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Research Article

Set-Theoretic Inequalities Based on Convex Multi-Argument Approximate Functions via Set Inclusion

Atiqe Ur Rahman,[1] **Muhammad Saeed**,[1] **Khuram Ali Khan**,[2] **and Rostin Matendo Mabela**[3]

[1]*Department of Mathematics, University of Management and Technology, Lahore 54000, Pakistan*
[2]*Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan*
[3]*Department of Maths and Computer Science, Faculty of Science, University of Kinshasa, Kinshasa, Democratic Republic of the Congo*

Correspondence should be addressed to Rostin Matendo Mabela; rostin.mabela@unikin.ac.cd

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Hypersoft set is a novel area of study which is established as an extension of soft set to handle its limitations. It employs a new approximate mapping called multi-argument approximate function which considers the Cartesian product of attribute-valued disjoint sets as its domain and the power set of universe as its co-domain. The domain of this function is broader as compared to the domain of soft approximate function. It is capable of handling the scenario where sub-attribute-valued sets are considered more significant than taking merely single set of attributes. In this study, notions of set inclusion, convex (concave) sets, strongly convex (concave) sets, strictly convex (concave) sets, convex hull, and convex cone are conceptualized for the multi-argument approximate function. Based on these characterized notions, some set-theoretic inequalities are established with generalized properties and results. The set-theoretic version of classical Jensen's type inequalities is also discussed with the help of proposed notions.

1. Introduction

In order to provide the parameterization tool to fuzzy set-like models [1–3] for dealing with uncertain data, in 1999, Molodtsov [4] employed the concept of soft approximate function (sa-function) for defining a novel model soft set (s-set). This set employs sa-function which maps single set of parameters to power set of initial universe. Many scholars discussed rudiments of s-sets but the contributions of Ali et al. [5], Babitha and Sunil [6, 7], Ge and Yang [8], Li [9], Maji et al. [10], Pei and Miao [11], and Sezgin and Atagün [12] are considered prominent regarding the characterization of elementary properties and set-theoretic operations of s-sets. Abbas et al. [13] investigated the notions of soft points and discussed their limitations, comparisons, and challenges. The hybridized structures of s-set with fuzzy set, intuitionistic fuzzy set, and neutrosophic set were developed by the authors [14–16]. In numerous real-life situations, parameters are to be further classified into their respective parametric-valued disjoint sets. The existing s-set model is deficient for dealing with such sort of partitioning of parameters. Hypersoft set (hs-set) [17] is developed to generalize s-set to manage real-life scenarios with subparametric disjoint sets. In order to utilize hs-set in various areas of knowledge, Abbas et al. [18] and Saeed et al. [19] investigated the elementary properties and operations of hs-set. Ajay [20] defined alpha open hs-sets for their further use in the characterization of relevant topological spaces. Debnath [21], Yolcu and Ozturk [22], Yolcu et al. [23], and Kamacı [24, 25] discussed the applications of hs-set in decision-making process by using its hybridized models. Martin and Smarandache [26] developed a novel outlook called concentric plithogenic hypergraph based on the combination of plithogenic set and hs-set. Musa and Asaad [27] discussed the bipolarity of hs-set and investigated some of its properties and aggregation operations.

1.1. Motivation. Convexity has a vital function in optimization, image processing and classification, pattern recognition, and other sub-fields of computational geometry. In order to tackle optimization and other related problems for vague and uncertain data, notions of classical convexity and concavity (Lara et al. [28, 29]) under s-set environment were characterized initially by Deli [30]; then, this work was further extended by Ihsan et al. [31, 32], Rahman et al. [33], and Salih and Sabir [34] for further investigations on certain variants of convexity. But these notions are not sufficient for handling information with entitlement of multi-argument approximate function (maa-function); therefore, Rahman et al. [35] conceptualized the notions of convexity cum concavity under hs-set scenarios. It can be easily observed that the aforementioned models focused on the consideration of sa-function and maa-function while defining certain variants of convexity cum concavity and ignored certain other important properties and aspects of convexity cum concavity which may have key role in convex optimization, computational geometry, and other related problems. Moreover, various classical inequalities like Jensen's inequalities are not discussed as particular cases. The main motive of the proposed study is to address the limitations and shortcomings of the above-mentioned literature.

The major contributions of the paper are outlined as follows:

(1) The classical notions of convex (concave) set, strictly convex (concave) set, strongly convex (concave) set, convex hull, and convex cone are generalized under hs-set environment with entitlement of maa-function.

(2) The concept of set inclusion for maa-function is characterized and then some set-theoretic inequalities are established based on these set inclusions along with other proposed notions. It is verified that the classical inequalities for soft approximate function are also valid under hs-set environment with entitlement of maa-function.

(3) Classical Jensen's convex inequalities are generalized for maa-function and discussed as particular cases of proposed set-theoretic inequalities.

(4) The vivid comparison is made with related existing research studies to judge the distinction of the proposed study.

The rest of the paper is organized as follows. Section 2 reviews the notions of some relevant definitions used in the proposed work. Section 3 presents notions of strictly and strongly convexity cum concavity on hs-set. It also presents some set-theoretic inequalities based on these notions. Section 4 concludes the paper with more future directions and scope.

2. Preliminaries

This section reviews few elementary definitions and results regarding s-set and hs-set. In order to deal with uncertainties, the models like Zadeh's fuzzy set [3], Atanassov's intuitionistic fuzzy set [1], and Smarandache's neutrosophic set [2] are considered significant, but these models face inadequacy regarding parameterization modes; therefore, Molodtsov [4] presented the concept of s-set as a new mathematical model to equip fuzzy set-like models with parameterization tool.

Definition 1 (see [4]). Let $\breve{X} = \{\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\}$ be an initial universe and $\mathscr{P} = \{\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_n\}$ be a set of parameters; then, a sa-function is a mapping

$$\mathfrak{F}_{\breve{\mathfrak{S}}} : \mathscr{Q} \longrightarrow \Gamma^{\breve{X}}, \tag{1}$$

and it is defined as

$$\mathfrak{F}_{\breve{\mathfrak{S}}}(\{\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_k\}) = \Gamma^{\{\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\}}, \tag{2}$$

where $\Gamma^{\breve{X}}$ denotes the power set of $\breve{X}$, $\mathscr{Q} \subseteq \mathscr{P}$ with $k \leq n$. The pair $(\mathscr{Q}, \mathfrak{F}_{\breve{\mathfrak{S}}})$ is known as s-set and represented by $\breve{\mathfrak{S}}$. The collection of s-sets is denoted by Ω_{ss}.

