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Research Article

Multicriteria Decision-Making Approach for Pythagorean Fuzzy Hypersoft Sets' Interaction Aggregation Operators

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In this paper, we examine the multicriteria decision-making (MCDM) difficulties for Pythagorean fuzzy hypersoft sets (PFHSSs). The PFHSSs are a suitable extension of the Pythagorean fuzzy soft sets (PFSSs) which deliberates the parametrization of multi-subattributes of considered parameters. It is a most substantial notion for describing fuzzy information in the decision-making (DM) procedure to accommodate more vagueness comparative to existing PFSSs and intuitionistic fuzzy hypersoft sets (IFHSSs). The core objective of this study is to plan some innovative operational laws considering the interaction for Pythagorean fuzzy hypersoft numbers (PFHSNs). Also, based on settled interaction operational laws, two aggregation operators (AOs) i.e., Pythagorean fuzzy hypersoft interaction weighted average (PFHSIWA) and Pythagorean fuzzy hypersoft interaction weighted geometric (PFHSIWG) operators for PFHSSs operators have been presented with their fundamental properties. Furthermore, an MCDM technique has been established using planned interaction AOs. To ensure the strength and practicality of the developed MCDM method, a mathematical illustration has been presented. The usefulness, influence, and versatility of the developed method have been demonstrated via comparative analysis with the help of some conventional studies.

1. Introduction

Multicriteria decision-making (MCDM) is a prerequisite for decision science. The goal is to distinguish between the most essential of the possible choices. The decision maker must assess the selection specified by different types of diagnostic circumstances such as intervals and numbers. However, in numerous circumstances, it is difficult for one person to do it because of various uncertainties within the data. One is because of the shortcoming of professional knowledge or contraventions. Hence, to measure given hazards and think about the method, a series of theories have been proposed. Zadeh presented the theory of fuzzy sets (FSs) [1] to resolve the complex problem of anxiety along with ambiguity. Usually, we need to observe membership as a nonmembership degree to indicate objects for which FSs cannot handle. To conquer the current concern, Atanassov anticipated the concept of intuitionistic fuzzy sets (IFSs) [2]. Atanassov's IFSs competently deal with insufficient data because of membership and nonmembership values, but IFSs are not able to influence incompatible and imprecise information. The theories declared over had been fairly advised by specialists, along with the sum up of two membership and nonmembership values cannot overreach one because the above work is regarded as to visualize the environment of linear inequality between the degree of membership (MD) and the degree of nonmembership (NMD). If the experts considered the MD and NMD such as MD = 0.4 and NDM = 0.7, then $0.4 + 0.7 \nleq 1$ and IFSs cannot handle the situation. Yager [3, 4] prolonged the idea of IFSs to Pythagorean fuzzy sets (PFSs) to overcome the above-discussed difficulties by amending $MD + NMD \leq 1$ to $MD^2 + NMD^2 \leq 1$. Succeeding the construction of PFSs, Zhang and Xu [5] planned operational rules for PFSs and set up the DM strategy to address the MCDM problem. Sanam et al. [6] presented the induced intuitionistic fuzzy Einstein hybrid AOs and discussed their desired properties. Wang and Li [7] offered some novel operational laws and AOs for PFSs considering the interaction with their desirable properties. Gao et al. [8] prolonged the notion of PFSs and developed numerous AOs considering the interaction. They also established a multiattribute decision-making (MADM) approach based on their established operators.

Wei [9] developed some novel operational laws for Pythagorean fuzzy numbers (PFNs) considering the interaction and proposed AOs for PFSs based on their developed operational laws. Talukdar et al. [10] utilized the linguistic PFSs for medical diagnoses and introduced some distance measures and accuracy function. They also proposed a DM technique to solve multiple criteria group decision-making (MCGDM) complications utilizing PFNs. Wang et al. [11] extended the concept of PFSs, proposed the interactive Hamacher AOs, and established a MADM method to resolve DM complications. Ejegwa et al. [12] established a correlation measure for IFSs and presented an MCDM approach. Peng and Yang [13] offered various essential operations for PFSs along with their basic characteristics. Garg [14] proposed some AOs for PFSs based on his developed logarithmic operational laws. Arora and Garg [15] introduced prioritized AOs for linguistic IFSs based on their developed operational laws. Ma and Xu [16] established novel AOs for PFSs and offered the comparison laws for PFNs. Current theories and their progressed DM strategies have been utilized in various aspects of life. However, these theories fail to cope with the parameters of alternatives.

The above-presented theories with their DM techniques are used in many fields of life such as medical diagnoses, artificial intelligence, and economics. But these theories have some limitations because of their inability with the parameterization tool. Molodtsov [17] introduced the notion of soft sets (SSs) to accommodate the abovementioned drawbacks considering the parameterization of the alternatives. Maji et al. [18] prolonged the idea of SSs with several necessary operations along with their appropriate possessions and established a DM method to resolve DM issues utilizing their developed operations [19]. Maji et al. [20] merged the two existing theories such as FSs and SSs and offered the concept of fuzzy soft sets (FSSs) with some elementary operations and their desired properties. Maji et al. [21] extended the notion of FSSs and proposed the idea of intuitionistic fuzzy soft sets with some operations and properties. Xu [22] introduced a method for IFSs to compare intuitionistic fuzzy numbers utilizing score and accuracy functions. Xu and Yager [23] proposed the weighted average and ordered weighted average operators for IFSs with their examples and properties. They also presented a DM approach to solve MADM complications utilizing their developed operators. Garg and Arora [24] proposed the generalized form of IFSSs with AOs and established a DM methodology based on their developed AOs to resolve DM issues. Garg and Arora [25] developed the correlation coefficient (CC) and weighted correlation coefficient (WCC) for IFSSs. They also presented the TOPSIS methodology to resolve MADM issues utilizing their developed correlation measures. Zulqarnain et al. [26] extended the notion of interval-valued IFSSs and proposed AOs for interval-valued IFSSs. They also presented the CC and WCC for interval-valued IFSSs and constructed the TOPSIS approach to resolve the MADM complications based on their presented correlation measures.

Peng et al. [27] introduced the theory of PFSSs by merging two existing theories such as PFSs and SSs. They also presented some fundamental operations of PFSSs and discussed their desirable properties. Athira et al. [28] extended the notion of PFSSs, introduced some novel distance measures for PFSSs, and established a DM method based on presented distance measures to solve complicated problems. Zulqarnain et al. [29] developed the operational laws for Pythagorean fuzzy soft numbers (PFSNs) and proposed the AOs for PFSNs. They also presented a MADM method to resolve DM concerns using their developed AOs. Riaz et al. [30] defined the concept of m polar PFSSs and developed the TOPSIS method to solve MCGDM problems. Riaz et al. [31] presented the similarity measures for PFSSs and discussed their essential properties. They also proposed the weighted AOs for m-polar PFSs [32] and established a decision-making approach to solve DM concerns. Zulqarnain et al. [33] extended the idea of PFSSs and developed the TOPSIS method based on the CC. They also presented an MCGDM approach and utilized their developed approach for the selection of suppliers in green supply chain management. Mehmood et al. [34] proposed the AOs for T-spherical fuzzy sets and developed a DM approach to solving MADM issues. Wang and Garg [35] introduced some novel operational laws considering the interaction and established the AOs based on their developed rules. Batool et al. [36] introduced the TOPSIS method for Pythagorean probabilistic hesitant fuzzy sets and entropy measures under considered environment. Ullah et al. [37] developed the complex PFSs with some novel distance measures and their desirable properties. Hussain et al. [38] introduced the soft rough PFSs and Pythagorean fuzzy soft rough set with some necessary operators and properties.

The existing studies are unable to accommodate the situation when any parameters of a set of attributes have corresponding subattributes. Smarandache [39] developed the concept of hypersoft sets (HSSs) which replace the function f of a parameter with a multi-subattribute, that is, characterized on the Cartesian product of n attributes. The developed HSS competently deals with the uncertainty and vagueness comparative to SS. He also presented many other extensions of HSS such as crisp HSS, fuzzy HSS, intuitionistic fuzzy HSS, neutrosophic HSS, and plithogenic HSS. Zulqarnain et al. [40] developed the theory of neutrosophic hypersoft matrices with some logical operators. They also proposed the MADM approach to solve DM concerns. The authors presented the generalized AOs for NHSSs [41]. Zulqarnain et al. [42] developed the CC and WCC for IFHSSs and proposed the TOPSIS method using developed CC. Zulqarnain et al. [43] proposed some AOs and CC for

PFHSSs and discussed their properties. They also developed the TOPSIS approach for PFHSSs based on their presented CC. However, the above-discussed theories only deal with the uncertainty utilizing MD and NMD of subattributes. If experts consider MD = 0.6 and NDM = 0.7, then $0.6 + 0.7 \geq 1$ of any subattribute of the alternatives. We will check that it cannot be addressed by the above strategies. To overwhelm the above restrictions, we introduced some AOs for PFHSSs by modifying the condition $\mathcal{T}_{\mathcal{F}(\check{d})}(\delta) + \mathcal{I}_{\mathcal{F}(\check{d})}(\delta) \leq 1$ to $(\mathcal{T}_{\mathcal{F}(\check{d})}(\delta))^2 + (\mathcal{I}_{\mathcal{F}(\check{d})}(\delta))^2 \leq 1$. The main purpose of the succeeding study is to originate new AOs for the PFHSSs considering interactions, which may also observe the assertions of PFHSs. Moreover, an MCDM method with a numerical example has been presented which shows the effectiveness of the planned methodology.

Supplier selection and valuation are a crucial prospect of business routine. Due to variations in management strategies, the selection of suppliers is considered from multiple perspectives, which included environmental and social necessities. Therefore, in the literature, this query is stated as a reference question for MCGDM as a sustainable supplier selection. Continuing, there are several papers [44–47] that carried the MCDM approach for the selection of sustainable suppliers according to relevant data and considerations that appropriately reflect the preferences of decision makers. However, all the above methods are not appropriate for summarizing the abovementioned methodologies and cannot deliberate the interaction among Mem and NMem functions. Particularly, we can say that the influence of other levels of Mem or NMem on the conforming geometric or average AOs does not have any influence on the aggregation process. In addition, it has been stated from the above-discussed models that the overall Mem (NMem) function level is independent of its corresponding NMem (Mem) function level. So, the consequences corresponding to those models are not favorable, so no reasonable order of preference is given for alternatives. Therefore, how to add these PFHSNs through interaction relations is an interesting topic. To solve this problem, in this article, we are going to develop some interaction AOs such as PFHSIWA and PFHSIWG operators for PFHSSs. An algorithm is planned to resolve the DM problem based on our established operators. A numerical example has been presented to ensure the practicality of the developed DM approach.

The rest of the research can be summarized as follows: In Section 2, we presented the necessary concepts such as SSs, FSSs, HSSs, IFHSSs, and PFHSSs which can support us to construct the subsequent research organization. In Section 3, we defined some novel operational laws for PFHSSs considering interaction and developed some AOs based on interaction operational laws such as PFHSIWA and PFHSIWG operators using presented operational laws with their desirable properties. In Section 4, an MCDM method is developed utilizing the proposed operators. A numerical example is provided to ensure the implementation of the setup MCDM method. Moreover, we used some of the existing methods to present comparative analysis with our planned approach. Also, we present the benefits, simplicity, flexibility, as well as effectiveness of the planned method in Section 5, and we organized a comprehensive debate and comparison among some available techniques and our established methodology.

