l_{p_\kappa}, l_{p_{\kappa+1}}, \dots, l_{p_n})\end{aligned} \tag{10}$$

for each strategy $l_{p_\tau} \in \mathfrak{F}_\tau$ of Player $\tau$ $(\tau = 1,2,\dots,\kappa-1,\kappa,\kappa+1,\dots,n)$.

**Definition 4.3.** Let

$$\ddot{\Gamma}_\kappa^n = \{((l_{p_1}, l_{p_2}, l_{p_3}, \dots, l_{p_n}), \gamma_{H_\kappa}^{\ n}(l_{p_1}, l_{p_2}, l_{p_3}, \dots, l_{p_n}))\colon (l_{p_1}, l_{p_2}, l_{p_3}, \dots, l_{p_n}) \in \otimes_{\kappa=1}^{n} \mathfrak{F}_\kappa\}$$

be an nphs-game. If for each player $\kappa = 1,2,\dots,n$, the following property hold

$$\begin{aligned}&\gamma_{H_\kappa}^{\ n}(l_{p_1}^{\#}, \dots, l_{p_{\kappa-1}}^{\#}, l_p, l_{p_{\kappa+1}}^{\#}, \dots, l_{p_n}^{\#})\\ &\subseteq \gamma_{H_\kappa}^{\ n}(l_{p_1}^{\#}, \dots, l_{p_{\kappa-1}}^{\#}, l_{p_\kappa}^{\#}, l_{p_{\kappa+1}}^{\#}, \dots, l_{p_n}^{\#})\end{aligned} \tag{11}$$

for each $l_p \in \mathfrak{F}_\kappa$ then $(l_{p_1}^{\#}, l_{p_2}^{\#}, \dots, l_{p_n}^{\#}) \in \otimes_{\kappa=1}^{n} \mathfrak{F}_\kappa$ is said to be a hypersoft Nash equilibrium of this nphs-game $\ddot{\Gamma}_\kappa^n$.

Note that if $(l_{p_1}^{\#}, l_{p_2}^{\#}, \dots, l_{p_n}^{\#}) \in \otimes_{\kappa=1}^{n} \mathfrak{F}_\kappa$ is a hypersoft Nash equilibrium of an nphs-game then Player 1 can gain at least $\gamma_{H_1}^{\ n}(l_{p_1}^{\#}, l_{p_2}^{\#}, \dots, l_{p_n}^{\#})$ by choosing the strategy $l_{p_1}^{\#} \in \mathfrak{F}_1$, Player 2 can gain at least $\gamma_{H_2}^{\ n}(l_{p_1}^{\#}, l_{p_2}^{\#}, \dots, l_{p_n}^{\#})$ by choosing the strategy $l_{p_2}^{\#} \in \mathfrak{F}_2$, and for each $\mu = 3,4,\dots,n$ Player $\mu$ can gain at least $\gamma_{H_\mu}^{\ n}(l_{p_1}^{\#}, l_{p_2}^{\#}, \dots, l_{p_n}^{\#})$ by choosing the strategy $l_{p_\mu}^{\#} \in \mathfrak{F}_\mu$. So, the hypersoft Nash equilibrium is an optimal action for the nphs-game, and thus $\gamma_{H_\kappa}^{\ n}(l_{p_1}^{\#}, l_{p_2}^{\#}, \dots, l_{p_n}^{\#})$ is a solution of the nphs-game for Player $\kappa$ $(\kappa = 1,2,\dots,n)$.

Now, we consider the following example for hypersoft Nash equilibrium of an nphs-game.

**Example 4.4.** Suppose that three restaurants in a city, such as Player 1, Player 2 and Player 3, want to increase the number of customers competitively. Let $\mathfrak{R} = \{\sigma_1 = \text{chicken dishes}, \sigma_2 = \text{meat dishes}, \sigma_3 = \text{fish dishes}, \sigma_4 = \text{juicy dishes}, \sigma_5 = \text{vegetable dishes}\}$ be a set of varieties of dishes on the menu of all three restaurants. By considering the following multi-valued attributes, Player 1, Player 2 and Player 3 determine their own strategies for this game.