Example 1. Let $\breve{X} = \{\widehat{x}_1, \widehat{x}_2, \widehat{x}_3, \widehat{x}_4, \widehat{x}_5, \widehat{x}_6\}$, $\mathscr{P} = \{\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_9\}$, and $\mathscr{Q} = \{\widehat{p}_1, \widehat{p}_2, \widehat{p}_3, \widehat{p}_4\}$; then, approximate elements of sa-function for s-set $\breve{\mathfrak{S}} = (\mathscr{Q}, \mathfrak{F}_{\breve{\mathfrak{S}}})$ are given as

$$\begin{aligned}
\mathfrak{F}_{\breve{\mathfrak{S}}}(\widehat{p}_1) &= \{\widehat{x}_1, \widehat{x}_3, \widehat{x}_6\},\\
\mathfrak{F}_{\breve{\mathfrak{S}}}(\widehat{p}_2) &= \{\widehat{x}_2, \widehat{x}_3, \widehat{x}_5\},\\
\mathfrak{F}_{\breve{\mathfrak{S}}}(\widehat{p}_3) &= \{\widehat{x}_4, \widehat{x}_5, \widehat{x}_6\},\\
\mathfrak{F}_{\breve{\mathfrak{S}}}(\widehat{p}_4) &= \{\widehat{x}_1, \widehat{x}_2, \widehat{x}_5\},
\end{aligned} \tag{3}$$

and s-set $\breve{\mathfrak{S}}$ is stated as $\breve{\mathfrak{S}} = \{\mathfrak{F}_{\breve{\mathfrak{S}}}(\widehat{p}_1), \mathfrak{F}_{\breve{\mathfrak{S}}}(\widehat{p}_2), \mathfrak{F}_{\breve{\mathfrak{S}}}(\widehat{p}_3), \mathfrak{F}_{\breve{\mathfrak{S}}}(\widehat{p}_4)\}$ or

$$\breve{\mathfrak{S}} = \{(\widehat{p}_1, \{\widehat{x}_1, \widehat{x}_3, \widehat{x}_6\}), (\widehat{p}_2, \{\widehat{x}_2, \widehat{x}_3, \widehat{x}_5\}), (\widehat{p}_3, \{\widehat{x}_4, \widehat{x}_5, \widehat{x}_6\}), (\widehat{p}_4, \{\widehat{x}_1, \widehat{x}_2, \widehat{x}_5\})\} \tag{4}$$

Many researchers investigated the basic properties, aggregation operations, and relations of s-set, but the contributions of authors [6, 11, 15] are more significant in this regard. Some properties of s-set are stated below for proper understanding.

Definition 2 (see [10]). If $\breve{\mathfrak{S}} = (\mathscr{Q}, \mathfrak{F}_{\breve{\mathfrak{S}}})$ and $\breve{\mathfrak{W}} = (\mathscr{R}, \mathfrak{F}_{\breve{\mathfrak{W}}}) \in \Omega_{ss}$, then

(1) $\breve{\mathfrak{S}} \,\widetilde{\subseteq}\, \breve{\mathfrak{W}}$ if $\mathfrak{F}_{\breve{\mathfrak{S}}}(\widehat{p}) \subseteq \mathfrak{F}_{\breve{\mathfrak{W}}}(\widehat{p}), \forall \widehat{p} \in \mathscr{Q}$ and $\mathscr{Q} \subseteq \mathscr{R}$.

(2) $\breve{\mathfrak{S}} \,\widetilde{\cup}\, \breve{\mathfrak{Q}}$ is stated as

$$\breve{\mathfrak{S}} \widetilde{\cup} \breve{\mathfrak{Q}} = \begin{cases} \mathfrak{F}_{\breve{\mathfrak{S}}}(\widehat{p}), & \widehat{p} \in \mathcal{Q} \setminus \mathcal{R}. \\ \mathfrak{F}_{\breve{\mathfrak{W}}}(\widehat{p}), & \widehat{p} \in \setminus \mathcal{Q}. \\ \mathfrak{F}_{\breve{\mathfrak{S}}}(\widehat{p}) \cup \mathfrak{F}_{\breve{\mathfrak{W}}}(\widehat{p}), & \widehat{p} \in \mathcal{Q} \cap \mathcal{R}. \end{cases} \tag{5}$$

(3) $\breve{\mathfrak{S}} \widetilde{\cap} \breve{\mathfrak{W}}$ is defined as

$$\mathfrak{F}_{\breve{\mathfrak{S}} \widetilde{\cap} \breve{\mathfrak{W}}}(\widehat{p}) = \mathfrak{F}_{\breve{\mathfrak{S}}}(\widehat{p}) \cap \mathfrak{F}_{\breve{\mathfrak{W}}}(\widehat{p}), \quad \forall \widehat{p} \in \mathcal{Q} \cap \mathcal{R}. \tag{6}$$

In many decision-making circumstances, it is observed that parameters need to be further partitioned into their respective parametric-valued sets. The s-set and its hybrids are not capable to tackle such kind of parametric classification. So, Smarandache [17] introduced a new model called hs-set which is capable to deal with such classification by employing a new approximate mapping, i.e., maa-function.

Definition 3 (see [17]). Let $\breve{X} = \{\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\}$ be an initial universe and $\mathcal{P} = \{\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_n\}$ be a set of parameters. The respective attribute-valued non-overlapping sets of each element of $\mathcal{P}$ are

$$\begin{aligned} \mathcal{Q}_1 &= \{\widehat{q}_{11}, \widehat{q}_{12}, \ldots, \widehat{q}_{1n}\} \\ \mathcal{Q}_2 &= \{\widehat{q}_{21}, \widehat{q}_{22}, \ldots, \widehat{q}_{2n}\} \\ \mathcal{Q}_3 &= \{\widehat{q}_{31}, \widehat{q}_{32}, \ldots, \widehat{q}_{3n}\} \\ &\cdots \\ &\cdots \\ &\cdots \\ \mathcal{Q}_n &= \{\widehat{q}_{n1}, \widehat{q}_{n2}, \ldots, \widehat{q}_{nn}\}, \end{aligned} \tag{7}$$

and $\mathcal{Q} = \mathcal{Q}_1 \times \mathcal{Q}_2 \times \mathcal{Q}_3 \times \cdots \times \mathcal{Q}_n = \{\widehat{q}_1, \widehat{q}_2, \widehat{q}_3, \ldots, \widehat{q}_r\}$ where every $\widehat{q}_i\,(i = 1, 2, \ldots, r)$ is a n-tuple element of $\mathcal{Q}$ with $r = \prod_{i=1}^{n} |\mathcal{Q}_i|$, $|\cdot|$ denotes the cardinality of $\mathcal{Q}_i$ then a maa-function is a mapping

$$\mathfrak{F}_{\breve{\mathfrak{H}}}: \mathcal{V} \longrightarrow \Gamma^{\breve{X}}, \tag{8}$$

and it is defined as

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\{\widehat{q}_1, \widehat{q}_2, \ldots, \widehat{q}_k\}) = \Gamma^{\{\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n\}}, \tag{9}$$

where $\Gamma^{\breve{X}}$ denotes the power set of $\breve{X}$, $\mathcal{V} \subseteq \mathcal{Q}$ with $k \le r$. The pair $(\mathcal{V}, \mathfrak{F}_{\breve{\mathfrak{H}}})$ is known as hs-set and represented by $\breve{\mathfrak{H}}$. The collection of all hs-sets is symbolized as $\mho_{\text{hss}}$.