2. Preliminaries

In this section, we recollect some fundamental notions such as SSs, FSSs, HSSs, IFHSSs, and PFHSSs.

Definition 1 (see [17]). Let $\mathcal{U}$ and $\mathcal{E}$ be the universe of discourse and set of attributes, respectively. Let $\mathcal{P}(\mathcal{U})$ be the power set of $\mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{E}$. A pair $(\mathcal{F}, \mathcal{A})$ is called SSs over $\mathcal{U}$, and its mapping is expressed as follows:

$$\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{P}(\mathcal{U}). \tag{1}$$

Also, it can be defined as follows:

$$(\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{A}}) = \{\boldsymbol{\mathcal{F}}(\mathbf{e}) \in \boldsymbol{\mathcal{P}}(\boldsymbol{\mathcal{U}})\colon \mathbf{e} \in \boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{F}}(\mathbf{e}) = \varnothing, \mathbf{if}\ \mathbf{e} \notin \boldsymbol{\mathcal{A}}\}. \tag{2}$$

Definition 2 (see [20]). Let $\mathcal{U}$ and $\mathcal{E}$ be a universe of discourse and set of attributes, respectively, and $\mathcal{F}(\mathcal{U})$ be a power set of $\mathcal{U}$. Let $\mathcal{A} \subseteq \mathcal{E}$; then, $(\mathcal{F}, \mathcal{A})$ is FSSs over $\mathcal{U}$, and its mapping can be stated as follows:

$$\mathsf{F}\colon \mathcal{A} \longrightarrow F(\mathcal{U}). \tag{3}$$

Definition 3 (see [39]). Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$, $k = \{k_1, k_2, k_3, \ldots, k_n\}$, $n \geq 1$, and K_i represents the set of attributes and their corresponding subattributes such as $K_i \cap K_j = \varphi$, where $i \neq j$ for each $n \geq 1$ and i and $j \in \{1, 2, 3, \ldots, n\}$. Assume $K_1 \times K_2 \times K_3 \times \cdots \times K_n = \dddot{\mathcal{A}} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of subattributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$. Then, the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \cdots \times K_n = (\mathcal{F}, \mathcal{A})$ is known as HSSs, defined as follows:

$$\mathcal{F}\colon K_1 \times K_2 \times K_3 \times \cdots \times K_n = \dddot{\mathcal{A}} \longrightarrow \mathcal{P}(\mathcal{U}). \tag{4}$$

It is also defined as

$$(\mathcal{F}, \dddot{\mathcal{A}}) = \left\{\check{d}, \mathcal{F}_{\ddot{A}}(\check{d})\colon \check{d} \in \dddot{\mathcal{A}}, \mathcal{F}_{\ddot{A}}(\check{d}) \in \mathcal{P}(\mathcal{U})\right\}. \tag{5}$$

Definition 4 (see [39]). Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ be a power set of $\mathcal{U}$, $k = \{k_1, k_2, k_3, \ldots, k_n\}$, $n \geq 1$, and K_i represents the set of attributes and their corresponding subattributes such as $K_i \cap K_j = \varphi$ where $i \neq j$ for each $n \geq 1$ and i and $j \in \{1, 2, 3, \ldots, n\}$. Assume $K_1 \times K_2 \times K_3 \times \cdots \times K_n = \mathcal{A} = \{d_{1h} \times d_{2k} \times \cdots \times d_{nl}\}$ is a collection of subattributes, where $1 \leq h \leq \alpha$, $1 \leq k \leq \beta$, $1 \leq l \leq \gamma$, and α, β, and $\gamma \in \mathbb{N}$, and let $\mathrm{IFS}^{\mathcal{U}}$ be a collection of all fuzzy subsets over $\mathcal{U}$. Then, the pair $(\mathcal{F}, K_1 \times K_2 \times K_3 \times \cdots \times K_n = (\mathcal{F}, \mathcal{A})$ is known as IFHSSs, defined as follows:

$$\mathcal{F}\colon K_1 \times K_2 \times K_3 \times \cdots \times K_n = \dddot{\mathcal{A}} \longrightarrow \mathrm{IFS}^{\mathcal{U}}. \tag{6}$$

It is also defined as $(\mathcal{F}, \dddot{\mathcal{A}}) = \left\{(\check{d}, \mathcal{F}_{\dddot{\mathcal{A}}}(\check{d}))\colon \check{d} \in \dddot{\mathcal{A}}, \mathcal{F}_{\dddot{\mathcal{A}}}(\check{d}) \in \mathrm{IFS}^{\mathcal{U}} \in [0, 1]\right\}$, where $\mathcal{F}_{\dddot{\mathcal{A}}}(\check{d}) = \left\{\delta, \mathcal{T}_{\mathcal{F}(\check{d})}(\delta), \mathcal{I}_{\mathcal{F}(\check{d})}\right.$

(δ): $\delta \in \mathcal{U}\}$, where $\mathcal{T}_{\mathcal{F}(\check{d})}(\delta)$ and $\mathcal{J}_{\mathcal{F}(\check{d})}(\delta)$ signify the Mem and NMem values of the attributes:

$$\mathcal{T}_{\mathcal{F}(\check{d})}(\delta), \mathcal{J}_{\mathcal{F}(\check{d})}(\delta) \in [0, 1], 0 \leq \mathcal{T}_{\mathcal{F}(\check{d})}(\delta) + \mathcal{J}_{\mathcal{F}(\check{d})}(\delta) \leq 1. \tag{7}$$

Remark 1. If $(\mathcal{T}_{\mathcal{F}(\check{d})}(\delta))^2 + (\mathcal{J}_{\mathcal{F}(\check{d})}(\delta))^2 \leq 1$ and $\mathcal{T}_{\mathcal{F}(\check{d})}(\delta) + \mathcal{J}_{\mathcal{F}(\check{d})}(\delta) \leq 1$ are satisfied, then PFHSSs are reduced to IFHSSs [42].

The PFHSNs $\mathcal{F}_{\delta_i}(\check{d}_j) = \{(\mathcal{T}_{\mathcal{F}(\check{d}_j)}(\delta_i), \mathcal{J}_{\mathcal{F}(\check{d}_j)}(\delta_i))| \delta_i \in \mathcal{U}\}$ can be express as $\mathfrak{J}_{\check{d}_{ij}} = \mathcal{T}_{\mathcal{F}(\check{d}_{ij})}, \mathcal{J}_{\mathcal{F}(\check{d}_{ij})}$. To compute the alternatives, ranking score function of $\mathfrak{J}_{\check{d}_{ij}}$ can be defined as follows:

$$\mathbb{S}\left(\mathfrak{J}_{\check{d}_{ij}}\right) = \mathcal{T}_{\mathcal{F}\left(\check{d}_{ij}\right)}{}^2 - \mathcal{J}_{\mathcal{F}\left(\check{d}_{ij}\right)}{}^2, \mathbb{S}\left(\mathfrak{J}_{\check{d}_{ij}}\right) \in [-1, 1]. \tag{8}$$

But, sometimes the scoring function such as $\mathfrak{J}_{\check{d}_{11}} = \langle 0.4, 0.7\rangle$ and $\mathfrak{J}_{\check{d}_{12}} = \langle 0.5, 0.8\rangle$ cannot compare two PFHSNs. It is impossible to claim that which alternative is most suitable $\mathbb{S}(\mathfrak{J}_{\check{d}_{11}}) = 0.3 = \mathbb{S}(\mathfrak{J}_{\check{d}_{12}})$. To overcome such difficulties, we need to introduce the accuracy function as follows:

$$H\left(\mathfrak{J}_{\check{d}_{ij}}\right) = \mathcal{T}_{\mathcal{F}\left(\check{d}_{ij}\right)}{}^2 + \mathcal{J}_{\mathcal{F}\left(\check{d}_{ij}\right)}{}^2, \quad H\left(\mathfrak{J}_{\check{d}_{ij}}\right) \in [0, 1]. \tag{9}$$

Hence, some rules have been introduced in the following for the comparison among two PFHSNs $\mathfrak{J}_{\check{d}_{ij}}$ and $\mathfrak{T}_{\check{d}_{ij}}$.

(1) If $\mathbb{S}(\mathfrak{J}_{\check{d}_{ij}}) > \mathbb{S}(\mathfrak{T}_{\check{d}_{ij}})$, then $\mathfrak{J}_{\check{d}_{ij}} > \mathfrak{T}_{\check{d}_{ij}}$

(2) If $\mathbb{S}(\mathfrak{J}_{\check{d}_{ij}}) = \mathbb{S}(\mathfrak{T}_{\check{d}_{ij}})$, then

If $H(\mathfrak{J}_{\check{d}_{ij}}) > H(\mathfrak{T}_{\check{d}_{ij}})$, then $\mathfrak{J}_{\check{d}_{ij}} > \mathfrak{T}_{\check{d}_{ij}}$

If $H(\mathfrak{J}_{\check{d}_{ij}}) = H(\mathfrak{T}_{\check{d}_{ij}})$, then $\mathfrak{J}_{\check{d}_{ij}} = \mathfrak{T}_{\check{d}_{ij}}$

Observe that the overall difference between PFHSNs and IFHSNs lies in their distinguishing limits. The Pythagorean membership degree area is larger than either the intuitionistic membership degree area. PFHSNs cannot only model IFHSNs' ability to capture DM scenarios anywhere the sum of Mem as well as NMem of subattributes of the considered parameters is equal to or less than 1 but it is also unable to handle the circumstances where IFHSNs are not able to characterize the sum of Mem as well as NMem of multi-subattributes of the considered attributes exceeding 1. On the contrary, PFHSNs accommodate more uncertainty considering Mem as well as NMem of multi-subattributes of the considered attributes, and the sum of their squares is equal to or less than 1.