$\mathfrak{a}_1 = flavour$ and $\mathfrak{A}_1 = \{\varkappa_1^1 = taste, \varkappa_2^1 = smell\}$
$\mathfrak{a}_2 = presentation$ and $\mathfrak{A}_2 = \{\varkappa_1^2 = service, \varkappa_2^2 = attitude\}$
$\mathfrak{a}_3 = performance$ and $\mathfrak{A}_3 = \{\varkappa_1^3 = price, \varkappa_2^3 = time\}$
Player 1 determines the strategies as $l_{1_1} = (\varkappa_1^1, \varkappa_1^2, \varkappa_1^3)$, $l_{1_2} = (\varkappa_1^1, \varkappa_1^2, \varkappa_2^3)$, and thus creates the strategy set as $\mathfrak{F}_1 = \{l_{1_1}, l_{1_2}\}$.
Player 2 determines the strategies as $l_{2_1} = (\varkappa_1^1, \varkappa_2^2, \varkappa_1^3)$, $l_{2_2} = (\varkappa_1^1, \varkappa_2^2, \varkappa_2^3)$, and thus creates the strategy set as $\mathfrak{F}_2 = \{l_{2_1}, l_{2_2}\}$.

Player 3 determines the strategies as $l_{3_1} = (\varkappa_1^1, \varkappa_2^2, \varkappa_1^3)$, $l_{3_2} = (\varkappa_2^1, \varkappa_1^2, \varkappa_1^3)$, $l_{3_3} = (\varkappa_1^1, \varkappa_1^2, \varkappa_2^3)$, and thus creates the strategy set as $\mathfrak{F}_3 = \{l_{3_1}, l_{3_2}, l_{3_3}\}$.

Then, 3phs-games of Player 1, Player 2 and Player 3 are presented in Table 12.

Table 12. The 3phs-games of Player 1, Player 2 and Player 3.

| The 3phs-game of Player $\kappa$ | | $\boldsymbol{l_{2_1}}$ | | | $\boldsymbol{l_{2_2}}$ | | |
|---|---|---|---|---|---|---|---|
| | | $\boldsymbol{l_{3_1}}$ | $\boldsymbol{l_{3_2}}$ | $\boldsymbol{l_{3_3}}$ | $\boldsymbol{l_{3_1}}$ | $\boldsymbol{l_{3_2}}$ | $\boldsymbol{l_{3_3}}$ |
| $\ddot{\Gamma}_1^3$ | $\boldsymbol{l_{1_1}}$ | $\{\sigma_1,\sigma_4,\sigma_5\}$ | $\{\sigma_1,\sigma_5\}$ | $\{\sigma_2,\sigma_4,\sigma_5\}$ | $\{\sigma_1,\sigma_4\}$ | $\{\sigma_3\}$ | $\{\sigma_4,\sigma_5\}$ |
| (for $\kappa=1$) | $\boldsymbol{l_{1_2}}$ | $\{\sigma_3,\sigma_5\}$ | $\{\sigma_1,\sigma_3,\sigma_5\}$ | $\{\sigma_3\}$ | $\{\sigma_2,\sigma_5\}$ | $\{\sigma_2,\sigma_4,\sigma_5\}$ | $\{\sigma_3\}$ |
| $\ddot{\Gamma}_2^3$ | $\boldsymbol{l_{1_1}}$ | $\{\sigma_3\}$ | $\{\sigma_2,\sigma_4,\sigma_5\}$ | $\{\sigma_1,\sigma_2,\sigma_5\}$ | $\{\sigma_1,\sigma_4,\sigma_5\}$ | $\{\sigma_1,\sigma_4,\sigma_5\}$ | $\{\sigma_3,\sigma_5\}$ |
| (for $\kappa=2$) | $\boldsymbol{l_{1_2}}$ | $\{\sigma_2,\sigma_4,\sigma_5\}$ | $\{\sigma_1,\sigma_5\}$ | $\{\sigma_1,\sigma_2\}$ | $\{\sigma_3,\sigma_4\}$ | $\{\sigma_1,\sigma_5\}$ | $\{\sigma_3,\sigma_5\}$ |
| $\ddot{\Gamma}_3^3$ | $\boldsymbol{l_{1_1}}$ | $\{\sigma_1,\sigma_3\}$ | $\{\sigma_2\}$ | $\{\sigma_1,\sigma_4,\sigma_5\}$ | $\{\sigma_1\}$ | $\{\sigma_1,\sigma_3,\sigma_5\}$ | $\{\sigma_3,\sigma_4\}$ |
| (for $\kappa=3$) | $\boldsymbol{l_{1_2}}$ | $\{\sigma_4,\sigma_5\}$ | $\{\sigma_1,\sigma_4,\sigma_5\}$ | $\{\sigma_4\}$ | $\{\sigma_2,\sigma_5\}$ | $\{\sigma_3,\sigma_4\}$ | $\{\sigma_4\}$ |