See [18, 19] for more operational properties of hs-sets.

3. Notions of Convexity cum Concavity for maa-Function with Relevant Set-Theoretic Inequalities

In this section, we define Δ-sets and Ψ-sets and prove some results. Throughout the remaining paper, $\mathfrak{Q}$, $\text{II}^{\cdot}$, II°, and $\breve{X}$ will play the role of R^n, closed unit interval, open unit interval, and universal set, respectively. Let $\widehat{q}_1 = (\breve{a}_1, \breve{a}_2, \breve{a}_3, \ldots, \breve{a}_n)$, $\widehat{q}_2 = (\breve{b}_1, \breve{b}_2, \breve{b}_3, \ldots, \breve{b}_n) \in \mathfrak{Q}$, $\widehat{q}_1 \neq \widehat{q}_2$ where $\mathfrak{Q} = \mathfrak{Q}_1 \times \mathfrak{Q}_2 \times \mathfrak{Q}_3 \times \cdots \times \mathfrak{Q}_n$ with $\mathfrak{Q}_i \cap \mathfrak{Q}_j = \varnothing$, for $i \neq j$, and $i, j \in \{1, 2, 3, \ldots, n\}$; $\mathfrak{F}_{\breve{\mathfrak{H}}}: \mathfrak{Q} \longrightarrow \Gamma^{\breve{X}}$ and $\widehat{\theta} \in \text{II}^{\circ}$.

Considering Definition 2, Rahman et al. [35] extended the work of Deli [30] and introduced the following notions of convexity cum concavity with multi-argument approximate settings.

Definition 4 (see [35]). If $\breve{\delta} \widetilde{\subseteq} \breve{X}$, then the $\breve{\delta}$ – inclusion of a hs-set $\breve{\mathfrak{H}}$ is stated as

$$\breve{\mathfrak{H}}^{\breve{\delta}} = \left\{\kappa \in \mathfrak{Q}: \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa) \supseteq \breve{\delta}\right\}. \tag{10}$$

Definition 5 (see [35]). A hs-set $\breve{\mathfrak{H}}$ on $\mathfrak{Q}$ is known as convex hs-set (Δ-set) if

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{a}\kappa + (1 - \widehat{a})\tau) \supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\tau), \tag{11}$$

$\forall \kappa, \tau \in \mathfrak{Q}$ and $\widehat{a} \in \text{II}^{\cdot}$. Its collection is represented as $\mathbb{H}^{\Delta}$.

Definition 6 (see [35]). A hs-set $\breve{\mathfrak{H}}$ on $\mathfrak{Q}$ is known as concave hs-set (Ψ-set) if

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{a}\kappa + (1 - \widehat{a})\tau) \subseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa) \cup \mathfrak{F}_{\breve{\mathfrak{H}}}(\tau), \tag{12}$$

$\forall \kappa, \tau \in \mathfrak{Q}$ and $\widehat{a} \in \text{II}^{\cdot}$.

Salih and Sabir [34] discussed the properties of strongly and strictly convexity cum concavity by using approximate function of s-set, but in case of multi-argument approximate function, their presented work is not sufficient; therefore, the following generalized definitions are characterized with the entitlement of multi-argument approximate function $\mathfrak{F}_{\breve{\mathfrak{H}}}$ which considers $\mathfrak{Q}$ (a Cartesian product of attribute-valued disjoint sets) as its domain and maps it to power set of initial universe.

Definition 7. The hs-set $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ on $\mathfrak{Q}$ is called a strongly convex hs-set (Ω_{schss}) if

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_1 + (1 - \widehat{\theta})\widehat{q}_2\right) \supset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2), \tag{13}$$

for every $\widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}, \widehat{q}_1 \neq \widehat{q}_2$ and $\widehat{\theta} \in \text{II}^{\circ}$.

Definition 8. The hs-set $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ on $\mathfrak{Q}$ is called a strongly concave hs-set (Ω_{schss}) if

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_1 + (1 - \widehat{\theta})\widehat{q}_2\right) \subset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cup \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2), \tag{14}$$

for every $\widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}, \widehat{q}_1 \neq \widehat{q}_2$ and $\widehat{\theta} \in \text{II}^{\circ}$.

Definition 9. The hs-set $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ on $\mathfrak{Q}$ is called a strictly convex hs-set (Θ_{schss}) if

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_1 + (1 - \widehat{\theta})\widehat{q}_2\right) \supset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2), \tag{15}$$

for every $\widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}, \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \neq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$ and $\widehat{\theta} \in \text{II}^{\circ}$.

Definition 10. The hs-set $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ on $\mathfrak{Q}$ is called a strictly concave hs-set (Θ_{schss}) if

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2\right) \subset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cup \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2), \tag{16}$$

for every $\widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}$, $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \neq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$ and $\widehat{\theta} \in \text{II}^{\circ}$.

Note. (i) If we replace $\supseteq, \subseteq, \supset, \subset, \cap$, and $\cup$ by $\geq, \leq, >, <, \wedge$, and $\vee$, respectively, in above equations, i.e., from (11) to (16), we get the same notions for fuzzy sets as described by Zadeh [3] and Chaudhuri [36] where $\wedge$ and $\vee$ are min and max operators. (ii) Similarly, if we replace $\mathfrak{Q}$ (Cartesian product of attribute-valued sets) by $\mathscr{P}$ (a set of attributes) in $\mathfrak{F}_{\breve{\mathfrak{H}}}$, we get simple approximate function $\mathfrak{F}_{\breve{\mathfrak{S}}}$ of s-set as given in Definition 1 and consequently the above definitions will be treated for s-set as discussed in [30, 34].