Definition 5 (see [43]). Let $\mathfrak{J}_{\check{d}_k} = (\mathcal{T}_{\check{d}_k}, \mathcal{J}_{\check{d}_k})$, $\mathfrak{J}_{\check{d}_{11}} = (\mathcal{T}_{\check{d}_{11}}, \mathcal{J}_{\check{d}_{11}})$, and $\mathfrak{J}_{\check{d}_{12}} = (\mathcal{T}_{\check{d}_{12}}, \mathcal{J}_{\check{d}_{12}})$ be three PFHSNs and α be a positive real number; by algebraic norms, we have

(1) $\mathfrak{J}_{\check{d}_{11}} \oplus \mathfrak{J}_{\check{d}_{12}} = \left\langle \sqrt{\mathcal{T}^2_{\check{d}_{11}} + \mathcal{T}^2_{\check{d}_{12}} - \mathcal{T}^2_{\check{d}_{11}}\mathcal{T}^2_{\check{d}_{12}}}, \mathcal{J}_{\check{d}_{11}}\mathcal{J}_{\check{d}_{12}} \right\rangle$

(2) $\mathfrak{J}_{\check{d}_{11}} \otimes \mathfrak{J}_{\check{d}_{12}} = \left\langle \mathcal{T}_{\check{d}_{11}}\mathcal{T}_{\check{d}_{12}}, \sqrt{\mathcal{J}^2_{\check{d}_{11}} + \mathcal{J}^2_{\check{d}_{12}} - \mathcal{J}^2_{\check{d}_{11}}\mathcal{J}^2_{\check{d}_{12}}} \right\rangle$

(3) $\alpha\mathfrak{J}_{\check{d}_k} = \left\langle \sqrt{1 - (1 - \mathcal{T}^2_{\check{d}_k})^\alpha}, \mathcal{J}^\alpha_{\check{d}_k} \right\rangle$

(4) $\mathfrak{J}^\alpha_{\check{d}_k} = \left\langle \mathcal{T}^\alpha_{\check{d}_k}, \sqrt{1 - (1 - \mathcal{J}^2_{\check{d}_k})^\alpha} \right\rangle$

For the collection of PFHSNs $\mathfrak{J}_{\check{d}_{ij}}$, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, $\Omega = (\Omega_1, \Omega_1, \ldots, \Omega_n)^T$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_m)^T$ be weight vectors for experts and attributes $\Omega_i > 0$, $\sum_{i=1}^n \Omega_i = 1$, $\gamma_j > 0$, and $\sum_{j=1}^m \gamma_j = 1$, respectively. Zulqarnain et al. [43] presented the averaging and geometric AOs as follows:

$$\text{PFHSWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \left\langle \sqrt{1 - \prod_{j=1}^m \left(\prod_{I=1}^n \left(1 - \mathcal{T}_{\check{d}_{ij}}{}^2\right)^{\Omega_i}\right)^{\gamma_j}}, \prod_{j=1}^m \left(\prod_{I=1}^n \left(\mathcal{J}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \right\rangle, \tag{10}$$

$$\text{PFHSWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \left\langle \prod_{j=1}^m \left(\prod_{I=1}^n \left(\mathcal{T}_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}, \sqrt{1 - \prod_{j=1}^m \left(\prod_{I=1}^n \left(1 - \mathcal{J}_{\check{d}_{ij}}{}^2\right)^{\Omega_i}\right)^{\gamma_j}} \right\rangle. \tag{11}$$

3. Interaction Aggregation Operators for Pythagorean Fuzzy Hypersoft Numbers

In this section, we introduce interaction AOs for PFHSNs. In it, some fundamental properties have been discussed based on defined interaction PFHSIWA and PFHSIWG operators for PFHSNs.

3.1. Interaction Operational Laws for PFHSNs

Definition 6. Let $\mathfrak{J}_{\check{d}_k} = (\mathcal{T}_{\check{d}_k}, \mathcal{J}_{\check{d}_k})$, $\mathfrak{J}_{\check{d}_{11}} = (\mathcal{T}_{\check{d}_{11}}, \mathcal{J}_{\check{d}_{11}})$, and $\mathfrak{J}_{\check{d}_{12}} = (\mathcal{T}_{\check{d}_{12}}, \mathcal{J}_{\check{d}_{12}})$ be three PFHSNs and α be a positive real number; by algebraic norms, considering the interaction, we have

(1) $\mathfrak{J}_{\check{d}_{11}} \oplus \mathfrak{J}_{\check{d}_{12}} = \left\langle \sqrt{\mathcal{T}^2_{\check{d}_{11}} + \mathcal{T}^2_{\check{d}_{12}} - \mathcal{T}^2_{\check{d}_{11}}\mathcal{T}^2_{\check{d}_{12}}}, \sqrt{\mathcal{J}^2_{\check{d}_{11}} + \mathcal{J}^2_{\check{d}_{12}} - \mathcal{J}^2_{\check{d}_{11}}\mathcal{J}^2_{\check{d}_{12}} - \mathcal{J}^2_{\check{d}_{11}}\mathcal{T}^2_{\check{d}_{12}} - \mathcal{T}^2_{\check{d}_{11}}\mathcal{J}^2_{\check{d}_{12}}} \right\rangle$

(2) $\mathfrak{J}_{\check{d}_{11}} \otimes \mathfrak{J}_{\check{d}_{12}} = \left\langle \sqrt{\mathcal{T}^2_{\check{d}_{11}} + \mathcal{T}^2_{\check{d}_{12}} - \mathcal{T}^2_{\check{d}_{11}} \quad \mathcal{T}^2_{\check{d}_{12}} - \mathcal{T}^2_{\check{d}_{11}} \mathcal{J}^2_{\check{d}_{12}} - \mathcal{J}^2_{\check{d}_{11}}\mathcal{T}^2_{\check{d}_{12}}}, \sqrt{\mathcal{J}^2_{\check{d}_{11}} + \mathcal{J}^2_{\check{d}_{12}} - \mathcal{J}^2_{\check{d}_{11}}\mathcal{J}^2_{\check{d}_{12}}} \right\rangle$

(3) $\alpha\mathfrak{J}_{\check{d}_k} = \left\langle \sqrt{1 - (1 - \mathcal{T}^2_{\check{d}_k})^\alpha}, \sqrt{(1 - \mathcal{T}^2_{\check{d}_k})^\alpha - [1 - (\mathcal{T}^2_{\check{d}_k} + \mathcal{J}^2_{\check{d}_k})]^\alpha} \right\rangle$

(4) $\mathfrak{J}_{\check{d}_k}^{\alpha} = \left\langle \sqrt{(1-\mathcal{I}_{\check{d}_k}^2)^{\alpha} - [1-(\mathcal{T}_{\check{d}_k}^2 + \mathcal{I}_{\check{d}_k}^2)]^{\alpha}}, \sqrt{1-(1-\mathcal{I}_{\check{d}_k}^2)^{\alpha}} \right\rangle$

Based on the above-defined operational laws, now we introduce some interaction AOs for PFHSNs' Δ.

Definition 7. Let $\mathfrak{J}_{\check{d}_{ij}} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{I}_{\check{d}_{ij}})$ be PFHSNs and Ω_i and γ_j represent the weights of expert's and multi-subattributes along with stated conditions $\Omega_i > 0$, $\sum_{i=1}^{n} \Omega_i = 1$, $\gamma_j > 0$, and $\sum_{j=1}^{m} \gamma_j = 1$. Then, PFHSIWA: $\Delta^n \longrightarrow \Delta$ is defined as follows:

$$\text{PFHSIWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \oplus_{j=1}^{m} \gamma_j \left(\oplus_{i=1}^{n} \Omega_i \mathfrak{J}_{\check{d}_{ij}}\right). \tag{12}$$

Theorem 1. *Let $\mathfrak{J}_{\check{d}_{ij}} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{I}_{\check{d}_{ij}})$ be PFHSNs, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Then, the attained aggregated values using equation (12) is also a PFHSN and*

$$\text{PFHSIWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \left\langle \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i} \right)^{\gamma_j}}, \sqrt{\prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left(1 - \mathcal{T}_{\check{d}_{ij}}{}^2\right)^{\Omega_i} \right)^{\gamma_j} - \prod_{j=1}^{m} \left(\prod_{i=1}^{n} \left[1 - \left(\mathcal{T}_{\check{d}_{ij}}{}^2 + \mathcal{I}_{\check{d}_{ij}}{}^2\right)\right]^{\Omega_i} \right)^{\gamma_j}} \right\rangle, \tag{13}$$

where Ω_i and γ_j represent the expert's and subattributes' weights with certain circumstances $\Omega_i > 0$, $\sum_{i=1}^{n} \Omega_i = 1$, $\gamma_j > 0$, and $\sum_{j=1}^{m} \gamma_j = 1$.

Proof. The PFHSIWA operator can be proved using the principle of mathematical induction as follows:

For $n = 1$, we get $\Omega_1 = 1$. Then, we have

$$\begin{aligned}
\text{PFHSIWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) &= \oplus_{j=1}^{m} \gamma_j \mathfrak{J}_{\check{d}_{ij}}, \\
\text{PFHSIWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) &= \left\langle \sqrt{1 - \prod_{j=1}^{m} \left(\left(1 - \mathcal{T}_{\check{d}_{ij}}{}^2\right)\right)^{\gamma_j}}, \sqrt{\prod_{j=1}^{m} \left(1 - \mathcal{T}_{\check{d}_{ij}}{}^2\right)^{\gamma_j} - \prod_{j=1}^{m} \left(1 - \left(\mathcal{T}_{\check{d}_{ij}}{}^2 + \mathcal{I}_{\check{d}_{ij}}{}^2\right)\right)^{\gamma_j}} \right\rangle, \\
&= \left\langle \sqrt{1 - \prod_{j=1}^{m} \left(\prod_{i=1}^{1} \left(1 - \mathcal{T}_{\check{d}_{ij}}{}^2\right)^{\Omega_i} \right)^{\gamma_j}}, \right. \\
&\quad \left. \sqrt{\prod_{j=1}^{m} \left(\prod_{i=1}^{1} \left(1 - \mathcal{T}_{\check{d}_{ij}}{}^2\right)^{\Omega_i} \right)^{\gamma_j} - \prod_{j=1}^{m} \left(\prod_{i=1}^{1} \left[1 - \left(\mathcal{T}_{\check{d}_{ij}}{}^2 + \mathcal{I}_{\check{d}_{ij}}{}^2\right)\right]^{\Omega_i} \right)^{\gamma_j}} \right\rangle.
\end{aligned} \tag{14}$$

For $m = 1$, we get $\gamma_1 = 1$. Then, we have

$$\begin{aligned}
\text{PFHSIWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) &= \oplus_{i=1}^{n} \Omega_i \mathfrak{J}_{\check{d}_{11}} \\
&= \left\langle \sqrt{1 - \prod_{i=1}^{n} \left(1 - \mathcal{T}_{\check{d}_{i1}}^2\right)^{\Omega_i}}, \sqrt{\prod_{1=1}^{n} \left(1 - \mathcal{T}_{\check{d}_{i1}}^2\right)^{\Omega_i} - \prod_{i=1}^{n} \left[1 - \left(\mathcal{T}_{\check{d}_{i1}}^2 + \mathcal{I}_{\check{d}_{i1}}^2\right)\right]^{\Omega_i}} \right\rangle \\
&= \left\langle \sqrt{1 - \prod_{j=1}^{1} \left(\prod_{i=1}^{n} \left(1 - \mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i} \right)^{\gamma_j}}, \right. \\
&\quad \left. \sqrt{\prod_{j=1}^{1} \left(\prod_{i=1}^{n} \left(1 - \mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i} \right)^{\gamma_j} - \prod_{j=1}^{1} \left(\prod_{i=1}^{n} \left[1 - \left(\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{I}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i} \right)^{\gamma_j}} \right\rangle.
\end{aligned} \tag{15}$$

The above justification shows that equation (13) holds for $n = 1$ and $m = 1$. Now, assume that equation (13) also holds for $m = \beta_1 + 1$, $n = \beta_2$, $m = \beta_1$, and $n = \beta_2 + 1$:

$$\oplus_{j=1}^{\beta_1+1}\gamma_j\left(\oplus_{i=1}^{\beta_2}\Omega_i\mathfrak{F}_{\check{d}_{ij}}\right) = \left\langle \sqrt{1-\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(1-\mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}}, \sqrt{\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(1-\mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j} - \prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left[1-\left(\mathcal{T}_{\check{d}_{ij}}^2+\mathcal{I}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}}\right\rangle,$$

$$\oplus_{j=1}^{\beta_1}\gamma_j\left(\oplus_{i=1}^{\beta_2+1}\Omega_i\mathfrak{F}_{\check{d}_{ij}}\right) = \left\langle \sqrt{1-\prod_{j=1}^{\beta_1}\left(\prod_{i=1}^{\beta_2+1}\left(1-\mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}}, \sqrt{\prod_{j=1}^{\beta_1}\left(\prod_{i=1}^{\beta_2+1}\left(1-\mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j} - \prod_{j=1}^{\beta_1}\left(\prod_{i=1}^{\beta_2+1}\left[1-\left(\mathcal{T}_{\check{d}_{ij}}^2+\mathcal{I}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}}\right\rangle. \tag{16}$$