From Table 12, we obtain that

i. $\gamma_{H_1^3}(l_{1_s}, l_{2_1}, l_{3_2}) \subseteq \gamma_{H_1^3}(l_{1_2}, l_{2_1}, l_{3_2})$ for each $l_{1_s} \in \mathfrak{F}_1$ $(s=1,2)$

(That is, $\gamma_{H_1^3}(l_{1_1}, l_{2_1}, l_{3_2}) = \{\sigma_1,\sigma_5\}) \subseteq \gamma_{H_1^3}(l_{1_2}, l_{2_1}, l_{3_2}) = \{\sigma_1,\sigma_3,\sigma_5\}$ and $\gamma_{H_1^3}(l_{1_2}, l_{2_1}, l_{3_2}) = \gamma_{H_1^3}(l_{1_2}, l_{2_1}, l_{3_2}) = \{\sigma_1,\sigma_3,\sigma_5\}$)

ii. $\gamma_{H_2^3}(l_{1_2}, l_{2_t}, l_{3_2}) \subseteq \gamma_{H_2^3}(l_{1_2}, l_{2_1}, l_{3_2})$ for each $l_{2_t} \in \mathfrak{F}_1$ $(t=1,2)$

ii. $\gamma_{H_3^3}(l_{1_2}, l_{2_1}, l_{3_v}) \subseteq \gamma_{H_3^3}(l_{1_2}, l_{2_1}, l_{3_2})$ for each $l_{2_v} \in \mathfrak{F}_1$ $(v=1,2)$

then $(l_{1_2}, l_{2_1}, l_{3_2}) \in \bigotimes_{\kappa=1}^{3} \mathfrak{F}_\kappa$ is a hypersoft Nash equilibrium of this 3phs-game. Hence, $\gamma_{H_1^3}(l_{1_2}, l_{2_1}, l_{3_2}) = \{\sigma_1,\sigma_3,\sigma_5\}$, $\gamma_{H_2^3}(l_{1_2}, l_{2_1}, l_{3_2}) = \{\sigma_1,\sigma_5\}$ and $\gamma_{H_3^3}(l_{1_2}, l_{2_1}, l_{3_2}) = \{\sigma_1,\sigma_4,\sigma_5\}$ is the solution of the 3phs-game for Player 1, Player 2 and Player 3, respectively. That is, Player 1 can gain at least $\{\sigma_1,\sigma_3,\sigma_5\}$ by choosing the strategy $l_{1_2} \in \mathfrak{F}_1$, Player 2 can gain at least $\{\sigma_1,\sigma_5\}$ by choosing the strategy $l_{2_1} \in \mathfrak{F}_2$, and Player 3 can gain at least $\{\sigma_1,\sigma_4,\sigma_5\}$ by choosing the strategy $l_{3_2} \in \mathfrak{F}_3$. Here, for example, the sentence "Player 1 can gain at least $\{\sigma_1,\sigma_3,\sigma_5\}$ by choosing the strategy $l_{1_2} \in \mathfrak{F}_1$" means that if the first restaurant (Player 1) chooses the strategy $l_{1_2} = (\varkappa_1^1, \varkappa_1^2, \varkappa_2^3)$ (i.e. gives importance to taste, service and time) then it can find/attract more customers for the dishes $\{\sigma_1,\sigma_3,\sigma_5\}$ than other restaurants.