Example 2. Suppose an educational institution wants to recruit some teachers with the help of some defined indicators. Consider there are 10 candidates who applied for the job and thus form a universe of discourse $\breve{X} = \{\breve{c}_1, \breve{c}_2, \breve{c}_3, \ldots, \breve{c}_{10}\}$. The candidates will be evaluated with the help of the following attributes:

$$\begin{aligned} \breve{a}_1 &= \text{total experience in years}, \\ \breve{a}_2 &= \text{total number of publications}, \\ \breve{a}_3 &= \text{ability to demonstrate}, \end{aligned} \tag{17}$$

and their respective attribute-valued sets are

$$\begin{aligned} \mathfrak{Q}_1 &= \{11 \text{ years}, 12 \text{ years}, 13 \text{ years}, 14 \text{ years}, 15 \text{ years}\}, \\ \mathfrak{Q}_2 &= \{16, 17, 18, 19, 20\}, \\ \mathfrak{Q}_3 &= \{\text{excellent}(5), \text{very good}(4), \text{good}(3), \text{average}(2), \text{bad}(1)\}. \end{aligned} \tag{18}$$

For simplicity, we write

$$\begin{aligned} \mathfrak{Q}_1 &= \{11, 12, 13, 14, 15\}, \\ \mathfrak{Q}_2 &= \{16, 17, 18, 19, 20\}, \\ \mathfrak{Q}_3 &= \{5, 4, 3, 2, 1\}. \end{aligned} \tag{19}$$

The hs-set $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ is a function defined by the mapping $\mathfrak{F}_{\breve{\mathfrak{H}}}: \mathfrak{Q} \longrightarrow \Gamma^{\breve{X}}$ where $\mathfrak{Q} = \mathfrak{Q}_1 \times \mathfrak{Q}_2 \times \mathfrak{Q}_3$.

Since the Cartesian product of $\mathfrak{Q}_1 \times \mathfrak{Q}_2 \times \mathfrak{Q}_3$ is a 3-tuple, we consider $\widehat{q}_1 = (12, 16, 3)$; then, the function becomes $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) = \mathfrak{F}_{\breve{\mathfrak{H}}}(12, 16, 3) = \{\breve{c}_1, \breve{c}_5\}$. Also, consider $\widehat{q}_2 = (13, 17, 4)$; then, the function becomes $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2) = \mathfrak{F}_{\breve{\mathfrak{H}}}(13, 17, 4) = \{\breve{c}_1, \breve{c}_3, \breve{c}_4\}$.

Now,

$$\begin{aligned} \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2) &= \mathfrak{F}_{\breve{\mathfrak{H}}}(\{12, 16, 3\}) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\{13, 17, 4\}) \\ &= \{\breve{c}_1, \breve{c}_5\} \cap \{\breve{c}_1, \breve{c}_3, \breve{c}_4\} = \{\breve{c}_1\}. \end{aligned} \tag{20}$$

Let $\widehat{\theta} = 0.6 \in \text{II}^{\circ}$; then, we have

$$\begin{aligned} \widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2 &= 0.6(12, 16, 3) + (1-0.6)(13, 17, 4) = 0.6(12, 16, 3) + 0.4(13, 17, 4) \\ &= (7.2, 9.6, 1.8) + (5.2, 6.8, 1.6) = (7.2 + 5.2, 9.6 + 6.8, 1.8 + 1.6) = (12.4, 16.4, 3.4), \end{aligned} \tag{21}$$

which is again a 3-tuple. By using the decimal round off property, we get $(12, 16, 3)$.

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2\right) = \mathfrak{F}_{\breve{\mathfrak{H}}}(12, 16, 3) = \{\breve{c}_1, \breve{c}_5\}, \tag{22}$$

and from equations (20) and (22), we have

$$\begin{aligned} &\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2\right) \supset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2) \\ &\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cup \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2) = \mathfrak{F}_{\breve{\mathfrak{H}}}(\{12, 16, 3\}) \cup \mathfrak{F}_{\breve{\mathfrak{H}}}(\{13, 17, 4\}) = \{\breve{c}_1, \breve{c}_5\} \cup \{\breve{c}_1, \breve{c}_3, \breve{c}_4\} = \{\breve{c}_1, \breve{c}_3, \breve{c}_4, \breve{c}_5\}, \end{aligned} \tag{23}$$

and from equations (22) and (23), we have

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2\right) \subset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cup \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2). \tag{24}$$

Definition 11. The convex hull $\breve{\widehat{H}}_{\Delta}$ of a hs-set $\breve{\mathfrak{H}} \in \mho_{\text{hss}}$ is the smallest Δ-set over $\breve{X}$ which contains $\breve{\mathfrak{H}}$, i.e.,

$$\widehat{H}_{\Delta} = \bigcap_{\breve{\mathfrak{K}}_i \supseteq \breve{\mathfrak{H}}} \left\{ \breve{\mathfrak{K}}_i : \breve{\mathfrak{K}}_i \in \mathbb{H}^{\Delta} \right. . \tag{25}$$

Definition 12. A hs-set $\breve{\mathfrak{H}} \in \mho_{\text{hss}}$ is called a cone if for all $\kappa \in \mathfrak{Q}$ and $\widehat{a} > 0$, we have

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{a}\kappa) \supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa). \tag{26}$$

If $\breve{\mathfrak{H}}$ is a Δ-set, then $\breve{\mathfrak{H}}$ is called convex cone hs-set (cchs-set). The collection of all cchs-sets is represented by Ω^{cchs}.

Definition 13. The convex cone $\widehat{\mathbb{C}}_{\Delta}$ of a hs-set $\breve{\mathfrak{H}} \in \mho_{\text{hss}}$ is the smallest cchs-set which contains $\breve{\mathfrak{H}}$, i.e.,

$$\widehat{\mathbb{C}}_{\Delta} = \bigcap_{\breve{\mathfrak{L}}_i \supseteq \breve{\mathfrak{H}}} \left\{ \breve{\mathfrak{L}}_i : \breve{\mathfrak{L}}_i \in \Omega^{\text{cchs}} \right. . \tag{27}$$

3.1. Set-Theoretic Inequalities for maa-Function. In this section, some set-theoretic inequalities are presented with discussion on their generalized results.