For $m = \beta_1 + 1$ and $n = \beta_2 + 1$, we have

$$\begin{aligned}
\oplus_{j=1}^{\beta_1+1}\gamma_j\left(\oplus_{i=1}^{\beta_2+1}\Omega_i\mathfrak{F}_{\check{d}_{ij}}\right) &= \oplus_{j=1}^{\beta_1+1}\gamma_j\left(\oplus_{i=1}^{\beta_2}\Omega_i\mathfrak{F}_{\check{d}_{ij}}\oplus\Omega_{\beta_2+1}\mathfrak{F}_{\check{d}_{(\beta_2+1)j}}\mathfrak{F}_{\check{d}_{(\beta_2+1)j}}\right),\\
&= \oplus_{j=1}^{\beta_1+1}\oplus_{i=1}^{\beta_2}\gamma_j\Omega_i\mathfrak{F}_{\check{d}_{ij}}\oplus_{j=1}^{\beta_1+1}\gamma_j\Omega_{\beta_2+1}\mathfrak{F}_{\check{d}_{(\beta_2+1)j}},\\
&= \left\langle \sqrt{1-\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(1-\mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}} \oplus \sqrt{1-\prod_{j=1}^{\beta_1+1}\left(\left(1-\mathcal{T}_{\check{d}_{(\beta_2+1)j}}^2\right)^{\Omega_{\beta_2+1}}\right)^{\gamma_j}}\right.\\
&\quad \sqrt{\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(1-\mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j} - \prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left[1-\left(\mathcal{T}_{\check{d}_{ij}}^2+\mathcal{I}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}}\\
&\quad \left.\oplus\sqrt{\prod_{j=1}^{\beta_1+1}\left(\left(1-\mathcal{T}_{\check{d}_{(\beta_2+1)j}}^2\right)^{\Omega_{\beta_2+1}}\right)^{\gamma_j} - \prod_{j=1}^{\beta_1+1}\left(\left[1-\left(\mathcal{T}_{\check{d}_{(\beta_2+1)j}}^2+\mathcal{I}_{\check{d}_{(\beta_2+1)j}}^2\right)\right]^{\Omega_{\beta_2+1}}\right)^{\gamma_j}}\right\rangle,\\
&= \left\langle \sqrt{1-\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2+1}\left(1-\mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}}, \sqrt{\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2+1}\left(1-\mathcal{T}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j} - \prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2+1}\left[1-\left(\mathcal{T}_{\check{d}_{ij}}^2+\mathcal{I}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}}\right\rangle.
\end{aligned} \tag{17}$$

Hence, it is true for $m = \beta_1 + 1$ and $n = \beta_2 + 1$. □

Example 1. Let $\mathcal{U}\{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3\}$ be a set of experts whose weights are given as $\Omega_i = (0.243, 0.514, 0.343)^T$. Experts evaluate the beauty of a house under a considered set of attributes $\mathfrak{F}' = \{d_1 = \text{lawn}, d_2 = \text{security system}\}$ with their corresponding subattributes lawn $= d_1 = \{d_{11} = \text{with grass}, d_{12} = \text{without grass}\}$ and security system $d_2 = \{d_{21} = \text{guards}, d_{22} = \text{cameras}\}$. Let $\mathfrak{F}' = d_1 \times d_2$ be a set of subattributes $\mathfrak{F}' = d_1 \times d_2 = \{d_{11}, d_{12}\} \times \{d_{21}, d_{22}\} = \{(d_{11}, d_{21}), (d_{11}, d_{22}), (d_{12}, d_{21}), (d_{12}, d_{22})\}$ and $\mathfrak{F}' = \{\check{d}_1, \check{d}_2, \check{d}_3, \check{d}_4\}$ represent the set subattributes with weights $\gamma_j = (0.25, 0.15, 0.2, 0.4)^T$. Experts' opinion for each multi-subattribute in the form of PFHSNs $(\mathfrak{F}, \mathfrak{F}') = \left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{I}_{\check{d}_{ij}} \right\rangle_{3\times 4}$ is given as follows:

$$(\mathfrak{F}, \mathfrak{F}') = \begin{bmatrix} (0.3, 0.8) & (0.4, 0.6) & (0.3, 0.6) & (0.5, 0.6)\\ (0.8, 0.3) & (0.7, 0.4) & (0.7, 0.3) & (0.4, 0.8)\\ (0.3, 0.6) & (0.5, 0.7) & (0.6, 0.5) & (0.5, 0.4) \end{bmatrix}. \tag{18}$$

By using equation (13),

$$\begin{aligned}
\text{PFHSIWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) &= \left\langle \sqrt{1 - \prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(1 - \mathscr{T}_{\check{d}_{ij}}^{2}\right)^{\Omega_i}\right)^{\gamma_j}}, \sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(1 - \mathscr{T}_{\check{d}_{ij}}^{2}\right)^{\Omega_i}\right)^{\gamma_j} - \prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left[1 - \left(\mathscr{T}_{\check{d}_{ij}}^{2} + \mathscr{I}_{\check{d}_{ij}}^{2}\right)\right]^{\Omega_i}\right)^{\gamma_j}} \right\rangle, \\
&= \left\langle \sqrt{1 - \left(\left\{(0.91)^{0.243}(0.36)^{.514}(0.91)^{.343}\right\}^{0.25}\left\{(0.84)^{0.243}(0.51)^{0.514}(0.75)^{0.343}\right\}^{0.15}\left\{(0.91)^{0.243}(0.51)^{0.514}(0.64)^{0.343}\right\}^{.2}\left\{(0.75)^{0.243}(0.84)^{0.514}(0.75)^{0.343}\right\}^{.4}\right)} \right. \\
&\sqrt{\left(\left\{(0.91)^{0.243}(0.36)^{0.514}(0.91)^{0.343}\right\}^{0.25}\left\{(0.84)^{0.243}(0.51)^{0.514}(0.75)^{0.343}\right\}^{0.15}\left\{(0.91)^{0.243}(0.51)^{0.514}(0.64)^{0.343}\right\}^{0.2}\left\{(0.75)^{0.243}(0.84)^{0.514}(0.75)^{0.343}\right\}^{0.4}\right) -} \\
&\left. \sqrt{\left(\left\{(0.27)^{0.243}(0.27)^{0.514}(0.55)^{0.343}\right\}^{0.25}\left\{(0.48)^{0.243}(0.35)^{0.514}(0.26)^{.343}\right\}^{0.15}\left\{(0.55)^{0.243}(0.42)^{0.514}(0.39)^{0.343}\right\}^{0.2}\left\{(0.39)^{0.243}(0.20)^{0.514}(0.59)^{0.343}\right\}^{0.4}\right)} \right\rangle, \\
&= 0.58759, 0.58241.
\end{aligned} \tag{19}$$

Hence, some fundamental properties utilizing the planned PFHSIWA operator for the collection of PFHSNs are established based on Theorem 1.

3.2. Properties of PFHSIWA Operator

3.2.1. Idempotency. If $\mathfrak{J}_{\check{d}_{ij}} = \mathfrak{J}_{\check{d}} = (\mathscr{T}_{\check{d}_{ij}}, \mathscr{I}_{\check{d}_{ij}}), \forall i, j$, then

$$\text{PFHSIWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \mathfrak{J}_{\check{d}}. \tag{20}$$

Proof. As we know that all $\mathfrak{J}_{\check{d}_{ij}} = \mathfrak{J}_{\check{d}} = (\mathscr{T}_{\check{d}_{ij}}, \mathscr{I}_{\check{d}_{ij}})$, using equation (13), we have

$$\begin{aligned}
\text{PFHSIWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) &= \left\langle \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \mathscr{T}_{\check{d}_{ij}}^{2}\right)^{\Omega_i}\right)^{\gamma_j}}, \right. \\
&\left. \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1 - \mathscr{T}_{\check{d}_{ij}}^{2}\right)^{\Omega_i}\right)^{\gamma_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left[1 - \left(\mathscr{T}_{\check{d}_{ij}}^{2} + \mathscr{I}_{\check{d}_{ij}}^{2}\right)\right]^{\Omega_i}\right)^{\gamma_j}} \right\rangle \\
&= \left\langle \sqrt{1 - \left(\left(1 - \mathscr{T}_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j}}, \right. \\
&\left. \sqrt{\left(\left(1 - \mathscr{T}_{\check{d}_{ij}}^{2}\right)^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j} - \left(\left[1 - \left(\mathscr{T}_{\check{d}_{ij}}^{2} + \mathscr{I}_{\check{d}_{ij}}^{2}\right)\right]^{\sum_{i=1}^{n}\Omega_i}\right)^{\sum_{j=1}^{m}\gamma_j}} \right\rangle \\
&= \left\langle \sqrt{1 - \left(1 - \mathscr{T}_{\check{d}_{ij}}^{2}\right)}, \sqrt{\left(1 - \mathscr{T}_{\check{d}_{ij}}^{2}\right) - \left[1 - \left(\mathscr{I}_{\check{d}_{ij}}^{2} + \mathscr{T}_{\check{d}_{ij}}^{2}\right)\right]} \right\rangle = \left(\mathscr{T}_{\check{d}_{ij}}, \mathscr{I}_{\check{d}_{ij}}\right) = \mathfrak{J}_{\check{d}}.
\end{aligned} \tag{21}$$

□

3.2.2. Boundedness. Let $\mathfrak{J}_{\check{d}_{ij}}$ be a collection of PFHSNs, $\mathfrak{J}_{\check{d}_{ij}}^{-} = \min_j \min_i \left\{\mathscr{T}_{\check{d}_{ij}}\right\}, \max_j \max_i \left\{\mathscr{I}_{\check{d}_{ij}}\right\}$, and $\mathfrak{J}_{\check{d}_{ij}}^{+} = \max_j \max_i \left\{\mathscr{T}_{\check{d}_{ij}}\right\}, \min_j \min_i \left\{\mathscr{I}_{\check{d}_{ij}}\right\}$; then, $\mathfrak{J}_{\check{d}_{ij}}^{-} \leq \text{PFHSIWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) \leq \mathfrak{J}_{\check{d}_{ij}}^{+}$.

