Soft Computing
https://doi.org/10.1007/s00500-022-06857-8

FOUNDATIONS

A theoretical and analytical approach to the conceptual framework of convexity cum concavity on fuzzy hypersoft sets with some generalized properties

Atiqe Ur Rahman[1] · **Muhammad Saeed**[1] · **Florentin Smarandache**[2]

Accepted: 22 January 2022

Abstract
Fuzzy hypersoft set (FHS-set) is an effective and flexible model as it not only minimizes the complexities of fuzzy set for dealing uncertainties, but also fulfills the parameterization requirements of soft set and fuzzy soft set. FHS-set is projected to address the limitations of these models regarding the entitlement of multi-argument approximate function. This kind of function maps the sub-parametric tuples to power set of universe. It emphasizes the partitioning of each attribute into its attribute-valued set that is missing in existing soft set-like structures. These features make it a completely new mathematical tool for solving problems dealing with uncertainties. As convexity has an essential function in optimization and control, pattern classification and recognition, image processing and in different fields of operation research, numerical analysis, etc. In order to tackle the various features of classical convexity (concavity) with uncertain environment of multi-argument approximate function, an articulate cum mathematical technique is utilized to develop a theoretical framework of convexity cum concavity on fuzzy hypersoft set which is more generalized and effective concept to deal with optimization relating problems. Moreover, some generalized properties like strictly convex (concave), strongly convex (concave), δ-inclusion and aggregation operations are established. The proposed study is authenticated with the provision of daily-life application based on proposed decision-making algorithm. Lastly, the features of proposed study are compared with the some existing relevant models to show its meritorious impact.

Keywords Convex soft set · Concave soft set · Hypersoft set · Fuzzy hypersoft set · Convex fuzzy hypersoft set · Concave fuzzy hypersoft set

1 Introduction

Zadeh (1965) initiated the concept of fuzzy sets. The theories like theory of probability, theory of fuzzy sets, and the interval mathematics are considered as mathematical means to tackle many intricate problems involving various uncertainties, in different fields of mathematical sciences. These theories have their own complexities which restrain them to solve these problems successfully. The reason for these hurdles is, possibly, the inadequacy of the parametrization tool. A mathematical tool is needed for dealing with uncertainties which should be free of all such impediments. Molodtsov (1999) has the honor to introduce such mathematical tool called soft sets in the literature as a new parameterized family of subsets of the universe of discourse. Maji et al. (2003) extended the concept and introduced some fundamental terminologies and operations like equality of two soft sets, subset and super set of a soft set, complement of a soft set, null soft set, absolute soft set, AND, OR and also the operations of union and intersection. They verified De Morgan's laws and a number of other results too. They also defined fuzzy soft set in Maji et al. (2001) and successfully applied it in decision making. Pei and Miao (2005) discussed the relationship between soft sets and information systems. They showed the soft set as a class of special information systems. Ali et al. (2009)

✉ Atiqe Ur Rahman
aurkhb@gmail.com

Muhammad Saeed
muhammad.saeed@umt.edu.pk

Florentin Smarandache
smarand@unm.edu

1 Department of Mathematics, University of Management and Technology, Lahore 54000, Pakistan

2 Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA

Published online: 24 February 2022

pointed several assertions in previous work of Maji et al. They defined new notions such as the restricted intersection, the restricted union, the restricted difference and the extended intersection of two soft sets. Babitha and Sunil (2010, 2011) introduced the concepts of soft set relation as a sub-soft set of the Cartesian product of the soft sets and also discussed many related concepts such as equivalent soft set relation, partition, composition and function. Sezgin and Atagün (2011), Ge and Yang (2011) and Li (2011) gave some modifications in the work of Maji et al. They also established some new results. Alcantud et al. (2017) employed the theory of valuation fuzzy soft set in decision making to valuate the assets. Feng et al. (2010), Liu et al. (2018) and Zhan and Zhu (2017) combined fuzzy soft set with rough set to tackle the vague information. Guan et al. (2013) discussed the order relations of fuzzy soft set along with some essential properties and applications. Hassan et al. (2017) initiated the gluing model of fuzzy soft expert set and applied it in the prediction of coronary artery disease. Khameneh and Kılıçman (2018) used fuzzy soft sets in three-way decision system for parametric reduction. Paik and Mondal (2020) employed the distance-similarity measures of fuzzy soft sets in decision-making process. Xiao (2018) and Zhang et al. (2020) discussed decision-making applications based on fuzzy soft set and fuzzy soft logic.
Deli (2019) defined convexity cum concavity on soft set and fuzzy soft set. Majeed (2016) investigated some more properties of convex soft sets. She developed the convex hull and the cone of a soft set with their generalized results. Salih and Sabir (2018) defined strictly and strongly convexity cum concavity on soft sets and they discussed their properties.
In some daily-life scenarios, it is necessitated to classify parameters into their respective parametric values in the form of non-overlapping sets. Soft set is inadequate for such scenarios. Smarandache (2018) introduced the concept of hypersoft set to tackle such scenarios with uncertain data. Saeed et al. (2021a) extended the concept and discussed the fundamentals of hypersoft set such as hypersoft subset, complement, not hypersoft set, aggregation operators along with hypersoft set relation, subrelation, complement relation, function, matrices and operations on hypersoft matrices. Abbas et al. (2020) investigated some properties of hypersoft points and hypersoft functions. They applied them to develop hypersoft function spaces. Rahman et al. (2021a, 2020, 2021b) developed hybridized structures of hypersoft set e.g., fuzzy parameterized hypersoft set, hypersoft hybrids with complex sets and bijective hypersoft sets, respectively. Saeed et al. (2021b) developed complex multi-fuzzy hypersoft set and discussed application in multi-criteria decision making based on its entropy and similarity measures. Martin and Smarandache (2020) initiated the concept of combined plithogenic hypersoft sets and discussed their properties with certain aggregations. Kamacı and Saqlain (2021) and Ihsan et al. (2021a, b) combined the theory of hypersoft sets with expert sets to tackle multi-decisive opinions of experts under uncertain environments. Debnath (2021) investigated the various rudiments of fuzzy hypersoft sets with numerical examples. Ahsan et al. (2021) employed an abstract approach to develop a framework of fuzzy hypersoft classes with certain properties.

1.1 Research gap and motivation

In many daily-life decision-making problems, we encounter with some scenarios where each attribute is required to be further classified into its respective attribute-valued set. Some examples of such scenarios are given below:

1. **Recruitment Process**: In this process, decision-makers usually use **qualification**, **age**, **experience** etc., as evaluating attributes. Since different candidates have different ages, qualifications and experiences so it is much pertinent to classify these attributes into their respective attribute values, i.e., ages (20 years, 25 years, etc.), qualifications (Graduate, Undergraduate, etc.) and experiences (5 years, 10 years, etc.).
2. **Product Selection**: In order to select a mobile from a mobile market, we usually prefer **RAM**, **ROM**, **Camera Resolution** etc., for its evaluation. As different mobile models are available with different RAMs, ROMs and Camera Resolutions so it is much better to classify these parameters into their respective sub-parametric valued disjoint sets, i.e., RAM (2 GB, 4 GB, 8 GB, etc.), ROM (32 GB, 64 GB, 128 GB, etc.) and Camera Resolution (5 Mega Pixels, 7 Mega pixels, etc.).
3. **Medical Diagnosis**: In order to diagnose heart diseases in patients, doctors (decision-makers) usually prefer **chest pain type**, **resting blood pressure**, **serum cholesterol** etc., as diagnostic parameters. After keen analysis, it is vivid that these parameters are required to be further partitioning into their sub-parametric values, i.e., chest pain type (typical angina, atypical angina, etc.), resting blood pressure (110 mmHg, 150 mmHg, 180 mmHg, etc.) and serum cholesterol (210 mg/dl, 320 mg/dl, 430 mg/dl, etc.).

In order to tackle such scenarios, hypersoft set is projected which employs the Cartesian product of disjoint attribute-valued sets as domain of approximate function (i.e., multi-argument approximate function). Fuzzy hypersoft set, a hybridized structure of fuzzy set and hypersoft set, assigns a fuzzy membership degree to each element in the universal set. The existing models like fuzzy set, soft set and fuzzy soft set are insufficient to deal uncertainties with such kind of approximate function. The vivid difference of fuzzy soft set and fuzzy hypersoft set is shown in Fig. 1 with the support of product selection decision making.

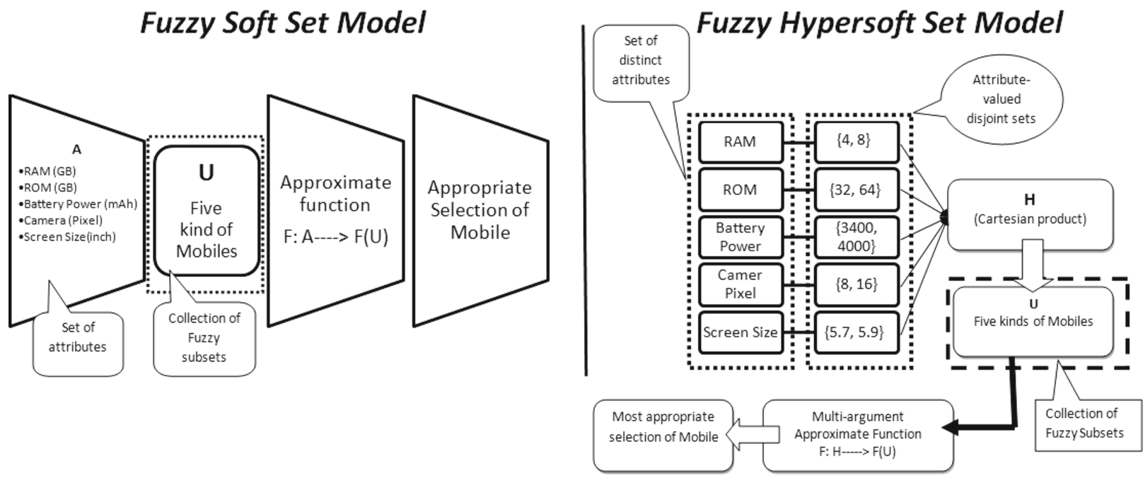

Fig. 1 Comparison of Fuzzy Soft Set and Fuzzy Hypersoft Set

Convexity is an essential concept in optimization, recognition and classification of certain patterns, processing and decomposition of images, discrete event simulation, duality problems and many other related topics in operation research, mathematical economics, numerical analysis and other mathematical sciences. Zadeh introduced the classical convexity under uncertain environment (fuzzy convex set) but it lacked parameterization tool so Deli resolved the problem and translated fuzzy convexity under soft and fuzzy soft set environment and Salih et al. extended the concept with the development of certain variants of convexity like strictly convexity, strongly convexity, etc., under soft set environment. These concepts are not capable to cope the convex optimization relating problems having further partitioning of parameters into sub-parametric values and multi-argument approximate function. Hence, it is the need of the literature to carve out a conceptual framework for solving such kind of problems under more generalized version, i.e., fuzzy hypersoft set. Therefore, to meet this demand, an abstract cum analytical approach is utilized to develop a basic framework of convexity and concavity on fuzzy hypersoft sets along with some important results inspiring from the above literature in general, and from Deli (2019), Salih and Sabir (2018) in specific. Pictorial version and examples of convexity and concavity on fuzzy hypersoft sets are presented first time in the literature.

1.2 Main contributions

The main contributions of this paper are summarized as follows.

1. A novel conceptual framework of classical convexity cum concavity under fuzzy hypersoft set is proposed. It tackles data or information in those convex optimization relating problems which involve uncertain fuzzy values attached with approximate elements. This leads to more precise results as compared to previous frameworks and contributes to make more reliable decisions.
2. In this proposed framework, some classical properties and results are investigated and proved under uncertain environment of fuzzy hypersoft set.
3. This proposed framework is equipped with strict and strong nature of classical convexity (concavity) with the provision of proofs of their essential properties.
4. The proposed framework is further authenticated with the help of proposed algorithm and applied in decision making to solve real-world problem.
5. The advantageous aspects of proposed framework are presented through its comparison with some existing relevant models.
6. Future directions and scope of the proposed work are mentioned with brief description on its implementation.