Theorem 1. *A hs-set* $\breve{\mathfrak{H}}$ *is a* Δ*-set if and only if for all* $\kappa_1, \kappa_2, \ldots, \kappa_n \in \mathfrak{Q}$ *and* $\widehat{a}_1, \widehat{a}_2, \ldots, \widehat{a}_n \in II^{\cdot}$ *such that* $\sum_{i=1}^{n} \widehat{a}_i = 1$*, we have*

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\sum_{i=1}^{n} \widehat{a}_i \kappa_i\right) \supseteq \bigcap_{i=1}^{n} \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa_i). \tag{28}$$

Proof. Let the sufficient condition hold. In particular, for any two elements $\kappa_1, \kappa_2 \in \mathfrak{Q}$ and $\widehat{a} \in II^{\cdot}$, then we have $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{a}\kappa_1 + (1-\widehat{a})\kappa_2) \supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa_2)$ which implies that $\breve{\mathfrak{H}}$ is a Δ-set according to Definition 5. Conversely, let $\breve{\mathfrak{H}}$ be a Δ-set; then, for any $n \in \mathbb{N}$, define a set $\mathbb{B}(\kappa, n)$ such that

$$\mathbb{B}(\kappa, n) = \left\{\{\kappa_1, \kappa_2, \ldots, \kappa_n\} \subset \mathfrak{Q} \colon \exists \widehat{a}_i \in II^{\bullet}; \sum_{i=1}^{n} \widehat{a}_i = 1, \kappa = \sum_{i=1}^{n} \widehat{a}_i \kappa_i\right\}. \tag{29}$$

By using induction on $n \in \mathbb{N}$, we prove the necessary condition as follows:

C-1: let $n = 2$; then, $\exists\{\kappa_1, \kappa_2\} \in \mathbb{B}(\kappa, 2)$ and $\widehat{a} \in II^{\cdot}$ such that $\kappa = \widehat{a}\kappa_1 + (1-\widehat{a})\kappa_2$. As $\breve{\mathfrak{H}}$ is a Δ-set, $\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa) \supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa_2) = \cap_{i=1}^{2} \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa_i)$.

C-2: let the result (29) be valid for $n = k$ which implies that for $\{\kappa_1, \kappa_2, \ldots, \kappa_k\} \in \mathbb{B}(\kappa, k)$ and $\kappa = \sum_{i=1}^{k} \widehat{a}_i \kappa_i$ with $\sum_{i=1}^{k} \widehat{a}_i = 1$, we have

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa) \supseteq \bigcap_{i=1}^{k} \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa_i). \tag{30}$$

Now prove the result (29) for $n = k+1$ with conditions $\{\kappa_1, \kappa_2, \ldots, \kappa_{k+1}\} \in \mathbb{B}(\kappa, k+1)$ and $\kappa = \sum_{i=1}^{k+1} \widehat{a}_i \kappa_i$ with $\sum_{i=1}^{k+1} \widehat{a}_i = 1$. Suppose there exists one $\widehat{a} \in II^{\cdot}$ (say $\widehat{a}_1$) such that $\widehat{a}_1 \neq 1$. Let $\tau = \ddot{a}_2\kappa_2 + \ddot{a}_3\kappa_3 + \cdots + \ddot{a}_{k+1}\kappa_{k+1}$ where $\ddot{a} = (\widehat{a}_i/(1-\widehat{a}_1)) \geq 0$ for i_2^{k+1}. Therefore, $\ddot{a}_2 + \ddot{a}_3 + \cdots + \ddot{a}_{k+1} = ((\widehat{a}_2 + \widehat{a}_3 + \cdots + \widehat{a}_{k+1})/1 - \widehat{a}_1) = ((1-\widehat{a}_1)/(1-\widehat{a}_1)) = 1$.

Hence, $\{\kappa_2, \ldots, \kappa_{k+1}\} \in \mathbb{B}(\tau, k)$, and equation (30) implies that

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa) \supseteq \bigcap_{i=2}^{k+1} \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa_i). \tag{31}$$

Now, we apply C-1 for $\tau, \kappa_1 \in \mathfrak{Q}$, i.e.,

$$\begin{aligned} &\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{a}\kappa_1 + (1-\widehat{a})\tau) \supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\tau), \\ &\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{a}\kappa_1 + (1-\widehat{a})\tau) \supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\sum_{i=2}^{k+1} \frac{\widehat{a}_i}{1-\widehat{a}_1}\kappa_i\right), \end{aligned} \tag{32}$$

which implies

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa) \supseteq \bigcap_{i=1}^{k+1} \mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa_i). \tag{33}$$

□

Remark 1

1. The result presented in Theorem 1 may or may not be valid for $n \longrightarrow \infty$.
(2) If, in case, the result presented in Theorem 1 is valid for $n \longrightarrow \infty$, then this result will be considered as generalized version of countably sub-additive sets under hs-set environment with entitlement of maa-function.
(3) If only equality holds for countable case, then this result will be considered as generalized version of countably additive sets under hs-set environment with entitlement of maa-function.

Theorem 2. *Let* $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ *be* Θ_{schss}*. If there exists* $\widehat{\theta} \in II^{\cdot}, \forall \widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}$ *such that* $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2) \supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$*, then* $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ *is a* Δ*-set.*

Proof. Assume that $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \subseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$ and $(\exists \widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}), \widehat{\theta}_1 \in II^{\cdot}$ such that

$$\breve{X} \backslash \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \supseteq \breve{X} \backslash \left\{\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)\right\}. \tag{34}$$

If $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \subset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$, then (34) contradicting $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ is a Θ_{schss}.

If $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) = \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$ and $\widehat{\theta}_1 \in [0, \widehat{\theta}]$, let $\widehat{q}_3 = (\widehat{\theta}_1/\widehat{\theta})(\widehat{q}_1) + (1-(\widehat{\theta}_1/\widehat{\theta}))\widehat{q}_2$ and $\widehat{\theta}_2 = ((1/\widehat{\theta}) - 1)((1/\widehat{\theta}_1) - 1)^{-1}$. Thus, by hypothesis,

$$\begin{aligned} \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) &= \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\left(\frac{\widehat{\theta}_1}{\widehat{\theta}}(\widehat{q}_1) + \left(1-\frac{\widehat{\theta}_1}{\widehat{\theta}}\right)\widehat{q}_2\right) + (1-\widehat{\theta})\widehat{q}_2\right) \\ &= \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_3 + (1-\widehat{\theta})\widehat{q}_2\right) \\ \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) &\supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_3). \end{aligned} \tag{35}$$

Now

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_3) = \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\frac{\widehat{\theta}_1}{\widehat{\theta}}(\widehat{q}_1) + \left(1-\frac{\widehat{\theta}_1}{\widehat{\theta}}\right)\widehat{q}_2\right) = \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_3\widehat{q}_1 + \left(1-\widehat{\theta}_3\right)\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right)\right). \tag{36}$$

From (14), (15), and $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) = \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$, it follows that

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_3). \tag{37}$$

From (34), (36), $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) = \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$, and Θ_{schss} condition, it follows that

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_3) \supset \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right), \tag{38}$$

or

$$\breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \supseteq \breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_3). \tag{39}$$

Hence, (37) and (39) give a contradiction.