Proof. As we know that $\mathfrak{J}_{\check{d}_{ij}} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{I}_{\check{d}_{ij}})$ is a collection of PFHSNs, then

$$\begin{aligned}
&\min_j \min_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\} \leq \mathcal{T}^2_{\check{d}_{ij}} \leq \max_j \max_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\} \\
&\Rightarrow 1 - \max_j \max_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\} \leq 1 - \mathcal{T}^2_{\check{d}_{ij}} \leq 1 - \min_j \min_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\} \\
&\Leftrightarrow \left(1 - \max_j \max_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\}\right)^{\Omega_i} \leq \left(1 - \mathcal{T}^2_{\check{d}_{ij}}\right) \leq \left(1 - \min_j \min_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\}\right)^{\Omega_i} \\
&\Leftrightarrow \left(1 - \max_j \max_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\}\right)^{\sum_{i=1}^n \Omega_i} \leq \prod_{i=1}^n \left(1 - \mathcal{T}^2_{\check{d}_{ij}}\right)^{\Omega_i} \leq \left(1 - \min_j \min_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\}\right)^{\sum_{i=1}^n \Omega_i} \\
&\Leftrightarrow \left(1 - \max_j \max_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\}\right)^{\sum_{j=1}^m \gamma_j} \leq \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}^2_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \leq \left(1 - \min_j \min_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\}\right)^{\sum_{j=1}^m \gamma_j} \\
&\Leftrightarrow 1 - \max_j \max_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\} \leq \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}^2_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \leq 1 - \min_j \min_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\} \\
&\Leftrightarrow \min_j \min_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\} \leq 1 - \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}^2_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} \leq \max_j \max_i \left\{\mathcal{T}^2_{\check{d}_{ij}}\right\} \\
&\Leftrightarrow \min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} \leq \sqrt{1 - \prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}^2_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}} \leq \max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}.
\end{aligned} \tag{22}$$

Similarly,

$$\min_j \min_i \left\{\mathcal{I}_{\check{d}_{ij}}\right\} \leq \sqrt{\prod_{j=1}^m \left(\prod_{i=1}^n \left(1 - \mathcal{T}^2_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} - \prod_{j=1}^m \left(\prod_{i=1}^n \left[1 - \left(\mathcal{I}_{\check{d}_{ij}}\right)\right]^{\Omega_i}\right)^{\gamma_j}} \leq \max_j \max_i \left\{\mathcal{I}_{\check{d}_{ij}}\right\}. \tag{23}$$

Let $\text{PFHSIWA}(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}) = \mathcal{T}_\delta$ and $\mathcal{I}_\delta = \mathfrak{J}_\delta$; then, inequalities (22) and (23) can be changed into the subsequent arrangement $\min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} \leq \mathcal{T}_\delta \leq \max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}$ and $\min_j \min_i \left\{\mathcal{I}_{\check{d}_{ij}}\right\} \leq \mathcal{I}_\delta \leq \max_j \max_i \left\{\mathcal{I}_{\check{d}_{ij}}\right\}$, respectively.

Operating equation (8), we get

$$\begin{aligned}
\mathcal{S}(\mathfrak{J}_\delta) &= \mathcal{T}^2_\delta - \mathcal{I}^2_\delta \leq \max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} - \min_j \min_i \left\{\mathcal{I}_{\check{d}_{ij}}\right\} = \mathcal{S}\left(\mathfrak{J}^+_{\check{d}_{ij}}\right), \\
\mathcal{S}(\mathfrak{J}_\delta) &= \mathcal{T}^2_\delta - \mathcal{I}^2_\delta \geq \min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\} - \max_j \max_i \left\{\mathcal{I}_{\check{d}_{ij}}\right\} = \mathcal{S}\left(\mathfrak{J}^-_{\check{d}_{ij}}\right).
\end{aligned} \tag{24}$$

Then, by order relation among two PFSNs, we have

$$\mathfrak{J}^-_{\check{d}_{ij}} \leq \text{PFHSIWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) \leq \mathfrak{J}^+_{\check{d}_{ij}}. \tag{25}$$

□

3.2.3. Homogeneity. Prove that $\text{PFHSIWA}(\alpha\mathfrak{J}_{\check{d}_{11}}, \alpha\mathfrak{J}_{\check{d}_{12}}, \ldots, \alpha\mathfrak{J}_{\check{d}_{nm}}) = \alpha \cdot \text{PFHSIWA}(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}})$ for any positive real number α.

Proof. Let $\mathfrak{J}_{\check{d}_{ij}}$ be a collection of PFHSNs and >0; then, by using Definition 6 (10), we have

$$\alpha\mathfrak{J}_{\check{d}_{ij}} = \left\langle \sqrt{1-\left(1-\mathcal{T}^2_{\check{d}_{ij}}\right)^{\alpha}}, \sqrt{\left(1-\mathcal{T}^2_{\check{d}_{ij}}\right)^{\alpha} - \left[1-\left(\mathcal{T}^2_{\check{d}_{ij}}+\mathcal{J}^2_{\check{d}_{ij}}\right)\right]^{\alpha}} \right\rangle. \tag{26}$$

So,

$$\begin{aligned} \text{PFHSIWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) &= \left\langle \sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}^2_{\check{d}_{ij}}\right)^{\alpha\Omega_i}\right)^{\gamma_j}}, \right. \\ &\left. \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}^2_{\check{d}_{ij}}\right)^{\alpha\Omega_i}\right)^{\gamma_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left[1-\left(\mathcal{T}^2_{\check{d}_{ij}}+\mathcal{J}^2_{\check{d}_{ij}}\right)\right]^{\alpha\Omega_i}\right)^{\gamma_j}} \right\rangle \\ &= \left\langle \sqrt{1-\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}^2_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right)^{\alpha}}, \right. \\ &\left. \sqrt{\left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{T}^2_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}\right)^{\alpha} - \left(\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left[1-\left(\mathcal{T}^2_{\check{d}_{ij}}+\mathcal{J}^2_{\check{d}_{ij}}\right)\right]^{\Omega_i}\right)^{\gamma_j}\right)^{\alpha}} \right\rangle \\ &= \alpha\text{PFHSIWA}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right). \end{aligned} \tag{27}$$

The proof is completed. □

Definition 8. Let $\mathfrak{J}_{\check{d}_{ij}} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}})$ be PFHSNs and Ω_i and γ_j represent the weights of expert's and multi-subattributes along with stated conditions $\Omega_i > 0$, $\sum_{i=1}^{n}\Omega_i = 1$, $\gamma_j > 0$, and $\sum_{j=1}^{m}\gamma_j = 1$. Then, PFHSIWG: $\Delta^n \longrightarrow \Delta$ is defined as follows:

$$\text{PFHSIWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \otimes_{j=1}^{m}\left(\otimes_{i=1}^{n}\mathfrak{J}^{\Omega_i}_{\check{d}_{nm}}\right)^{\gamma_j}. \tag{28}$$

Theorem 2. *Let $\mathfrak{J}_{\check{d}_{ij}} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}})$ be a collection of PFHSNs, where $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Then, utilizing equation (28), we get PFHSN and*

$$\text{PFHSIWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \left\langle \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{J}^2_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left[1-\left(\mathcal{T}^2_{\check{d}_{ij}}+\mathcal{J}^2_{\check{d}_{ij}}\right)\right]^{\Omega_i}\right)^{\gamma_j}}, \sqrt{1-\prod_{j=1}^{m}\left(\prod_{i=1}^{n}\left(1-\mathcal{J}^2_{\check{d}_{ij}}\right)^{\Omega_i}\right)^{\gamma_j}} \right\rangle, \tag{29}$$

where Ω_i and γ_j represent the expert's and subattributes' weights with certain circumstances $\Omega_i > 0$, $\sum_{i=1}^{n}\Omega_i = 1$, $\gamma_j > 0$, and $\sum_{j=1}^{m}\gamma_j = 1$.

Proof. The PFHSIWG operator can be proved using the principle of mathematical induction as follows:

For $n = 1$, we get $\Omega_1 = 1$. Then, we have

$$\text{PFHSIWG}\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) = \otimes_{j=1}^{m} \mathfrak{I}_{\check{d}_{1j}}^{\gamma_j},$$

$$\text{PFHSIWG}\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) = \left\langle \sqrt{\prod_{j=1}^{m}\left(1 - \mathcal{I}_{\check{d}_{1j}}^2\right)^{\gamma_j} - \prod_{j=1}^{m}\left(1 - \left(\mathcal{T}_{\check{d}_{1j}}^2 + \mathcal{I}_{\check{d}_{1j}}^2\right)\right)^{\gamma_j}}, \sqrt{1 - \prod_{j=1}^{m}\left(\left(1 - \mathcal{I}_{\check{d}_{1j}}^2\right)\right)^{\gamma_j}} \right\rangle,$$

$$= \left\langle \sqrt{\prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1 - \mathcal{I}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j} - \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left[1 - \left(\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{I}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}}, \right.$$

$$\left. \sqrt{1 - \prod_{j=1}^{m}\left(\prod_{i=1}^{1}\left(1 - \mathcal{I}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}} \right\rangle. \tag{30}$$

For $m = 1$, we get $\gamma_1 = 1$. Then, we have

$$\text{PFHSIWG}\left(\mathfrak{I}_{\check{d}_{11}}, \mathfrak{I}_{\check{d}_{12}}, \ldots, \mathfrak{I}_{\check{d}_{nm}}\right) = \otimes_{i=1}^{n} \Omega_i \mathfrak{I}_{\check{d}_{11}}$$

$$= \left\langle \sqrt{\prod_{1=1}^{n}\left(1 - \mathcal{I}_{\check{d}_{i1}}^2\right)^{\Omega_i} - \prod_{i=1}^{n}\left[1 - \left(\mathcal{T}_{\check{d}_{i1}}^2 + \mathcal{I}_{\check{d}_{i1}}^2\right)\right]^{\Omega_i}}, \sqrt{1 - \prod_{i=1}^{n}\left(1 - \mathcal{I}_{\check{d}_{i1}}^2\right)^{\Omega_i}} \right\rangle \tag{31}$$

$$= \left\langle \sqrt{\prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1 - \mathcal{I}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j} - \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left[1 - \left(\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{I}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}}, \sqrt{1 - \prod_{j=1}^{1}\left(\prod_{i=1}^{n}\left(1 - \mathcal{I}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}} \right\rangle.$$

The above justification shows that equation (10) holds for $n = 1$ and $m = 1$. Now, assume that equation (10) also holds for $m = \beta_1 + 1$, $n = \beta_2$, $m = \beta_1$, and $n = \beta_2 + 1$:

$$\otimes_{j=1}^{\beta_1+1} \gamma_j \left(\otimes_{i=1}^{\beta_2} \Omega_i \mathfrak{I}_{\check{d}_{ij}}\right) = \left\langle \sqrt{\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(1 - \mathcal{I}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j} - \prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left[1 - \left(\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{I}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}}, \sqrt{1 - \prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(1 - \mathcal{I}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}} \right\rangle,$$

$$\otimes_{j=1}^{\beta_1} \gamma_j \left(\otimes_{i=1}^{\beta_2+1} \Omega_i \mathfrak{I}_{\check{d}_{ij}}\right) = \left\langle \sqrt{\prod_{j=1}^{\beta_1}\left(\prod_{i=1}^{\beta_2+1}\left(1 - \mathcal{I}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j} - \prod_{j=1}^{\beta_1}\left(\prod_{i=1}^{\beta_2+1}\left[1 - \left(\mathcal{T}_{\check{d}_{ij}}^2 + \mathcal{I}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}}, \sqrt{1 - \prod_{j=1}^{\beta_1}\left(\prod_{i=1}^{\beta_2+1}\left(1 - \mathcal{I}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}} \right\rangle. \tag{32}$$