1.3 Organization of the paper

The rest of this article is structured as follows: In Sect. 2, some basic definitions and terms have been recalled from the literature to support main results. In Sect. refs3, convex and concave fuzzy hypersoft sets have been introduced along with some generalized results. In Sect. 4, strictly and strongly convex (concave) fuzzy hypersoft sets have been introduced along with some generalized results. In Sect. 5, an algorithm is proposed to solve real-life decision-making problem. In Sect. 6, the proposed study is compared with some existing relevant models. In Sect. 7, the paper has been concluded along with future directions and scope. Throughout the paper, G , $J^{\bullet}$, J°, $\sqcup$ and $P(\sqcup)$ will play the role of R^n, unit closed interval, unit open interval, universal set and power set, respectively.

2 Preliminaries

In this section, some fundamental terms regarding fuzzy set, soft set, hypersoft set, fuzzy hypersoft set and their convexity cum concavity are presented.

Definition 1 Zadeh (1965)
Suppose a universal set $\sqcup$ and a *fuzzy set* X is written as $X = \{(x, \alpha_X(x)) | x \in \sqcup\}$ such that

$$\alpha_X : \sqcup \rightarrow [0, 1]$$

where $\alpha_X(x)$ describes the membership percentage of $x \in \sqcup$.

Definition 2 Molodtsov (1999)
Let $\sqcup$ be an initial universe set and let E be a set of parameters. A pair (ζ_S, E) is called a *soft set* over $\sqcup$, where ζ_S is a mapping given by $\zeta_S : E \rightarrow P(\sqcup)$. In other words, a soft set (ζ_S, E) over $\sqcup$ is a parameterized family of subsets of $\sqcup$. For $\omega \in E$, $\zeta_S(\omega)$ may be considered as the set of ω-elements or ω-approximate elements of the soft set (ζ_S, E).

Definition 3 Maji et al. (2003)
Let (Φ_S, A) and (Ψ_S, B) be two soft sets over a common universe $\sqcup$,

1. we say that (Φ_S, A) is a *soft subset* of (Ψ_S, B) denoted by $(\Phi_S, A) \subseteq (\Psi_S, B)$ if
   i $A \subseteq B$, and
   ii $\forall \omega \in A$, $\Phi_S(\omega)$ and $\Psi_S(\omega)$ are identical approximations.
2. the *union* of (Φ_S, A) and (Ψ_S, B), denoted by $(\Phi_S, A) \cup (\Psi_S, B)$, is a soft set (ζ_S, C), where $C = A \cup B$ and $\omega \in C$,

$$\zeta_S(\omega) = \begin{cases} \Phi_S(\omega), & \omega \in A - B \\ \Psi_S(\omega), & \omega \in B - A \\ \Phi_S(\omega) \cup \Psi_S(\omega), & \omega \in A \cap B \end{cases}$$

3. the *intersection* of (Φ_S, A) and (Ψ_S, B) denoted by $(\Phi_S, A) \cap (\Psi_S, B)$, is a soft set (ζ_S, C), where $C = A \cap B$ and $\omega \in C$, $\zeta_S(\omega) = \Phi_S(\omega)$ or $\Psi_S(\omega)$ (as both are same set).

Definition 4 Maji et al. (2003)
The *complement* of a soft set (ζ_S, A), denoted by $(\zeta_S, A)^c$, is defined as $(\zeta_S, A)^c = ({\zeta_S}^c, \neg A)$ where

$${\zeta_S}^c : \neg A \rightarrow P(\sqcup)$$

is a mapping given by

$${\zeta_S}^c(\omega) = \sqcup - \zeta_S(\neg\omega) \forall \omega \in \neg A.$$

Definition 5 Maji et al. (2001)
Let $F(\sqcup)$ be the collection of all fuzzy sets over $\sqcup$; then, a pair (ζ_{FSS}, A) is called a fuzzy soft set over $\sqcup$, where

$$\zeta_{FSS} : A \rightarrow F(\sqcup)$$

is a mapping from A into $F(\sqcup)$.

Definition 6 Smarandache (2018)
Let $\sqcup$ be a universe of discourse, $P(\sqcup)$ the power set of $\sqcup$. Let $a_1, a_2, a_3, ..., a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are, respectively, the sets $A_1, A_2, A_3, ..., A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., n\}$. Then, the pair (ζ, G), where $G = A_1 \times A_2 \times A_3 \times ..... \times A_n$ and $\zeta : G \rightarrow P(\sqcup)$, is called a *hypersoft Set* over $\sqcup$.

Definition 7 Saeed et al. (2021a)
Let (Φ, G_1) and (Ψ, G_2) be two hypersoft sets over the same universal set $\sqcup$; then, their *union* $(\Phi, G_1) \cup (\Psi, G_2)$ is hypersoft set (ζ, G_3), where $G_3 = G_1 \cup G_2$; $G_1 = A_1 \times A_2 \times A_3 \times ..... \times A_n$, $G_2 = B_1 \times B_2 \times B_3 \times ..... \times B_n$ and $\forall\, e \in G_3$ with

$$\zeta(e) = \begin{cases} \Phi(e), & e \in G_1 - G_2 \\ \Psi(e), & e \in G_2 - G_1 \\ \Phi(e) \cup \Psi(e), & e \in G_2 \cap G_1 \end{cases}$$

Definition 8 Saeed et al. (2021a)
Let (Φ, G_1) and (Ψ, G_2) be two hypersoft sets over the same universal set $\sqcup$; then, their *intersection* $(\Phi, G_1) \cap (\Psi, G_2)$ is hypersoft set (ζ, G_3), where $G_3 = G_1 \cap G_2$; where $G_1 = A_1 \times A_2 \times A_3 \times ..... \times A_n$, $G_2 = B_1 \times B_2 \times B_3 \times ..... \times B_n$. and $\forall\, e \in G_3$ with $\zeta(e) = \Phi(e) \cap \Psi(e)$.

Definition 9 Smarandache (2018)
A hypersoft set (Φ, G) over a fuzzy universe of discourse is called *fuzzy hypersoft set* and denoted by (Φ_{FHS}, G).

More definitions and examples can be seen from Abbas et al. (2020) and Saeed et al. (2021a).

Definition 10 Deli (2019)
The δ-*inclusion* of a soft set (ζ_S, A) (where $\delta \subseteq \sqcup$) is defined by

$$(\zeta_S, A)^{\delta} = \{\omega \in A : h_S(\omega) \supseteq \delta\}$$

Definition 11 Deli (2019)
The soft set (ζ_S, A) on A is called a *convex soft set* if

$$\zeta_S(\epsilon\omega + (1 - \epsilon)\mu) \supseteq \zeta_S(\omega) \cap \zeta_S(\mu)$$

for every $\omega, \mu \in A$ and $\epsilon \in J^{\bullet}$.

Definition 12 Deli (2019)
The soft set (ζ_S, A) on A is called a *concave soft set* if

$$\zeta_S(\epsilon\omega + (1-\epsilon)\mu) \subseteq \zeta_S(\omega) \cup \zeta_S(\mu)$$

for every $\omega, \mu \in A$ and $\epsilon \in J^{\bullet}$.

Definition 13 Deli (2019)
The fuzzy soft set (ζ_{FS}, B) on B is called a *convex fuzzy soft set* if

$$\zeta_{FS}(\epsilon\omega + (1-\epsilon)\mu) \supseteq \zeta_{FS}(\omega) \cap \zeta_{FS}(\mu)$$

for every $\omega, \mu \in A$ and $\epsilon \in J^{\bullet}$.

Definition 14 Deli (2019)
The fuzzy soft set (ζ_{FS}, B) on B is called a *concave fuzzy soft set* if

$$\zeta_{FS}(\epsilon\omega + (1-\epsilon)\mu) \subseteq \zeta_{FS}(\omega) \cup \zeta_{FS}(\mu)$$

for every $\omega, \mu \in A$ and $\epsilon \in J^{\bullet}$.

Definition 15 Salih and Sabir (2018)
The soft set (ζ_S, A) on A is called a *strictly convex soft set* if

$$\zeta_S(\alpha\beta + (1-\alpha)\theta) \supset \zeta_S(\beta) \cap \zeta_S(\theta)$$

for every $\beta, \theta \in A$, $\zeta_S(\beta) \neq \zeta_S(\theta)$ and $\alpha \in J^{\circ} = (0, 1)$.

Definition 16 Salih and Sabir (2018)
The soft set (ζ_S, A) on A is called a *strictly concave soft set* if

$$\zeta_S(\alpha\beta + (1-\alpha)\theta) \subset \zeta_S(\beta) \cup \zeta_S(\theta)$$

for every $\beta, \theta \in A$, $\zeta_S(\beta) \neq \zeta_S(\theta)$ and $\alpha \in J^{\circ}$.

Definition 17 Salih and Sabir (2018)
The soft set (ζ_S, A) on A is called a *strongly convex soft set* if

$$\zeta_S(\alpha\beta + (1-\alpha)\theta) \supset \zeta_S(\beta) \cap \zeta_S(\theta)$$

for every $\beta, \theta \in A$, $\beta \neq \theta$ and $\alpha \in J^{\circ}$.

Definition 18 Salih and Sabir (2018)
The soft set (ζ_S, A) on A is called a *strongly concave soft set* if

$$\zeta_S(\alpha\beta + (1-\alpha)\theta) \subset \zeta_S(\beta) \cup \zeta_S(\theta)$$

for every $\beta, \theta \in A$, $\beta \neq \theta$ and $\alpha \in J^{\circ}$.

More about convex soft sets can be seen from Deli (2019), Majeed (2016).

3 Convex and concave fuzzy hypersoft sets

Here, convex fuzzy hypersoft sets and concave fuzzy hypersoft sets are defined and some important results are proved.

Definition 19 Let $F(\sqcup)$ be the collection of all fuzzy sets over $\sqcup$. Let $A_1, A_2, A_3, ..., A_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are, respectively, the sets $A_1, A_2, A_3, ..., A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., n\}$. Then, a *fuzzy hypersoft set* (ζ_{FHS}, G) over $\sqcup$ is defined by the set of ordered pairs as follows:

$$(\zeta_{FHS}, G) = \{(\underline{x}, \zeta_{FHS}(\underline{x})) : \underline{x} \in G, \zeta_{FHS}(\underline{x}) \in F(\sqcup)\}$$

where $\zeta_{FHS} : G \to F(\sqcup)$ and for all $\underline{x} \in G = A_1 \times A_2 \times A_3 \times ..... \times A_n$

$$\zeta_{FHS}(\underline{x}) = \{\mu_{\zeta_{FHS}(\underline{x})}(u)/u : u \in \sqcup, \mu_{\zeta_{FHS}(\underline{x})}(u) \in [0, 1]\}$$

is a fuzzy set over $\sqcup$.

Above definition is a modified version of fuzzy hypersoft set given in Smarandache (2018).

Definition 20 The $\delta-inclusion$ of a fuzzy hypersoft set (ζ_{FHS}, G) (where $\delta \subseteq \sqcup$) is defined by

$$(\zeta_{FHS}, G)^{\delta} = \left\{\underline{\omega} \in G : \zeta_{FHS}\left(\underline{\omega}\right) \supseteq \delta\right\}$$

Definition 21 The fuzzy hypersoft set (ζ_{FHS}, G) on $\sqcup$ is called a *convex fuzzy hypersoft set* if

$$\zeta_{FHS}\left(\epsilon\underline{\omega} + (1-\epsilon)\underline{\mu}\right) \supseteq \zeta_{FHS}\left(\underline{\omega}\right) \cap \zeta_{FHS}\left(\underline{\mu}\right)$$

for every $\underline{\omega}$, $\underline{\mu} \in G$ where $G = A_1 \times A_2 \times A_3 \times ..... \times A_n$ with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., n\}$; $\zeta_{FHS} : G \to F(\sqcup)$ and $\epsilon \in J^{\bullet}$.