If $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) = \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$ and $\widehat{\theta}_1 \in [\widehat{\theta}, 1]$, let $\underline{\rho} = ((\widehat{\theta}_1 - \widehat{\theta})/1 - \widehat{\theta})\widehat{q}_1 + ((1-\widehat{\theta}_1)/1-\widehat{\theta})\widehat{q}_2$. Thus, by hypothesis,

$$\begin{aligned} &\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) = \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\underline{\rho}\right), \\ &\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\underline{\rho}\right). \end{aligned} \tag{40}$$

From (34), (40), and $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) = \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$, it follows that

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\underline{\rho}\right). \tag{41}$$

On the other hand, $\widehat{\theta}_1\widehat{q}_1 + (1-\widehat{\theta}_1)\widehat{q}_2 = \widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\underline{\rho}$ gives

$$\begin{aligned} \underline{\rho} &= \left(\frac{1}{1-\widehat{\theta}}\right)\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) - \left(\frac{\widehat{\theta}}{\widehat{\theta}-1}\right)\widehat{q}_1, \\ \underline{\rho} &= \left(\frac{1}{1-\widehat{\theta}}\right)\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) - \left(\frac{\widehat{\theta}}{\widehat{\theta}-1}\right)\left(\frac{1}{\widehat{\theta}_1}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) - \frac{1-\widehat{\theta}_1}{\widehat{\theta}_1}\widehat{q}_2\right), \\ \underline{\rho} &= \left(\frac{\widehat{\theta}_1 - \widehat{\theta}}{\left(1-\widehat{\theta}_1\right)\widehat{\theta}_1}\right)\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) + \left(1 - \frac{\widehat{\theta}_1 - \widehat{\theta}}{\left(1-\widehat{\theta}_1\right)\widehat{\theta}_1}\right)\widehat{q}_2. \end{aligned} \tag{42}$$

From (34), (42), $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) = \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$, and Θ_{schss} condition, it follows that

$$\breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \supseteq \breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\underline{\rho}\right). \tag{43}$$

Hence, (41) and (43) give a contradiction. □

Theorem 3. *Let $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ be a Θ_{scohss}. If there exists $\widehat{\theta} \in II^{\cdot}, \forall \widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}$ such that $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2) \subseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cup \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$, then $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ is a Ψ-set.*

Proof. This can easily be proved by following the procedure discussed in Theorem 2. □

Theorem 4. *Let $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ be a Δ-set. If there exists $\widehat{\theta} \in II^{\cdot}, \forall \widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}, \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \neq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$ such that $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2) \supset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$, then $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ is a Θ_{scohss}.*

Proof. Assume that $(\exists \widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}), \widehat{\theta}_1 \in II^{\cdot}$ such that

$$\breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \supset \breve{X}\backslash\left\{\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)\right\}. \tag{44}$$

If $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \supset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$, then (44) gives

$$\breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \supset \breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2). \tag{45}$$

On the other hand, from the Δ-set condition, we have that

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \supseteq \left\{\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)\right\}. \tag{46}$$

From (44) and (46), it follows that

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) = \left\{\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)\right\}, \tag{47}$$

which, together with $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \supset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$, yields

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) = \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2), \tag{48}$$

or

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \subset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1). \tag{49}$$

Thus, from (49) and the hypothesis,

$$\begin{aligned} &\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right)\right) \\ &\quad \supset \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right). \end{aligned} \tag{50}$$

More generally, for $n \in \{1, 2, 3, \ldots\}$, it can easily be shown that

$$\begin{aligned} &\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}^n\widehat{q}_1 + \left(1-\widehat{\theta}^n\right)\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right)\right) \\ &\quad \supset \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right). \end{aligned} \tag{51}$$

Let $\widehat{q}_3 = \widehat{\theta}_2\widehat{q}_1 + (1-\widehat{\theta}_2)\widehat{q}_2$ where $\widehat{\theta}_2 = \widehat{\theta}_1 - \widehat{\theta}^n\widehat{\theta}_1 + \widehat{\theta}^n \in II^{\cdot}$ for some n. Then, from (51), we see that

$$\begin{aligned} &\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_3) = \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_2\widehat{q}_1 + \left(1-\widehat{\theta}_2\right)\widehat{q}_2\right), \\ &\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_3) = \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}^n\widehat{q}_1 + \left(1-\widehat{\theta}^n\right)\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right)\right), \\ &\mathfrak{F}_{\breve{\mathfrak{H}}}(\nu) \supset \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right). \end{aligned} \tag{52}$$

Also, let $\underline{\rho} = \widehat{\theta}_3\widehat{q}_1 + (1-\widehat{\theta}_3)\widehat{q}_2$ where $\widehat{\theta}_3 = \widehat{\theta}_1 - \widehat{\theta}^n + (1/(1-\widehat{\theta})) + (\widehat{\theta}^n\widehat{\theta}_1/(1-\widehat{\theta})) \in \mathrm{II}^{\cdot}$ for some n.

Then,

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) = \mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_3 + (1-\widehat{\theta})\underline{\rho}\right). \quad (53)$$

Now, if $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_3) \subseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\underline{\rho})$, then (53) and $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$, which is a Δ-set, imply that

$$\breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_3) \supset \breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right), \quad (54)$$

which contradicts (52).

If $\breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_3) \subseteq \breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}(\underline{\rho})$, then (53) and the hypothesis of the theorem imply that

$$\begin{aligned}&\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \supset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_3) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\underline{\rho}),\\ &\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \supseteq \left(\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)\right) \cap \left(\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)\right) = \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1).\end{aligned} \quad (55)$$

This contradicts (49). □

Theorem 5. *Let $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ be a Ψ-set. If there exists $\widehat{\theta} \in II^{\cdot}, \forall \widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}, \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \neq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$ such that $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2) \subset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cup \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$, then $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ is a Θ_{scohss}.*

Proof. This can easily be proved by following the procedure discussed in Theorem 4. □

Theorem 6. *Let $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ be a Ω_{schss}. If there exists $\widehat{\theta} \in II^{\cdot}, \forall \widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}$, such that*

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2\right) \supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2), \quad (56)$$

then $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ is a Δ-set.

Proof. Assume that $(\exists \widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}), \widehat{\theta}_1 \in \mathrm{II}^{\cdot}$ such that

$$\breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \supseteq \breve{X}\backslash\left\{\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)\right\}. \quad (57)$$

If $\widehat{q}_1 \neq \widehat{q}_2$, then (56) contradicts that $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ is a Ω_{schss}.

If $\widehat{q}_1 = \widehat{q}_2$, then choose $\widehat{\theta}_1 \neq \widehat{\theta}_2 \in \mathrm{II}^{\circ}$ such that $\widehat{\theta}_1 = \widehat{\theta}\widehat{\theta}_2 + (1-\widehat{\theta})\widehat{\theta}_2$.