For $m = \beta_1 + 1$ and $n = \beta_2 + 1$, we have

$$
\begin{aligned}
\otimes_{j=1}^{\beta_1+1}\gamma_j\left(\otimes_{i=1}^{\beta_2+1}\Omega_i\mathfrak{J}_{\check{d}_{ij}}\right) &= \otimes_{j=1}^{\beta_1+1}\gamma_j\left(\otimes_{i=1}^{\beta_2}\Omega_i\mathfrak{J}_{\check{d}_{ij}}\otimes\Omega_{\beta_2+1}\mathfrak{J}_{\check{d}_{(\beta_2+1)j}}\right)\\
&= \otimes_{j=1}^{\beta_1+1}\otimes_{i=1}^{\beta_2}\gamma_j\Omega_i\mathfrak{J}_{\check{d}_{ij}}\otimes_{j=1}^{\beta_1+1}\gamma_j\Omega_{\beta_2+1}\mathfrak{J}_{\check{d}_{(\beta_2+1)j}}\\
&= \left\langle\sqrt{\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(1-\mathcal{J}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}-\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left[1-\left(\mathcal{T}_{\check{d}_{ij}}^2+\mathcal{J}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}}\right.\\
&\quad\oplus\sqrt{\prod_{j=1}^{\beta_1+1}\left(\left(1-\mathcal{J}_{\check{d}_{(\beta_2+1)j}}^2\right)^{\Omega_{\beta_2+1}}\right)^{\gamma_j}-\prod_{j=1}^{\beta_1+1}\left(\left[1-\left(\mathcal{T}_{\check{d}_{(\beta_2+1)j}}^2+\mathcal{J}_{\check{d}_{(\beta_2+1)j}}^2\right)\right]^{\Omega_{\beta_2+1}}\right)^{\gamma_j}}\\
&\quad\left.\sqrt{1-\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2}\left(1-\mathcal{J}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}}\oplus\sqrt{1-\prod_{j=1}^{\beta_1+1}\left(\left(1-\mathcal{J}_{\check{d}_{(\beta_2+1)j}}^2\right)^{\Omega_{\beta_2+1}}\right)^{\gamma_j}}\right\rangle\\
&= \left\langle\sqrt{\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2+1}\left(1-\mathcal{J}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}-\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2+1}\left[1-\left(\mathcal{T}_{\check{d}_{ij}}^2+\mathcal{J}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}},\right.\\
&\quad\left.\sqrt{1-\prod_{j=1}^{\beta_1+1}\left(\prod_{i=1}^{\beta_2+1}\left(1-\mathcal{J}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}}\right\rangle.
\end{aligned}
\tag{33}
$$

Hence, it is true for $m = \beta_1 + 1$ and $n = \beta_2 + 1$. □

Example 2. Let $\mathcal{U}\{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3\}$ be a set of experts whose weights are given as $\Omega_i = (0.243, 0.514, 0.343)^T$. Experts evaluate the beauty of a house under a considered set of attributes $\mathfrak{J}' = \{d_1 = \text{lawn}, d_2 = \text{security system}\}$ with their corresponding subattributes lawn $= d_1 = \{d_{11} = \text{with grass}, d_{12} = \text{without grass}\}$ and security system $= d_2 = \{d_{21} = \text{guards}, d_{22} = \text{cameras}\}$. Let $\mathfrak{J}' = d_1 \times d_2$ be a set of subattributes $\mathfrak{J}' = d_1 \times d_2 = \{d_{11}, d_{12}\} \times \{d_{21}, d_{22}\} = \{(d_{11}, d_{21}), (d_{11}, d_{22}), (d_{12}, d_{21}), (d_{12}, d_{22})\}$ and $\mathfrak{J}' = \{\check{d}_1, \check{d}_2, \check{d}_3, \check{d}_4\}$ represents the set subattributes with weights $\gamma_j = (0.25, 0.15, 0.2, .4)^T$. Experts' opinion for each multi-subattribute in the form of PFHSNs $(\mathfrak{J}, \mathfrak{J}') = \left\langle \mathcal{T}_{\check{d}_{ij}}, \mathcal{J}_{\check{d}_{ij}} \right\rangle_{3\times 4}$ is given as follows:

$$
(\mathfrak{J}, \mathfrak{J}') = \begin{bmatrix} (0.3, 0.8) & (0.4, 0.6) & (0.3, 0.6) & (0.5, 0.6) \\ (0.8, 0.3) & (0.7, 0.4) & (0.7, 0.3) & (0.4, 0.8) \\ (0.3, 0.6) & (0.5, 0.7) & (0.6, 0.5) & (0.5, 0.4) \end{bmatrix}. \tag{34}
$$

By using equation (13),

$$
\begin{aligned}
&\text{PFHSIWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \left\langle\sqrt{\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(1-\mathcal{J}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}-\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left[1-\left(\mathcal{T}_{\check{d}_{ij}}^2+\mathcal{J}_{\check{d}_{ij}}^2\right)\right]^{\Omega_i}\right)^{\gamma_j}}, \sqrt{1-\prod_{j=1}^{4}\left(\prod_{i=1}^{3}\left(1-\mathcal{J}_{\check{d}_{ij}}^2\right)^{\Omega_i}\right)^{\gamma_j}}\right\rangle,\\
&= \left\langle\sqrt{\left(\left\{(0.36)^{0.243}(0.91)^{0.514}(0.64)^{0.343}\right\}^{0.25}\left\{(0.64)^{0.243}(0.84)^{0.514}(0.51)^{0.343}\right\}^{0.15}\left\{(0.64)^{0.243}(0.91)^{0.514}(0.75)^{0.343}\right\}^{0.2}\left\{(0.64)^{0.243}(0.36)^{0.514}(0.84)^{0.343}\right\}^{0.4}\right)-}\right.\\
&\sqrt{\left(\left\{(0.27)^{0.243}(0.27)^{0.514}(0.55)^{0.343}\right\}^{0.25}\left\{(0.48)^{0.243}(0.35)^{0.514}(0.26)^{0.343}\right\}^{0.15}\left\{(0.55)^{0.243}(0.42)^{0.514}(0.39)^{0.343}\right\}^{0.2}\left\{(0.39)^{0.243}(0.20)^{0.514}(0.59)^{0.343}\right\}^{0.4}\right)}\\
&\left.\sqrt{1-\left(\left\{(0.36)^{0.243}(0.91)^{0.514}(0.64)^{0.343}\right\}^{0.25}\left\{(0.64)^{0.243}(0.84)^{0.514}(0.51)^{0.343}\right\}^{0.15}\left\{(0.64)^{0.243}(0.91)^{0.514}(0.75)^{0.343}\right\}^{0.2}\left\{(0.64)^{0.243}(0.36)^{0.514}(0.84)^{0.343}\right\}^{0.4}\right)}\right\rangle,\\
&\langle 0.53653, 0.62976.
\end{aligned}
\tag{35}
$$

Hence, some basic properties for PFHSNs using the PFHSWG operator are established using Theorem 2.

3.3. Properties of PFHSIWG Operator

3.3.1. Idempotency. $\mathfrak{J}_{\check{d}_{ij}} = \mathfrak{J}_{\check{d}} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{I}_{\check{d}_{ij}}), \forall i, j$, then

$$\text{PFHSIWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) = \mathfrak{J}_{\check{d}_{\delta}}. \quad (36)$$

3.3.2. Boundedness. Let $\mathfrak{J}_{\check{d}_{ij}}$ be a collection of PFHSNs, $\mathfrak{J}^{-}_{\check{d}_{ij}} = \min_j \min_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}, \max_j \max_i \left\{\mathcal{I}_{\check{d}_{ij}}\right\}$, and $\mathfrak{J}^{+}_{\check{d}_{ij}} = \max_j \max_i \left\{\mathcal{T}_{\check{d}_{ij}}\right\}, \min_j \min_i \left\{\mathcal{I}_{\check{d}_{ij}}\right\}$; then,

$$\mathfrak{J}^{-}_{\check{d}_{ij}} \leq \text{PFHSIWG}\left(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}}\right) \leq \mathfrak{J}^{+}_{\check{d}_{ij}}. \quad (37)$$

3.3.3. Homogeneity. Prove that $\text{PFHSIWG}(\alpha\mathfrak{J}_{\check{d}_{11}}, \alpha\mathfrak{J}_{\check{d}_{12}}, \ldots, \alpha\mathfrak{J}_{\check{d}_{nm}}) = \alpha \cdot \text{PFHSIWG}(\mathfrak{J}_{\check{d}_{11}}, \mathfrak{J}_{\check{d}_{12}}, \ldots, \mathfrak{J}_{\check{d}_{nm}})$ for any positive real number α.

4. An MCDM Approach Based on Interaction Aggregation Operators for PFHSSs

An MCDM approach is established here under the developed operators and presented a comprehensive comparative analysis to prove the usefulness and practicality of our established method.

4.1. Proposed MCDM Approach. Consider $\chi = \{\chi^{(1)}, \chi^{(2)}, \chi^{(3)}, \ldots, \chi^{(s)}\}$ to be a set of s alternatives and $\mathcal{U} = \{\delta_1, \delta_2, \delta_3, \ldots, \delta_n\}$ to be a set n experts. The weights of experts are given as $\Omega = (\Omega_1, \Omega_1, \ldots, \Omega_n)^T$ and $\Omega_i > 0$, $\sum_{i=1}^{n} \Omega_i = 1$. Let $\mathfrak{L} = \{d_1, d_2, \ldots, d_m\}$ represent the set attributes with their corresponding multi-subattributes such as $\mathfrak{J}' = \left\{(d_{1\rho} \times d_{2\rho} \times \cdots \times d_{m\rho}) \text{for all } \rho \in \{1, 2, \ldots, t\}\right\}$ with weights $\gamma = (\gamma_{1\rho}, \gamma_{2\rho}, \gamma_{3\rho}, \ldots, \gamma_{m\rho})^T$ such as $\gamma_\rho > 0$, $\sum_{\rho=1}^{t} \gamma_\rho = 1$, and can be stated as $\mathfrak{J}' = \left\{\check{d}_\partial : \partial \in \{1, 2, \ldots, m\}\right\}$. The group of experts $\{\kappa^i : i = 1, 2, \ldots, n\}$ assess the alternatives $\left\{\aleph^{(z)} : z = 1, 2, \ldots, s\right\}$ under the chosen subattributes $\left\{\check{d}_\partial : \partial = 1, 2, \ldots, k\right\}$ in the form of PFHSNs such as $(\chi^{(z)}_{\check{d}_{ik}})_{n\times m} = (\mathcal{T}_{\check{d}_{ij}}, \mathcal{I}_{\check{d}_{ij}})_{n\times m}$ where $0 \leq \mathcal{T}_{\check{d}_{ij}}, \mathcal{I}_{\check{d}_{ij}} \leq 1$, and $0 \leq (\mathcal{T}_{\check{d}_{ij}})^2 + (\mathcal{I}_{\check{d}_{ij}})^2 \leq 1$ for all i and k. Utilizing the proposed PFHSIWA, PFHSIWG operators develop aggregated PFHSNs $\mathcal{L}_\phi$ for each alternative according to the expert's preferences. Finally, utilizing equation (8), compute the score function. The above-presented approach can be concise as follows:

Step 1. Develop decision matrices for each alternative $\{D^{(z)} : z = 1, 2, \ldots, s\}$ as follows:

$$\left(\chi^{(z)}, \mathfrak{J}'\right)_{n\times\partial} = \begin{matrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{matrix} \begin{pmatrix} \left(\mathcal{T}^{(z)}_{\check{d}_{11}}, \mathcal{I}^{(z)}_{\check{d}_{11}}\right) & \left(\mathcal{T}^{(z)}_{\check{d}_{12}}, \mathcal{I}^{(z)}_{\check{d}_{12}}\right) & \cdots & \left(\mathcal{T}^{(z)}_{\check{d}_{1\partial}}, \mathcal{I}^{(z)}_{\check{d}_{1\partial}}\right) \\ \left(\mathcal{T}^{(z)}_{\check{d}_{21}}, \mathcal{I}^{(z)}_{\check{d}_{21}}\right) & \left(\mathcal{T}^{(z)}_{\check{d}_{22}}, \mathcal{I}^{(z)}_{\check{d}_{22}}\right) & \cdots & \left(\mathcal{T}^{(z)}_{\check{d}_{2\partial}}, \mathcal{I}^{(z)}_{\check{d}_{2\partial}}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \left(\mathcal{T}^{(z)}_{\check{d}_{n1}}, \mathcal{I}^{(z)}_{\check{d}_{n1}}\right) & \left(\mathcal{T}^{(z)}_{\check{d}_{n2}}, \mathcal{I}^{(z)}_{\check{d}_{n2}}\right) & \cdots & \left(\mathcal{T}^{(z)}_{\check{d}_{n\partial}}, \mathcal{I}^{(z)}_{\check{d}_{n\partial}}\right) \end{pmatrix}. \quad (38)$$

Step 2. Obtain normalized decision matrices for alternatives utilizing the normalization rule:

$$H_{ij} = \begin{cases} \mathfrak{J}^{c}_{\check{d}_{ij}}; \text{cost type parameter}, \\ \mathfrak{J}_{\check{d}_{ij}}; \text{benefit type parameter}. \end{cases} \quad (39)$$

Step 3. Establish a collective decision matrix $\mathcal{L}_k$ for each alternative using developed AOs

Step 4. Using equation (8), compute the score values for each alternative

Step 5. Select the most suitable alternative with the maximum score value

Step 6. Rank the alternatives

The graphical representation of the presented approach can be expressed in following Figure 1.

4.2. Case Study. The problem of supplier selection is an essential part at both a logical and practical level. This is an ongoing problem for the organization because the most suitable choice of suppliers is the basis for effective supply chain management and also the basis of reasonable benefit, which includes environmental management standards and includes more features of sustainable improvement in environmental management standards and supplier selection procedures. Depending on the visible horizon of substantial or social activities, supplier selection is typically known as "sustainable supplier selection" in the literature. This is a multidimensional consequence along with conflicting specifications. The self-assessment process needs to deliberate several features. From these perspectives, the issue of supplier selection is often considered a "reference" issue in the literature, with a wide range of methods used to support incorporative decisions. The problem of choosing and assessing a sustainable supplier is solved in lots of the best ways. This example of sustainable supplier selection results in a set of five parameters, using the analysis of [44–53]. These are d_1, supremacy of service, d_2, delivery, d_3, environmental efficiently, d_4, troposphere, and d_5, corporate social concern.

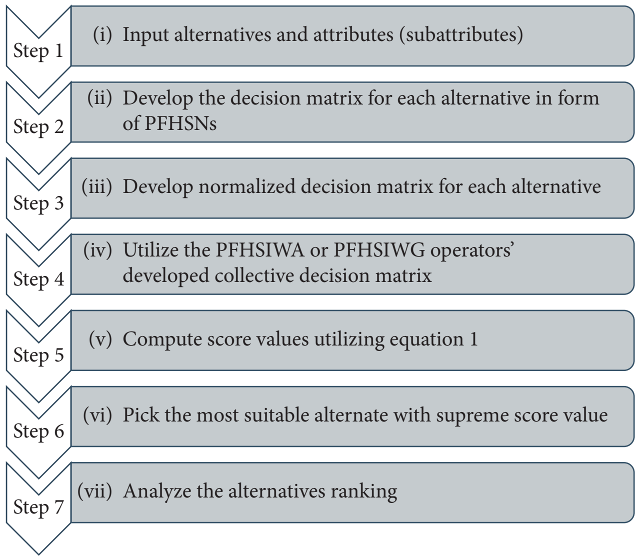

Figure 1: Flowchart of presented PFHSIWA or PFHSIWG operators.

Consider $\{\chi^{(1)}, \chi^{(2)}, \chi^{(3)}, \chi^{(4)}, \chi^{(5)}\}$ to be a set of alternatives, and $\mathfrak{L} = \{d_1 =$ supremacy, $d_2 =$ delivery, $d_3 =$ environmental efficiently, $d_4 =$ troposphere, $d_5 =$ corporate societal concern$\}$ represents the collection of considered parameters prearranged as supremacy $= d_1 = \{d_{11} =$ national level, $d_{12} =$ international level$\}$, delivery $= d_2 = \{d_{21} =$ by carriar, $d_{22} =$ by hand$\}$, environmental efficiently $= d_3 = \{d_{31} =$ environmental efficiently$\}$, troposphere $= d_4 = \{d_{41} =$ friendly, $d_{42} =$ non serious$\}$, and corporate social concern $= d_5 = \{d_{51} =$ corporate social concern$\}$. Let $\mathfrak{J}' = d_1 \times d_2 \times d_3 \times d_4 \times d_5$ be a set of subattributes:

$$\begin{aligned}\mathfrak{J}' &= d_1 \times d_2 \times d_3 \times d_4 \times d_5 = \{d_{11}, d_{12}\} \times \{d_{21}, d_{22}\} \times d\{d_{31}, d_{32}\} \times \{d_{41}\} \times \{d_{51}\},\\ &= \{(d_{11}, d_{21}, d_{31}, d_{41}, d_{51}), (d_{11}, d_{21}, d_{32}, d_{41}, d_{51}), (d_{11}, d_{22}, d_{31}, d_{41}, d_{51}), (d_{11}, d_{22}, d_{32}, d_{41}, d_{51}),\\ &\quad (d_{12}, d_{21}, d_{31}, d_{41}, d_{51}), (d_{12}, d_{21}, d_{32}, d_{41}, d_{51}), (d_{12}, d_{22}, d_{31}, d_{41}, d_{51}), (d_{12}, d_{22}, d_{32}, d_{41}, d_{51})\},\end{aligned} \tag{40}$$

where $\mathfrak{J}' = \{\breve{d}_1, \breve{d}_2, \breve{d}_3, \breve{d}_4, \breve{d}_5, \breve{d}_6, \breve{d}_7, \breve{d}_8\}$ represents the set of all subattributes with weights $(0.12, 0.18, 0.1, 0.15, 0.05, 0.22, 0.08, 0.1)^T$. Let $\{u_1, u_2, u_3\}$ be a set of experts with weights $(0.243, 0.514, 0.343)^T$ to evaluate the optimum alternative. Specialists contribute their predilections in the form of PFHSNs under multi-subattributes of considered attributes.

4.2.1. By Using PFHSIWA Operator

Step 1. Experts access the matters to illustrate the PFHSN. A summary of the many subattributes of the perceived attributes as well as their score values is given in Tables 1–3 .

Step 2. All attributes are of the same type, so no need to normalize them.

Step 3. Experts' opinion can be summarized utilizing equation (13) as follows:

$\mathscr{L}_1 = 0.6009, 0.4342$, $\mathscr{L}_2 = 0.6499, 0.4078$, $\mathscr{L}_3 = 0.6179, 0.3506$, $\mathscr{L}_4 = 0.6076, 0.3527$, and $\mathscr{L}_5 = 0.5493, 0.4345$.

Step 4. Compute the score values using equation (8):

$\mathbb{S}(\mathscr{L}_1) = 0.1667$, $\mathbb{S}(\mathscr{L}_2) = 0.2421$, $\mathbb{S}(\mathscr{L}_3) = 0.2673$, $\mathbb{S}(\mathscr{L}_4) = 0.2549$, and $\mathbb{S}(\mathscr{L}_5) = 0.1148$.

Step 5. $\chi^{(3)}$ has greatest score value, so $\chi^{(3)}$ is the finest option.

Step 6. Alternatives' ranking using the PFHSIWA operator is given as follows:

$\mathbb{S}(\mathscr{L}_3) > \mathbb{S}(\mathscr{L}_4) > \mathbb{S}(\mathscr{L}_2) > \mathbb{S}(\mathscr{L}_1) > \mathbb{S}(\mathscr{L}_5)$. So, $\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(1)} > \chi^{(5)}$.

4.2.2. By Using PFHSIWG Operator

Step 1 and Step 2. They are the same as Section 4.2.1.

Step 3. Experts' opinion can be summarized utilizing equation (29) as follows:

$\mathscr{L}_1 = 0.4679, 0.5590$, $\mathscr{L}_2 = 0.5157, 0.5289$, $\mathscr{L}_3 = 0.4892, 0.4387$, $\mathscr{L}_4 = 0.4910, 0.4751$, and $\mathscr{L}_5 = 0.4440, 0.6407$

Step 4. Compute the score values using equation (8):

$\mathbb{S}(\mathscr{L}_1) = -0.0911$, $\mathbb{S}(\mathscr{L}_2) = -0.0132$, $\mathbb{S}(\mathscr{L}_3) = 0.0505$, $\mathbb{S}(\mathscr{L}_4) = 0.0159$, and $\mathbb{S}(\mathscr{L}_5) = -0.1967$

Table 1: PFHS decision matrix for u_1.

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$	$\check{d}_5$	$\check{d}_6$	$\check{d}_7$	$\check{d}_8$
$\chi^{(1)}$	(0.3, 0.8)	(0.7, 0.3)	(0.6, 0.7)	(0.5, 0.4)	(0.2, 0.4)	(0.4, 0.6)	(0.5, 0.8)	(0.9, 0.3)
$\chi^{(2)}$	(0.6, 0.7)	(0.4, 0.6)	(0.3, 0.4)	(0.9, 0.2)	(0.3, 0.8)	(0.2, 0.4)	(0.7, 0.5)	(0.4, 0.5)
$\chi^{(3)}$	(0.7, 0.3)	(0.2, 0.5)	(0.1, 0.6)	(0.3, 0.4)	(0.4, 0.6)	(0.8, 0.4)	(0.6, 0.7)	(0.2, 0.5)
$\chi^{(4)}$	(0.8, 0.4)	(0.2, 0.9)	(0.2, 0.4)	(0.4, 0.6)	(0.6, 0.5)	(0.5, 0.6)	(0.4, 0.5)	(0.8, 0.3)
$\chi^{(5)}$	(0.5, 0.7)	(0.8, 0.5)	(0.7, 0.4)	(0.4, 0.3)	(0.4, 0.9)	(0.2, 0.4)	(0.8, 0.4)	(0.7, 0.5)

Table 2: PFHS decision matrix for u_2.