Theorem 1 *$(f_{FHS}, S) \cap (g_{FHS}, T)$ is a convex fuzzy hypersoft set when both (f_{FHS}, S) and (g_{FHS}, T) are convex fuzzy hypersoft sets.*

Proof Suppose that $(f_{FHS}, S) \cap (g_{FHS}, T) = (\zeta_{FHS}, G)$ with $G = S \cap T$, for $\underline{\omega}_1, \underline{\omega}_2 \in G$; $\epsilon \in J^{\bullet}$, we have then

$$\zeta_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) = f_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \cap g_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right)$$

As (f_{FHS}, S) and (g_{FHS}, T) are convex fuzzy hypersoft sets,

$$f_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \supseteq f_{FHS}\left(\underline{\omega}_1\right) \cap f_{FHS}\left(\underline{\omega}_2\right)$$
$$g_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \supseteq g_{FHS}\left(\underline{\omega}_1\right) \cap g_{FHS}\left(\underline{\omega}_2\right)$$

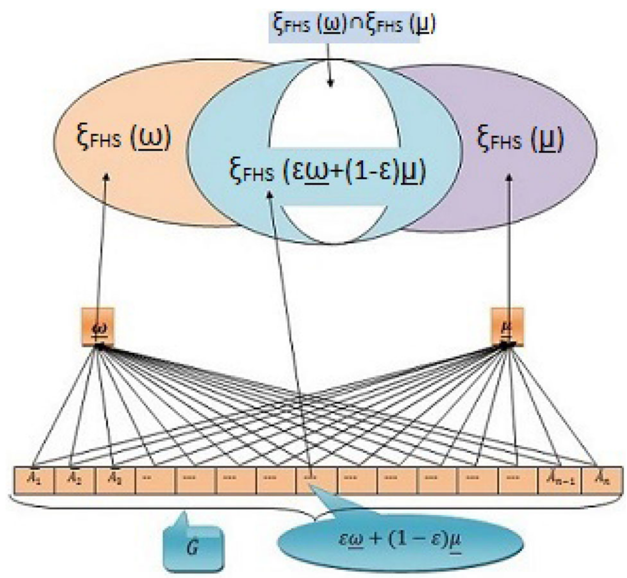

Fig. 2 Convex fuzzy hypersoft Set

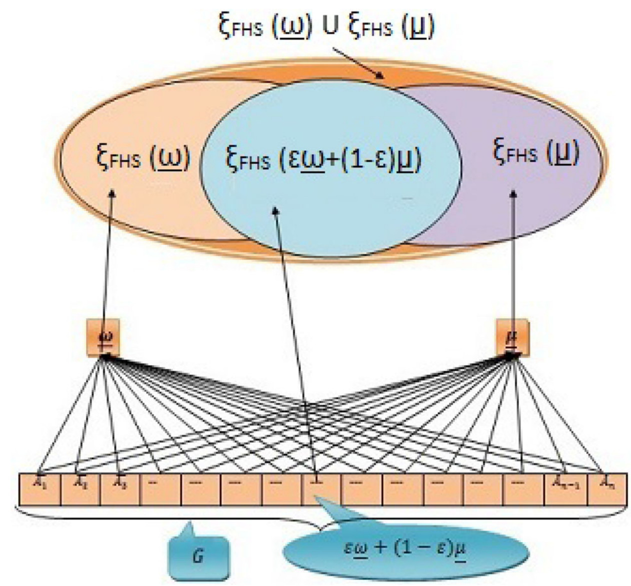

Fig. 3 Concave fuzzy hypersoft Set

which implies

$$\zeta_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \supseteq \left(f_{FHS}\left(\underline{\omega}_1\right) \cap f_{FHS}\left(\underline{\omega}_2\right)\right) \cap \left(g_{FHS}\left(\underline{\omega}_1\right) \cap g_{FHS}\left(\underline{\omega}_2\right)\right)$$

and thus

$$\zeta_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \supseteq \zeta_{FHS}\left(\underline{\omega}_1\right) \cap \zeta_{FHS}\left(\underline{\omega}_2\right)$$

□

Remark 1 If $\left\{(h^i{}_{FHS}, G_i) : i \in \{1, 2, 3, ...\}\right\}$ is any family of convex fuzzy hypersoft sets, then the intersection $\bigcap_{i\in I}(h^i{}_{FHS}, G_i)$ is a convex fuzzy hypersoft set.

Remark 2 The union of any family

$$\left\{(h^i{}_{FHS}, G_i) : i \in \{1, 2, 3, ...\}\right\}$$

of convex fuzzy hypersoft sets is not necessarily a convex fuzzy hypersoft set.

Theorem 2 (ζ_{FHS}, G) *is convex fuzzy hypersoft set iff for every* $\epsilon \in J^{\bullet}$ *and* $\delta \in F(\sqcup)$, $(\zeta_{FHS}, G)^{\delta}$ *is convex fuzzy hypersoft set.*

Proof Suppose (ζ_{FHS}, G) is convex fuzzy hypersoft set. If $\underline{\omega}, \underline{\mu} \in G$ and $\delta \in F(\sqcup)$, then $\zeta_{FHS}\left(\underline{\omega}\right) \supseteq \delta$ and $\zeta_{FHS}\left(\underline{\mu}\right) \supseteq \delta$, it implies that $\zeta_{FHS}\left(\underline{\omega}\right) \cap \zeta_{FHS}\left(\underline{\mu}\right) \supseteq \delta$. So we have,

$$\zeta_{FHS}\left(\epsilon\underline{\omega} + (1-\epsilon)\,\underline{\mu}\right) \supseteq \zeta_{FHS}\left(\underline{\omega}\right) \cap \zeta_{FHS}\left(\underline{\mu}\right) \supseteq \delta$$

$$\Rightarrow \quad \zeta_{FHS}\left(\epsilon\underline{\omega} + (1-\epsilon)\,\underline{\mu}\right) \supseteq \delta$$

thus $(\zeta_{FHS}, G)^{\delta}$ is convex fuzzy hypersoft set.
Conversely suppose that $(\zeta_{FHS}, G)^{\delta}$ is convex fuzzy hypersoft set for every $\epsilon \in J^{\bullet}$. For $\underline{\omega}, \underline{\mu} \in G$, $(\zeta_{FHS}, G)^{\delta}$ is convex fuzzy hypersoft set with $\delta = \zeta_{FHS}\left(\underline{\omega}\right) \cap \zeta_{FHS}\left(\underline{\mu}\right)$. Since $\zeta_{FHS}\left(\underline{\omega}\right) \supseteq \delta$ and $\zeta_{FHS}\left(\underline{\mu}\right) \supseteq \delta$, we have $\underline{\omega} \in (\zeta_{FHS}, G)^{\delta}$ and $\underline{\mu} \in (\zeta_{FHS}, G)^{\delta}$,
$\Rightarrow \quad \epsilon\underline{\omega} + (1-\epsilon)\,\underline{\mu} \in (\zeta_{FHS}, G)^{\delta}$.
Therefore,

$$\zeta_{FHS}\left(\epsilon\underline{\omega} + (1-\epsilon)\,\underline{\mu}\right) \supseteq \delta$$

So

$$\zeta_{FHS}\left(\epsilon\underline{\omega} + (1-\epsilon)\,\underline{\mu}\right) \supseteq \zeta_{FHS}\left(\underline{\omega}\right) \cap \zeta_{FHS}\left(\underline{\mu}\right),$$

Hence, (ζ_{FHS}, G) is convex fuzzy hypersoft set. □

Definition 22 The fuzzy hypersoft set (ζ_{FHS}, G) on $\sqcup$ is called a *concave fuzzy hypersoft set* if

$$\zeta_{FHS}\left(\epsilon\underline{\omega} + (1-\epsilon)\,\underline{\mu}\right) \subseteq \zeta_{FHS}\left(\underline{\omega}\right) \cup \zeta_{FHS}\left(\underline{\mu}\right)$$

for every $\underline{\omega} = (a_1, a_2, a_3, ..., a_n)$, $\underline{\mu} = (b_1, b_2, b_3, ..., b_n) \in G$ where, $G = A_1 \times A_2 \times A_3 \times ..... \times A_n$ with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., n\}$; $\zeta_{FHS} : G \to F(\sqcup)$ and $\epsilon \in J^{\bullet}$.

Example 1 Consider a set of mobiles as a universe of discourse $\sqcup = \{m_1, m_2, m_3, ..., m_5\}$. The attributes of mobiles under consideration is the set $Y = \{y_{11}, y_{12}, y_{13}\}$, where

y_{11} = Company
y_{12} = Resolution of Camera in MP
y_{13} = RAM/Storage in GB

such that the attribute values against these attributes, respectively, are the sets given as

$$Z_{11} = \{C_1, C_2, C_3, C_4, C_5\} = \{1, 2, 3, 4, 5\}$$
$$Z_{12} = \{6, 7, 8, 9, 10\}$$
$$Z_{13} = \{1/16, 2/32, 3/64, 4/128, 5/256\}$$

The fuzzy hypersoft set (ζ_{FHS}, G) is a function defined by the mapping $\zeta_{FHS} : G \to F(\sqcup)$ where $G = Z_{11} \times Z_{12} \times Z_{13}$. Consider $\underline{\beta} = (2, 6, 3/64)$, then

$\zeta_{FHS}(\underline{\beta}) = \zeta_{FHS}(2, 6, 3/64) = \{0.01/m_1, 0.05/m_5\}$.
Also, consider $\underline{\theta} = (3, 7, 2/32)$, then
$\zeta_{FHS}(\underline{\theta}) = \zeta_{FHS}(3, 7, 2/32) = \{0.01/m_1, 0.03/m_3, 0.04/m_4\}$

Now,
$\zeta_{FHS}(\underline{\beta}) \cap \zeta_{FHS}(\underline{\theta}) = \zeta_{FHS}(2, 6, 3/64) \cap \zeta_{FHS}(3, 7, 2/32)$
$\zeta_{FHS}(\underline{\beta}) \cap \zeta_{FHS}(\underline{\theta}) = \{0.01/m_1, 0.05/m_5\} \cap \{0.01/m_1, 0.03/m_3, 0.04/m_4\}$

$$\zeta_{FHS}(\underline{\beta}) \cap \zeta_{FHS}(\underline{\theta}) = \{0.01/m_1\} \tag{1}$$

Let $\alpha = 0.1 \in J^\bullet$; then, we have
$\alpha\underline{\beta} + (1-\alpha)\underline{\theta} = 0.1(2, 6, 3/64) + (1-0.1)(3, 7, 2/32) = 0.1(2, 6, 3/64) + 0.9(3, 7, 2/32) = (0.2, 0.6, 0.3/64) + (2.7, 6.3, 1.8/32) = (0.2+2.7, 0.6+6.3, 0.3/64+1.8/32) = (2.9, 6.9, 3.9/64)$
By using the decimal round off property, we get $(3, 7, 4/64) = (3, 7, 2/32)$

$$\zeta_{FHS}(\alpha\underline{\beta} + (1-\alpha)\underline{\theta}) = \zeta_{FHS}(3, 7, 2/32) = \{0.01/m_1, 0.03/m_3, 0.04/m_4\} \tag{2}$$

it is vivid from equations (1) and (2), we have

$$\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\underline{\theta}\right) \supseteq \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)$$

Similarly,
$\zeta_{FHS}(\underline{\beta}) \cup \zeta_{FHS}(\underline{\theta}) = \zeta_{FHS}(2, 6, 3/64) \cup \zeta_{FHS}(3, 7, 2/32)$
$\zeta_{FHS}(\underline{\beta}) \cap \zeta_{FHS}(\underline{\theta}) =$
$\{0.01/m_1, 0.05/m_5\} \cup \{0.01/m_1, 0.03/m_3, 0.04/m_4\}$

$$\zeta_{FHS}(\underline{\beta}) \cap \zeta_{FHS}(\underline{\theta}) = \{0.01/m_1, 0.03/m_3, 0.04/m_4, 0.05/m_5\} \tag{3}$$

From (2) and (3), we have

$$\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\underline{\theta}\right) \subseteq \zeta_{FHS}\left(\underline{\beta}\right) \cup \zeta_{FHS}\left(\underline{\theta}\right)$$