**Note 7.** Considering the formula (Eq. (1)) in Definition 2.8, a ratio for each player in Example 4.4 to win from this game can also be given. For example; for Player 1, Player 2, and Player 3 in Example 4.4, it is obtained as $\frac{|\{\sigma_1,\sigma_3,\sigma_5\}|}{|\Re|} = \frac{3}{5} = 0.6$ (60%), $\frac{|\{\sigma_1,\sigma_5\}|}{|\Re|} = 0.4$ (or 40%) and $\frac{|\{\sigma_1,\sigma_4,\sigma_5\}|}{|\Re|} = 0.6$ (60%), respectively.

## 5 Conclusions

In this article, the notion of the hypersoft game was newly introduced to address multi-valued attribute-based strategies with hypersoft payoffs. Then, we presented the tphs-games and their various properties with concrete examples. We also proposed different types of solution methods for the tphs-games and gave a real-life application based on the hypersoft elimination method. Finally, the tphs-game was extended by introducing the nphs-game. In conclusion, the main contributions can be summarized in the following.

(1) The procedures of tphs-games are established and the properties are adressed. Thus, the solution methods are developed for two-person soft games containing multi-valued attribute-based strategies, that is, the solutions can be obtained for complicated game theory problems in the soft set environment. In the other words, soft game theory is advanced from a different perspective based on multiple strategies.

(2) When hs-games are played between three or more players, it is discussed how to follow the solution steps and thus the principles of the nphs-game are presented. This provides the freedom to have as many players as desired in hs-games.

We hope that the findings in this study will offer new perspectives to researchers addressing various real-world problems. In the future, we may consider some possible studies in more detail. To more accurately express the thinking mode of human beings, we can develop some new extensions of hypersoft games, such as fuzzy hypersoft games, intuitionistic hypersoft games, neutrosophic hypersoft games, pilthogenic fuzzy hypersoft games. We can also study uncertainty-based hybrid forms of hypersoft game theory and their applications.

## 6 Author contribution statement

In the scope of this study, the authors contributed equally to formation of the idea, literature review, data collection, supplying the materials, conducting the analyses, design, writing and assessment of the obtained results.

## 7 Ethics committee approval and conflict of interest statement

There is no need to obtain permission from the ethics committee for the article prepared.

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# 8 References


[1] Zadeh LA. "Fuzzy sets". *Information and Control*, 8, 338-353, 1965.

[2] Atanassov KT. "Intuitionistic fuzzy sets". *Fuzzy Sets and Systems,* 20, 87-96, 1986.

[3] Smarandache F. "Neutrosophic set-a generalization of the intuitionistic fuzzy set". *International Journal of Pure and Applied Mathematics,* 24, 287-297, 2005.

[4] Kara A, Masri A, Kaya GK. "New branch location selection with AHP, ARAS and fuzzy TOPSIS: an example of a supplier company in the maritime industry". *Pamukkale University Journal of Engineering Sciences,* 28(1), 148-159, 2022.

[5] Otay İ, Kahraman C. "A novel circular intuitionistic fuzzy AHP&VIKOR methodology: An application to a multi-expert supplier evaluation problem". *Pamukkale University Journal of Engineering Sciences,* 28(1), 194-207, 2022.

[6] Oturakçı M, Yıldırım RB. "Analysis of supply chain risks by structural equation model and fuzzy analytical hierarchy process". *Pamukkale University Journal of Engineering Sciences,* 28(1), 117-127, 2022.

[7] Öztayşi B, Onar SC, Gündoğdu FK, Kahraman C. "Location-based advertisement selection using spherical fuzzy AHP-VIKOR". *Journal of Multiple-Valued Logic and Soft Computing,* 35, 5-23, 2020.

[8] Molodtsov D. "Soft set theory-first results". *Computers & Mathematics with Applications,* 37, 19-31, 1999.

[9] Maji PK, Biswas R, Roy AR. "Soft set theory". *Computers & Mathematics with Applications,* 45, 555-562, 2003.