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$	$\check{d}_5$	$\check{d}_6$	$\check{d}_7$	$\check{d}_8$
$\chi^{(1)}$	(0.7, 0.6)	(0.3, 0.4)	(0.6, 0.5)	(0.3, 0.9)	(0.5, 0.4)	(0.4, 0.6)	(0.7, 0.5)	(0.4, 0.8)
$\chi^{(2)}$	(0.8, 0.5)	(0.7, 0.4)	(0.9, 0.2)	(0.7, 0.4)	(0.4, 0.5)	(0.9, 0.3)	(0.2, 0.7)	(0.3, 0.8)
$\chi^{(3)}$	(0.3, 0.7)	(0.4, 0.5)	(0.4, 0.8)	(0.3, 0.4)	(0.6, 0.7)	(0.3, 0.4)	(0.9, 0.2)	(0.7, 0.2)
$\chi^{(4)}$	(0.5, 0.4)	(0.7, 0.6)	(0.9, 0.3)	(0.8, 0.5)	(0.9, 0.2)	(0.2, 0.4)	(0.4, 0.6)	(0.6, 0.5)
$\chi^{(5)}$	(0.8, 0.5)	(0.7, 0.4)	(0.8, 0.5)	(0.5, 0.2)	(0.5, 0.7)	(0.7, 0.5)	(0.7, 0.6)	(0.6, 0.4)

Table 3: PFHS decision matrix for u_3.

	$\check{d}_1$	$\check{d}_2$	$\check{d}_3$	$\check{d}_4$	$\check{d}_5$	$\check{d}_6$	$\check{d}_7$	$\check{d}_8$
$\chi^{(1)}$	(0.5, 0.7)	(0.8, 0.5)	(0.7, 0.4)	(0.4, 0.3)	(0.4, 0.9)	(0.2, 0.4)	(0.8, 0.4)	(0.7, 0.5)
$\chi^{(2)}$	(0.8, 0.5)	(0.7, 0.4)	(0.8, 0.5)	(0.5, 0.2)	(0.5, 0.7)	(0.7, 0.5)	(0.7, 0.6)	(0.6, 0.4)
$\chi^{(3)}$	(0.6, 0.8)	(0.4, 0.5)	(0.6, 0.5)	(0.6, 0.4)	(0.7, 0.5)	(0.8, 0.4)	(0.5, 0.8)	(0.4, 0.5)
$\chi^{(4)}$	(0.5, 0.7)	(0.9, 0.3)	(0.3, 0.5)	(0.5, 0.7)	(0.3, 0.5)	(0.8, 0.5)	(0.7, 0.5)	(0.2, 0.5)
$\chi^{(5)}$	(0.5, 0.4)	(0.4, 0.8)	(0.5, 0.6)	(0.3, 0.4)	(0.7, 0.6)	(0.7, 0.5)	(0.4, 0.9)	(0.5, 0.2)

Table 4: Comparison of PFHSSs with some prevailing models.

	Set	Truthiness	Falsity	Parametrization	Attributes	Subattributes
Zadeh [1]	FS	✓	×	×	✓	×
Atanassov [2]	IFS	✓	✓	×	✓	×
Maji et al. [21]	IFSS	✓	✓	✓	✓	×
Peng et al. [27]	PFSS	✓	✓	✓	✓	×
Zulqarnain et al. [42]	IFHSS	✓	✓	✓	✓	✓
Proposed approach	PFHSS	✓	✓	✓	✓	✓

Step 5. $\chi^{(3)}$ has the greatest score value, so $\chi^{(3)}$ is the finest option

Step 6. Alternatives' ranking using the PFHSIWG operator is given as follows:

$\mathbb{S}(\mathscr{L}_3) > \mathbb{S}(\mathscr{L}_4) > \mathbb{S}(\mathscr{L}_2) > \mathbb{S}(\mathscr{L}_1) > \mathbb{S}(\mathscr{L}_5)$. So, $\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(1)} > \chi^{(5)}$.

5. Comparative Analysis and Discussion

In the following section, we will discuss quality, naivety, and tractability by means of the planned method. We also gave a brief overview of the following: the proposed approach with some existing methods.

5.1. Superiority of the Proposed Method. Through this study, along with association, it is resolute that the concerns attained with the proposed method are rather extrageneral than either technique. However, the developed MCDM approach has been provided more information to cope with the hesitancy in the DM procedure related to the existing MCDM strategies. Besides, the numerous mixed structures of FSs have grown into a unique feature of PFHSSs; after adding some proper conditions, the general facts about the component can be stated precisely and logically, as shown in Table 4. It is observed that the obtained results deliver extrainformation comparative to existing studies. The developed PFHSSs accurately accommodate more information considering the multi-subattributes of the parameters. It is quite an easy tool to mix inexact and unsure information within the DM process. Hence, the proposed methodology is pragmatic, diffident, and distinctive from available hybrid structures of fuzzy sets.

5.2. Discussion. Zadeh's [1] FSs only addressed the rough and vague facts using MD considering the subattributes for each alternative. But, the FSs are unable to deal with the

Table 5: Comparative analysis with existing operators.

Method	Score values for alternatives $\chi^{(1)}$	$\chi^{(2)}$	$\chi^{(3)}$	$\chi^{(4)}$	$\chi^{(5)}$	Ranking order
PFIWA [8]	0.55374	0.33901	0.60019	0.52007	0.36813	$\chi^{(3)} > \chi^{(1)} > \chi^{(4)} > \chi^{(5)} > \chi^{(2)}$
PFIWG [9]	0.49325	0.41837	0.73000	0.48906	0.46524	$\chi^{(3)} > \chi^{(1)} > \chi^{(4)} > \chi^{(5)} > \chi^{(2)}$
PFSWA [10]	0.21173	0.22017	0.33215	0.27008	0.21893	$\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(5)} > \chi^{(1)}$
PFSWG [10]	0.20587	0.23066	0.32902	0.25462	0.21727	$\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(5)} > \chi^{(1)}$
PFEWA [54]	0.51686	0.54833	0.60467	0.59021	0.51235	$\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(1)} > \chi^{(5)}$
PFEWG [54]	0.54219	0.56597	0.62190	0.59381	0.52209	$\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(1)} > \chi^{(5)}$
SPFWA [16]	0.08158	0.07674	0.14762	0.09959	0.07985	$\chi^{(3)} > \chi^{(4)} > \chi^{(1)} > \chi^{(5)} > \chi^{(2)}$
IFHSWA [55]	0.49830	0.41735	0.40935	0.46175	0.43247	$\chi^{(3)} > \chi^{(2)} > \chi^{(4)} > \chi^{(1)} > \chi^{(5)}$
IFHSWG [55]	0.42615	0.36175	0.35635	0.40790	0.40635	$\chi^{(3)} > \chi^{(2)} > \chi^{(4)} > \chi^{(1)} > \chi^{(5)}$
Proposed PFHSIWA operator	0.1667	0.2421	0.2673	0.2549	0.1148	$\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(1)} > \chi^{(5)}$
Proposed PFHSIWG operator	−0.0911	−0.0132	0.0505	0.0159	−0.1967	$\chi^{(3)} > \chi^{(4)} > \chi^{(2)} > \chi^{(1)} > \chi^{(5)}$

NMD of parameters. Atanassov's [2] IFSs accommodate the unclear and undefined objects using MD and NMD. However, IFSs are unable to handle the circumstances when $MD + NMD > 1$; on the contrary, our presented idea expertly compacts with such complications. Maji et al. [21] proposed the theory of IFSSs; the presented idea conducts the anxiety of the object in which the characteristics of MD and NMD can be used appropriately along with their parameterization with the following condition $MD + NMD \leq 1$. To handle these consequences, Peng et al. [27] suggested the idea of PFSSs by amending the condition $MD + NMD \leq 1$ to $MD^2 + NMD^2 \leq 1$ with their parametrization. But there is no information about the subattributes of the attributes under consideration in all the above studies. Therefore, the above theories are unable to address the situation when their subattributes are associated with the attributes. All prevailing hybrid structures of FSs cannot handle the NMem values of subattributes of considered n-tuple attributes. Zulqarnain et al. [42] extended the IFSSs to IFHSSs and proposed the CC and WCC for IFHSSs in which $MD + NMD \leq 1$ for each subattribute. But IFHSSs cannot provide any information on the Mem and NMem values of the subattribute of the considered attribute when $MD + NMD \geq 1$. It can be seen the finest choice of the projected approach simulates itself and ensures the success of the developed method as well as the responsibility.

5.3. Comparative Analysis. We endorse a new algorithmic rule for PFHSSs using developed PFHSIWA and PFHSIWG operators within the succeeding section. Consequently, we used the proposed algorithmic rule for any veridical problem, that is to say, supplier selection in SSCM. Results demonstrate that algorithmic governance is effective as well as sensible. From the above calculation, it can be observed that $\chi^{(3)}$ supplier is the premium alternative for SSCM. From the exploration, it is terminated that the results attained from the proposed viewpoint are more than the consequences of the planned theories. Thus, compared to available techniques, established AOs addressed unsure and unclear information efficiently. However, under available MCDM methods, the most important benefit of the proposed approach is that it can serve more information in the data than the available methodology. The comparison between existing AOs and our developed operators is given in following Table 5. The presented approach contemplates the interaction among the Mem and NMem function of PFHSNs, which can attain the more realistic decision effects considering the parametric values of the multi-subattributes of the parameters.

The existing PFIWA [8], PFIWG [9], PFEWA, PFEWG [54], and SPFWA [16] operators are not capable of dealing with the parametrization of the alternatives. The PFSWA and PFSWG [10] operators handle the parametric values of the alternatives but these operators cannot accommodate the multi-subattributes of the considered parameters. The prevailing IFHSWA and IFHSWG [55] operators competently deals the multi-sub attributes of the parameters comparative to above discuss operators. But IFHSSs cannot handle the situation when the sum of Mem and NMem values of the subattribute of the considered attribute exceeds 1. On the contrary, our proposed PFHSIWA and PFHSIWG operators competently accommodate the abovementioned shortcomings. So, we claim that our established operators are extraordinary than existing operators to solve imprecise as well as vague facts in DM procedure. The assistance of the deliberated approach along with related measures over present approaches is evading conclusions grounded on adverse reasons. Therefore, it is a useful tool for combining inaccurate and uncertain information in the DM process.

6. Conclusion

In this article, PFHSSs consider solving the complexities of information related to unsatisfactory, instability, and deviation by considering MD and NMD on the n-tuple subattributes of the suggested attributes. The core objective of this research is to propose novel operational laws considering the interaction. We also presented interaction aggregation operators, i.e., PFHSIWA and PFHSIWG, utilizing developed operational laws and discussed their desirable properties. Furthermore, based on developed interaction AOs, an MCDM approach has been established to solve real-life complications. To certify the applicability and practicality of our anticipated method, we planned an ephemeral comparative analysis of our developed methodology with some existing studies. From the obtained results, it can be decided absolutely that the predetermined methodology indicates that the experts have high stability and accessibility in the process of DM. The subsequent study will clarify the presentation of DM techniques using a number of other initiatives under PFHSSs, such as entropy and similarity measures. Furthermore, many other structures can be established and proposed, such as topological structure, algebraic structure, and configurable structure. In the future, PFHSSs can be extended to q-rung orthopair fuzzy hypersoft sets and spherical and T-spherical fuzzy hypersoft sets with their several AOs and decision-making approaches.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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