Theorem 3 *$(f_{FHS}, S) \cup (g_{FHS}, T)$ is a concave fuzzy hypersoft set when both (f_{FHS}, S) and (g_{FHS}, T) are concave fuzzy hypersoft sets.*

Proof Suppose that $(f_{FHS}, S) \cup (g_{FHS}, T) = (\zeta_{FHS}, G)$ with $G = S \cup T$, for $\underline{\omega}_1, \underline{\omega}_2 \in G$; $\epsilon \in J^\bullet$, we have then

$$\zeta_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) = f_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \cup g_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right)$$

As (f_{FHS}, S) and (g_{FHS}, T) are concave fuzzy hypersoft sets,

$$f_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq f_{FHS}\left(\underline{\omega}_1\right) \cup f_{FHS}\left(\underline{\omega}_2\right)$$
$$g_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq g_{FHS}\left(\underline{\omega}_1\right) \cup g_{FHS}\left(\underline{\omega}_2\right)$$

which implies

$$\zeta_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq (f_{FHS}\left(\underline{\omega}_1\right) \cup f_{FHS}\left(\underline{\omega}_2\right)) \cup (g_{FHS}\left(\underline{\omega}_1\right) \cup g_{FHS}\left(\underline{\omega}_2\right))$$

and thus

$$\zeta_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq \zeta_{FHS}\left(\underline{\omega}_1\right) \cup \zeta_{FHS}\left(\underline{\omega}_2\right)$$

□

Remark 3 If $\left\{(\breve{h}^i{}_{FHS}, G_i) : i \in \{1, 2, 3, ...\}\right\}$ is any family of concave fuzzy hypersoft sets, then the union $\bigcup_{i\in I}(\breve{h}^i{}_{FHS}, G_i)$ is a concave fuzzy hypersoft set.

Theorem 4 *$(f_{FHS}, S) \cap (g_{FHS}, T)$ is a concave fuzzy hypersoft set when both (f_{FHS}, S) and (g_{FHS}, T) are concave fuzzy hypersoft sets.*

Proof Suppose that $(f_{FHS}, S) \cap (g_{FHS}, T) = (\zeta_{FHS}, G)$ with $G = S \cap T$, for $\underline{\omega}_1, \underline{\omega}_2 \in G$; $\epsilon \in J^\bullet$, we have then

$$\zeta_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) = f_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \cap g_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right)$$

As (f_{FHS}, S) and (g_{FHS}, T) are concave fuzzy hypersoft sets,

$$f_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq f_{FHS}\left(\underline{\omega}_1\right) \cup f_{FHS}\left(\underline{\omega}_2\right)$$
$$g_{FHS}\left(\epsilon\underline{\omega}_1 + (1-\epsilon)\underline{\omega}_2\right) \subseteq g_{FHS}\left(\underline{\omega}_1\right) \cup g_{FHS}\left(\underline{\omega}_2\right)$$

which implies

$$\zeta_{FHS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\subseteq\left(f_{FHS}\left(\underline{\omega}_1\right)\cup f_{FHS}\left(\underline{\omega}_2\right)\right)\cap\left(g_{FHS}\left(\underline{\omega}_1\right)\cup g_{FHS}\left(\underline{\omega}_2\right)\right)$$

and thus

$$\zeta_{FHS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\subseteq\zeta_{FHS}\left(\underline{\omega}_1\right)\cup\zeta_{FHS}\left(\underline{\omega}_2\right)$$

□

Remark 4 The intersection of any family

$$\left\{(\breve{h}^i{}_{FHS},G_i):i\in\{1,2,3,...\}\right\}$$

of concave fuzzy hypersoft sets is a concave fuzzy hypersoft set.

Theorem 5 *$(\zeta_{FHS},G)^c$ is a convex fuzzy hypersoft set when (ζ_{FHS},G) is a concave fuzzy hypersoft set.*

Proof Suppose that for $\underline{\omega}_1,\ \underline{\omega}_2\in G,\ \epsilon\in J^\bullet$ and (ζ_{FHS},G) be concave fuzzy hypersoft set.
Since (ζ_{FHS},G) is concave fuzzy hypersoft set,

$$\zeta_{FHS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\subseteq\zeta_{FHS}\left(\underline{\omega}_1\right)\cup\zeta_{FHS}\left(\underline{\omega}_2\right)$$

or

$$\sqcup\backslash\,\zeta_{FHS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\supseteq\sqcup\backslash\left\{\zeta_{FHS}\left(\underline{\omega}_1\right)\cup\zeta_{FHS}\left(\underline{\omega}_2\right)\right\}$$

If $\zeta_{FHS}\left(\underline{\omega}_1\right)\supset\zeta_{FHS}\left(\underline{\omega}_2\right)$ then
$\zeta_{FHS}\left(\underline{\omega}_1\right)\cup\zeta_{FHS}\left(\underline{\omega}_2\right)=\zeta_{FHS}\left(\underline{\omega}_1\right)$
Therefore,

$$\sqcup\backslash\,\zeta_{FHS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\supseteq\sqcup\backslash\ \zeta_{FHS}\left(\underline{\omega}_1\right). \tag{4}$$

If $\zeta_{FHS}\left(\underline{\omega}_1\right)\subset\zeta_{FHS}\left(\underline{\omega}_2\right)$ then
$\zeta_{FHS}\left(\underline{\omega}_1\right)\cup\zeta_{FHS}\left(\underline{\omega}_2\right)=\zeta_{FHS}\left(\underline{\omega}_2\right)$
Therefore,

$$\sqcup\backslash\,\zeta_{FHS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\supseteq\sqcup\backslash\ \zeta_{FHS}\left(\underline{\omega}_2\right). \tag{5}$$

From (4) and (5) , we have

$$\sqcup\backslash\,\zeta_{FHS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\supseteq(\sqcup\backslash\zeta_{FHS}\left(\underline{\omega}_1\right))\cap(\sqcup\backslash\zeta_M\left(\underline{\omega}_2\right)).$$

So, $(\zeta_{FHS},G)^c$ is a convex fuzzy hypersoft set. □

Theorem 6 *$(\zeta_{FHS},G)^c$ is a concave fuzzy hypersoft set when (ζ_{FHS},G) is a convex fuzzy hypersoft set.*

Proof Suppose that for $\underline{\omega}_1,\ \underline{\omega}_2\in G,\ \epsilon\in J^\bullet$ and (ζ_{FHS},G) be convex fuzzy hypersoft set.
since (ζ_{FHS},G) is convex fuzzy hypersoft set,

$$\zeta_{FHS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\supseteq\zeta_{FHS}\left(\underline{\omega}_1\right)\cap\zeta_{FHS}\left(\underline{\omega}_2\right)$$

or

$$\sqcup\backslash\,\zeta_{FHS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\subseteq\sqcup\backslash\left\{\zeta_{FHS}\left(\underline{\omega}_1\right)\cap\zeta_{FHS}\left(\underline{\omega}_2\right)\right\}$$

If $\zeta_{FHS}\left(\underline{\omega}_1\right)\supset\zeta_{FHS}\left(\underline{\omega}_2\right)$ then
$\zeta_{FHS}\left(\underline{\omega}_1\right)\cap\zeta_{FHS}\left(\underline{\omega}_2\right)=\zeta_{FHS}\left(\underline{\omega}_2\right)$
Therefore,

$$\sqcup\backslash\,\zeta_{FHS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\subseteq\sqcup\backslash\ \zeta_{FHS}\left(\underline{\omega}_2\right). \tag{6}$$

If $\zeta_{FHS}\left(\underline{\omega}_1\right)\subset\zeta_{FHS}\left(\underline{\omega}_2\right)$ then
$\zeta_{FHS}\left(\underline{\omega}_1\right)\cap\zeta_{FHS}\left(\underline{\omega}_2\right)=\zeta_{FHS}\left(\underline{\omega}_1\right)$
Therefore,

$$\sqcup\backslash\,\zeta_{FHS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\subseteq\sqcup\backslash\ \zeta_{FHS}\left(\underline{\omega}_1\right). \tag{7}$$

From (6) and (7) , we have

$$\sqcup\backslash\,\zeta_{FHS}\left(\epsilon\underline{\omega}_1+(1-\epsilon)\underline{\omega}_2\right)\supseteq(\sqcup\backslash\zeta_{FHS}\left(\underline{\omega}_1\right))\cup(\sqcup\backslash\zeta_M\left(\underline{\omega}_2\right)).$$

So $(\zeta_{FHS},G)^c$ is a concave fuzzy hypersoft set. □

Theorem 7 *(ζ_{FHS},G) is concave fuzzy hypersoft set iff for every $\epsilon\in J^\bullet$ and $\delta\in F(\sqcup)$, $(\zeta_{FHS},G)^\delta$ is concave fuzzy hypersoft set.*

Proof Suppose (ζ_{FHS},G) is concave fuzzy hypersoft set. If $\underline{\omega},\ \underline{\mu}\in G$ and $\delta\in F(\sqcup)$, then $\zeta_{FHS}\left(\underline{\omega}\right)\supseteq\delta$ and $\zeta_{FHS}\left(\underline{\mu}\right)\supseteq\delta$, it implies that $\zeta_{FHS}\left(\underline{\omega}\right)\cup\zeta_{FHS}\left(\underline{\mu}\right)\supseteq\delta$. So we have,

$$\delta\subseteq\zeta_{FHS}\left(\underline{\omega}\right)\cap\zeta_{FHS}\left(\underline{\mu}\right)\subseteq\zeta_{FHS}\left(\epsilon\underline{\omega}+(1-\epsilon)\,\underline{\mu}\right)\subseteq\zeta_{FHS}\left(\underline{\omega}\right)\cup\zeta_{FHS}\left(\underline{\mu}\right)$$

$\Rightarrow\quad\delta\subseteq\zeta_{FHS}\left(\epsilon\underline{\omega}+(1-\epsilon)\,\underline{\mu}\right)$
thus $(\zeta_{FHS},G)^\delta$ is concave fuzzy hypersoft set.
Conversely, suppose that $(\zeta_{FHS},G)^\delta$ is concave fuzzy hypersoft set for every $\epsilon\in J^\bullet$. For $\underline{\omega},\ \underline{\mu}\in G$, $(\zeta_{FHS},G)^\delta$ is concave fuzzy hypersoft set with $\delta=\zeta_{FHS}\left(\underline{\omega}\right)\cup\zeta_{FHS}\left(\underline{\mu}\right)$.
Since $\zeta_{FHS}\left(\underline{\omega}\right)\subseteq\delta$ and $\zeta_{FHS}\left(\underline{\mu}\right)\subseteq\delta$, we have
$\underline{\omega}\in(\zeta_{FHS},G)^\delta$ and $\underline{\mu}\in(\zeta_{FHS},G)^\delta$,
$\Rightarrow\quad\epsilon\underline{\omega}+(1-\epsilon)\,\underline{\mu}\in(\zeta_{FHS},G)^\delta$.

Therefore,

$$\zeta_{FHS}\left(\epsilon\underline{\omega}+(1-\epsilon)\,\underline{\mu}\right)\subseteq\ \delta$$

So

$$\zeta_{FHS}\left(\epsilon\underline{\omega}+(1-\epsilon)\,\underline{\mu}\right)\subseteq\ \zeta_{FHS}\left(\underline{\omega}\right)\cup\zeta_{FHS}\left(\underline{\mu}\right),$$

Hence, (ζ_{FHS}, G) is concave fuzzy hypersoft set. □

4 Strongly and strictly convex cum concave fuzzy hypersoft sets

Here, strongly and strictly convex (concave) fuzzy hypersoft sets are defined, and some results are generalized with proofs.