[10] Çağman N, Enginoğlu S. "Soft set theory and uni-int decision making". *European Journal of Operational Research,* 207, 848-855, 2010.

[11] Aygün E, Kamacı H. "Some generalized operations in soft set theory and their role in similarity and decision making". *Journal of Intelligent & Fuzzy Systems,* 36, 6537-6547, 2019.

[12] Riaz M, Naeem K, Ahmad MO. "Novel concepts of soft sets with applications". *Annals of Fuzzy Mathematics and Informatics,* 13(2), 239-251, 2017.

[13] Atagün AO, Kamacı H, Oktay O. "Reduced soft matrices and generalized products with applications in decision making". *Neural Computing and Applications,* 29, 445-456, 2018.

[14] Kamacı H. "Similarity measure for soft matrices and its applications". *Journal of Intelligent & Fuzzy Systems,* 36(4), 3061-3072, 2019.

[15] Kamacı H, Atagün AO, Sönmezoğlu A. "Row-products of soft matrices with applications in multiple-disjoint decision making". *Applied Soft Computing***,** 62, 892-914, 2018.

[16] Petchimuthu S, Kamacı H. "The row-products of inverse soft matrices in multicriteria decision making". *Journal of Intelligent & Fuzzy Systems,* 36, 6425-6441, 2019.

[17] Çağman N, Karataş S. "Intuitionistic fuzzy soft set theory and its decision making". *Journal of Intelligent & Fuzzy Systems,* 24(4), 829-836, 2013.

[18] Karaaslan F. "Neutrosophic soft sets with applications in decision making". *International Journal of Information Science and Intelligent System***,** 4(2), 1-20, 2015.

[19] Deli I. "Interval-valued neutrosophic soft sets and its decision making". *International Journal of Machine Learning and Cybernetics,* 8, 665-676, 2017.

[20] Peng X, Yang Y, Song J, Jiang Y. "Pythagorean fuzzy soft set and its application". *Computer Engineering,* 41, 224-229, 2015.

[21] Akçetin E, Kamacı H. "Three-valued soft set and its multi-criteria group decision making via TOPSIS and ELECTRE". *Scientia Iranica E,* 28(6), 3719-3742, 2021.

[22] Fatimah F, Rosadi D, Hakim RBF, Alcantud JCR. "N-soft sets and their decision making algorithms". *Soft Computing,* 22, 3829-3842, 2018.

[23] Kamacı H. "Introduction to N-soft algebraic structures". *Turkish Journal of Mathematics,* 44(6), 2356-2379, 2020.

[24] Zhang H, Jia-Hua D, Yan C. "Multi-attribute group decision-making methods based on Pythagorean fuzzy N-soft sets". *IEEE Access,* 8, 62298-62309, 2020.

[25] Riaz M, Naeem K, Zareef I, Afzal D. "Neutrosophic N-soft sets with TOPSIS method for multiple attribute decision making". *Neutrosophic Sets and Systems,* 32, 1-24, 2020.

[26] Dalkılıç O, Demirtaş N, "VFP-soft sets and its application on decision making problems". *Journal of Polytechnic,* 24(4), 1391-1399, 2021.

[27] Smarandache F. "Extension of soft set to hypersoft set, and then to plithgenic hypersoft set". *Neutrosophic Sets and Systems,* 22, 168-170, 2018.

[28] Jafar MN, Saeed M, Saqlain M, Yang MS. "Trigonometric similarity measures for neutrosophic hypersoft sets with application to renewable energy source selection". *IEEE Access,* 9, 129178-129187, 2021.

[29] Kamacı H. "On hybrid structures of hypersoft sets and rough sets". *International Journal of Modern Science and Technology,* 6, 69-82, 2021.

[30] Martin N, Smarandache F. "Introduction to combined plithgenic hypersoft sets". *Neutrosophic Sets and Systems,* 35, 503-510, 2020.

[31] Saeed M, Ahsan M, Siddique MK, Ahmad MR. "A study of the fundamentals of hypersoft set theory". *International Journal of Scientific and Engineering Research,* 11: 320-329, 2020.