Let $\widehat{q}_1 = \widehat{q}_2 = \widehat{\theta}_2\widehat{q}_1 + (1-\widehat{\theta}_2)\widehat{q}_2$. Then, (56) implies that

$$\breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \supseteq \breve{X}\backslash\left\{\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)\right\}, \quad (58)$$

$$\breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2) \supseteq \breve{X}\backslash\left\{\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)\right\}. \quad (59)$$

According to (56), (58), and (59), we have

$$\begin{aligned}\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2\right) &\supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)\\ &\subset \left(\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)\right) \cap \left(\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)\right)\\ &= \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2),\end{aligned} \quad (60)$$

which contradicts that $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ is a Ω_{schss}. □

Theorem 7. *Let $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ be a Ω_{schss}. If there exists $\widehat{\theta} \in II^{\cdot}, \forall \widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}$, such that $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2) \subseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cup \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$, then $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ is a Ψ-set.*

Proof. This can easily be proved by following the procedure discussed in Theorem 6. □

Theorem 8. *Let $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ be a Δ-set. If there exists $\widehat{\theta} \in II^{\cdot}, \forall \widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}\widehat{q}_1 \neq \widehat{q}_2$, such that $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2) \supset \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$, then $(\mathfrak{F}_{\breve{\mathfrak{H}}}, \mathfrak{Q})$ is a Ω_{schss}.*

Proof. Assume that $\exists(\widehat{q}_1 \neq \widehat{q}_2)], \widehat{q}_1, \widehat{q}_2 \in \mathfrak{Q}), \widehat{\theta}_1 \in \mathrm{II}^{\cdot}$ such that

$$\breve{X}\backslash\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) \supseteq \breve{X}\backslash\left\{\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)\right\}. \quad (61)$$

Thus, from (61) and the Δ-set condition, we get that

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2\right) = \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2). \quad (62)$$

Furthermore, it can be easily seen that

$$\widehat{\theta}\widehat{q}_1 + (1-\widehat{\theta})\widehat{q}_2 = \widehat{\theta}_1\widehat{q}_1 + \left(1-\widehat{\theta}_1\right)\widehat{q}_2, \quad (63)$$

where both $\widehat{q}_1$ and $\widehat{q}_2$ are of the forms $\widehat{q}_1 = \widehat{\theta}_1\widehat{q}_1 + (1-\widehat{\theta}_1)\widehat{q}_2$ and $\widehat{q}_2 = \widehat{\theta}_1\widehat{q}_1 + (1-\widehat{\theta}_1)\widehat{q}_2$ for choosing $\widehat{\theta}_1 \in \mathrm{II}^{\circ}$.

On the other hand, from the Δ-set condition and our definition of $\widehat{q}_1$ and $\widehat{q}_2$, we get

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2), \quad (64)$$

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2) \supseteq \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1) \cap \mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2). \quad (65)$$

Therefore, from (63)–(65) and the hypothesis of the theorem, we get that

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}_1\widehat{q}_1+\left(1-\widehat{\theta}_1\right)\widehat{q}_2\right)=\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{\theta}\widehat{q}_1+\left(1-\widehat{\theta}_1\right)\widehat{q}_2\right)\supset\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{q}_1\right)\cap\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{q}_2\right)$$
$$\supseteq\left(\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{q}_1\right)\cap\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{q}_2\right)\right)\cap\left(\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{q}_1\right)\cap\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{q}_2\right)\right)=\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{q}_1\right)\cap\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\widehat{q}_2\right). \tag{66}$$

This contradicts (62). □

Theorem 9. *Let* $(\mathfrak{F}_{\breve{\mathfrak{H}}},\mathfrak{Q})$ *be a* Ψ*-set. If* $\exists\widehat{\theta}\in II^{\cdot},\forall\widehat{q}_1,\widehat{q}_2\in\mathfrak{Q},\widehat{q}_1\neq\widehat{q}_2$, *such that* $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{\theta}\widehat{q}_1+(1-\widehat{\theta})\widehat{q}_2)\subset\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1)\cup\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$, *then* $(\mathfrak{F}_{\breve{\mathfrak{H}}},\mathfrak{Q})$ *is a* Ω_{scohss}.

Proof. This can easily be proved by following the procedure discussed in Theorem 8. □

Theorem 10. *A hs-set* $\breve{\mathfrak{H}}$ *is a convex cone iff for* $\kappa,\tau\in\mathfrak{Q}$, $\widehat{a}>0$,

(i) $\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{a}\kappa)\supseteq\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa)$.
(ii) $\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa+\tau)\supseteq\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa)\cap\mathfrak{F}_{\breve{\mathfrak{H}}}(\tau)$.

Proof. Let $\breve{\mathfrak{H}}$ be a cchs-set over $\breve{X}$; then, part (i) is obviously verified. Consider $\kappa,\tau\in\mathfrak{Q}\&\widehat{a}>0$. As $\breve{\mathfrak{H}}$ is a cchs-set, by using Definition 5 with $\widehat{a}=(1/2)$, we have

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\frac{1}{2}\kappa+\frac{1}{2}\tau\right)\supseteq\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa)\cap\mathfrak{F}_{\breve{\mathfrak{H}}}(\tau), \tag{67}$$

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(2\left(\frac{1}{2}\kappa+\frac{1}{2}\tau\right)\right)\supseteq\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\frac{1}{2}\kappa+\frac{1}{2}\tau\right), \tag{68}$$

and combining equations (67) and (68), we get

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa+\tau)=\mathfrak{F}_{\breve{\mathfrak{H}}}\left(2\left(\frac{1}{2}\kappa+\frac{1}{2}\tau\right)\right)\supseteq\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\frac{1}{2}\kappa+\frac{1}{2}\tau\right)$$
$$\supseteq\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa)\cap\mathfrak{F}_{\breve{\mathfrak{H}}}(\tau), \tag{69}$$

which implies

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa+\tau)\supseteq\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa)\cap\mathfrak{F}_{\breve{\mathfrak{H}}}(\tau). \tag{70}$$

Conversely, let the given conditions (i) and (ii) be satisfied. According to condition (i), $\breve{\mathfrak{H}}$ is a cone. Now using both conditions for $\kappa,\tau\in\mathfrak{Q}\&\widehat{a}>0$, we have

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{a}\kappa+(1-\widehat{a})\tau)\supseteq\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{a}\kappa)\cap\mathfrak{F}_{\breve{\mathfrak{H}}}((1-\widehat{a})\tau)$$
$$\supseteq\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa)\cap\mathfrak{F}_{\breve{\mathfrak{H}}}(\tau), \tag{71}$$

which shows that $\breve{\mathfrak{H}}$ is a Δ-set. Hence, $\breve{\mathfrak{H}}$ is cchs-set. □