Definition 23 The fuzzy hypersoft set (ζ_{FHS}, G) is called a *strongly convex fuzzy hypersoft set* if

$$\zeta_{FHS}\left(\alpha\underline{\beta}+(1-\alpha)\,\underline{\theta}\right)\supset\zeta_{FHS}\left(\underline{\beta}\right)\cap\zeta_{FHS}\left(\underline{\theta}\right)$$

for every

$$\underline{\beta}=(a^{11},a^{12},a^{13},.....,a^{1n})$$
$$\underline{\theta}=(a^{21},a^{22},a^{23},.....,a^{2n})\in G,\ \underline{\beta}\neq\underline{\theta}$$

where $G = A^1 \times A^2 \times A^3 \times ..... \times A^n$ with $A^i \cap A^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., n\}$; $\zeta_{FHS} : G \to P(\sqcup)$ and $\alpha \in J^\circ$.

Definition 24 The fuzzy hypersoft set (ζ_{FHS}, G) on is called a *strongly concave fuzzy hypersoft set* if

$$\zeta_{FHS}\left(\alpha\underline{\beta}+(1-\alpha)\,\underline{\theta}\right)\subset\zeta_{FHS}\left(\underline{\beta}\right)\cup\zeta_{FHS}\left(\underline{\theta}\right)$$

for every $\underline{\beta}$, $\underline{\theta} \in G$, $\underline{\beta} \neq \underline{\theta}$ where, $G = A^1\times A^2\times A^3\times.....\times A^n$ with $A^i \cap A^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., n\}$; $\zeta_{FHS} : G \to P(\sqcup)$ and $\alpha \in J^\circ$.

Definition 25 The fuzzy hypersoft set (ζ_{FHS}, G) is called a *strictly convex fuzzy hypersoft set* if

$$\zeta_{FHS}\left(\alpha\underline{\beta}+(1-\alpha)\,\underline{\theta}\right)\supset\zeta_{FHS}\left(\underline{\beta}\right)\cap\zeta_{FHS}\left(\underline{\theta}\right)$$

for every $\underline{\beta}$, $\underline{\theta} \in G$ where $G = A^1 \times A^2 \times A^3 \times ..... \times A^n$ with $A^i \cap A^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., n\}$; $\zeta_{FHS} : G \to P(\sqcup)$, $\zeta_{FHS}\left(\underline{\beta}\right) \neq \zeta_{FHS}\left(\underline{\theta}\right)$ and $\alpha \in J^\circ$.

Definition 26 The fuzzy hypersoft set (ζ_{FHS}, G) is called a *strictly concave fuzzy hypersoft set* if

$$\zeta_{FHS}\left(\alpha\underline{\beta}+(1-\alpha)\,\underline{\theta}\right)\subset\zeta_{FHS}\left(\underline{\beta}\right)\cup\zeta_{FHS}\left(\underline{\theta}\right)$$

for every $\underline{\beta}$, $\underline{\theta} \in G$ where, $G = A^1 \times A^2 \times A^3 \times ..... \times A^n$ with $A^i \cap A^j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, 3, ..., n\}$; $\zeta_{FHS} : G \to P(\sqcup)$, $\zeta_{FHS}\left(\underline{\beta}\right) \neq \zeta_{FHS}\left(\underline{\theta}\right)$ and $\alpha \in J^\circ$.

Example 2 Suppose a university wants to observe (evaluate) the characteristics of its teachers by some defined indicators. For this purpose, consider a set of teachers as a universe of discourse $\sqcup = \{t_1, t_2, t_3, ..., t_{10}\}$. The attributes of the teachers under consideration are the set $A = \{a^{11}, a^{12}, a^{13}\}$, where

a^{11} = Total experience in years
a^{12} = Total no. of publications
a^{13} = Student's evaluation against each teacher

such that the attribute values against these attributes, respectively, are the sets given as

$$A^{11}=\{1year,2years,3years,4years,5years\}$$
$$A^{12}=\{1,2,3,4,5\}$$
$$A^{13}=\left\{\begin{matrix}Excellent(1), verygood(2), good(3),\\ average(4), bad(5)\end{matrix}\right\}$$

For simplicity, we write

$$A^{11}=\{1,2,3,4,5\}$$
$$A^{12}=\{1,2,3,4,5\}$$
$$A^{13}=\{1,2,3,4,5\}$$

The fuzzy hypersoft set (ζ_{FHS}, G) is a function defined by the mapping $\zeta_{FHS} : G \to F(\sqcup)$ where $G = A^{11} \times A^{12} \times A^{13}$.
Since the elements of $A^{11} \times A^{12} \times A^{13}$ is a 3-tuple, we consider $\underline{\beta} = (2, 1, 3)$; then, the function becomes $\zeta_{FHS}(\underline{\beta}) = \zeta_{FHS}(2, 1, 3) = \{0.1/t_1, 0.5/t_5\}$. Also, consider $\underline{\theta} = (3, 2, 2)$, then the function becomes
$\zeta_{FHS}(\underline{\theta}) = \zeta_{FHS}(3, 2, 2) = \{0.1/t_1, 0.3/t_3, 0.4/t_4\}$
Now,
$\zeta_{FHS}(\underline{\beta}) \cap \zeta_{FHS}(\underline{\theta}) = \zeta_{FHS}(2, 1, 3) \cap \zeta_{FHS}(3, 2, 2)$
$\zeta_{FHS}(\underline{\beta}) \cap \zeta_{FHS}(\underline{\theta}) = \{0.1/t_1, 0.5/t_5\} \cap \{0.1/t_1, 0.3/t_3, 0.4/t_4\}$

$$\zeta_{FHS}(\underline{\beta})\cap\zeta_{FHS}(\underline{\theta})=\{0.1/t_1\} \quad (8)$$

Let $\alpha = 0.6 \in J^\circ$, then, we have
$\alpha\underline{\beta} + (1 - \alpha)\underline{\theta} = 0.6(2, 1, 3) + (1 - 0.6)(3, 2, 2) = 0.6(2, 1, 3) + 0.4(3, 2, 2)$
$= (1.2, 0.6, 1.8) + (1.2, 0.8, 0.8) = (1.2 + 1.2, 0.6 + 0.8, 1.8 + 0.8) = (2.4, 1.4, 2.6)$

which is again a 3-tuple. By using the decimal round off property, we get (2, 1, 3)

$$\zeta_{FHS}(\alpha\underline{\beta} + (1-\alpha)\underline{\theta}) = \zeta_{FHS}(2,1,3) = \{0.1/t_1, 0.5/t_5\} \tag{9}$$

It is vivid from equations (8) and (9), we have

$$\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\,\underline{\theta}\right) \supset \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)$$

Similarly,
$\zeta_{FHS}(\underline{\beta}) \cup \zeta_{FHS}(\underline{\theta}) = \zeta_{FHS}(2,1,3) \cup \zeta_{FHS}(3,2,2)$
$\zeta_{FHS}(\underline{\beta}) \cap \zeta_{FHS}(\underline{\theta}) = \{0.1/t_1, 0.5/t_5\} \cup \{0.1/t_1, 0.3/t_3, 0.4/t_4\}$

$$\zeta_{FHS}(\underline{\beta}) \cap \zeta_{FHS}(\underline{\theta}) = \{0.1/t_1, 0.3/t_3, 0.4/t_4, 0.5/t_5\} \tag{10}$$

From (9) and (10), we have

$$\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\,\underline{\theta}\right) \subset \zeta_{FHS}\left(\underline{\beta}\right) \cup \zeta_{FHS}\left(\underline{\theta}\right)$$

Theorem 8 *(ζ_{FHS}, G) be a strictly convex fuzzy hypersoft set.If there exists $\alpha \in J^{\bullet}$, $\forall\, \underline{\beta}, \underline{\theta} \in G$ such that*

$$\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\,\underline{\theta}\right) \supseteq \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)$$

then (ζ_{FHS}, G) is a convex fuzzy hypersoft set.

Proof Assume that $\zeta_{FHS}\left(\underline{\beta}\right) \subseteq \zeta_{FHS}\left(\underline{\theta}\right)$ and $\exists\, \underline{\beta}, \underline{\theta} \in G), \alpha_1 \in J^{\bullet}$ such that

$$\begin{aligned} &\sqcup\backslash\, \zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \\ &\quad \supseteq \sqcup\backslash \left\{\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)\right\} \end{aligned} \tag{11}$$

If $\zeta_{FHS}\left(\underline{\beta}\right) \subset \zeta_{FHS}\left(\underline{\theta}\right)$, then (11) contradicting (ζ_{FHS}, G) is a strictly convex fuzzy hypersoft set.
If $\zeta_{FHS}\left(\underline{\beta}\right) = \zeta_{FHS}\left(\underline{\theta}\right)$ and $\alpha_1 \in [0, \alpha]$, let $\underline{\nu} = \frac{\alpha_1}{\alpha}(\underline{\beta}) + (1-\frac{\alpha_1}{\alpha})\underline{\theta}$ and $\alpha_2 = (\frac{1}{\alpha}-1)(\frac{1}{\alpha_1}-1)^{-1}$. Thus, by hypothesis,

$$\begin{aligned} &\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \\ &= \zeta_{FHS}\left(\alpha(\frac{\alpha_1}{\alpha}(\underline{\beta}) + (1-\frac{\alpha_1}{\alpha})\underline{\theta}) + (1-\alpha)\underline{\theta}\right) \end{aligned}$$

$$= \zeta_{FHS}\left(\alpha\underline{\nu} + (1-\alpha)\underline{\theta}\right)$$

$$\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \supseteq \zeta_{FHS}\left(\underline{\theta}\right) \cap \zeta_{FHS}\left(\underline{\nu}\right) \tag{12}$$

Now,

$$\zeta_{FHS}\left(\underline{\nu}\right) = \zeta_{FHS}\left(\tfrac{\alpha_1}{\alpha}(\underline{\beta}) + (1-\tfrac{\alpha_1}{\alpha})\underline{\theta}\right)$$

$$\zeta_{FHS}\left(\underline{\nu}\right) = \zeta_{FHS}\left(\alpha_3\underline{\beta} + (1-\alpha_3)(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta})\right) \tag{13}$$

From (11), (12) and $\zeta_{FHS}\left(\underline{\beta}\right) = \zeta_{FHS}\left(\underline{\theta}\right)$, it follows that

$$\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \supseteq \zeta_{FHS}\left(\underline{\nu}\right) \tag{14}$$

From (11), (13), $\zeta_{FHS}\left(\underline{\beta}\right) = \zeta_{FHS}\left(\underline{\theta}\right)$ and strictly convex fuzzy hypersoft set condition, it follows that

$$\zeta_{FHS}\left(\underline{\nu}\right) \supset \zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \tag{15}$$

or

$$\sqcup\backslash\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \supseteq \sqcup\backslash\zeta_{FHS}\left(\underline{\nu}\right) \tag{16}$$

Hence, (14) and (16) gives contradicts.
If $\zeta_{FHS}\left(\underline{\beta}\right) = \zeta_{FHS}\left(\underline{\theta}\right)$ and $\alpha_1 \in [\alpha, 1]$, let $\underline{\rho} = (\frac{\alpha_1-\alpha}{1-\alpha})\underline{\beta} + (\frac{1-\alpha_1}{1-\alpha})\underline{\theta}$. Thus, by hypothesis,

$$\begin{aligned} &\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) = \zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\underline{\rho}\right) \\ &\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \supseteq \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\rho}\right) \end{aligned} \tag{17}$$

From (11), (17) and $\zeta_{FHS}\left(\underline{\beta}\right) = \zeta_{FHS}\left(\underline{\theta}\right)$, it follows that

$$\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \supseteq \zeta_{FHS}\left(\underline{\rho}\right) \tag{18}$$

On the other hand, $\alpha_1\underline{\beta} + (1-\alpha_1)\,\underline{\theta} = \alpha\underline{\beta} + (1-\alpha)\,\underline{\rho}$ gives

$$\begin{aligned} \underline{\rho} &= \left(\frac{1}{1-\alpha}\right)\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) - \left(\frac{\alpha}{\alpha-1}\right)\underline{\beta} \\ \underline{\rho} &= \left(\frac{1}{1-\alpha}\right)\left(\alpha_1\underline{\beta} + (1-\alpha_1)\,\underline{\theta}\right) \\ &\quad - \left(\frac{\alpha}{\alpha-1}\right)\left(\frac{1}{\alpha_1}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\,\underline{\theta}\right) - \frac{1-\alpha_1}{\alpha_1}\underline{\theta}\right) \\ \underline{\rho} &= \left(\frac{\alpha_1-\alpha}{(1-\alpha)\,\alpha_1}\right)\left(\alpha_1\underline{\beta} + (1-\alpha_1)\,\underline{\theta}\right) \\ &\quad + \left(1 - \frac{\alpha_1-\alpha}{(1-\alpha)\,\alpha_1}\right)\underline{\theta} \end{aligned} \tag{19}$$

From (11), (19), $\zeta_{FHS}\left(\underline{\beta}\right) = \zeta_{FHS}\left(\underline{\theta}\right)$ and strictly convex fuzzy hypersoft set condition, it follows that

$$\sqcup\backslash\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \supseteq \sqcup\backslash\zeta_{FHS}\left(\underline{\rho}\right) \tag{20}$$

Hence, (18) and (20) gives a contradict. □

Theorem 9 *(ζ_{FHS}, G) be a strictly concave fuzzy hypersoft set.If there exists $\alpha \in J^\bullet$, $\forall\, \underline{\beta},\ \underline{\theta} \in\ G$ such that*

$$\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\,\underline{\theta}\right) \subseteq \zeta_{FHS}\left(\underline{\beta}\right) \cup \zeta_{FHS}\left(\underline{\theta}\right)$$

then (ζ_{FHS}, G) is a concave fuzzy hypersoft set.