[32] Neumann J, Morgenstern O. *Theory of games and economic behavior*. Princeton University Press, New Jersey, USA, 1944.

[33] Nash J. "Noncooperative games". *Annals of Mathematics,* 54, 289-295, 1954.

[34] Campos L. "Fuzzy linear programming models to solve fuzzy matrix games". *Fuzzy Sets and Systems,* 32, 275-289, 1989.

[35] Seikh MR, Nayak PK, Pal M. "Matrix games with intuitionistic fuzzy pay-offs". *Journal of Information and Optimization Sciences,* 36, 159-181, 2015.

[36] Khalifa HA. "An approach for solving two-person zero-sum matrix games in neutrosophic environment". *Journal of Industrial and Systems Engineering,* 12, 186-198, 2019.

[37] Li S, Tu G. "Probabilistic linguistic matrix game based on fuzzy envelope and prospect theory with its application". *Mathematics,* 10, 1070, 2022.

[38] Xiao H, Zhang X, Lin D, Khalifa HAEW, Edalatpanah SA. "A new methodology for solving piecewise quadratic fuzzy cooperative continuous static games". *Advances in Mathematical Physics,* 2022, 8 pages, 2022.

[39] Scalzo V. "Other-regarding behaviour in fuzzy non-cooperative games: Existence of altruistic equilibria". *Fuzzy Sets and Systems,* in press, 2022. https://doi.org/10.1016/j.fss.2022.07.013

[40] Li S, Tu G. "Bi-matrix games with general intuitionistic fuzzy payoffs and application in corporative environmental behavior". *Symmetry,* 14, 671, 2022.

[41] Brikaa MG, Zheng Z, Dagestani AA, Ammar E-S, AlNemer G, Zakarya M. "Ambika approach for solving matrix games with payoffs of single-valued trapezoidal neutrosophic numbers". *Journal of Intelligent and Fuzzy Systems,* 42, 5139-5153, 2022.
[42] Deli I. Matrix Games with Simplified Neutrosophic Payoffs. Editors: Kahraman C, Otay I. Fuzzy Multi-Criteria Decision-Making Using Neutrosophic Sets. Studies in Fuzziness and Soft Computing, Springer, 369, 2019. https://doi.org/10.1007/978-3-030-00045-5_10
[43] Martinez RCJ, Paucar CEP, Arboleda JIC, Lierena MAG, Caballero EG. "Neutrosophic matrix games to solve project management conflicts". *Neutrosophic Sets and Systems,* 44, 10-17, 2021.
[44] Deli I, Çağman N. "Application of soft sets in decision making based on game theory". *Annals of Fuzzy Mathematics and Informatics,* 11, 425-438, 2016.
[45] Deli I, Çağman N. "Fuzzy soft games". *Filomat,* 29, 1901-1917, 2015.
[46] Mukherjee A, Debnath S. "Intuitionistic fuzzy soft game theory". *Songklanakarin Journal of Science and Technology,* 40, 409-417, 2018.
[47] Selvakumari K, Lavanya S. "Neutrosophic fuzzy soft game". *International Journal of Engineering and Technology,* 7, 667-669, 2018.
[48] Kamacı H. "Linguistic single-valued neutrosophic soft sets with applications in game theory". *International Journal of Intelligent Systems,* 36, 3917-3960, 2021.
[49] Kamacı H. "Games based on simplified neutrosophic multiplicative soft sets and their applications". *Neutrosophic Sets and Systems,* 47, 491-510, 2021.
[50] Gulistan M, Hassan N. "A generalized approach towards soft expert sets via neutrosophic cubic sets with applications in games". *Symmetry,* 11, 289, 2019.
[51] Abbas M, Murtaza G, Smarandache F. "Basic operations on hypersoft sets and hypersoftpoint". *Neutrosophic Sets and Systems*, 35, 407-421, 2020.
[52] Khosrow-Pour DBAM. "Encyclopedia of Information Science and Technology, Four Edition (10 Volumes, 8104 pages)". *IGI Global Publisher, USA.*