Theorem 11. *A hs-set* $\breve{\mathfrak{H}}$ *is a convex cone iff for* $\kappa_1,\kappa_2,\ldots,\kappa_n\in\mathfrak{Q}$, $\widehat{a}>0$,

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\sum_{i=1}^{n}\widehat{a}_i\kappa_i\right)\supseteq\bigcap_{i=1}^{n}\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa_i). \tag{72}$$

Proof. By induction on n, $\breve{\mathfrak{H}}$ is a cone when we take $n=1$, i.e.,

$$\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{a}_1\kappa_1)\supseteq\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa_1). \tag{73}$$

The remaining part can be easily proved with the help of Theorem 1. □

3.2. Jensen-Convex (Mid-Convex) Set and Its Inverse for maa-Function. In this part of the paper, we present classical versions of Jensen-convex sets and their inverse for multi-argument approximate functions. It can be seen that the nature of Jensen-convex set and its inverse is interchanged for maa-function in set-theoretic approach.

Definition 14. A Δ-set is said to be J-convex hs-set (Δ_J-set) over $\mathfrak{Q}$ if

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\frac{\kappa+\tau}{2}\right)\supseteq\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa)\cap\mathfrak{F}_{\breve{\mathfrak{H}}}(\tau), \tag{74}$$

$\forall\kappa,\tau\in\mathfrak{Q}$.

Definition 15. A Ψ-set is said to be J-concave hs-set (Ψ_J-set) over $\mathfrak{Q}$ if

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\frac{\kappa+\tau}{2}\right)\subseteq\mathfrak{F}_{\breve{\mathfrak{H}}}(\kappa)\cup\mathfrak{F}_{\breve{\mathfrak{H}}}(\tau), \tag{75}$$

$\forall\kappa,\tau\in\mathfrak{Q}$.

Remark 2. If $\subseteq$ is replaced with $\leq$ and $\cup$ by $+$ in equation (75), we get classical version of Jensen inequality.

Definition 16. A Ω_{schss} on $\mathfrak{Q}$ is called a strongly J-convex hs-set (Ω^J_{schss}) if

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\frac{\widehat{q}_1+\widehat{q}_2}{2}\right)\supset\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1)\cap\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2), \tag{76}$$

for every $\widehat{q}_1,\widehat{q}_2\in\mathfrak{Q},\widehat{q}_1\neq\widehat{q}_2$.

Definition 17. A Ω_{schss} on $\mathfrak{Q}$ is called a strongly J-concave hs-set (Ω^J_{schss}) if

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\frac{\widehat{q}_1+\widehat{q}_2}{2}\right)\subset\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1)\cup\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2), \tag{77}$$

for every $\widehat{q}_1,\widehat{q}_2\in\mathfrak{Q},\widehat{q}_1\neq\widehat{q}_2$.

Remark 3. If $\subset$ is replaced with $<$ and $\cup$ by $+$ in equation (77), we obtain classical version of Jensen inequality for strong convexity.

Definition 18. A Θ_{schss} on $\mathfrak{Q}$ is called a strictly J-convex hs-set (Θ^J_{schss}) if

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\frac{\widehat{q}_1+\widehat{q}_2}{2}\right)\supset\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1)\cap\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2), \tag{78}$$

for every $\widehat{q}_1,\widehat{q}_2\in\mathfrak{Q},\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1)\neq\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$.

Table 1: Comparison of the proposed approach with existing relevant approaches.

References	EDOA	CMAAF	CSAF	DCIPC	DCH	DCC	SCSCx	SCSCc
Deli [30]	×	×	✓	×	×	×	✓	✓
Ihsan et al. [31]	×	×	✓	×	×	×	×	×
Ihsan et al. [32]	×	×	✓	×	×	×	×	×
Majeed [37]	×	×	✓	×	✓	✓	×	×
Rahman et al. [35]	✓	✓	✓	×	×	×	×	×
Salih and Sabir [34]	×	×	✓	×	×	×	✓	✓
Proposed study	✓	✓	✓	✓	✓	✓	✓	✓

Definition 19. A Θ_{schss} on $\mathfrak{Q}$ is called a strictly J-concave hs-set ($\Theta^{J}_{\text{schss}}$) if

$$\mathfrak{F}_{\breve{\mathfrak{H}}}\left(\frac{\widehat{q}_1+\widehat{q}_2}{2}\right)\subset\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1)\cup\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2), \tag{79}$$

for every $\widehat{q}_1,\widehat{q}_2\in\mathfrak{Q},\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_1)\neq\mathfrak{F}_{\breve{\mathfrak{H}}}(\widehat{q}_2)$.

Remark 4. If $\subset$ is replaced with $<$ and $\cup$ by $+$ in equation (79), we obtain classical version of Jensen inequality for strict convexity.

3.3. Comparative Analysis. In this section, the proposed study is compared professionally with some existing relevant works in order to judge its advantageous aspects. This comparison is based on some important evaluating features like emphasis on deep observation of attributes (EDOA), consideration of maa-function (CMAAF), sa-function (CSAF), discussion on classical inequalities as particular cases (DCIPC), discussion on convex hull (DCH), discussion on convex cone (DCC), strictly cum strongly convexity (SCSCx), and strictly cum strongly concavity (SCSCc) which depict the advantages of the proposed study over the existing approaches which are presented in Table 1.

4. Conclusion

In this paper, the classical notions of convex (concave) sets, strictly convexity (concavity), strongly convexity (concavity), convex hull, convex cone, and set inclusion, are generalized for hypersoft set environment. Some set-theoretic inequalities based on these notions are established and discussed by using certain aspects of set inclusion. Although the proposed study provides a conceptual framework for handling convex optimization problems under uncertain scenario with deep observation of parameters, it has limitations for hypersoft set environment with fuzzy, intuitionistic fuzzy, and neutrosophic settings in the domain as well as in the range of maa-function. Therefore, future work may include the following:

(1) The establishment of modular inequalities by considering the interval nature of uncertain data for maa-function.

(2) The establishment of classical inequalities such as Ostrowski-type inequalities, Hadamard-type inequalities, Minkowski's inequalities, and others under hs-set environment.

(3) The extension of the proposed study for hybridized structures of hs-set like fhs-set, ifhs-set, nhs-set, and so on.

(4) The extension of the proposed study for m-convex and m-concave sets under hs-set environment.

(5) The implementation of the proposed framework for solving convex optimization problems and decision-making problems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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