Proof This can easily be proved by following the procedure discussed in Theorem (8). □

Theorem 10 *Let (ζ_{FHS}, G) be a convex fuzzy hypersoft set. If there exists $\alpha \in J^\bullet$, $\forall\, \underline{\beta},\ \underline{\theta} \in\ G$, $\zeta_{FHS}\left(\underline{\beta}\right) \neq \zeta_{FHS}\left(\underline{\theta}\right)$ such that*

$$\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\,\underline{\theta}\right) \supset \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)$$

then (ζ_{FHS}, G) is a strictly convex fuzzy hypersoft set.

Proof Assume that $\exists\ \underline{\beta}, \underline{\theta} \in G), \alpha_1 \in J^\bullet$ such that

$$\begin{aligned} &\sqcup\backslash\, \zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \\ &\quad \supset \sqcup\backslash \left\{\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)\right\} \end{aligned} \tag{21}$$

If $\zeta_{FHS}\left(\underline{\beta}\right) \supset \zeta_{FHS}\left(\underline{\theta}\right)$, then (21) gives

$$\sqcup\backslash\, \zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \supset \sqcup\backslash\zeta_{FHS}\left(\underline{\theta}\right) \tag{22}$$

On the other hand, from the convex fuzzy hypersoft set condition, we have that

$$\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \supseteq \left\{\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)\right\} \tag{23}$$

From (21) and (23), it follows that

$$\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) = \left\{\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)\right\} \tag{24}$$

which together with $\zeta_{FHS}\left(\underline{\beta}\right) \supset \zeta_{FHS}\left(\underline{\theta}\right)$, getting that

$$\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) = \zeta_{FHS}\left(\underline{\theta}\right) \tag{25}$$

or

$$\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \subset \zeta_{FHS}\left(\underline{\beta}\right) \tag{26}$$

Thus, from (26) and the hypothesis,

$$\begin{aligned} &\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta})\right) \supset \\ &\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \end{aligned} \tag{27}$$

More generally, for $n \in \{1, 2, 3, ...\}$ can easily show that

$$\begin{aligned} &\zeta_{FHS}\left(\alpha^n\underline{\beta} + (1-\alpha^n)(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta})\right) \supset \\ &\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \end{aligned} \tag{28}$$

Let $\underline{\nu} = \alpha_2\underline{\beta} + (1-\alpha_2)\underline{\theta}$ where $\alpha_2 = \alpha_1 - \alpha^n\alpha_1 + \alpha^n \in J^\bullet$ for some n . Then, from (28), we see that

$$\begin{aligned} &\zeta_{FHS}\left(\underline{\nu}\right) = \zeta_{FHS}\left(\alpha_2\underline{\beta} + (1-\alpha_2)\underline{\theta}\right) \\ &\zeta_{FHS}\left(\underline{\nu}\right) = \zeta_{FHS}\left(\alpha^n\underline{\beta} + (1-\alpha^n)(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta})\right) \\ &\zeta_{FHS}\left(\nu\right) \supset \zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \end{aligned} \tag{29}$$

Also, let $\underline{\rho} = \alpha_3\underline{\beta} + (1-\alpha_3)\underline{\theta}$ where $\alpha_3 = \alpha_1 - \alpha^n + \frac{1}{1-\alpha} + \frac{\alpha^n\alpha_1}{1-\alpha} \in J^\bullet$ for some n.
Then,

$$\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) = \zeta_{FHS}\left(\alpha\underline{\nu} + (1-\alpha)\underline{\rho}\right) \tag{30}$$

Now, if $\zeta_{FHS}\left(\underline{\nu}\right) \subseteq \zeta_{FHS}\left(\underline{\rho}\right)$, then (30) and (ζ_{FHS}, G) is a convex fuzzy hypersoft set implies that

$$\sqcup\backslash\, \zeta_{FHS}\left(\underline{\nu}\right) \supset \sqcup\backslash\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right)$$

this contradicts (29).

If $\sqcup\backslash\, \zeta_{FHS}\left(\underline{\nu}\right) \subseteq \sqcup\backslash\, \zeta_{FHS}\left(\underline{\rho}\right)$, then (30) and the hypothesis of the theorem implies that

$$\begin{aligned} &\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \supset \zeta_{FHS}\left(\underline{\nu}\right) \cap \zeta_{FHS}\left(\underline{\rho}\right) \\ &\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \supseteq (\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)) \cap \\ &(\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)) = \zeta_{FHS}\left(\underline{\beta}\right) \end{aligned}$$

This contradicts (26). □

Theorem 11 *Let (ζ_{FHS}, G) be a concave fuzzy hypersoft set. If there exists $\alpha \in J^\bullet$, $\forall\, \underline{\beta},\ \underline{\theta} \in\ G$, $\zeta_{FHS}\left(\underline{\beta}\right) \neq \zeta_{FHS}\left(\underline{\theta}\right)$ such that*

$$\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\,\underline{\theta}\right) \subset \zeta_{FHS}\left(\underline{\beta}\right) \cup \zeta_{FHS}\left(\underline{\theta}\right)$$

then (ζ_{FHS}, G) is a strictly concave fuzzy hypersoft set.

Proof This can easily be proved by following the procedure discussed in Theorem (10). □

Theorem 12 *Let (ζ_{FHS}, G) be a strongly convex fuzzy hypersoft set. If there exists $\alpha \in J^\bullet$, $\forall\, \underline{\beta},\ \underline{\theta} \in\ G$, such that*

$$\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\,\underline{\theta}\right) \supseteq \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right) \tag{31}$$

then (ζ_{FHS}, G) *is a convex fuzzy hypersoft set.*

Proof Assume that $\exists\ \underline{\beta}, \underline{\theta} \in G), \alpha_1 \in J^{\bullet}$ such that

$$\sqcup\backslash\, \zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \supseteq \sqcup\backslash \left\{\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)\right\} \quad (32)$$

If $\underline{\beta} \neq \underline{\theta}$, then (31) contradicting that (ζ_{FHS}, G) is a strongly convex fuzzy hypersoft set.

If $\underline{\beta} = \underline{\theta}$, then choose $\alpha_1 \neq \alpha_2 \in J^{\circ}$ such that $\alpha_1 = \alpha\alpha_2 + (1-\alpha)\alpha_2$.

Let $\underline{\beta} = \underline{\theta} = \alpha_2\underline{\beta} + (1-\alpha_2)\underline{\theta}$. Then, (31) implies that

$$\sqcup\backslash\, \zeta_{FHS}\left(\underline{\beta}\right) \supseteq \sqcup\backslash \left\{\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)\right\} \quad (33)$$

$$\sqcup\backslash\, \zeta_{FHS}\left(\underline{\theta}\right) \supseteq \sqcup\backslash \left\{\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)\right\} \quad (34)$$

According to (31), (33) and (34), we have

$$\begin{aligned}&\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\,\underline{\theta}\right) \supseteq \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)\\ &\subset (\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)) \cap (\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right))\\ &= \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)\end{aligned}$$

which contradicts that (ζ_{FHS}, G) is a strongly convex fuzzy hypersoft set. □

Theorem 13 *Let* (ζ_{FHS}, G) *be a strongly concave fuzzy hypersoft set. If there exists* $\alpha \in J^{\bullet}$, $\forall\, \underline{\beta}, \underline{\theta} \in G$, *such that*

$$\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\,\underline{\theta}\right) \subseteq \zeta_{FHS}\left(\underline{\beta}\right) \cup \zeta_{FHS}\left(\underline{\theta}\right)$$

then (ζ_{FHS}, G) *is a concave fuzzy hypersoft set.*

Proof This can easily be proved by following the procedure discussed in Theorem (12). □

Theorem 14 *Let* (ζ_{FHS}, G) *be a convex fuzzy hypersoft set. If there exists* $\alpha \in J^{\bullet}$, $\forall\, \underline{\beta}, \underline{\theta} \in G$, $\underline{\beta} \neq \underline{\theta}$, *such that*

$$\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\,\underline{\theta}\right) \supset \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)$$

then (ζ_{FHS}, G) *is a strongly convex fuzzy hypersoft set.*

Proof Assume that $\exists\ (\underline{\beta} \neq \underline{\theta})],\ \underline{\beta}, \underline{\theta} \in G), \alpha_1 \in J^{\bullet}$ such that

$$\sqcup\backslash\, \zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) \supseteq \sqcup\backslash \left\{\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)\right\} \quad (35)$$

Thus, from (35) and the convex fuzzy hypersoft set condition, we get that

$$\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right) = \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right) \quad (36)$$

Furthermore, it can be easily seen that

$$\alpha\underline{\beta} + (1-\alpha)\underline{\theta} = \alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta} \quad (37)$$

where both $\underline{\beta}$ and $\underline{\theta}$ are of the form $\underline{\beta} = \alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}$ and $\underline{\theta} = \alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}$ for choosing $\alpha_1 \in J^{\circ}$

On the other hand, from the convex fuzzy hypersoft set condition and our definition of $\underline{\beta}$ and $\underline{\theta}$, getting

$$\zeta_{FHS}\left(\underline{\beta}\right) \supseteq \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right) \quad (38)$$

$$\zeta_{FHS}\left(\underline{\theta}\right) \supseteq \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right) \quad (39)$$

Therefore, from (37), (38), (39) and the hypothesis of the theorem, we get that

$$\begin{aligned}&\zeta_{FHS}\left(\alpha_1\underline{\beta} + (1-\alpha_1)\underline{\theta}\right)\\ &\quad= \zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\underline{\theta}\right) \supset \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)\\ &\quad\supseteq (\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)) \cap (\zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right))\\ &\quad= \zeta_{FHS}\left(\underline{\beta}\right) \cap \zeta_{FHS}\left(\underline{\theta}\right)\end{aligned}$$

This contradicts (36). □

Theorem 15 *Let* (ζ_{FHS}, G) *be a concave fuzzy hypersoft set. If there exists* $\alpha \in J^{\bullet}$, $\forall\, \underline{\beta}, \underline{\theta} \in G$, $\underline{\beta} \neq \underline{\theta}$, *such that*

$$\zeta_{FHS}\left(\alpha\underline{\beta} + (1-\alpha)\,\underline{\theta}\right) \subset \zeta_{FHS}\left(\underline{\beta}\right) \cup \zeta_{FHS}\left(\underline{\theta}\right)$$

then (ζ_{FHS}, G) *is a strongly concave fuzzy hypersoft set.*

Proof This can easily be proved by following the procedure discussed in Theorem (14). □

5 Application in decision making

In this section, we first propose an algorithm based on proposed study and then apply it generally to solve a real-world problem through decision making.

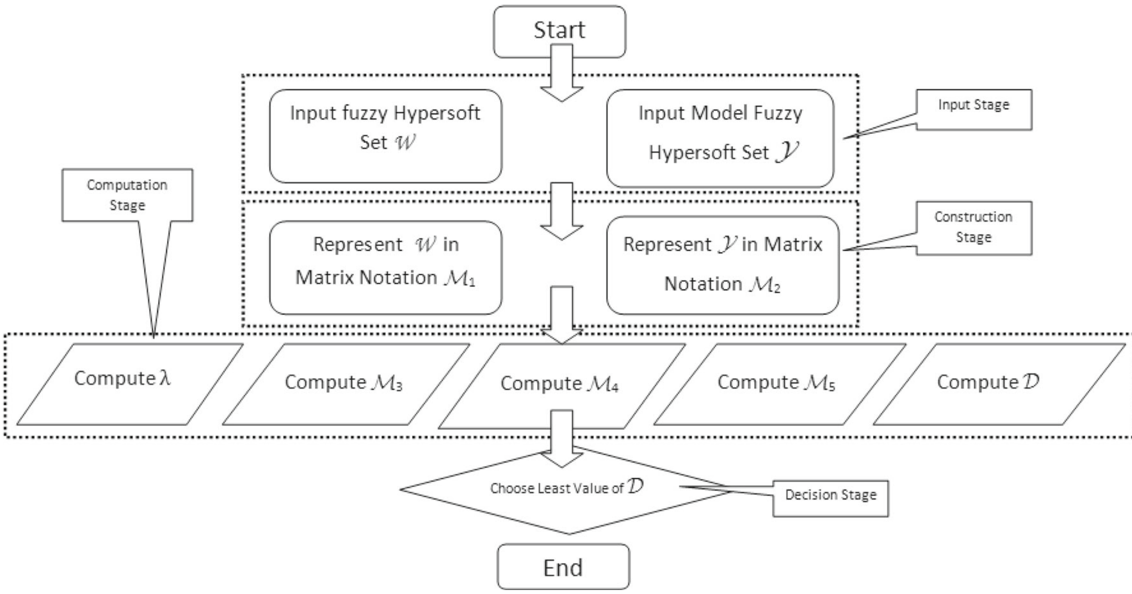

Fig. 4 Flowchart of Proposed Algorithm

Algorithm: Appropriate Selection of Hand-Sanitizer

▹ **Start**
▹ **Input:**
1. Input Fuzzy Hypersoft set $\mathscr{W}$
2. Input Model Fuzzy Hypersoft set $\mathscr{Y}$
▹ **Construction:**
3. Represent $\mathscr{W}$ in matrix notation $\mathscr{M}_1$
4. Represent $\mathscr{Y}$ in matrix notation $\mathscr{M}_2$
▹ **Computation:**
5. Compute λ by taking arithmetic mean of all fuzzy values used in $\mathscr{W}$ and $\mathscr{Y}$
6. Compute $\mathscr{M}_3 = \lambda\mathscr{M}_1 \bigvee (1-\lambda)\mathscr{M}_2$
7. Compute $\mathscr{M}_4 = \lambda\mathscr{M}_1 \bigwedge (1-\lambda)\mathscr{M}_2$
8. Compute $\mathscr{M}_5 = \mathscr{M}_3 \setminus \mathscr{M}_4$
9. Compute Score $\mathscr{D}$ corresponding to each member in the universe.
▹ **Output:**
10. Choose least score as final decision
▹ **End**

The flowchart of above algorithm is shown in Fig. 4.

5.1 Statement of the problem

Professor John is the head of an educational institution. He is very concerned about the health of the students as well as the faculty members of his institution in view of the current coronary epidemic. He wants to buy good and useful Hand Sanitizers for his institution but he is also worried about the non-standard Hand Sanitizers available in the market. So he decides to call for bids from different potential suppliers for this purchase to fulfil the departmental official compliances and to avoid any expected loss. Some suppliers are scrutinized by adopting proper procedure already framed by relevant department. For the sake of satisfaction, he constitutes a committee consisting of some staff members with good procurement experience to evaluate the items (Hand Sanitizers) offered by scrutinized suppliers. The following example elaborates the whole procedure of such evaluation:

Example 3 Suppose there are six kinds of Hand Sanitizer (options) which form the set of discourse $\sqcup = \{\sqcup_1, \sqcup_2\}$ where $\sqcup_1 = \{\mathscr{H}^1, \mathscr{H}^2, \mathscr{H}^3\}$ and $\sqcup_2 = \{\mathscr{H}^4, \mathscr{H}^5, \mathscr{H}^6\}$ are the collections of Hand Sanitizers made by manufacturers X_1 and X_2, respectively. With their mutual consensus, the committee members (experts) agreed on a set of parameters after observing various attributes for this evaluation. The finalized evaluating attributes are : b^1 = Manufacturer, b^2 = Quantity of Ethanol (percentage), b^3 = Quantity of Distilled Water (percentage), b^4 = Quantity of Glycerol (percentage), and b^5 = Quantity of Hydrogen peroxide (percentage). After observing the opinions of various professionals and other relevant sources on the composition of Hand Sanitizers, these attributes are further classified into attribute-valued sets which are given as:

$$B^1 = \{b^{11} = X_1, b^{12} = X_2\}$$
$$B^2 = \{b^{21} = 75.15, b^{22} = 80\}$$

$B^3 = \{b^{31} = 23.425, b^{32} = 18.425\}$
$B^4 = \{b^{41} = 1.30, b^{42} = 1.45\}$
$B^5 = \{b^{51} = 0.125\}$

then $\mathscr{Q} = B^1 \times B^2 \times B^3 \times B^4 \times B^5$
$\mathscr{Q} = \{q^1, q^2, q^3, q^4, ..., q^{16}\}$ where each q^i, $i = 1, 2, ..., 16$, is a 5-tuple element. For convenience, take
$\mathscr{R} = \{q^1, q^4, q^7, q^9, q^{13}, q^{16}\} \subseteq \mathscr{Q}$.

Input Stage (1-2):
Then, the fuzzy hypersoft set $\mathscr{W}$ and model fuzzy hypersoft set $\mathscr{Y}$ corresponding to adopted $\mathscr{R}$ are constructed as: $\mathscr{W} =$

$$\left\{\begin{array}{l} \left(q^1, \mathscr{H}^1/0.1, \mathscr{H}^2/0.2, \mathscr{H}^3/0.3, \mathscr{H}^4/0.4, \mathscr{H}^5/0.5, \mathscr{H}^6/0.6\right), \\ \left(q^4, \mathscr{H}^1/0.2, \mathscr{H}^2/0.3, \mathscr{H}^3/0.4, \mathscr{H}^4/0.5, \mathscr{H}^5/0.6, \mathscr{H}^6/0.7\right), \\ \left(q^7, \mathscr{H}^1/0.3, \mathscr{H}^2/0.4, \mathscr{H}^3/0.5, \mathscr{H}^4/0.6, \mathscr{H}^5/0.7, \mathscr{H}^6/0.8\right), \\ \left(q^9, \mathscr{H}^1/0.4, \mathscr{H}^2/0.5, \mathscr{H}^3/0.6, \mathscr{H}^4/0.7, \mathscr{H}^5/0.8, \mathscr{H}^6/0.9\right), \\ \left(q^{13}, \mathscr{H}^1/0.5, \mathscr{H}^2/0.6, \mathscr{H}^3/0.7, \mathscr{H}^4/0.8, \mathscr{H}^5/0.9, \mathscr{H}^6/0.1\right), \\ \left(q^{16}, \mathscr{H}^1/0.6, \mathscr{H}^2/0.7, \mathscr{H}^3/0.8, \mathscr{H}^4/0.9, \mathscr{H}^5/0.1, \mathscr{H}^6/0.2\right) \end{array}\right\}$$

and $\mathscr{Y} =$

$$\left\{\begin{array}{l} \left(q^1, \mathscr{H}^1/0.4, \mathscr{H}^2/0.4, \mathscr{H}^3/0.4, \mathscr{H}^4/0.5, \mathscr{H}^5/0.5, \mathscr{H}^6/0.5\right), \\ \left(q^4, \mathscr{H}^1/0.3, \mathscr{H}^2/0.3, \mathscr{H}^3/0.3, \mathscr{H}^4/0.6, \mathscr{H}^5/0.6, \mathscr{H}^6/0.6\right), \\ \left(q^7, \mathscr{H}^1/0.2, \mathscr{H}^2/0.2, \mathscr{H}^3/0.2, \mathscr{H}^4/0.7, \mathscr{H}^5/0.7, \mathscr{H}^6/0.7\right), \\ \left(q^9, \mathscr{H}^1/0.5, \mathscr{H}^2/0.5, \mathscr{H}^3/0.5, \mathscr{H}^4/0.3, \mathscr{H}^5/0.3, \mathscr{H}^6/0.3\right), \\ \left(q^{13}, \mathscr{H}^1/0.1, \mathscr{H}^2/0.1, \mathscr{H}^3/0.1, \mathscr{H}^4/0.8, \mathscr{H}^5/0.8, \mathscr{H}^6/0.8\right), \\ \left(q^{16}, \mathscr{H}^1/0.6, \mathscr{H}^2/0.7, \mathscr{H}^3/0.8, \mathscr{H}^4/0.1, \mathscr{H}^5/0.1, \mathscr{H}^6/0.1\right) \end{array}\right\}$$

Construction Stage (3-4):
The matrix notations $\mathscr{M}_1 = [a_{ij}]_{(6\times6)}$ and $\mathscr{M}_2 = [b_{ij}]_{(6\times6)}$ of Fuzzy Hypersoft Sets $\mathscr{W}$ and $\mathscr{Y}$ are presented below where $\mathscr{H}^i$ are arranged in columns and opted attribute-valued tuples q^j are arranged in rows.

$$\mathscr{M}_1 = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 \\ 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 \\ 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 \\ 0.4 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 \\ 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 0.1 \\ 0.6 & 0.7 & 0.8 & 0.9 & 0.1 & 0.2 \end{bmatrix}$$

$$\mathscr{M}_2 = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.5 & 0.5 & 0.5 \\ 0.3 & 0.3 & 0.3 & 0.6 & 0.6 & 0.6 \\ 0.2 & 0.2 & 0.2 & 0.7 & 0.7 & 0.7 \\ 0.5 & 0.5 & 0.5 & 0.3 & 0.3 & 0.3 \\ 0.1 & 0.1 & 0.1 & 0.8 & 0.8 & 0.8 \\ 0.6 & 0.7 & 0.8 & 0.1 & 0.1 & 0.1 \end{bmatrix}$$

Computation Stage (5-9):
By taking arithmetic mean of all fuzzy values of $\mathscr{W}$ and $\mathscr{Y}$, we obtain the value of λ that is 0.4792 (rounded off to 4 decimal places). Now, we have

$$\mathscr{M}_3 = \begin{bmatrix} 0.25624 & 0.30416 & 0.35208 & 0.45208 & 0.5 & 0.54792 \\ 0.25208 & 0.3 & 0.34792 & 0.55208 & 0.6 & 0.3604 \\ 0.289584 & 0.337504 & 0.385424 & 0.65208 & 0.41428 & 0.4604 \\ 0.45208 & 0.5 & 0.54792 & 0.20416 & 0.25208 & 0.3 \\ 0.29168 & 0.3396 & 0.1 & 0.51248 & 0.5604 & 0.60832 \\ 0.6 & 0.41248 & 0.51248 & 0.19584 & 0.24376 & 0.29168 \end{bmatrix}$$

$$\text{and}\,\mathscr{M}_4 = \begin{bmatrix} -0.1604 & -0.11248 & -0.06456 & -0.06872 & -0.0208 & 0.02712 \\ -0.0604 & -0.01248 & 0.03544 & -0.07288 & -0.02496 & -0.26456 \\ -0.00206 & 0.045856 & 0.093776 & -0.07704 & -0.31664 & -0.26872 \\ -0.06872 & -0.0208 & 0.02712 & -0.10832 & -0.0604 & -0.01248 \\ 0.18752 & 0.23544 & -0.00416 & -0.3208 & -0.27288 & -0.22496 \\ -0.02496 & -0.31664 & -0.3208 & 0.09168 & 0.1396 & 0.18752 \end{bmatrix}.$$

Now, we compute $\mathscr{M}_5$ that is given as

$$\mathscr{M}_5 = \begin{bmatrix} 0.41664 & 0.41664 & 0.41664 & 0.5208 & 0.5208 & 0.5208 \\ 0.31248 & 0.31248 & 0.31248 & 0.62496 & 0.62496 & 0.62496 \\ 0.291648 & 0.291648 & 0.291648 & 0.72912 & 0.72912 & 0.72912 \\ 0.5208 & 0.5208 & 0.5208 & 0.31248 & 0.31248 & 0.31248 \\ 0.10416 & 0.10416 & 0.10416 & 0.83328 & 0.83328 & 0.83328 \\ 0.62496 & 0.72912 & 0.83328 & 0.10416 & 0.10416 & 0.10416 \end{bmatrix}.$$

It is pertinent to mention here that $\bigvee$ and $\bigwedge$ represents the ordinary sum and subtraction of matrices, whereas \ denotes ordinary numerical difference. Hence, score values corresponding to each element of $\sqcup$ are given as

$$\mathscr{D} = \left\{ \begin{array}{l} \mathscr{H}^1/2.270688, \mathscr{H}^2/2.374848.\mathscr{H}^3/2.479008, \\ \mathscr{H}^4/3.1248, \mathscr{H}^5/3.1248, \mathscr{H}^6/3.1248 \end{array} \right\}.$$

Out Stage:

10. As $\mathscr{H}^1$ has attained the least score value 2.270688 that means its nature is more convex as compared to other so it is selected as most appropriate product.

6 Comparison analysis

In this section, the proposed study is compared with relevant models. The concepts presented by Deli (2019) and Salih and Sabir (2018) are the most relevant to this proposed study for comparison. Table 1 shows that the proposed study is more flexible as it addresses the insufficiencies of existing structures. As in the proposed study, attributes are observed deeply through entitlement of their respective sub-attributive values, the decision-making process becomes more reliable and authentic to have precise and accurate results while dealing with optimization-relating problems.

7 Conclusion

In this study, convexity cum concavity on fuzzy hypersoft sets is introduced by adopting an abstract cum analytical technique. This is novel addition in the literature and may enable the researchers to deal with important applications of convexity under fuzzy and hypersoft environment with precise results. Moreover, strictly and strongly convexity cum concavity on fuzzy hypersoft sets are also conceptualized along with generalized results. Future directions and scope are stated below:

1. A hypothetical data is used to validate the proposed study in decision making. This may be applied to real-life scenarios by using real data as a case study in medical sciences (diagnostic study), pattern recognition, image processing and operation research (convex optimization).
2. It is convenient to develop convex hull, convex cone and many other types of convexity like (m, n)-convexity, ϕ-convexity, graded convexity, triangular convexity, concavo-convexity, etc., on fuzzy hypersoft set by using proposed study.
3. Special conditions can be applied to the aggregation operations discussed in the proposed study to have more generalized results, e.g., considering the modified ver-

Table 1 Comparison of Proposed Study with Existing Relevant Structures

Authors	Structures	Focus on attributes	Focus on sub-attributive values	Nature of Approximate Function	Fuzzy Membership	Numerical Examples	Real-world Application
Deli (2019)	Convex (Concave) Soft Set	Yes	No	Single-Argument	No	No	No
Deli (2019)	Convex (Concave) Fuzzy Soft Set	Yes	No	Single-Argument	Yes	No	No
Salih and Sabir (2018)	Strictly and Strongly Convex (Concave) Soft Set	Yes	No	Single-Argument	No	No	No
Proposed Study	Convex (Concave) Fuzzy Hypersoft Set	Yes	Yes	Multi-Argument	Yes	Yes	Yes

sions of complement, intersection and union as discussed in Ali et al. (2009) and Maji et al. (2003).

4. This study may further be employed to establish certain mathematical inequalities, e.g., Ostrowski's Inequalities, Hardi's inequalities, etc., by introducing convex and convex hypersoft functions.
5. This may also be applied in algebraic structures like topological spaces, vector spaces, etc., to modify various important results.
6. As the proposed study emphasizes on membership degree, it can further be extended for considering non-membership and indeterminacy grades.

Acknowledgements Authors are thankful to referees for improving the quality of this paper their their valuable suggestions.

Author Contributions Conceptualization was done by A.U.R. and M.S.; methodology was done by A.U.R. and F.S.; software was done by A.U.R.; validation was done by A.U.R., M.S. and F.S.; formal analysis was carried out by A.U.R. and M.S.; investigation was done by M.S. and F.S.; visualization was done by A.U.R., M.S. and F.S.; supervision was done by M.S. and F.S. All authors have read and agreed to the published version of the manuscript.

Funding The authors received no specific funding for this study.

Data Availability This study has no associated data.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Informed consent Informed consent was obtained from all individual participants included in the study.

References

Abbas M, Murtaza G, Smarandache F (2020) Basic operations on hypersoft sets and hypersoft point. Neutrosophic Sets Syst 35:407–421. https://doi.org/10.5281/zenodo.3951694

Ahsan M, Saeed M, Rahman AU (2021) A theoretical and analytical approach for fundamental framework of composite mappings on fuzzy hypersoft classes. Neutrosophic Sets and Syst 45:268–285. https://doi.org/10.5281/zenodo.5486295

Alcantud JCR, Rambaud SC, Torrecillas MJM (2017) Valuation fuzzy soft sets: a flexible fuzzy soft set based decision making procedure for the valuation of assets. Symmetry 9(11):253. https://doi.org/10.3390/sym9110253

Ali MI, Feng F, Liu X, Min WK, Sabir M (2009) On some new operations in soft set theory. Comput Math Appl 57:1547–1553. https://doi.org/10.1016/j.camwa.2008.11.009

Babitha KV, Sunil JJ (2010) Soft set relations and functions. Comput Math Appl 60:1840–1849. https://doi.org/10.1016/j.camwa.2010.07.014

Babitha KV, Sunil JJ (2011) Transitive closure and ordering in soft set. Comput Math Appl 61:2235–2239. https://doi.org/10.1016/j.camwa.2011.07.010

Debnath S (2021) Fuzzy hypersoft sets and its weightage operator for decision making. J Fuzzy Ext Appl 2(2):163–170. https://doi.org/10.22105/jfea.2021.275132.1083

Deli I (2019) Convex and concave sets based on soft sets and fuzzy soft sets. J New Theory 29:101–110

Feng F, Li C, Davvaz B, Ali MI (2010) Soft sets combined with fuzzy sets and rough sets: a tentative approach. Soft Comput 14(9):899–911. https://doi.org/10.1007/s00500-009-0465-6

Ge X, Yang S (2011) Investigations on some operations of soft sets. World Acad Sci Eng Technol 75:1113–1116

Guan X, Li Y, Feng F (2013) A new order relation on fuzzy soft sets and its application. Soft Comput 17(1):63–70. https://doi.org/10.1007/s00500-012-0903-8

Hassan N, Sayed OR, Khalil AM, Ghany MA (2017) Fuzzy soft expert system in prediction of coronary artery disease. Int J Fuzzy Syst 19(5):1546–1559. https://doi.org/10.1007/s40815-016-0255-0

Ihsan M, Rahman AU, Saeed M (2021) Hypersoft expert set with application in decision making for recruitment process. Neutrosophic Sets Syst 42:191–207. https://doi.org/10.5281/zenodo.4711524

Ihsan M, Rahman AU, Saeed M (2021) Fuzzy hypersoft expert set with application in decision making for the best selection of product. Neutrosophic Sets and Syst 46:318. https://doi.org/10.5281/zenodo.5553534

Kamacı H, Saqlain M (2021) n-ary fuzzy hypersoft expert sets. Neutrosophic Sets Syst 43:180–211. https://doi.org/10.5281/zenodo.4914849

Khameneh AZ, Kılıçman A (2018) Parameter reduction of fuzzy soft sets: an adjustable approach based on the three-way decision. Int J Fuzzy Syst 20(3):928–942. https://doi.org/10.1007/s40815-017-0409-8

Li F (2011) Notes on soft set operations. ARPN J Syst softw 1(6):205–208

Liu Y, Qin K, Martínez L (2018) Improving decision making approaches based on fuzzy soft sets and rough soft sets. Appl Soft Comput 65:320–332. https://doi.org/10.1016/j.asoc.2018.01.012

Majeed SN (2016) Some notions on convex soft sets. Ann Fuzzy Math Inform 12(4):517–526

Maji PK, Biswas R, Roy AR (2003) Soft set theory. Comput Math Appl 45:555–562. https://doi.org/10.1016/S0898-1221(03)00016-6

Maji PK, Biswas R, Roy AR (2001) Fuzzy soft sets. J Fuzzy Math 9(3):589–602

Molodtsov D (1999) Soft set theory - first results. Comput Math Appl 37:19–31. https://doi.org/10.1016/S0898-1221(99)00056-5

Martin N, Smarandache F (2020) Introduction to combined plithogenic hypersoft sets. Neutrosophic Sets Syst 35:503–510. https://doi.org/10.5281/zenodo.3951708

Paik B, Mondal SK (2020) A distance-similarity method to solve fuzzy sets and fuzzy soft sets based decision-making problems. Soft Comput 24(7):5217–5229. https://doi.org/10.1007/s00500-019-04273-z

Pei D, Miao D (2005) From soft set to information system. In international conference of granular computing IEEE 2:617–621. https://doi.org/10.1109/GRC.2005.1547365

Rahman AU, Saeed M, Zahid S (2021) Application in decision making based on fuzzy parameterized hypersoft set theory. Asia Mathematika 5(1):19–27. https://doi.org/10.5281/zenodo.4721481

Rahman AU, Saeed M, Smarandache F, Ahmad MR (2020) Development of hybrids of hypersoft set with complex fuzzy set, complex intuitionistic fuzzy set and complex neutrosophic set. Neutrosophic Sets Syst 38:335–354. https://doi.org/10.5281/zenodo.4300520

Rahman AU, Saeed M, Hafeez A (2021) Theory of bijective hypersoft set with application in decision making. Punjab Univ J Math 53(7):511–526. https://doi.org/10.52280/pujm.2021.530705

Salih HM, Sabir PO (2018) Some new properties of convex and concave soft sets. Diyala J Pure Sci 15(9):84–94. https://doi.org/10.24237/djps.1501.471B

Saeed M, Rahman AU, Ahsan M, Smarandache F (2021) An inclusive study on fundamentals of hypersoft set. In: Smarandache F, Saeed M, Abdel-Baset M, Saqlain M (eds) Theory Appl Hypersoft Set. Pons Publishing House, Brussels, pp 1–23

Saeed M, Ahsan M, Abdeljawad T (2021) A development of complex multi-fuzzy hypersoft set with application in MCDM based on entropy and similarity measure. IEEE Access 9:60026–60042. https://doi.org/10.1109/ACCESS.2021.3073206

Sezgin A, Atagün AO (2011) On operations of soft sets. Comput Math Appl 61(5):1457–1467. https://doi.org/10.1016/j.camwa.2011.01.018

Smarandache F (2018) Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. Neutrosophic Sets Syst 22:168–170

Xiao F (2018) A hybrid fuzzy soft sets decision making method in medical diagnosis. IEEE Access 6:25300–25312. https://doi.org/10.1109/ACCESS.2018.2820099

Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353. https://doi.org/10.1016/S0019-9958(65)90241-X

Zhan J, Zhu K (2017) A novel soft rough fuzzy set: z-soft rough fuzzy ideals of hemirings and corresponding decision making. Soft Comput 21(8):1923–1936. https://doi.org/10.1007/s00500-016-2119-9

Zhang J, Wu X, Lu R (2020) Decision analysis methods combining quantitative logic and fuzzy soft sets. Int J Fuzzy Syst 22(6):1801–1814. https://doi.org/10.1007/s40815-020-00899-6